Topological properties of convolutor spaces

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We will discuss topological properties of a general class of convolutor spaces.

Our results are quantified versions of classical results, mainly motivated by:

1. Schwartz' convolution description of $\mathcal{D}'_{L^1}$.
2. Grothendieck's results on the completeness of $\mathcal{O}_C$ and the (ultra-)bornologicity of $\mathcal{O}'_C$.

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- The space $\mathcal{B}$ consists of all $\varphi \in C^\infty(\mathbb{R}^d)$ such that

$$\|\partial^\alpha \varphi\|_{L^\infty} < \infty, \qquad \forall \alpha \in \mathbb{N}^d.$$

- The space $\mathcal{B}$ is a Fréchet space.
- The space $\dot{\mathcal{B}}$ is given by the closure of $\mathcal{D}(\mathbb{R}^d)$ in $\mathcal{B}$, i.e., all $\varphi \in C^\infty(\mathbb{R}^d)$ such that

$$\lim_{|x|\to\infty} \partial^\alpha \varphi(x) = 0, \qquad \forall \alpha \in \mathbb{N}^d.$$

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Two natural topologies on $\mathcal{D}'_{L^1}$:

1. The strong topology $b(\mathcal{D}'_{L^1}, \dot{\mathcal{B}})$.
2. The initial topology *op* w.r.t. the mapping

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Theorem (Schwartz, 1950)

The spaces $\mathcal{D}'_{L^1,b}$ and $\mathcal{D}'_{L^1,op}$ have the same bounded sets and null sequences.

Question

Do the topologies b and op coincide on $\mathcal{D}'_{L^1}$?

Is $\mathcal{D}'_{L^1,op}$ (ultra-)bornological? If yes, this answers positively the above question.

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- $\mathcal{B}'$ stands for the space of bounded distributions, dual of

$$\mathcal{D}_{L^1} = \{\varphi : \partial^\alpha \varphi \in L^1, \forall \alpha \in \mathbb{N}^d\}.$$

- A distribution f belongs to $\mathcal{O}'_C$ if $(1 + |x|^2)^k f \in \mathcal{B}'$, for all $k \in \mathbb{N}$.

Theorem (Schwartz: $\mathcal{O}'_C$ is the space of convolutors of $\mathcal{S}(\mathbb{R}^d)$)

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- $\mathcal{O}_C$ is an (LF)-space (countable inductive limit of Fréchet spaces).
- The space $\mathcal{O}'_C$ of rapidly decreasing distributions is given by the topological dual of $\mathcal{O}_C$.
- Schwartz wrote in his book: "the space $\mathcal{O}_C$ seems not to play any important role".
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The space of rapidly decreasing distributions

Define the topologies b and op on $\mathcal{O}'_C$ as before.

Theorem (Grothendieck, 1955)

The space $\mathcal{O}'_{C,op}$ is complete, semi-reflexive, and (ultra-)bornological (hence reflexive).

Consequently, $\mathcal{O}'_{C,b} = \mathcal{O}'_{C,op}$ and the (LF)-space $\mathcal{O}_C$ is complete.

Grothendieck method:

- He showed that $\mathcal{O}'_{C,op}$ is isomorphic to a complemented subspace of $s\widehat{\otimes}s'$.
- Then proved that that $s\widehat{\otimes}s'$ is bornological.
- Moreover, he showed that $(\mathcal{O}'_{C,op})'_b = \mathcal{O}_C$.

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There have been attempts to generalize Grothendieck's result. Let $\mathcal{W} = (w_N)_N$ be an increasing sequence of positive continuous functions such that

$$\forall N\, \exists M \;:\; \lim_{|x|\to\infty} \omega_N(x)/\omega_M(x) = 0.$$

- Define the Fréchet space $\mathcal{K}_{\mathcal{W}}(= \mathcal{K}\{w_N\}) = \{\varphi \in C^\infty(\mathbb{R}^d) : w_N \partial^\alpha \varphi \in L^\infty, \forall N, \alpha\}$
- E.g. $w_N(x) = (1 + |x|)^N$ leads to $\mathcal{S}$, while if $w_N(x) = e^{N|x|}$ one obtains the space of exponentially decreasing smooth functions $\mathcal{K}_1$.
- Associated convolutor space: $\mathcal{O}'_C(\mathcal{K}_{\mathcal{W}}) = \{f \in \mathcal{K}'_{\mathcal{W}} \;:\; f * \varphi \in \mathcal{K}_{\mathcal{W}} \text{ for all } \varphi \in \mathcal{K}_{\mathcal{W}}\}$.

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Is $\mathcal{O}'_{C,op}(\mathcal{K}_{\mathcal{W}})$ (ultra-)bornological?

- Zielezny claims to have shown this for $\mathcal{O}'_{C,op}(\mathcal{K}_1)$.

  Studia Math. 31 (1968), 111–124.

  His proofs seem to contain major gaps.
- Abdullah even claims in

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  that $\mathcal{O}'_{C,op}(\mathcal{K}_{\mathcal{W}})$ is always ultrabornological when the family of weights is of the form

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  with ω a positive increasing convex function tending to ∞.
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- On the other hand, we showed Zielezny's claim was true.

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Goals

- Show the full topological identity $\mathcal{D}'_{L^1,b} = \mathcal{D}'_{L^1,op}$ and extend it to weighted $\mathcal{D}'_{L^1}$ spaces.
- Unified approach for $\mathcal{D}'_{L^1}$ and $\mathcal{O}'_C$, or more generally, convolutor spaces for Gelfand-Shilov spaces $\mathcal{K}_{\mathcal{W}}$.
- Analyze completeness of weighted inductive limits of spaces of smooth functions (In particular first direct proof of completeness of $\mathcal{O}_C$).
- To this end, we study structural and topological properties of a general class of weighted L^1 convolutor spaces.
- Our arguments are based on the mapping properties of the short-time Fourier transform. Inspired by:

C. Bargetz, N. Ortner, Characterization of L. Schwartz' convolutor and multiplier spaces $\mathcal{O}'_C$ and $\mathcal{O}_M$ by the short-time Fourier transform, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Math. RACSAM 108 (2014), 833–847.

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Goals

- Show the full topological identity $\mathcal{D}'_{L^1,b} = \mathcal{D}'_{L^1,op}$ and extend it to weighted $\mathcal{D}'_{L^1}$ spaces.
- Unified approach for $\mathcal{D}'_{L^1}$ and $\mathcal{O}'_C$, or more generally, convolutor spaces for Gelfand-Shilov spaces $\mathcal{K}_{\mathcal{W}}$.
- Analyze completeness of weighted inductive limits of spaces of smooth functions (In particular first direct proof of completeness of $\mathcal{O}_C$).
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The short-time Fourier transform (STFT)

- $T_x f := f(\cdot - x)$ and $M_\xi f := e^{2\pi i \xi t} f(t)$ for $x, \xi \in \mathbb{R}^d$.
- The STFT of $f \in L^2(\mathbb{R}^d)$ w.r.t. a window function $\psi \in L^2(\mathbb{R}^d) \backslash \{0\}$ is defined as

$$V_\psi f(x,\xi) := (f, M_\xi T_x \psi)_{L^2} = \int_{\mathbb{R}^d} f(t) \overline{\psi(t-x)} e^{-2\pi i \xi t} \mathrm{d}t.$$

- The mapping $V_\psi : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$ is continuous.
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Example: The equality $\mathcal{D}'_{L^1,b} = \mathcal{D}'_{L^1,op}$

- Define $C_{\text{pol}}(\mathbb{R}^d)$ as the space consisting of all $\varphi \in C(\mathbb{R}^d)$ such that there is $N \in \mathbb{N}$ for which

$$\sup_{x \in \mathbb{R}^d} \frac{|\varphi(x)|}{(1+|x|)^N} < \infty.$$

- $C_{\text{pol}}(\mathbb{R}^d)$ is an (LB)-space.

Theorem

Let $\psi \in \mathcal{D}(\mathbb{R}^d)\backslash\{0\}$ and let $\tau = b$ or op. Then,

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Weighted inductive limits of smooth functions

- Let $\mathcal{W} = (w_N)_N$ be an increasing sequence of continuous functions.
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Completeness of $\mathcal{B}_{\mathcal{W}}$ and $\dot{\mathcal{B}}_{\mathcal{W}}$

- Assume that $\mathcal{W} = (w_N)_N$ satisfies

$$\forall N \, \exists M \geq N \, \exists C > 0 \, \forall x \in \mathbb{R}^d : \sup_{y \in [-1,1]^d} w_N(x+y) \leq C w_M(x).$$

Theorem (Debrouwere and V., 2018)

TFAE:

- $\dot{\mathcal{B}}_{\mathcal{W}}$ *is complete.*
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- Define $L^1_{\mathcal{W}}$ as the space consisting of all measurable functions f on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} f(x) w_N(x) \mathrm{d}x < \infty, \qquad \forall N \in \mathbb{N}.$$

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$$\mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}}) := \{ f \in \mathcal{D}'(\mathbb{R}^d) \mid f * \varphi \in L^1_{\mathcal{W}} \text{ for all } \varphi \in \mathcal{D}(\mathbb{R}^d) \}$$

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Weighted L^1 convolutor spaces

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The main result

Theorem (Debrouwere and V., 2018)

The equality $(\dot{\mathcal{B}}_{\mathcal{W}})' = \mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}})$ *always holds algebraically.*
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- $(\dot{\mathcal{B}}_{\mathcal{W}})'_b = \mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}})$.

In such a case, the bidual of $\dot{\mathcal{B}}_{\mathcal{W}}$ *is (topologically)* $\mathcal{B}_{\mathcal{W}}$.

- Suppose $\forall N\, \exists M\ :\ \lim_{|x|\to\infty} \omega_N(x)/\omega_M(x) = 0$. Then $\mathcal{O}'_{C,op}(\mathcal{K}_{\mathcal{W}}) = \mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}})$ and we have then settled when the spaces of convolutors of Gelfand-Shilov spaces are bornological.
- For weight sequences of the form
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Particular case: $\mathcal{D}'_{L^1}$ weighted spaces

Let w be a positive measurable function and set

$$\mathcal{D}'_{L^1_w} = (\dot{\mathcal{B}}_w)'_b.$$

Corollary

Assume that

$$\operatorname*{ess\,sup}_{x \in \mathbb{R}^d} \frac{w(x+\cdot)}{w(x)} \in L^\infty_{\mathrm{loc}}. \tag{1}$$

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It is worth noticing: the hypothesis (1) is equivalent to L^1_w being translation-invariant.

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