The absence of remainders in the Wiener-Ikehara theorem

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The Wiener-Ikehara theorem is a landmark in 20th century analysis. It states,

Theorem (Wiener-Ikehara)

Let S be a non-decreasing function and suppose that

$$G(s) := \int_1^\infty S(x) x^{-s-1} \mathrm{d}x \text{ converges for } \Re e\, s > 1$$

and that there exists A such that $G(s) - A/(s-1)$ admits a continuous extension to $\Re e\, s \geq 1$, then

$$S(x) = Ax + o(x). \tag{1}$$

We discuss here whether it is possible to improve the remainder in (1) under an analytic continuation hypothesis.

We will give a negative answer to a conjecture of M. Müger.

The talk is based on collaborative work with Gregory Debruyne

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Remainders in the Wiener-Ikehara theorem

S non-decreasing with Mellin transform $G(s)$ on $\Re e\, s > 1$.

If one wishes to attain a stronger remainder than $o(x)$, i.e.,

$$S(x) = Ax + O(x\rho(x)) \qquad \text{with} \qquad \rho(x) = o(1),$$

it is natural to strengthen the regularity assumptions on

$$G(s) - \frac{A}{s-1}. \tag{2}$$

Our goal: to study the following hypothesis:

(2) has analytic continuation to $\Re e\, s > \alpha$, where $0 < \alpha < 1$.

Well-known: remainders can be obtained if bounds on (2) hold.

Theorem (Simplest example)

If $G(s) - \frac{A}{s-1} \ll (1 + |\Im m\, s|)^{N-1}$ *on the strip* $\alpha < \Re e\, s < 2$,

$$S(x) = Ax + O(x^{\frac{N+1+\alpha}{N+2}}).$$

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A conjecture under merely analytic continuation

M. Müger raised the question of whether it is still possible to obtain error terms without the bounds on the analytic continuation of $G(s) - a/(s-1)$. He actually conjectured one could get the error term

Conjecture (Müger, 2017)

Let $0 < \alpha < 1$ and $a > 0$. If $G(s) - \dfrac{a}{s-1}$ has analytic continuation to $\Re e\, s > \alpha$, then

$$S(x) = ax + O_{\varepsilon}(x^{\frac{\alpha+2}{3}+\varepsilon}), \quad \forall \varepsilon > 0.$$

We show in this talk that the latter conjecture is false; in fact, we report the following more general result:

Negative general answer

No reasonably good remainder can be expected in the Wiener-Ikehara theorem, with solely the analyticity on $\Re e\, s > \alpha$.

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Absence of remainders in Wiener-Ikehara theorem

Theorem (Debruyne and V., 2018)

Let ρ be a positive function, $A > 0$, and $0 < \alpha < 1$. Suppose that every non-decreasing function S on $[1, \infty)$, whose Mellin transform $G(s)$ is such that

$$G(s) - \frac{A}{s-1}$$

admits an analytic extension to $\Re e\, s > \alpha$, satisfies

$$S(x) = Ax + O(x\rho(x)).$$

Then, one must necessarily have

$$\rho(x) = \Omega(1).$$

(the latter means $\rho(x) \not\to 0$.)

The rest of the talk is devoted to outline the proof of this result.

First reduction

We will use functional analysis and need to make a vector space out of our problem.

- As the "Tauberian theorem hypothesis" holds for some $A > 0$, it holds $\forall A > 0$.

Set $T(x) = S(x) - Ax$.

- The Mellin transform $G_T(s) := \int_1^\infty x^{-1-s} T(x) \mathrm{d}x$ has analytic continuation to $\Re e\, s > \alpha$.
- If T is absolutely continuous, $T'(x)$ is bounded from below.
- The asymptotic formula for S becomes $T(x) \ll x\rho(x)$.

We shall use less to show our original result, i.e., it is contained in:

Theorem

If $T(x) = O(x\rho(x))$ for any T with $T' \in L^\infty(1, \infty)$ such that $G_T(s)$ has analytic continuation to $\Re e\, s > \alpha$, then

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The proof: open mapping theorem argument

If $T(x) = O(x\rho(x))$ for any T with $T' \in L^{\infty}(1,\infty)$ such that $G_T(s)$ has analytic continuation to $\Re e\, s > \alpha$, then $\rho(x) = \Omega(1)$.

- Let Y be the Fréchet space of Lipschitz continuous functions T on $[1,\infty)$ such that $G_T(s)$ can be analytically continued to $\Re e\, s > \alpha$ and continuously extended to $\Re e\, s \geq \alpha$.
- The natural topology of Y is given by the seminorms

$$\|T\|_{Y,n} = \operatorname{ess\,sup}_{x\geq 1} |T'(x)| + \sup_{\Re e\, s\geq\alpha,\ |\Im m\, s|\leq n} |G_T(s)|, \qquad n = 1,2,\dots$$

- The second Fréchet space $Z \subseteq Y$ is defined via the norms

$$\|T\|_{Z,n} = \sup_{x\geq 1} |T(x)/(x\rho(x))| + \|T\|_{Y,n}, \qquad n = 1,2,\dots$$

- The inclusion $Z \to Y$ is continuous and our hypothesis is $Z = Y$.

The open mapping theorem implies there are $N, C > 0$ such that

$$\sup_{x\geq 1} \left| \frac{T(x)}{x\rho(x)} \right| \leq C \|T\|_{Y,N}$$

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The proof: using the inequality

The key inequality

$$\sup_{x\geq 1}\left|\frac{T(x)}{x\rho(x)}\right| \leq C\left(\operatorname{ess\,sup}_{x\geq 1}|T'(x)| + \sup_{\Re e\, s\geq\alpha,\ |\Im m\, s|\leq N}|G_T(s)|\right)$$

extends to the completion of Y with respect to $\|\cdot\|_{Y,N}$.

Any T for which $T'(x)=o(1)$, $T(1)=0$, and whose Mellin transform has analytic continuation in a neighborhood of $\{s:\Re e\, s\geq\alpha, |\Im m\, s|\leq N\}$ is in that completion.

What remains to be done?

- We further proceed by contradiction and assume that $\rho(x)\to 0$.
- We construct a T with these properties such that when inserted in the key inequality contradicts $\rho(x)\to 0$.

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- We construct a T with these properties such that when inserted in the key inequality contradicts $\rho(x)\to 0$.

The proof: a construction based on a majorant lemma

- Since $\rho(x) \to 0$, we can choose a positive non-increasing function $\ell(x) \to 0$ such that $\ell(\log x)/\rho(x) \to \infty$.

Lemma

Let ℓ be a positive non-increasing function such that $\ell(x) = o(1)$. Then, there is a positive function L such that

$$\ell(x) \ll L(x) = o(1)$$

and an angle $\pi/2 < \theta < \pi$ such that $\mathcal{L}\{L; s\} = \int_0^\infty L(x)e^{-sx}\mathrm{d}x$ has analytic continuation to the sector $-\theta < \arg s < \theta$

We choose L as in this lemma. If we manage to show

$$L(\log x) \ll \rho(x),$$

this contradicts $\ell(\log x)/\rho(x) \to \infty$ and hence one must have $\rho(x) \nrightarrow 0$.

The proof: final step

We now consider

$$T_b(x) := \int_1^x L(\log u)\cos(b\log u)\mathrm{d}u.$$

Its Mellin transform

$$G_{T_b}(s) = \frac{1}{2s}\left(\mathcal{L}\{L; s-1+ib\} + \mathcal{L}\{L; s-1-ib\}\right).$$

is analytic in $\{s : \Re e\, s \geq \alpha, |\Im m\, s| \leq N\}$ for sufficiently large b. We have the right to apply the key inequality to T_b

$$\sup_{x\geq 1}\left|\frac{T_b(x)}{x\rho(x)}\right| \leq C\left(\operatorname*{ess\,sup}_{x\geq 1}|T_b'(x)| + \sup_{\Re e\, s\geq\alpha,\ |\Im m\, s|\leq N}\left|G_{T_b}(s)\right|\right)$$

Further manipulations of this inequality and studying some asymptotics for T_b lead to

$$L(\log x) \ll \rho(x).$$

Some references

This talk is based on our article:

- G. Debruyne, J. V., Note on the absence of remainders in the Wiener-Ikehara theorem, Proc. Amer. Math. Soc. 146 (2018), 5097–5103.

For sharp versions of the W-I theorem without remainder, see

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Remainders might be deduced from shape of region of analytic continuation + bounds, see e.g. the recent works:

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