Wavelets and Gelfand-Shilov spaces

Jasson Vindas
jasson.vindas@UGent.be

Ghent University

Logic and Analysis Seminar
Ghent, March 11, 2020
In this talk we discuss approximation properties of MRA and wavelets in the so-called Gelfand-Shilov spaces.

I will talk about:

1. Some classes of ‘highly regular’ MRA and wavelets.
2. Their connection with Gevrey and Gelfand-Shilov spaces.
3. Approximation properties of these highly regular MRA and wavelets.
4. Some mapping properties of the wavelet transform.

The talk is based on collaborative works with Dušan Rakić, Stevan Pilipović, and Nenad Teofanov.
In this talk we discuss approximation properties of MRA and wavelets in the so-called Gelfand-Shilov spaces.

I will talk about:

1. Some classes of ‘highly regular’ MRA and wavelets.
2. Their connection with Gevrey and Gelfand-Shilov spaces.
3. Approximation properties of these highly regular MRA and wavelets.
4. Some mapping properties of the wavelet transform.

The talk is based on collaborative works with Dušan Rakić, Stevan Pilipović, and Nenad Teofanov.
In this talk we discuss approximation properties of MRA and wavelets in the so-called Gelfand-Shilov spaces.

I will talk about:

1. Some classes of ‘highly regular’ MRA and wavelets.
2. Their connection with Gevrey and Gelfand-Shilov spaces.
3. Approximation properties of these highly regular MRA and wavelets.
4. Some mapping properties of the wavelet transform.

The talk is based on collaborative works with Dušan Rakić, Stevan Pilipović, and Nenad Teofanov.
A function $\psi \in L^2(\mathbb{R})$ is called an orthonormal wavelet if

$$
\psi_{n,m}(x) = 2^{m/2} \psi(2^m x - n), \quad n, m \in \mathbb{Z},
$$

is an orthonormal basis of $L^2(\mathbb{R})$. Given any $f \in L^2(\mathbb{R})$ we have the wavelet series expansion

$$
f = \sum_{n,m \in \mathbb{Z}} c_{n,m} \psi_{n,m},
$$

where

$$
c_{n,m} = \int_{-\infty}^{\infty} f(x) \overline{\psi_{n,m}(x)} dx.
$$
A function $\psi \in L^2(\mathbb{R})$ is called an orthonormal wavelet if

$$\psi_{n,m}(x) = 2^{m/2}\psi(2^m x - n), \quad n, m \in \mathbb{Z},$$

is an orthonormal basis of $L^2(\mathbb{R})$. Given any $f \in L^2(\mathbb{R})$ we have the wavelet series expansion

$$f = \sum_{n,m\in\mathbb{Z}} c_{n,m}\psi_{n,m},$$

where

$$c_{n,m} = \int_{-\infty}^{\infty} f(x)\overline{\psi}_{n,m}(x)dx.$$
A function $\psi \in L^2(\mathbb{R})$ is called an orthonormal wavelet if

$$\psi_{n,m}(x) = 2^{m/2}\psi(2^m x - n), \quad n, m \in \mathbb{Z},$$

is an orthonormal basis of $L^2(\mathbb{R})$. Given any $f \in L^2(\mathbb{R})$ we have the wavelet series expansion

$$f = \sum_{n,m \in \mathbb{Z}} c_{n,m} \psi_{n,m},$$

where

$$c_{n,m} = \int_{-\infty}^{\infty} f(x) \overline{\psi_{n,m}(x)} dx.$$
One effective way to construct wavelets is via the next concept.

**Definition**

A multiresolution analysis (MRA) is an increasing sequence \( \{ V_m \}_{m \in \mathbb{Z}} \) of closed subspaces of \( L^2(\mathbb{R}^d) \) such that:

(i) \( \bigcap_{m \in \mathbb{Z}} V_m = \{0\} \) and \( \bigcup_{m \in \mathbb{Z}} V_m \) is dense in \( L^2(\mathbb{R}^d) \);

(ii) \( f(x) \in V_m \iff f(2x) \in V_{m+1}, \ m \in \mathbb{Z} \);

(iii) \( f(x) \in V_0 \iff f(x - n) \in V_0, \ n \in \mathbb{Z}^d \);

(iv) there exists \( \phi \in L^2(\mathbb{R}^d) \) such that \( \{ \phi(x - n) \}_{n \in \mathbb{Z}^d} \) is an orthonormal basis of \( V_0 \).

The function \( \phi \) from (iv) is called a scaling function for the MRA.
One effective way to construct wavelets is via the next concept.

**Definition**

A multiresolution analysis (MRA) is an increasing sequence \{V_m\}_{m \in \mathbb{Z}} of closed subspaces of \(L^2(\mathbb{R}^d)\) such that:

- (i) \(\bigcap_{m \in \mathbb{Z}} V_m = \{0\}\) and \(\bigcup_{m \in \mathbb{Z}} V_m\) is dense in \(L^2(\mathbb{R}^d)\);
- (ii) \(f(x) \in V_m \iff f(2x) \in V_{m+1}, \ m \in \mathbb{Z}\);
- (iii) \(f(x) \in V_0 \iff f(x - n) \in V_0, \ n \in \mathbb{Z}^d\);
- (iv) there exists \(\phi \in L^2(\mathbb{R}^d)\) such that \(\{\phi(x - n)\}_{n \in \mathbb{Z}^d}\) is an orthonormal basis of \(V_0\).

The function \(\phi\) from (iv) is called a scaling function for the MRA.
One effective way to construct wavelets is via the next concept.

**Definition**

A multiresolution analysis (MRA) is an increasing sequence \( \{ V_m \}_{m \in \mathbb{Z}} \) of closed subspaces of \( L^2(\mathbb{R}^d) \) such that:

(i) \( \bigcap_{m \in \mathbb{Z}} V_m = \{ 0 \} \) and \( \bigcup_{m \in \mathbb{Z}} V_m \) is dense in \( L^2(\mathbb{R}^d) \);

(ii) \( f(x) \in V_m \iff f(2x) \in V_{m+1}, \ m \in \mathbb{Z} \);

(iii) \( f(x) \in V_0 \iff f(x - n) \in V_0, \ n \in \mathbb{Z}^d \);

(iv) there exists \( \phi \in L^2(\mathbb{R}^d) \) such that \( \{ \phi(x - n) \}_{n \in \mathbb{Z}^d} \) is an orthonormal basis of \( V_0 \).

The function \( \phi \) from (iv) is called a scaling function for the MRA.
One effective way to construct wavelets is via the next concept.

Definition

A multiresolution analysis (MRA) is an increasing sequence \( \{ V_m \}_{m \in \mathbb{Z}} \) of closed subspaces of \( L^2(\mathbb{R}^d) \) such that:

(i) \( \bigcap_{m \in \mathbb{Z}} V_m = \{0\} \) and \( \bigcup_{m \in \mathbb{Z}} V_m \) is dense in \( L^2(\mathbb{R}^d) \);

(ii) \( f(x) \in V_m \iff f(2x) \in V_{m+1}, \ m \in \mathbb{Z} \);

(iii) \( f(x) \in V_0 \iff f(x - n) \in V_0, \ n \in \mathbb{Z}^d \);

(iv) there exists \( \phi \in L^2(\mathbb{R}^d) \) such that \( \{ \phi(x - n) \}_{n \in \mathbb{Z}^d} \) is an orthonormal basis of \( V_0 \).

The function \( \phi \) from (iv) is called a scaling function for the MRA.
One effective way to construct wavelets is via the next concept.

Definition

A multiresolution analysis (MRA) is an increasing sequence $\{V_m\}_{m \in \mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^d)$ such that:

(i) $\bigcap_{m \in \mathbb{Z}} V_m = \{0\}$ and $\bigcup_{m \in \mathbb{Z}} V_m$ is dense in $L^2(\mathbb{R}^d)$;

(ii) $f(x) \in V_m \iff f(2x) \in V_{m+1}$, $m \in \mathbb{Z}$;

(iii) $f(x) \in V_0 \iff f(x - n) \in V_0$, $n \in \mathbb{Z}^d$;

(iv) there exists $\phi \in L^2(\mathbb{R}^d)$ such that $\{\phi(x - n)\}_{n \in \mathbb{Z}^d}$ is an orthonormal basis of $V_0$.

The function $\phi$ from (iv) is called a scaling function for the MRA.
One effective way to construct wavelets is via the next concept.

**Definition**

A multiresolution analysis (MRA) is an increasing sequence \( \{ V_m \}_{m \in \mathbb{Z}} \) of closed subspaces of \( L^2(\mathbb{R}^d) \) such that:

(i) \( \bigcap_{m \in \mathbb{Z}} V_m = \{0\} \) and \( \bigcup_{m \in \mathbb{Z}} V_m \) is dense in \( L^2(\mathbb{R}^d) \);

(ii) \( f(x) \in V_m \iff f(2x) \in V_{m+1}, \ m \in \mathbb{Z} \);

(iii) \( f(x) \in V_0 \iff f(x - n) \in V_0, \ n \in \mathbb{Z}^d \);

(iv) there exists \( \phi \in L^2(\mathbb{R}^d) \) such that \( \{ \phi(x - n) \}_{n \in \mathbb{Z}^d} \) is an orthonormal basis of \( V_0 \).

The function \( \phi \) from (iv) is called a scaling function for the MRA.
Let \( \{ V_m \}_{m \in \mathbb{Z}} \) be an MRA with scaling function \( \phi \).

- Write \( V_1 = V_0 \oplus W_0 \).
- \( V_{m+1} = V_m \oplus W_m \) with \( W_m = \{ f : f(2^{-m}x) \in W_0 \} \).
- \( V_m = W_{m-1} \oplus W_{m-2} \oplus \ldots \).
- \( L^2(\mathbb{R}) = \bigoplus_{m \in \mathbb{Z}} W_m \).

**Conclusion:** We get an orthonormal wavelet if we find \( \psi \in W_0 \) such that \( \{ \psi(x - n) \}_{n \in \mathbb{Z}} \) is an orthonormal basis of \( W_0 \).

\( \phi(x/2) \in V_{-1} \) has expansion in terms of \( \{ \phi(x - n) \}_{n \in \mathbb{N}} \). Fourier transforming we find a \( 2\pi \)-periodic function \( m_0 \) with

\[
\hat{\phi}(2\xi) = m_0(\xi)\hat{\phi}(\xi).
\]

**Characterization:** \( \{ \psi(x - n) \}_{n \in \mathbb{Z}} \) is an orthonormal basis of \( W_0 \) iff there is a \( 2\pi \)-periodic function \( \nu \in L^2[-\pi, \pi] \) such that

\[
\hat{\psi}(2\xi) = e^{i\xi} \nu(2\xi)\overline{m_0(\xi + \pi)}\hat{\phi}(\xi).
\]
Let \( \{ V_m \}_{m \in \mathbb{Z}} \) be an MRA with scaling function \( \phi \).

- Write \( V_1 = V_0 \oplus W_0 \).
- \( V_{m+1} = V_m \oplus W_m \) with \( W_m = \{ f : f(2^{-m}x) \in W_0 \} \).
- \( V_m = W_{m-1} \oplus W_{m-2} \oplus \ldots \).
- \( L^2(\mathbb{R}) = \bigoplus_{m \in \mathbb{Z}} W_m \).

**Conclusion:** We get an orthonormal wavelet if we find \( \psi \in W_0 \) such that \( \{ \psi(x - n) \}_{n \in \mathbb{Z}^d} \) is an orthonormal basis of \( W_0 \).

\( \phi(x/2) \in V_{-1} \) has expansion in terms of \( \{ \phi(x - n) \}_{n \in \mathbb{N}} \). Fourier transforming we find a \( 2\pi \)-periodic function \( m_0 \) with
\[
\hat{\phi}(2\xi) = m_0(\xi)\hat{\phi}(\xi).
\]

**Characterization:** \( \{ \psi(x - n) \}_{n \in \mathbb{Z}} \) is an orthonormal basis of \( W_0 \) iff there is a \( 2\pi \)-periodic function \( \nu \in L^2[-\pi, \pi] \) such that
\[
\hat{\psi}(2\xi) = e^{i\xi}\nu(2\xi)\overline{m_0(\xi + \pi)}\hat{\phi}(\xi).
\]
Let \( \{ V_m \}_{m \in \mathbb{Z}} \) be an MRA with scaling function \( \phi \).

- Write \( V_1 = V_0 \oplus W_0 \).
- \( V_{m+1} = V_m \oplus W_m \) with \( W_m = \{ f : f(2^{-m}x) \in W_0 \} \).
- \( V_m = W_{m-1} \oplus W_{m-2} \oplus \ldots \).
- \( L^2(\mathbb{R}) = \bigoplus_{m \in \mathbb{Z}} W_m \).

**Conclusion:** We get an orthonormal wavelet if we find \( \psi \in W_0 \) such that \( \{ \psi(x - n) \}_{n \in \mathbb{Z}^d} \) is an orthonormal basis of \( W_0 \).

\( \phi(x/2) \in V_{-1} \) has expansion in terms of \( \{ \phi(x - n) \}_{n \in \mathbb{N}} \). Fourier transforming we find a \( 2\pi \)-periodic function \( m_0 \) with

\[
\hat{\phi}(2\xi) = m_0(\xi)\hat{\phi}(\xi).
\]

**Characterization:** \( \{ \psi(x - n) \}_{n \in \mathbb{Z}} \) is an orthonormal basis of \( W_0 \) iff there is a \( 2\pi \)-periodic function \( \nu \in L^2[-\pi, \pi] \) such that

\[
\hat{\psi}(2\xi) = e^{i\xi} \nu(2\xi)m_0(\xi + \pi)\hat{\phi}(\xi).
\]
Construction of wavelets from an MRA

Let \( \{ V_m \}_{m \in \mathbb{Z}} \) be an MRA with scaling function \( \phi \).

- Write \( V_1 = V_0 \oplus W_0 \).
- \( V_{m+1} = V_m \oplus W_m \) with \( W_m = \{ f : f(2^{-m}x) \in W_0 \} \).
- \( V_m = W_{m-1} \oplus W_{m-2} \oplus \ldots \).
- \( L^2(\mathbb{R}) = \bigoplus_{m \in \mathbb{Z}} W_m \).

Conclusion: We get an orthonormal wavelet if we find \( \psi \in W_0 \) such that \( \{ \psi(x - n) \}_{n \in \mathbb{Z}^d} \) is an orthonormal basis of \( W_0 \).

\( \phi(x/2) \in V_{-1} \) has expansion in terms of \( \{ \phi(x - n) \}_{n \in \mathbb{N}} \). Fourier transforming we find a \( 2\pi \)-periodic function \( m_0 \) with

\[
\hat{\phi}(2\xi) = m_0(\xi)\hat{\phi}(\xi).
\]

Characterization: \( \{ \psi(x - n) \}_{n \in \mathbb{Z}} \) is an orthonormal basis of \( W_0 \) iff there is a \( 2\pi \)-periodic function \( \nu \in L^2[-\pi, \pi] \) such that

\[
\hat{\psi}(2\xi) = e^{i\xi}\nu(2\xi)m_0(\xi + \pi)\hat{\phi}(\xi).
\]
Construction of wavelets from an MRA

Let \( \{ V_m \}_{m \in \mathbb{Z}} \) be an MRA with scaling function \( \phi \).

- Write \( V_1 = V_0 \oplus W_0 \).
- \( V_{m+1} = V_m \oplus W_m \) with \( W_m = \{ f : f(2^{-m}x) \in W_0 \} \).
- \( V_m = W_{m-1} \oplus W_{m-2} \oplus \ldots \).
- \( L^2(\mathbb{R}) = \bigoplus_{m \in \mathbb{Z}} W_m \).

**Conclusion:** We get an orthonormal wavelet if we find \( \psi \in W_0 \) such that \( \{ \psi(x - n) \}_{n \in \mathbb{Z}^d} \) is an orthonormal basis of \( W_0 \).

\( \phi(x/2) \in V_{-1} \) has expansion in terms of \( \{ \phi(x - n) \}_{n \in \mathbb{N}} \). Fourier transforming we find a \( 2\pi \)-periodic function \( m_0 \) with

\[
\hat{\phi}(2\xi) = m_0(\xi)\hat{\phi}(\xi).
\]

**Characterization:** \( \{ \psi(x - n) \}_{n \in \mathbb{Z}} \) is an orthonormal basis of \( W_0 \) iff there is a \( 2\pi \)-periodic function \( \nu \in L^2[-\pi, \pi] \) such that

\[
\hat{\psi}(2\xi) = e^{i\xi \nu(2\xi)} m_0(\xi + \pi)\hat{\phi}(\xi).
\]
Construction of wavelets from an MRA

Let \( \{ V_m \}_{m \in \mathbb{Z}} \) be an MRA with scaling function \( \phi \).

- Write \( V_1 = V_0 \oplus W_0 \).
- \( V_{m+1} = V_m \oplus W_m \) with \( W_m = \{ f : f(2^{-m}x) \in W_0 \} \).
- \( V_m = W_{m-1} \oplus W_{m-2} \oplus \ldots \).
- \( L^2(\mathbb{R}) = \bigoplus_{m \in \mathbb{Z}} W_m \).

**Conclusion:** We get an orthonormal wavelet if we find \( \psi \in W_0 \) such that \( \{ \psi(x - n) \}_{n \in \mathbb{Z}^d} \) is an orthonormal basis of \( W_0 \).

\( \phi(x/2) \in V_{-1} \) has expansion in terms of \( \{ \phi(x - n) \}_{n \in \mathbb{N}} \). Fourier transforming we find a \( 2\pi \)-periodic function \( m_0 \) with

\[ \hat{\phi}(2\xi) = m_0(\xi)\hat{\phi}(\xi). \]

**Characterization:** \( \{ \psi(x - n) \}_{n \in \mathbb{Z}} \) is an orthonormal basis of \( W_0 \) iff there is a \( 2\pi \)-periodic function \( \nu \in L^2[-\pi, \pi] \) such that

\[ \hat{\psi}(2\xi) = e^{i\xi} \nu(2\xi) m_0(\xi + \pi)\hat{\phi}(\xi). \]
Construction of wavelets from an MRA

Let \( \{V_m\}_{m \in \mathbb{Z}} \) be an MRA with scaling function \( \phi \).

- Write \( V_1 = V_0 \oplus W_0 \).
- \( V_{m+1} = V_m \oplus W_m \) with \( W_m = \{f : f(2^{-m}x) \in W_0\} \).
- \( V_m = W_{m-1} \oplus W_{m-2} \oplus \ldots \).
- \( L^2(\mathbb{R}) = \bigoplus_{m \in \mathbb{Z}} W_m \).

**Conclusion:** We get an orthonormal wavelet if we find \( \psi \in W_0 \) such that \( \{\psi(x - n)\}_{n \in \mathbb{Z}^d} \) is an orthonormal basis of \( W_0 \).

\( \phi(x/2) \in V_{-1} \) has expansion in terms of \( \{\phi(x - n)\}_{n \in \mathbb{N}} \). Fourier transforming we find a \( 2\pi \)-periodic function \( m_0 \) with

\[
\hat{\phi}(2\xi) = m_0(\xi)\hat{\phi}(\xi).
\]

**Characterization:** \( \{\psi(x - n)\}_{n \in \mathbb{Z}} \) is an orthonormal basis of \( W_0 \) iff there is a \( 2\pi \)-periodic function \( \nu \in L^2[-\pi, \pi] \) such that

\[
\hat{\psi}(2\xi) = e^{i\xi} \nu(2\xi)\overline{m_0(\xi + \pi)}\hat{\phi}(\xi).
\]
Construction of wavelets from an MRA

Let \( \{ V_m \}_{m \in \mathbb{Z}} \) be an MRA with scaling function \( \phi \).

- Write \( V_1 = V_0 \oplus W_0 \).
- \( V_{m+1} = V_m \oplus W_m \) with \( W_m = \{ f : f(2^{-m}x) \in W_0 \} \).
- \( V_m = W_{m-1} \oplus W_{m-2} \oplus \ldots \).
- \( L^2(\mathbb{R}) = \bigoplus_{m \in \mathbb{Z}} W_m \).

**Conclusion:** We get an orthonormal wavelet if we find \( \psi \in W_0 \) such that \( \{ \psi(x - n) \}_{n \in \mathbb{Z}^d} \) is an orthonormal basis of \( W_0 \).

\( \phi(x/2) \in V_{-1} \) has expansion in terms of \( \{ \phi(x - n) \}_{n \in \mathbb{N}} \). Fourier transforming we find a \( 2\pi \)-periodic function \( m_0 \) with

\[
\hat{\phi}(2\xi) = m_0(\xi)\hat{\phi}(\xi).
\]

**Characterization:** \( \{ \psi(x - n) \}_{n \in \mathbb{Z}} \) is an orthonormal basis of \( W_0 \) iff there is a \( 2\pi \)-periodic function \( \nu \in L^2[-\pi, \pi] \) such that

\[
\hat{\psi}(2\xi) = e^{i\xi} \nu(2\xi) \overline{m_0(\xi + \pi)} \hat{\phi}(\xi).
\]
Let \( \{ V_m \}_{m \in \mathbb{Z}} \) be an MRA with scaling function \( \phi \).

- Write \( V_1 = V_0 \oplus W_0 \).
- \( V_{m+1} = V_m \oplus W_m \) with \( W_m = \{ f : f(2^{-m}x) \in W_0 \} \).
- \( V_m = W_{m-1} \oplus W_{m-2} \oplus \ldots \).
- \( L^2(\mathbb{R}) = \bigoplus_{m \in \mathbb{Z}} W_m \).

Conclusion: We get an orthonormal wavelet if we find \( \psi \in W_0 \) such that \( \{ \psi(x - n) \}_{n \in \mathbb{Z}} \) is an orthonormal basis of \( W_0 \).

\( \phi(x/2) \in V_{-1} \) has expansion in terms of \( \{ \phi(x - n) \}_{n \in \mathbb{N}} \). Fourier transforming we find a \( 2\pi \)-periodic function \( m_0 \) with

\[
\hat{\phi}(2\xi) = m_0(\xi) \hat{\phi}(\xi).
\]

Characterization: \( \{ \psi(x - n) \}_{n \in \mathbb{Z}} \) is an orthonormal basis of \( W_0 \) iff there is a \( 2\pi \)-periodic function \( \nu \in L^2[-\pi, \pi] \) such that

\[
\hat{\psi}(2\xi) = e^{i\xi} \nu(2\xi) m_0(\xi + \pi) \hat{\phi}(\xi).
\]
MRA and wavelets are effective to approximate functions, and, in turn, to describe a large number of function and distribution spaces. This effectiveness: related to regularity properties of scaling function and wavelet. By regularity we mean: smoothness and decay. There is however a trade-off between smoothness and decay.

Here a well-known example of this interplay leading to conflicts:

There is no orthonormal wavelet $\psi$ sharing simultaneously these two properties:

1. $\psi(x) \ll e^{-c|x|}$ for some $c > 0$.
2. $\psi \in C^\infty(\mathbb{R})$, with all derivatives being bounded.
MRA and wavelets are effective to approximate functions, and, in turn, to describe a large number of function and distribution spaces. This effectiveness: related to regularity properties of scaling function and wavelet.

By regularity we mean: smoothness and decay. There is however a trade-off between smoothness and decay.

Here a well-known example of this interplay leading to conflicts:

There is no orthonormal wavelet $\psi$ sharing simultaneously these two properties:

1. $\psi(x) \ll e^{-c|x|}$ for some $c > 0$.
2. $\psi \in C^\infty(\mathbb{R})$, with all derivatives being bounded.
MRA and wavelets are effective to approximate functions, and, in turn, to describe a large number of function and distribution spaces. This effectiveness: related to regularity properties of scaling function and wavelet. By regularity we mean: smoothness and decay. There is however a trade-off between smoothness and decay.

Here a well-known example of this interplay leading to conflicts:

There is no orthonormal wavelet $\psi$ sharing simultaneously these two properties:

1. $\psi(x) \ll e^{-c|x|}$ for some $c > 0$.
2. $\psi \in C^\infty(\mathbb{R})$, with all derivatives being bounded.
MRA and wavelets are effective to approximate functions, and, in turn, to describe a large number of function and distribution spaces. This effectiveness: related to regularity properties of scaling function and wavelet. By regularity we mean: smoothness and decay.

There is however a trade-off between smoothness and decay.

Here a well-known example of this interplay leading to conflicts:

There is no orthonormal wavelet $\psi$ sharing simultaneously these two properties:

1. $\psi(x) \ll e^{-c|x|}$ for some $c > 0$.
2. $\psi \in C^\infty(\mathbb{R})$, with all derivatives being bounded.
Regularity of wavelets: smoothness vs decay

- MRA and wavelets are effective to approximate functions, and, in turn, to describe a large number of function and distribution spaces.
- This effectiveness: related to regularity properties of scaling function and wavelet.
- By regularity we mean: smoothness and decay.
- There is however a trade-off between smoothness and decay.

Here a well-known example of this interplay leading to conflicts:

There is no orthonormal wavelet \( \psi \) sharing simultaneously these two properties:

1. \( \psi(x) \ll e^{-c|x|} \) for some \( c > 0 \).
2. \( \psi \in C^\infty(\mathbb{R}) \), with all derivatives being bounded.
Let us fix $\psi \in L^1(\mathbb{R})$ with the second property from the last statement, that is,

$$\psi^{(n)} \in L^\infty(\mathbb{R}), \quad n = 0, 1, 2, \ldots. \quad (1)$$

We consider the decay (for a positive weight function $\omega$):

$$\psi(x) \ll e^{-\omega(|x|)}, \quad (2)$$

Under certain standard regularity assumptions $\omega$, one shows:

If there is an orthonormal wavelet $\psi$ satisfying (1) and (2) then

$$\int_{1}^{\infty} \frac{\omega(x)}{x^2} < \infty. \quad (3)$$

Conclusion: No wavelets with (1) and (2) such that (3) diverges.
Let us fix $\psi \in L^1(\mathbb{R})$ with the second property from the last statement, that is,

$$\psi^{(n)} \in L^\infty(\mathbb{R}), \quad n = 0, 1, 2, \ldots.$$  \hspace{1cm} (1)

We consider the decay (for a positive weight function $\omega$):

$$\psi(x) \ll e^{-\omega(|x|)},$$  \hspace{1cm} (2)

Under certain standard regularity assumptions $\omega$, one shows:

If there is an orthonormal wavelet $\psi$ satisfying (1) and (2) then

$$\int_{1}^{\infty} \frac{\omega(x)}{x^2} < \infty.$$  \hspace{1cm} (3)

Conclusion: No wavelets with (1) and (2) such that (3) diverges.
Let us fix $\psi \in L^1(\mathbb{R})$ with the second property from the last statement, that is,

$$\psi^{(n)} \in L^\infty(\mathbb{R}), \quad n = 0, 1, 2, \ldots$$  \hspace{1cm} (1)

We consider the decay (for a positive weight function $\omega$):

$$\psi(x) \ll e^{-\omega(|x|)}, \hspace{1cm} (2)$$

Under certain standard regularity assumptions $\omega$, one shows:

If there is an orthonormal wavelet $\psi$ satisfying (1) and (2) then

$$\int_1^\infty \frac{\omega(x)}{x^2} < \infty.$$ \hspace{1cm} (3)

Conclusion: No wavelets with (1) and (2) such that (3) diverges.
What we cannot get!

Let us fix $\psi \in L^1(\mathbb{R})$ with the second property from the last statement, that is,

$$\psi^{(n)} \in L^\infty(\mathbb{R}), \quad n = 0, 1, 2, \ldots \quad (1)$$

We consider the decay (for a positive weight function $\omega$):

$$\psi(x) \ll e^{-\omega(|x|)}, \quad (2)$$

Under certain standard regularity assumptions $\omega$, one shows:

If there is an orthonormal wavelet $\psi$ satisfying (1) and (2) then

$$\int_1^\infty \frac{\omega(x)}{x^2} < \infty. \quad (3)$$

Conclusion: No wavelets with (1) and (2) such that (3) diverges
Due to the constrains we have discussed so far, we might try to find smooth $\psi$ (with bounded derivatives) with decay

$$\psi(x) \ll e^{-\omega(x)}, \quad \text{where} \quad \int_{1}^{\infty} \frac{\omega(x)}{x^2} < \infty.$$ 

First try

$\omega(x) = n \log x$, so that $\psi(x) \ll |x|^{-n}$.

This works! Actually, Meyer did better in 1985 and found an orthonormal wavelet $\psi \in S(\mathbb{R})$. For future reference:

Properties of orthonormal wavelets $\psi \in S(\mathbb{R})$

1. $\psi$ is an MRA wavelet.
2. $\int_{-\infty}^{\infty} x^n \psi(x) \, dx = 0, \, n = 0, 1, \ldots$

We write $S_0(\mathbb{R})$ for the subspace of $S(\mathbb{R})$ consisting of functions whose all moments vanish.
What we can try to do!

Due to the constraints we have discussed so far, we might try to find smooth $\psi$ (with bounded derivatives) with decay

$$\psi(x) \ll e^{-\omega(x)}, \quad \text{where } \int_1^\infty \frac{\omega(x)}{x^2} < \infty.$$  

First try

$\omega(x) = n \log x$, so that $\psi(x) \ll |x|^{-n}$.

This works! Actually, Meyer did better in 1985 and found an orthonormal wavelet $\psi \in S(\mathbb{R})$. For future reference:

Properties of orthonormal wavelets $\psi \in S(\mathbb{R})$

1. $\psi$ is an MRA wavelet.
2. $\int_{-\infty}^{\infty} x^n \psi(x) \, dx = 0$, $n = 0, 1, \ldots$

We write $S_0(\mathbb{R})$ for the subspace of $S(\mathbb{R})$ consisting of functions whose all moments vanish.
What we can try to do!

Due to the constrains we have discussed so far, we might try to find smooth $\psi$ (with bounded derivatives) with decay

$$\psi(x) \ll e^{-\omega(x)}, \quad \text{where} \quad \int_1^\infty \frac{\omega(x)}{x^2} < \infty.$$ 

First try

$$\omega(x) = n \log x,$$

so that $\psi(x) \ll |x|^{-n}$.

This works! Actually, Meyer did better in 1985 and found an orthonormal wavelet $\psi \in S(\mathbb{R})$. For future reference:

Properties of orthonormal wavelets $\psi \in S(\mathbb{R})$

1. $\psi$ is an MRA wavelet.
2. $\int_{-\infty}^{\infty} x^n \psi(x) dx = 0, \ n = 0, 1, \ldots$

We write $S_0(\mathbb{R})$ for the subspace of $S(\mathbb{R})$ consisting of functions whose all moments vanish.
Due to the constrains we have discussed so far, we might try to find smooth $\psi$ (with bounded derivatives) with decay

$$\psi(x) \ll e^{-\omega(x)}, \quad \text{where} \quad \int_1^{\infty} \frac{\omega(x)}{x^2} < \infty.$$

First try

$\omega(x) = n \log x$, so that $\psi(x) \ll |x|^{-n}$.

This works! Actually, Meyer did better in 1985 and found an orthonormal wavelet $\psi \in S(\mathbb{R})$. For future reference:

Properties of orthonormal wavelets $\psi \in S(\mathbb{R})$

1. $\psi$ is an MRA wavelet.
2. $\int_{-\infty}^{\infty} x^n \psi(x) \, dx = 0$, $n = 0, 1, \ldots$

We write $S_0(\mathbb{R})$ for the subspace of $S(\mathbb{R})$ consisting of functions whose all moments vanish.
Due to the constrains we have discussed so far, we might try to find smooth $\psi$ (with bounded derivatives) with decay

$$\psi(x) \ll e^{-\omega(x)}, \quad \text{where } \int_1^\infty \frac{\omega(x)}{x^2} < \infty.$$ 

First try

$$\omega(x) = n \log x,$$

so that

$$\psi(x) \ll |x|^{-n}.$$ 

This works! Actually, Meyer did better in 1985 and found an orthonormal wavelet $\psi \in S(\mathbb{R})$. For future reference:

**Properties of orthonormal wavelets $\psi \in S(\mathbb{R})$**

1. $\psi$ is an MRA wavelet.
2. $$\int_{-\infty}^{\infty} x^n \psi(x) dx = 0, \ n = 0, 1, \ldots$$

We write $S_0(\mathbb{R})$ for the subspace of $S(\mathbb{R})$ consisting of functions whose all moments vanish.
What we can try to do!

Due to the constrains we have discussed so far, we might try to find smooth $\psi$ (with bounded derivatives) with decay

$$\psi(x) \ll e^{-\omega(x)}, \quad \text{where} \quad \int_1^\infty \frac{\omega(x)}{x^2} < \infty.$$ 

First try

$\omega(x) = n \log x$, so that $\psi(x) \ll |x|^{-n}$.

This works! Actually, Meyer did better in 1985 and found an orthonormal wavelet $\psi \in S(\mathbb{R})$. For future reference:

Properties of orthonormal wavelets $\psi \in S(\mathbb{R})$

1. $\psi$ is an MRA wavelet.
2. $\int_{-\infty}^{\infty} x^n \psi(x) \, dx = 0$, $n = 0, 1, \ldots$

We write $S_0(\mathbb{R})$ for the subspace of $S(\mathbb{R})$ consisting of functions whose all moments vanish.
We now try \( \psi(x) \ll e^{-c|x|^{1/t}} \). To match the integral constrain:

\[
\int_{1}^{\infty} |x|^{-(2-1/t)} < \infty, \quad \text{i.e.}, \quad t > 1.
\]

To make progress, note Meyer’s wavelets \( \psi \in \mathcal{S}(\mathbb{R}) \) satisfy:
- It is of Lemarié-Meyer type: \( \widehat{\psi} \) has compact support.
- Since \( \psi \) is band-limited, \( \psi \in \mathcal{S}(\mathbb{R}) \) iff \( \widehat{\psi} \in C^\infty(\mathbb{R}) \).
- The latter achieved by taking smooth ‘bell functions’.

### A real Paley-Wiener type theorem, \( t > 1 \)

A band-limited function \( g \) satisfies \( g(x) \ll e^{-c|x|^{1/t}} \) iff \( \widehat{g} \) belongs to the Gevrey class \( G^t(\mathbb{R}) \).

### Theorem (Dziubański-Hernández)

\textit{Given} \( t > 1 \) \textit{and} \( c > 0 \), there is a band-limited orthonormal wavelet with

\[
\psi(x) \ll e^{-c|x|^{1/t}}.
\]
We now try $\psi(x) \ll e^{-c|x|^{1/t}}$. To match the integral constrain:

$$\int_1^\infty |x|^{-(2-1/t)} < \infty, \quad \text{i.e.,} \quad t > 1.$$ 

To make progress, note Meyer’s wavelets $\psi \in S(\mathbb{R})$ satisfy:

- It is of Lemarié-Meyer type: $\hat{\psi}$ has compact support.
- Since $\psi$ is band-limited, $\psi \in S(\mathbb{R})$ iff $\hat{\psi} \in C^\infty(\mathbb{R})$.
- The latter achieved by taking smooth ‘bell functions’.

### A real Paley-Wiener type theorem, $t > 1$

A band-limited function $g$ satisfies $g(x) \ll e^{-c|x|^{1/t}}$ iff $\hat{g}$ belongs to the Gevrey class $G^t(\mathbb{R})$.

### Theorem (Dziubański-Hernández)

*Given $t > 1$ and $c > 0$, there is a band-limited orthonormal wavelet with

$$\psi(x) \ll e^{-c|x|^{1/t}}.$$*
We now try $\psi(x) \ll e^{-c|x|^{1/t}}$. To match the integral constrain:

$$\int_{1}^{\infty} |x|^{-(2-1/t)} < \infty, \quad \text{i.e.,} \quad t > 1.$$ 

To make progress, note Meyer’s wavelets $\psi \in S(\mathbb{R})$ satisfy:

- It is of Lemarié-Meyer type: $\hat{\psi}$ has compact support.
- Since $\psi$ is band-limited, $\psi \in S(\mathbb{R})$ iff $\hat{\psi} \in C^\infty(\mathbb{R})$.
- The latter achieved by taking smooth ‘bell functions’.

**A real Paley-Wiener type theorem, $t > 1$**

A band-limited function $g$ satisfies $g(x) \ll e^{-c|x|^{1/t}}$ iff $\hat{g}$ belongs to the Gevrey class $G^t(\mathbb{R})$.

**Theorem (Dziubański-Hernández)**

*Given $t > 1$ and $c > 0$, there is a band-limited orthonormal wavelet with*

$$\psi(x) \ll e^{-c|x|^{1/t}}.$$
We now try \( \psi(x) \ll e^{-c|x|^{1/t}} \). To match the integral constrain:

\[
\int_1^\infty |x|^{-(2-1/t)} < \infty, \quad \text{i.e.,} \quad t > 1.
\]

To make progress, note Meyer’s wavelets \( \psi \in S(\mathbb{R}) \) satisfy:

- It is of Lemarié-Meyer type: \( \hat{\psi} \) has compact support.
- Since \( \psi \) is band-limited, \( \psi \in S(\mathbb{R}) \) iff \( \hat{\psi} \in C^\infty(\mathbb{R}) \).
- The latter achieved by taking smooth ‘bell functions’.

A real Paley-Wiener type theorem, \( t > 1 \)

A band-limited function \( g \) satisfies \( g(x) \ll e^{-c|x|^{1/t}} \) iff \( \hat{g} \) belongs to the Gevrey class \( G^t(\mathbb{R}) \).

Theorem (Dziubański-Hernández)

Given \( t > 1 \) and \( c > 0 \), there is a band-limited orthonormal wavelet with

\[
\psi(x) \ll e^{-c|x|^{1/t}}.
\]
We now try $\psi(x) \ll e^{-c|x|^{1/t}}$. To match the integral constrain:

$$\int_{1}^{\infty} |x|^{-(2-1/t)} < \infty, \quad \text{i.e.,} \quad t > 1.$$  

To make progress, note Meyer’s wavelets $\psi \in S(\mathbb{R})$ satisfy:
- It is of Lemarié-Meyer type: $\hat{\psi}$ has compact support.
- Since $\psi$ is band-limited, $\psi \in S(\mathbb{R})$ iff $\hat{\psi} \in C^\infty(\mathbb{R})$.
- The latter achieved by taking smooth ‘bell functions’.

**A real Paley-Wiener type theorem, $t > 1$**

A band-limited function $g$ satisfies $g(x) \ll e^{-c|x|^{1/t}}$ iff $\hat{g}$ belongs to the Gevrey class $G^t(\mathbb{R})$.

**Theorem (Dziubański-Hernández)**

Given $t > 1$ and $c > 0$, there is a band-limited orthonormal wavelet with

$$\psi(x) \ll e^{-c|x|^{1/t}}.$$
We now try \( \psi(x) \ll e^{-c|x|^{1/t}} \). To match the integral constrain:

\[
\int_{1}^{\infty} |x|^{-(2-1/t)} < \infty, \quad \text{i.e.,} \quad t > 1.
\]

To make progress, note Meyer’s wavelets \( \psi \in S(\mathbb{R}) \) satisfy:

- It is of Lemarié-Meyer type: \( \hat{\psi} \) has compact support.
- Since \( \psi \) is band-limited, \( \psi \in S(\mathbb{R}) \) iff \( \hat{\psi} \in C^\infty(\mathbb{R}) \).
- The latter achieved by taking smooth ‘bell functions’.

**A real Paley-Wiener type theorem, \( t > 1 \)**

A band-limited function \( g \) satisfies \( g(x) \ll e^{-c|x|^{1/t}} \) iff \( \hat{g} \) belongs to the Gevrey class \( G^t(\mathbb{R}) \).

**Theorem (Dziubański-Hernández)**

*Given \( t > 1 \) and \( c > 0 \), there is a band-limited orthonormal wavelet with*

\[
\psi(x) \ll e^{-c|x|^{1/t}}.
\]
We now try $\psi(x) \ll e^{-c|x|^{1/t}}$. To match the integral constrain:

$$\int_{1}^{\infty} |x|^{-(2-1/t)} < \infty,$$

i.e., $t > 1$.

To make progress, note Meyer’s wavelets $\psi \in S(\mathbb{R})$ satisfy:

- It is of Lemarié-Meyer type: $\hat{\psi}$ has compact support.
- Since $\psi$ is band-limited, $\psi \in S(\mathbb{R})$ iff $\hat{\psi} \in C^\infty(\mathbb{R})$.
- The latter achieved by taking smooth ‘bell functions’.

### A real Paley-Wiener type theorem, $t > 1$

A band-limited function $g$ satisfies $g(x) \ll e^{-c|x|^{1/t}}$ iff $\hat{g}$ belongs to the Gevrey class $G^t(\mathbb{R})$.

### Theorem (Dziubański-Hernández)

Given $t > 1$ and $c > 0$, there is a band-limited orthonormal wavelet with

$$\psi(x) \ll e^{-c|x|^{1/t}}.$$
The Gevrey functions generalize real analytic functions.

A function \( f \) is real analytic in \( I \) iff for each compact subinterval there are \( A \) and \( C \) such that

\[
\sup_{x \in [a, b]} |f^{(n)}(x)| \leq CA^n n!, \quad n \in \mathbb{N}.
\]

Definition

\( f \in G^t(I) \) if \( \sup_{x \in [a, b]} |f^{(n)}(x)| \leq CA^n (n!)^t \) on each \([a, b] \subset I\).

Gevrey classes naturally arise in the analysis of PDE.

If \( t < 1 \), \( G^t(\mathbb{R}) \) consists of entire functions.

If \( t > 1 \), an example of \( f \in G^t(\mathbb{R}) \) is \((\alpha = 1/(t - 1))\)

\[
f(x) = e^{-(x+1)^{-\alpha} - (1-x)^{-\alpha}} \text{ if } |x| \leq 1 \quad \text{and otherwise } f(x) = 0.
\]

Conclusion: \( G^t(\mathbb{R}) \) contains non-trivial compactly supported functions if \( t > 1 \), we write \( G^t_c(\mathbb{R}) = G^t(\mathbb{R}) \cap C^\infty_c(\mathbb{R}) \).
**Gevrey classes**

- The Gevrey functions generalize **real analytic** functions.
- A function \( f \) is real analytic in \( I \) iff for each compact subinterval there are \( A \) and \( C \) such that

\[
\sup_{x \in [a,b]} |f^{(n)}(x)| \leq CA^n n!, \quad n \in \mathbb{N}.
\]

**Definition**

\( f \in G^t(I) \) if

\[
\sup_{x \in [a,b]} |f^{(n)}(x)| \leq CA^n (n!)^t
\]

on each \([a, b] \subset I\).

- Gevrey classes naturally arise in the analysis of PDE.
- If \( t < 1 \), \( G^t(\mathbb{R}) \) consists of entire functions.
- If \( t > 1 \), an example of \( f \in G^t(\mathbb{R}) \) is \( (\alpha = 1/(t-1)) \)

\[
f(x) = e^{-(x+1)^{-\alpha} - (1-x)^{-\alpha}} \text{ if } |x| \leq 1 \quad \text{and otherwise } f(x) = 0.
\]

**Conclusion:** \( G^t(\mathbb{R}) \) contains non-trivial compactly supported functions if \( t > 1 \), we write \( G^t_c(\mathbb{R}) = G^t(\mathbb{R}) \cap C^\infty_c(\mathbb{R}). \)
Gevrey classes

- The Gevrey functions generalize real analytic functions.
- A function $f$ is real analytic in $I$ iff for each compact subinterval there are $A$ and $C$ such that

$$\sup_{x \in [a, b]} |f^{(n)}(x)| \leq CA^n n!, \quad n \in \mathbb{N}.$$ 

**Definition**

$$f \in G^t(I) \text{ if } \sup_{x \in [a, b]} |f^{(n)}(x)| \leq CA^n (n!)^t \text{ on each } [a, b] \subset I.$$ 

- Gevrey classes naturally arise in the analysis of PDE.
- If $t < 1$, $G^t(\mathbb{R})$ consists of entire functions.
- If $t > 1$, an example of $f \in G^t(\mathbb{R})$ is $(\alpha = 1/(t - 1))$

$$f(x) = e^{-(x+1)^{-\alpha}-(1-x)^{-\alpha}} \text{ if } |x| \leq 1 \quad \text{ and otherwise } f(x) = 0.$$ 

**Conclusion:** $G^t(\mathbb{R})$ contains non-trivial compactly supported functions if $t > 1$, we write $G^t_c(\mathbb{R}) = G^t(\mathbb{R}) \cap C^\infty_c(\mathbb{R})$. 

J. Vindas  Wavelets and Gelfand-Shilov spaces
The Gevrey functions generalize real analytic functions. A function $f$ is real analytic in $I$ iff for each compact subinterval there are $A$ and $C$ such that

$$\sup_{x \in [a, b]} |f^{(n)}(x)| \leq CA^n n!, \quad n \in \mathbb{N}.$$ 

**Definition**

$f \in G^t(I)$ if

$$\sup_{x \in [a, b]} |f^{(n)}(x)| \leq CA^n (n!)^t \text{ on each } [a, b] \subset I.$$

Gevrey classes naturally arise in the analysis of PDE.

- If $t < 1$, $G^t(\mathbb{R})$ consists of entire functions.
- If $t > 1$, an example of $f \in G^t(\mathbb{R})$ is ($\alpha = 1/(t - 1)$)

$$f(x) = e^{-(x+1)^{-\alpha} - (1-x)^{-\alpha}} \text{ if } |x| \leq 1 \quad \text{and otherwise } f(x) = 0.$$  

**Conclusion:** $G^t(\mathbb{R})$ contains non-trivial compactly supported functions if $t > 1$, we write $G^t_c(\mathbb{R}) = G^t(\mathbb{R}) \cap C^\infty_c(\mathbb{R})$. 

---

J. Vindas  
Wavelets and Gelfand-Shilov spaces
Gevrey classes

- The Gevrey functions generalize real analytic functions.
- A function $f$ is real analytic in $I$ iff for each compact subinterval there are $A$ and $C$ such that

$$\sup_{x \in [a, b]} |f^{(n)}(x)| \leq CA^n n!, \quad n \in \mathbb{N}.$$ 

**Definition**

$f \in G^t(I)$ if \( \sup_{x \in [a, b]} |f^{(n)}(x)| \leq CA^n (n!)^t \) on each $[a, b] \subset I$.

- Gevrey classes naturally arise in the analysis of PDE.
- If $t < 1$, $G^t(\mathbb{R})$ consists of entire functions.
- If $t > 1$, an example of $f \in G^t(\mathbb{R})$ is $\left( \alpha = \frac{1}{(t - 1)} \right)$

$$f(x) = e^{-(x+1)^{-\alpha}-(1-x)^{-\alpha}} \text{ if } |x| \leq 1 \quad \text{and otherwise } f(x) = 0.$$ 

**Conclusion:** $G^t(\mathbb{R})$ contains non-trivial compactly supported functions if $t > 1$, we write $G^t_c(\mathbb{R}) = G^t(\mathbb{R}) \cap C^\infty_c(\mathbb{R})$. 

J. Vindas
The Gevrey functions generalize real analytic functions. A function $f$ is real analytic in $I$ iff for each compact subinterval there are $A$ and $C$ such that

$$\sup_{x \in [a,b]} |f^{(n)}(x)| \leq CA^n n!, \quad n \in \mathbb{N}.$$ 

**Definition**

$f \in G^t(I)$ if $\sup_{x \in [a,b]} |f^{(n)}(x)| \leq CA^n (n!)^t$ on each $[a, b] \subset I$.

- Gevrey classes naturally arise in the analysis of PDE.
- If $t < 1$, $G^t(\mathbb{R})$ consists of entire functions.
- If $t > 1$, an example of $f \in G^t(\mathbb{R})$ is ($\alpha = 1/(t-1)$)

$$f(x) = e^{-(x+1)^{-\alpha}-(1-x)^{-\alpha}} \text{ if } |x| \leq 1 \quad \text{and otherwise } f(x) = 0.$$ 

**Conclusion:** $G^t(\mathbb{R})$ contains non-trivial compactly supported functions if $t > 1$, we write $G^t_c(\mathbb{R}) = G^t(\mathbb{R}) \cap C^\infty_c(\mathbb{R})$. 

J. Vindas

Wavelets and Gelfand-Shilov spaces
Gevrey classes

- The Gevrey functions generalize real analytic functions.
- A function $f$ is real analytic in $I$ iff for each compact subinterval there are $A$ and $C$ such that
  \[
  \sup_{x \in [a, b]} |f^{(n)}(x)| \leq CA^n n!, \quad n \in \mathbb{N}.
  \]

**Definition**

$f \in G^t(I)$ if

\[
\sup_{x \in [a, b]} |f^{(n)}(x)| \leq CA^n (n!)^t
\]
on each $[a, b] \subset I$.

- Gevrey classes naturally arise in the analysis of PDE.
- If $t < 1$, $G^t(\mathbb{R})$ consists of entire functions.
- If $t > 1$, an example of $f \in G^t(\mathbb{R})$ is ($\alpha = 1/(t - 1)$)
  \[
f(x) = e^{-(x+1)^{-\alpha} - (1-x)^{-\alpha}} \text{ if } |x| \leq 1 \quad \text{and otherwise } f(x) = 0.
  \]

**Conclusion:** $G^t(\mathbb{R})$ contains non-trivial compactly supported functions if $t > 1$, we write $G^t_c(\mathbb{R}) = G^t(\mathbb{R}) \cap C^\infty_c(\mathbb{R})$. 
Define the class $\mathcal{E}\{M_n\}[a,b]$ of smooth functions such that

$$\sup_{x \in [a,b]} |f^{(n)}(x)| \leq C A^n M_n$$

(for some $C, A$).

One may assume $m_n = M_{n+1}/M_n$ increases (Cartan-Gorny theorem).

Hadamard’s problem, 1912

Characterize $\{M_n\}_{n \in \mathbb{N}}$ such that $\mathcal{E}\{M_n\}[a,b]$ contains non-trivial compactly supported functions in $(a,b)$ (= non-quasianalyticity).

Denjoy-Carleman theorem

Suppose $m_n = M_{n+1}/M_n$ is increasing. Then, $\mathcal{E}\{M_n\}[a,b]$ is non-quasianalytic iff

$$\sum_{n=0}^{\infty} 1/m_n < \infty.$$

Under ‘standard assumptions’, one adapts the Dziubański-Hernández construction to find a band-limited orthogonal wavelet with decay $\psi(x) \ll e^{-M(|x|)}$, where $M(x) = \sup_{n \in \mathbb{N}} \log_+(x^n/M_n)$.
Define the class $\mathcal{E}^{\{M_n\}}[a, b]$ of smooth functions such that
\[
\sup_{x \in [a, b]} |f^{(n)}(x)| \leq C A^n M_n \quad \text{(for some } C, A). \]

One may assume $m_n = M_{n+1}/M_n$ increases (Cartan-Gorny theorem).

**Hadamard’s problem, 1912**

Characterize $\{M_n\}_{n \in \mathbb{N}}$ such that $\mathcal{E}^{\{M_n\}}[a, b]$ contains non-trivial compactly supported functions in $(a, b)$ (= non-quasianalyticity).

**Denjoy-Carleman theorem**

Suppose $m_n = M_{n+1}/M_n$ is increasing. Then, $\mathcal{E}^{\{M_n\}}[a, b]$ is non-quasianalytic iff
\[
\sum_{n=0}^{\infty} \frac{1}{m_n} < \infty.
\]

Under ‘standard assumptions’, one adapts the Dziubański-Hernández construction to find a band-limited orthogonal wavelet with decay
\[
\psi(x) \ll e^{-M(|x|)}, \quad \text{where } M(x) = \sup_{n \in \mathbb{N}} \log_{+}(x^n/M_n).
\]
Define the class $\mathcal{E}^{\{M_n\}}[a, b]$ of smooth functions such that
\[
\sup_{x \in [a, b]} |f^{(n)}(x)| \leq C A^n M_n \quad \text{(for some } C, A)\).
\]
One may assume $m_n = M_{n+1}/M_n$ increases (Cartan-Gorny theorem).

**Hadamard’s problem, 1912**

Characterize $\{M_n\}_{n \in \mathbb{N}}$ such that $\mathcal{E}^{\{M_n\}}[a, b]$ contains non-trivial compactly supported functions in $(a, b)$ (= non-quasianalyticity).

**Denjoy-Carleman theorem**

Suppose $m_n = M_{n+1}/M_n$ is increasing. Then, $\mathcal{E}^{\{M_n\}}[a, b]$ is non-quasianalytic iff
\[
\sum_{n=0}^{\infty} 1/m_n < \infty.
\]

Under ‘standard assumptions’, one adapts the Dziubański-Hernández construction to find a band-limited orthogonal wavelet with decay $
\psi(x) \ll e^{-M(|x|)}$, where $M(x) = \sup_{n \in \mathbb{N}} \log_+(x^n/M_n)$.
The Denjoy-Carleman theorem

Define the class $\mathcal{E}^{\{M_n\}}[a, b]$ of smooth functions such that

$$\sup_{x \in [a, b]} |f^{(n)}(x)| \leq CA^nM_n$$

(for some $C, A$).

One may assume $m_n = M_{n+1}/M_n$ increases (Cartan-Gorny theorem).

**Hadamard’s problem, 1912**

Characterize $\{M_n\}_{n \in \mathbb{N}}$ such that $\mathcal{E}^{\{M_n\}}[a, b]$ contains non-trivial compactly supported functions in $(a, b)$ (= non-quasianalyticity).

**Denjoy-Carleman theorem**

Suppose $m_n = M_{n+1}/M_n$ is increasing. Then, $\mathcal{E}^{\{M_n\}}[a, b]$ is non-quasianalytic iff

$$\sum_{n=0}^{\infty} 1/m_n < \infty.$$
The Denjoy-Carleman theorem

Define the class $\mathcal{E}^{\{M_n\}}[a, b]$ of smooth functions such that
\[
\sup_{x \in [a, b]} |f^{(n)}(x)| \leq CA^n M_n \quad \text{(for some } C, A)\].

One may assume $m_n = M_{n+1}/M_n$ increases (Cartan-Gorny theorem).

### Hadamard’s problem, 1912

Characterize $\{M_n\}_{n \in \mathbb{N}}$ such that $\mathcal{E}^{\{M_n\}}[a, b]$ contains non-trivial compactly supported functions in $(a, b)$ (= non-quasianalyticity).

### Denjoy-Carleman theorem

Suppose $m_n = M_{n+1}/M_n$ is increasing. Then, $\mathcal{E}^{\{M_n\}}[a, b]$ is

non-quasianalytic iff $\sum_{n=0}^{\infty} 1/m_n < \infty$.

Under ‘standard assumptions’, one adapts the Dziubański-Hernández construction to find a band-limited orthogonal wavelet with decay
\[
\psi(x) \ll e^{-M(|x|)} \text{, where } M(x) = \sup_{n \in \mathbb{N}} \log_+(x^n/M_n)
\]
The Dziubański-Hernández wavelets belong to $\mathcal{F}(G^t_c(\mathbb{R}))$, where $\mathcal{F}$ stands for the Fourier transform.

Elements of $\mathcal{F}(G^t_c(\mathbb{R}))$ are determined by global estimates

$$|x^m f^{(n)}(x)| \ll B^{n+m} (m!)^t \quad x \in \mathbb{R}.$$ 

Definition

Let $t, s \geq 0$. The space $S^s_t(\mathbb{R})$ consists of all Schwartz functions such that, for some $B$,

$$|x^m f^{(n)}(x)| \ll B^{n+m} (n!)^s (m!)^t.$$ 

- Introduced by Gelfand-Shilov in connection with PDEs.
- $S^s_t(\mathbb{R}) \subset G^s(\mathbb{R})$, so $s$ measures Gevrey regularity.
- The parameter $t$ measures decay ($t > 0$): $f \in S^s_t(\mathbb{R})$ iff

$$|f^{(n)}(x)| \ll B^n (n!)^s e^{-c|x|^{1/t}}.$$
The Dziubański-Hernández wavelets belong to $\mathcal{F}(G^t_c(\mathbb{R}))$, where $\mathcal{F}$ stands for the Fourier transform.

Elements of $\mathcal{F}(G^t_c(\mathbb{R}))$ are determined by global estimates

$$|x^m f^{(n)}(x)| \ll B^{n+m}(m!)^t \quad x \in \mathbb{R}.$$ 

Definition

Let $t, s \geq 0$. The space $S^s_t(\mathbb{R})$ consists of all Schwartz functions such that, for some $B$,

$$|x^m f^{(n)}(x)| \ll B^{n+m}(n!)^s(m!)^t.$$ 

- Introduced by Gelfand-Shilov in connection with PDEs.
- $S^s_t(\mathbb{R}) \subset G^s(\mathbb{R})$, so $s$ measures Gevrey regularity.
- The parameter $t$ measures decay ($t > 0$): $f \in S^s_t(\mathbb{R})$ iff

$$|f^{(n)}(x)| \ll B^n(n!)^s e^{-c|x|^{1/t}}.$$ 

J. Vindas

Wavelets and Gelfand-Shilov spaces
The Dziubański-Hernández wavelets belong to $\mathcal{F}(G_c^t(\mathbb{R}))$, where $\mathcal{F}$ stands for the Fourier transform.

Elements of $\mathcal{F}(G_c^t(\mathbb{R}))$ are determined by global estimates

$$|x^m f^{(n)}(x)| \ll B^{n+m}(m!)^t \quad x \in \mathbb{R}.$$

**Definition**

Let $t, s \geq 0$. The space $S_t^s(\mathbb{R})$ consists of all Schwartz functions such that, for some $B$,

$$|x^m f^{(n)}(x)| \ll B^{n+m}(n!)^s(m!)^t.$$

- Introduced by Gelfand-Shilov in connection with PDEs.
- $S_t^s(\mathbb{R}) \subset G^s(\mathbb{R})$, so $s$ measures Gevrey regularity.
- The parameter $t$ measures decay ($t > 0$): $f \in S_t^s(\mathbb{R})$ iff

$$|f^{(n)}(x)| \ll B^n (n!)^s e^{-c|x|^{1/t}}.$$
The Dziubański-Hernández wavelets belong to $\mathcal{F}(G^t_c(\mathbb{R}))$, where $\mathcal{F}$ stands for the Fourier transform. Elements of $\mathcal{F}(G^t_c(\mathbb{R}))$ are determined by global estimates

$$|x^m f^{(n)}(x)| \ll B^{n+m}(m!)^t \quad x \in \mathbb{R}.$$

**Definition**

Let $t, s \geq 0$. The space $S^s_t(\mathbb{R})$ consists of all Schwartz functions such that, for some $B$,

$$|x^m f^{(n)}(x)| \ll B^{n+m}(n!)^s(m!)^t.$$

- Introduced by Gelfand-Shilov in connection with PDEs.
- $S^s_t(\mathbb{R}) \subset G^s(\mathbb{R})$, so $s$ measures Gevrey regularity.
- The parameter $t$ measures decay ($t > 0$): $f \in S^s_t(\mathbb{R})$ iff

$$|f^{(n)}(x)| \ll B^n (n!)^se^{-c|x|^{1/t}}.$$
The Dziubański-Hernández wavelets belong to $\mathcal{F}(G^t_c(\mathbb{R}))$, where $\mathcal{F}$ stands for the Fourier transform.

Elements of $\mathcal{F}(G^t_c(\mathbb{R}))$ are determined by global estimates

$$|x^m f^{(n)}(x)| \ll B^{n+m}(m!)^t \quad x \in \mathbb{R}.$$ 

**Definition**

Let $t, s \geq 0$. The space $S^s_t(\mathbb{R})$ consists of all Schwartz functions such that, for some $B$,

$$|x^m f^{(n)}(x)| \ll B^{n+m}(n!)^s(m!)^t.$$ 

- Introduced by Gelfand-Shilov in connection with PDEs.
- $S^s_t(\mathbb{R}) \subset G^s(\mathbb{R})$, so $s$ measures Gevrey regularity.
- The parameter $t$ measures decay ($t > 0$): $f \in S^s_t(\mathbb{R})$ iff

$$|f^{(n)}(x)| \ll B^n(n!)^s e^{-c|x|^{1/t}}.$$
Some properties of Gelfand-Shilov spaces

- The family $S^s_t(\mathbb{R})$ is increasing with respect to $s$ and $t$.
- $\mathcal{F}: S^s_t(\mathbb{R}) \rightarrow S^t_s(\mathbb{R})$ is an isomorphism.
- Fourier transform characterization: $f \in S^s_t(\mathbb{R})$ iff
  \[ |f(x)| \ll e^{-c|x|^{1/t}} \text{ and } |\hat{f}(\xi)| \ll e^{-c|\xi|^{1/s}}. \]

- The space $S^s_t(\mathbb{R})$ is non-trivial iff:
  
  either $s + t > 1$, or $s + t = 1$ and $s, t > 0$.

- $S^0_t(\mathbb{R}) = \mathcal{F}(G^t_{c}(\mathbb{R}))$ and thus $S^s_0(\mathbb{R}) = G^s_{c}(\mathbb{R})$.

- If $t > 0$, $S^1_t(\mathbb{R})$ consists of functions $f$ that can be extended analytically to some horizontal strip around $\mathbb{R}$ where it satisfies
  \[ |f(x + iy)| \ll e^{-c|x|^{1/t}} \text{ for } |y| < h \]

- If $s, t > 0$ and $s < 1$, then $f \in S^s_t(\mathbb{R})$ iff $f$ is entire and satisfies
  \[ |f(x + iy)| \ll \exp(-c|x|^{1/t} + c|y|^{s-1}). \]
Some properties of Gelfand-Shilov spaces

- The family $S_t^s(\mathbb{R})$ is increasing with respect to $s$ and $t$.
- $\mathcal{F} : S_t^s(\mathbb{R}) \to S_s^t(\mathbb{R})$ is an isomorphism.
- Fourier transform characterization: $f \in S_t^s(\mathbb{R})$ iff
  \[ |f(x)| \ll e^{-c|x|^{1/t}} \quad \text{and} \quad \hat{f}(\xi) | \ll e^{-c|\xi|^{1/s}}. \]

- The space $S_t^s(\mathbb{R})$ is non trivial iff:
  either $s + t > 1$, or $s + t = 1$ and $s, t > 0$.
- $S_t^0(\mathbb{R}) = \mathcal{F}(G_t^c(\mathbb{R}))$ and thus $S_0^s(\mathbb{R}) = G_c^s(\mathbb{R})$.
- If $t > 0$, $S_t^1(\mathbb{R})$ consists of functions $f$ that can be extended analytically to some horizontal strip around $\mathbb{R}$ where it satisfies
  \[ |f(x + iy)| \ll e^{-c|x|^{1/t}} \quad \text{for} \ |y| < h \]

- If $s, t > 0$ and $s < 1$, then $f \in S_t^s(\mathbb{R})$ iff $f$ is entire and satisfies
  \[ |f(x + iy)| \ll \exp(-c|x|^{1/t} + c|y|^{s-1}). \]
Some properties of Gelfand-Shilov spaces

- The family $S_t^s(\mathbb{R})$ is increasing with respect to $s$ and $t$.
- $\mathcal{F}: S_t^s(\mathbb{R}) \rightarrow S_s^t(\mathbb{R})$ is an isomorphism.
- Fourier transform characterization: $f \in S_t^s(\mathbb{R})$ iff
  $$|f(x)| \ll e^{-c|x|^{1/t}} \text{ and } |\hat{f}(\xi)| \ll e^{-c|\xi|^{1/s}}.$$ 

- The space $S_t^s(\mathbb{R})$ is non-trivial iff:
  either $s + t > 1$, or $s + t = 1$ and $s, t > 0$.

- $S_t^0(\mathbb{R}) = \mathcal{F}(G_t^c(\mathbb{R}))$ and thus $S_0^s(\mathbb{R}) = G_c^s(\mathbb{R})$.

- If $t > 0$, $S_t^1(\mathbb{R})$ consists of functions $f$ that can be extended analytically to some horizontal strip around $\mathbb{R}$ where it satisfies
  $$|f(x + iy)| \ll e^{-c|x|^{1/t}} \text{ for } |y| < h$$

- If $s, t > 0$ and $s < 1$, then $f \in S_t^s(\mathbb{R})$ iff $f$ is entire and satisfies
  $$|f(x + iy)| \ll \exp(-c|x|^{1/t} + c|y|^{s-1}).$$
Some properties of Gelfand-Shilov spaces

- The family $S^s_t(\mathbb{R})$ is increasing with respect to $s$ and $t$.
- $\mathcal{F} : S^s_t(\mathbb{R}) \to S^t_s(\mathbb{R})$ is an isomorphism.
- Fourier transform characterization: $f \in S^s_t(\mathbb{R})$ iff
  \[ |f(x)| \ll e^{-c|x|^{1/t}} \quad \text{and} \quad |\hat{f}(\xi)| \ll e^{-c|\xi|^{1/s}}. \]

- The space $S^s_t(\mathbb{R})$ is non trivial iff:
  either $s + t > 1$, or $s + t = 1$ and $s, t > 0$.
- $S^0_t(\mathbb{R}) = \mathcal{F}(G^t_c(\mathbb{R}))$ and thus $S^s_0(\mathbb{R}) = G^s_c(\mathbb{R})$.
- If $t > 0$, $S^1_t(\mathbb{R})$ consists of functions $f$ that can be extended analytically to some horizontal strip around $\mathbb{R}$ where it satisfies
  \[ |f(x + iy)| \ll e^{-c|x|^{1/t}} \quad \text{for} \quad |y| < h \]

- If $s, t > 0$ and $s < 1$, then $f \in S^s_t(\mathbb{R})$ iff $f$ is entire and satisfies
  \[ |f(x + iy)| \ll \exp(-c|x|^{1/t} + c|y|^{s-1}). \]
The family $S^s_t(\mathbb{R})$ is increasing with respect to $s$ and $t$.

$\mathcal{F}: S^s_t(\mathbb{R}) \rightarrow S^t_s(\mathbb{R})$ is an isomorphism.

Fourier transform characterization: $f \in S^s_t(\mathbb{R})$ iff

$$|f(x)| \ll e^{-c|x|^{1/t}} \quad \text{and} \quad |\hat{f}(\xi)| \ll e^{-c|\xi|^{1/s}}.$$ 

The space $S^s_t(\mathbb{R})$ is non trivial iff:

either $s + t > 1$, \quad or $s + t = 1$ and $s, t > 0$.

$S^0_t(\mathbb{R}) = \mathcal{F}(G^t_c(\mathbb{R}))$ and thus $S^s_0(\mathbb{R}) = G^s_c(\mathbb{R})$.

If $t > 0$, $S^1_t(\mathbb{R})$ consists of functions $f$ that can be extended analytically to some horizontal strip around $\mathbb{R}$ where it satisfies

$$|f(x + iy)| \ll e^{-c|x|^{1/t}} \quad \text{for} \quad |y| < h$$

If $s, t > 0$ and $s < 1$, then $f \in S^s_t(\mathbb{R})$ iff $f$ is entire and satisfies

$$|f(x + iy)| \ll \exp(-c|x|^{1/t} + c|y|^{s-1}).$$
Some properties of Gelfand-Shilov spaces

- The family $S^s_t(\mathbb{R})$ is increasing with respect to $s$ and $t$.
- $\mathcal{F} : S^s_t(\mathbb{R}) \to S^t_s(\mathbb{R})$ is an isomorphism.
- Fourier transform characterization: $f \in S^s_t(\mathbb{R})$ iff
  \[ |f(x)| \ll e^{-c|x|^{1/t}} \quad \text{and} \quad |\hat{f}(\xi)| \ll e^{-c|\xi|^{1/s}}. \]

- The space $S^s_t(\mathbb{R})$ is non trivial iff:
  - either $s + t > 1$, or $s + t = 1$ and $s, t > 0$.
- $S^0_t(\mathbb{R}) = \mathcal{F}(G^t_c(\mathbb{R}))$ and thus $S^s_0(\mathbb{R}) = G^s_c(\mathbb{R})$.
- If $t > 0$, $S^1_t(\mathbb{R})$ consists of functions $f$ that can be extended analytically to some horizontal strip around $\mathbb{R}$ where it satisfies
  \[ |f(x + iy)| \ll e^{-c|x|^{1/t}} \quad \text{for} \ |y| < h \]

- If $s, t > 0$ and $s < 1$, then $f \in S^s_t(\mathbb{R})$ iff $f$ is entire and satisfies
  \[ |f(x + iy)| \ll \exp(-c|x|^{1/t} + c|y|^{s^{-1}}). \]
If $\psi$ is a Dziubański-Hernández wavelet with $\psi(x) \ll e^{-c|x|^{1/\rho_2}}$, then $\psi \in S_{\rho_2}^{\rho_1}(\mathbb{R})$ for all $\rho_1 \geq 0$. They are examples of

**Definition**

Let $\rho_1 \geq 0$ and $\rho_2 > 1$. An orthonormal wavelet $\psi$ is $(\rho_1, \rho_2)$-regular if $\psi \in S_{\rho_2}^{\rho_1}(\mathbb{R})$.

**Definition**

Let $\rho_1 \geq 0$ and $\rho_2 > 1$. An MRA is called $(\rho_1, \rho_2)$-regular if it possesses a scaling function $\phi \in S_{\rho_2}^{\rho_1}(\mathbb{R})$.

**Remark**

It should be by now clear that $\rho_2 \leq 1$ is not admissible here.

**Open question**

Every $(\rho_1, \rho_2)$-regular is an MRA wavelet. Does it arise from a $(\rho_1, \rho_2)$-regular MRA?
If \( \psi \) is a Dziubański-Hernández wavelet with \( \psi(x) \ll e^{-c|x|^{1/\rho_2}} \), then \( \psi \in S_{\rho_1}(\mathbb{R}) \) for all \( \rho_1 \geq 0 \). They are examples of

**Definition**

Let \( \rho_1 \geq 0 \) and \( \rho_2 > 1 \). An orthonormal wavelet \( \psi \) is \((\rho_1, \rho_2)\)-regular if \( \psi \in S_{\rho_2}(\mathbb{R}) \).

**Definition**

Let \( \rho_1 \geq 0 \) and \( \rho_2 > 1 \). An MRA is called \((\rho_1, \rho_2)\)-regular if it possesses a scaling function \( \phi \in S_{\rho_2}(\mathbb{R}) \).

**Remark**

It should be by now clear that \( \rho_2 \leq 1 \) is not admissible here.

**Open question**

Every \((\rho_1, \rho_2)\)-regular is an MRA wavelet. Does it arise from a \((\rho_1, \rho_2)\)-regular MRA?
If $\psi$ is a Dziubański-Hernández wavelet with $\psi(x) \ll e^{-c|x|^{1/\rho_2}}$, then $\psi \in S_{\rho_2}^{\rho_1}(\mathbb{R})$ for all $\rho_1 \geq 0$. They are examples of

**Definition**

Let $\rho_1 \geq 0$ and $\rho_2 > 1$. An orthonormal wavelet $\psi$ is $(\rho_1, \rho_2)$-regular if $\psi \in S_{\rho_2}^{\rho_1}(\mathbb{R})$.

**Definition**

Let $\rho_1 \geq 0$ and $\rho_2 > 1$. An MRA is called $(\rho_1, \rho_2)$-regular if it possesses a scaling function $\phi \in S_{\rho_2}^{\rho_1}(\mathbb{R})$.

**Remark**

It should be by now clear that $\rho_2 \leq 1$ is not admissible here.

**Open question**

Every $(\rho_1, \rho_2)$-regular is an MRA wavelet. Does it arise from a $(\rho_1, \rho_2)$-regular MRA?
If \( \psi \) is a Dziubański-Hernández wavelet with \( \psi(x) \ll e^{-c|x|^{1/\rho_2}} \), then \( \psi \in S_{\rho_1}^{\rho_2}(\mathbb{R}) \) for all \( \rho_1 \geq 0 \). They are examples of

**Definition**

Let \( \rho_1 \geq 0 \) and \( \rho_2 > 1 \). An orthonormal wavelet \( \psi \) is \((\rho_1, \rho_2)\)-regular if \( \psi \in S_{\rho_2}(\mathbb{R}) \).

**Definition**

Let \( \rho_1 \geq 0 \) and \( \rho_2 > 1 \). An MRA is called \((\rho_1, \rho_2)\)-regular if it possesses a scaling function \( \phi \in S_{\rho_2}(\mathbb{R}) \).

**Remark**

It should be by now clear that \( \rho_2 \leq 1 \) is not admissible here.

**Open question**

Every \((\rho_1, \rho_2)\)-regular is an MRA wavelet. Does it arise from a \((\rho_1, \rho_2)\)-regular MRA?
If $\psi$ is a Dziubański-Hernández wavelet with $\psi(x) \ll e^{-c|x|^{1/\rho_2}}$, then $\psi \in S_{\rho_2}^{\rho_1}(\mathbb{R})$ for all $\rho_1 \geq 0$. They are examples of

**Definition**

Let $\rho_1 \geq 0$ and $\rho_2 > 1$. An orthonormal wavelet $\psi$ is $(\rho_1, \rho_2)$-regular if $\psi \in S_{\rho_2}^{\rho_1}(\mathbb{R})$.

**Definition**

Let $\rho_1 \geq 0$ and $\rho_2 > 1$. An MRA is called $(\rho_1, \rho_2)$-regular if it possesses a scaling function $\phi \in S_{\rho_2}^{\rho_1}(\mathbb{R})$.

**Remark**

It should be by now clear that $\rho_2 \leq 1$ is not admissible here.

**Open question**

Every $(\rho_1, \rho_2)$-regular is an MRA wavelet. Does it arise from a $(\rho_1, \rho_2)$-regular MRA?
Convergence of multiresolution expansions

**Theorem**

Let \( \{ V_m \}_{m \in \mathbb{Z}} \) be a \((\rho_1, \rho_2)\)-regular MRA with orthogonal projections \( E_m : L^2(\mathbb{R}) \to V_m \) and set \( \sigma = \rho_1 + \rho_2 - 1 \). Let \( s \geq \sigma \) and \( t \geq \rho_2 \). Then,

\[
\lim_{m \to \infty} E_m f = f \text{ in } S_t^s(\mathbb{R}),
\]

for each \( f \in S_t^{s-\sigma}(\mathbb{R}) \).

There is a loss of regularity measured by \( \sigma > 0 \). We wonder

1. Is \( \sigma \) optimal? We conjecture so ...
2. Are there special classes of MRA that avoid the loss of regularity?
Theorem

Let \( \{ V_m \}_{m \in \mathbb{Z}} \) be a \((\rho_1, \rho_2)\)-regular MRA with orthogonal projections

\[
E_m : L^2(\mathbb{R}) \to V_m
\]

and set \( \sigma = \rho_1 + \rho_2 - 1 \). Let \( s \geq \sigma \) and \( t \geq \rho_2 \). Then,

\[
\lim_{m \to \infty} E_m f = f \text{ in } S^s_t(\mathbb{R}),
\]

for each \( f \in S^{s-\sigma}_t(\mathbb{R}) \).

There is a loss of regularity measured by \( \sigma > 0 \). We wonder

1. Is \( \sigma \) optimal? We conjecture so ...
2. Are there special classes of MRA that avoid the loss of regularity?
Convergence of multiresolution expansions

**Theorem**

Let \( \{ V_m \}_{m \in \mathbb{Z}} \) be a \((\rho_1, \rho_2)\)-regular MRA with orthogonal projections

\[
E_m : L^2(\mathbb{R}) \rightarrow V_m
\]

and set \( \sigma = \rho_1 + \rho_2 - 1 \). Let \( s \geq \sigma \) and \( t \geq \rho_2 \). Then,

\[
\lim_{m \to \infty} E_m f = f \text{ in } S^s_t(\mathbb{R}),
\]

for each \( f \in S^{s-\sigma}_t(\mathbb{R}) \).

There is a loss of regularity measured by \( \sigma > 0 \). We wonder

1. Is \( \sigma \) optimal? We conjecture so ...
2. Are there special classes of MRA that avoid the loss of regularity?
Convergence of multiresolution expansions

Theorem

Let \( \{ V_m \}_{m \in \mathbb{Z}} \) be a \((\rho_1, \rho_2)\)-regular MRA with orthogonal projections

\[
E_m : L^2(\mathbb{R}) \rightarrow V_m
\]

and set \( \sigma = \rho_1 + \rho_2 - 1 \). Let \( s \geq \sigma \) and \( t \geq \rho_2 \). Then,

\[
\lim_{m \rightarrow \infty} E_m f = f \quad \text{in} \quad S^s_t(\mathbb{R}),
\]

for each \( f \in S^{s-\sigma}_t(\mathbb{R}) \).

There is a loss of regularity measured by \( \sigma > 0 \). We wonder

1. Is \( \sigma \) optimal? We conjecture so ...
2. Are there special classes of MRA that avoid the loss of regularity?
Theorem

Let \( \{ V_m \}_{m \in \mathbb{Z}} \) be a \((\rho_1, \rho_2)\)-regular MRA with orthogonal projections

\[
E_m : L^2(\mathbb{R}) \to V_m
\]

and set \( \sigma = \rho_1 + \rho_2 - 1 \). Let \( s \geq \sigma \) and \( t \geq \rho_2 \). Then,

\[
\lim_{m \to \infty} E_m f = f \text{ in } S_t^s(\mathbb{R}),
\]

for each \( f \in S_t^{s-\sigma}(\mathbb{R}) \).

There is a loss of regularity measured by \( \sigma > 0 \). We wonder

1. Is \( \sigma \) optimal? We conjecture so ...

2. Are there special classes of MRA that avoid the loss of regularity?
Theorem

Let \( \{ V_m \}_{m \in \mathbb{Z}} \) be a \((\rho_1, \rho_2)\)-regular MRA with orthogonal projections\( E_m : L^2(\mathbb{R}) \to V_m \) and set \( \sigma = \rho_1 + \rho_2 - 1 \). Let \( s \geq \sigma \) and \( t \geq \rho_2 \). Then,

\[
\lim_{m \to \infty} E_m f = f \text{ in } S^s_t(\mathbb{R}),
\]

for each \( f \in S^{s-\sigma}_t(\mathbb{R}) \).

There is a loss of regularity measured by \( \sigma > 0 \). We wonder:

1. Is \( \sigma \) optimal? We conjecture so ...
2. Are there special classes of MRA that avoid the loss of regularity?
Convergence of multiresolution expansions

**Theorem**

Let \( \{ V_m \}_{m \in \mathbb{Z}} \) be a \((\rho_1, \rho_2)\)-regular MRA with orthogonal projections

\[
E_m : L^2(\mathbb{R}) \to V_m
\]

and set \( \sigma = \rho_1 + \rho_2 - 1 \). Let \( s \geq \sigma \) and \( t \geq \rho_2 \). Then,

\[
\lim_{m \to \infty} E_m f = f \text{ in } S^s_t(\mathbb{R}),
\]

for each \( f \in S^{s-\sigma}_t(\mathbb{R}) \).

There is a **loss of regularity** measured by \( \sigma > 0 \). We wonder

1. Is \( \sigma \) optimal? We conjecture so ...
2. Are there special classes of MRA that avoid the loss of regularity?
Convergence of wavelet expansions

Write \( (S_t^s)_0(\mathbb{R}) = \{ f \in S_t^s(\mathbb{R}) : \int_{-\infty}^{\infty} x^n f(x) dx = 0, \; n = 0, 1, \ldots \} \).

A \((\rho_1, \rho_2)\)-regular wavelet automatically satisfies \( \psi \in (S_{\rho_2}^\rho_1)_0(\mathbb{R}) \).

**Theorem**

Let \( \psi \in (S_{\rho_2}^\rho_1)_0(\mathbb{R}) \) be a \((\rho_1, \rho_2)\)-regular orthonormal wavelet. Set \( \sigma = \rho_1 + \rho_2 - 1 \) and consider \( s > \sigma \) and \( t > \sigma + 1 \).

If \( f \in (S_t^{s-\sigma})_0(\mathbb{R}) \), then

\[
f = \sum_{n, m} \langle f, \overline{\psi}_{n,m} \rangle \psi_{n,m}
\]

converges in the space \((S_t^s)_0(\mathbb{R})\).

- Again we have lost regularity and the same questions as before make sense ...
- Our arguments here rely on mapping properties of wavelet transforms.
Convergence of wavelet expansions

Write \((S_t^s)_0(\mathbb{R}) = \{ f \in S_t^s(\mathbb{R}) : \int_{-\infty}^{\infty} x^n f(x) dx = 0, \ n = 0, 1, \ldots \}\).

A \((\rho_1, \rho_2)\)-regular wavelet automatically satisfies \(\psi \in (S_{\rho_2}^\rho)_0(\mathbb{R})\).

**Theorem**

Let \(\psi \in (S_{\rho_2}^{\rho_1})_0(\mathbb{R})\) be a \((\rho_1, \rho_2)\)-regular orthonormal wavelet. Set \(\sigma = \rho_1 + \rho_2 - 1\) and consider \(s > \sigma\) and \(t > \sigma + 1\).

If \(f \in (S_{t-\sigma}^{s-\sigma})_0(\mathbb{R})\), then

\[
    f = \sum_{n,m} \langle f, \overline{\psi}_{n,m} \rangle \psi_{n,m}
\]

converges in the space \((S_t^s)_0(\mathbb{R})\).

- Again we have lost regularity and the same questions as before make sense ...
- Our arguments here rely on mapping properties of wavelet transforms.
Convergence of wavelet expansions

Write \((S^s_t)_0(\mathbb{R}) = \{ f \in S^s_t(\mathbb{R}) : \int_{-\infty}^{\infty} x^n f(x) dx = 0, \ n = 0, 1, \ldots \}\).

A \((\rho_1, \rho_2)\)-regular wavelet automatically satisfies \(\psi \in (S^\rho_1^\rho_2)_0(\mathbb{R})\).

**Theorem**

Let \(\psi \in (S^\rho_1^\rho_2)_0(\mathbb{R})\) be a \((\rho_1, \rho_2)\)-regular orthonormal wavelet. Set \(\sigma = \rho_1 + \rho_2 - 1\) and consider \(s > \sigma\) and \(t > \sigma + 1\).

If \(f \in (S^s_t^{-\sigma})_0(\mathbb{R})\), then

\[ f = \sum_{n,m} \langle f, \psi_{n,m} \rangle \psi_{n,m} \quad \text{converges in the space } (S^s_t)_0(\mathbb{R}). \]

- Again we have lost regularity and the same questions as before make sense ...
- Our arguments here rely on mapping properties of wavelet transforms.
Convergence of wavelet expansions

Write \((S^s_t)_0(\mathbb{R}) = \{ f \in S^s_t(\mathbb{R}) : \int_{-\infty}^{\infty} x^n f(x) dx = 0, \ n = 0, 1, \ldots \}\).

A \((\rho_1, \rho_2)\)-regular wavelet automatically satisfies \(\psi \in (S_{\rho_2}^{\rho_1})_0(\mathbb{R})\).

Theorem

**Let** \(\psi \in (S_{\rho_2}^{\rho_1})_0(\mathbb{R})\) **be a** \((\rho_1, \rho_2)\)-**regular orthonormal wavelet.**

**Set** \(\sigma = \rho_1 + \rho_2 - 1\) **and consider** \(s > \sigma\) **and** \(t > \sigma + 1\).

**If** \(f \in (S_{t-\sigma}^{s-\sigma})_0(\mathbb{R})\), **then**

\[
f = \sum_{n,m} \langle f, \overline{\psi}_{n,m} \rangle \psi_{n,m} \text{ converges in the space } (S^s_t)_0(\mathbb{R}).
\]

- Again we have lost regularity and the same questions as before make sense ...
- Our arguments here rely on mapping properties of wavelet transforms.
Convergence of wavelet expansions

Write \((S^s_t)_0(\mathbb{R}) = \{ f \in S^s_t(\mathbb{R}) : \int_{-\infty}^{\infty} x^n f(x) dx = 0, \ n = 0, 1, \ldots \}\). 

A \((\rho_1, \rho_2)\)-regular wavelet automatically satisfies \(\psi \in (S^{\rho_1}_{\rho_2})_0(\mathbb{R})\).

**Theorem**

Let \(\psi \in (S^{\rho_1}_{\rho_2})_0(\mathbb{R})\) be a \((\rho_1, \rho_2)\)-regular orthonormal wavelet. Set \(\sigma = \rho_1 + \rho_2 - 1\) and consider \(s > \sigma\) and \(t > \sigma + 1\).

If \(f \in (S^{s-\sigma}_{t-\sigma})_0(\mathbb{R})\), then

\[
f = \sum_{n,m} \langle f, \psi_{n,m} \rangle \psi_{n,m}
\]

converges in the space \((S^s_t)_0(\mathbb{R})\).

- Again we have lost regularity and the same questions as before make sense ...
- Our arguments here rely on mapping properties of wavelet transforms.
We consider the wavelet transform

$$\mathcal{W}_\psi f(b, a) = \frac{1}{a} \int_{-\infty}^{\infty} f(x) \overline{\psi} \left( \frac{x - b}{a} \right) \, dx.$$ 

Denote $\mathbb{H} = \{(b, a) : a > 0\}$. The wavelet synthesis operator is

$$\mathcal{M}_\psi F(x) = \iint_{\mathbb{H}} F(b, a) \psi \left( \frac{x - b}{a} \right) \, \frac{db da}{a^2}.$$ 

The space of highly localized functions on $\mathbb{H}$ is

$$S(\mathbb{H}) = \{ F \in C^\infty(\mathbb{H}) : F(b, a) \ll (1+|b|)^{-n}(a+1/a)^{-n}, \forall n > 0 \}.$$ 

For a wavelet $\psi \in S_0(\mathbb{R})$, one gets continuity of

$$\mathcal{W}_\psi : S_0(\mathbb{R}) \to S(\mathbb{H}) \quad \text{and} \quad \mathcal{M}_\psi : S(\mathbb{H}) \to S_0(\mathbb{R}),$$

which yields a wavelet transform theory for distributions.
The wavelet transform: distribution case

We consider the wavelet transform

\[ \mathcal{W}_\psi f(b, a) = \frac{1}{a} \int_{-\infty}^{\infty} f(x) \overline{\psi \left( \frac{x - b}{a} \right)} \, dx. \]

Denote \( \mathbb{H} = \{(b, a) : a > 0\} \). The wavelet synthesis operator is

\[ \mathcal{M}_\psi F(x) = \int \int_{\mathbb{H}} F(b, a) \psi \left( \frac{x - b}{a} \right) \frac{dbda}{a^2}. \]

The space of highly localized functions on \( \mathbb{H} \) is

\[ S(\mathbb{H}) = \{ F \in C^\infty(\mathbb{H}) : F(b, a) \ll (1+|b|)^{-n}(a+1/a)^{-n}, \forall n > 0 \}. \]

For a wavelet \( \psi \in S_0(\mathbb{R}) \), one gets continuity of

\[ \mathcal{W}_\psi : S_0(\mathbb{R}) \to S(\mathbb{H}) \quad \text{and} \quad \mathcal{M}_\psi : S(\mathbb{H}) \to S_0(\mathbb{R}), \]

which yields a wavelet transform theory for distributions.
We consider the wavelet transform

\[ \mathcal{W}_\psi f(b, a) = \frac{1}{a} \int_{-\infty}^{\infty} f(x) \overline{\psi \left( \frac{x - b}{a} \right)} \, dx. \]

Denote \( \mathbb{H} = \{(b, a) : a > 0\} \). The wavelet synthesis operator is

\[ \mathcal{M}_\psi F(x) = \iint_{\mathbb{H}} F(b, a) \psi \left( \frac{x - b}{a} \right) \frac{db \, da}{a^2}. \]

The space of highly localized functions on \( \mathbb{H} \) is

\[ S(\mathbb{H}) = \{ F \in C^\infty(\mathbb{H}) : F(b, a) \ll (1 + |b|)^{-n}(a + 1/a)^{-n}, \forall n > 0 \}. \]

For a wavelet \( \psi \in S_0(\mathbb{R}) \), one gets continuity of

\[ \mathcal{W}_\psi : S_0(\mathbb{R}) \to S(\mathbb{H}) \quad \text{and} \quad \mathcal{M}_\psi : S(\mathbb{H}) \to S_0(\mathbb{R}), \]

which yields a wavelet transform theory for distributions.
We consider the wavelet transform

$$\mathcal{W}_\psi f(b, a) = \frac{1}{a} \int_{-\infty}^{\infty} f(x) \overline{\psi \left( \frac{x - b}{a} \right)} \, dx.$$ 

Denote $\mathbb{H} = \{(b, a) : a > 0\}$. The wavelet synthesis operator is

$$\mathcal{M}_\psi F(x) = \iint_{\mathbb{H}} F(b, a) \psi \left( \frac{x - b}{a} \right) \frac{db da}{a^2}.$$ 

The space of highly localized functions on $\mathbb{H}$ is

$$S(\mathbb{H}) = \{ F \in C^\infty(\mathbb{H}) : F(b, a) \ll (1+|b|)^{-n}(a+1/a)^{-n}, \forall n > 0 \}.$$ 

For a wavelet $\psi \in S_0(\mathbb{R})$, one gets continuity of

$$\mathcal{W}_\psi : S_0(\mathbb{R}) \to S(\mathbb{H}) \quad \text{and} \quad \mathcal{M}_\psi : S(\mathbb{H}) \to S_0(\mathbb{R}),$$

which yields a wavelet transform theory for distributions.
The wavelet transform in Gelfand-Shilov spaces

Let \( s, t, \tau_1, \tau_2 \). Define \( S^s_{t, \tau_1, \tau_2}(\mathbb{H}) \) as the space of smooth functions satisfying estimates

\[
\partial_a^m \partial_b^n F(b, a) \ll_m B^n(n!)^s \exp \left( -c \left( a^{1/\tau_1} + a^{-1/\tau_2} + |b|^{1/t} \right) \right)
\]

We have made a thorough analysis of wavelet transforms on Gelfand-Shilov spaces. In their simplest forms, our results yield:

**Theorem**

Let \( \psi \in (S^\rho_{\rho_2})_0(\mathbb{R}) \) where \( \rho_1 \geq 0 \) and \( \rho_2 > 1 \). Set \( \sigma = \rho_1 + \rho_2 - 1 \). If \( s > \sigma \) and \( t > \sigma + 1 \), the wavelet mappings

\[
\mathcal{W}_\psi : (S^{s-\sigma})_0(\mathbb{R}) \to S^s_{t, t-\rho_2, s-\rho_1}(\mathbb{H})
\]

and

\[
\mathcal{M}_\psi : S^s_{t, t-\rho_2, s-\rho_1}(\mathbb{H}) \to (S^s_t)_0(\mathbb{R})
\]

are continuous.
Let $s, t, \tau_1, \tau_2$. Define $S_{s,t,\tau_1,\tau_2}^s(\mathbb{H})$ as the space of smooth functions satisfying estimates

$$\partial_a^m \partial_b^n F(b, a) \ll_m B^n(n!)^s \exp \left(-c \left(\frac{a^{1/\tau_1}}{1} + \frac{a^{-1/\tau_2}}{1} + |b|^{1/t}\right)\right)$$

We have made a thorough analysis of wavelet transforms on Gelfand-Shilov spaces. In their simplest forms, our results yield:

**Theorem**

Let $\psi \in (S_{\rho_2}^{\rho_1})_0(\mathbb{R})$ where $\rho_1 \geq 0$ and $\rho_2 > 1$. Set $\sigma = \rho_1 + \rho_2 - 1$. If $s > \sigma$ and $t > \sigma + 1$, the wavelet mappings

$$\mathcal{W}_\psi : (S_{t-\sigma}^{s-\sigma})_0(\mathbb{R}) \rightarrow S_{t,t-\rho_2,s-\rho_1}^s(\mathbb{H})$$

and

$$\mathcal{M}_\psi : S_{t,t-\rho_2,s-\rho_1}^s(\mathbb{H}) \rightarrow (S_t^s)_0(\mathbb{R})$$

are continuous.
The wavelet transform in Gelfand-Shilov spaces

Let \( s, t, \tau_1, \tau_2 \). Define \( S_{s,t,\tau_1,\tau_2}^s(\mathbb{H}) \) as the space of smooth functions satisfying estimates

\[
\partial_a^m \partial_b^n F(b, a) \ll m \cdot B^n(n!)^s \exp \left( -c \left( a^{1/\tau_1} + a^{-1/\tau_2} + |b|^{1/t} \right) \right)
\]

We have made a thorough analysis of wavelet transforms on Gelfand-Shilov spaces. In their simplest forms, our results yield:

**Theorem**

Let \( \psi \in (S_{\rho_2}^{\rho_1})_0(\mathbb{R}) \) where \( \rho_1 \geq 0 \) and \( \rho_2 > 1 \). Set \( \sigma = \rho_1 + \rho_2 - 1 \). If \( s > \sigma \) and \( t > \sigma + 1 \), the wavelet mappings

\[
\mathcal{W}_\psi : (S_{t-\sigma}^{s-\sigma})_0(\mathbb{R}) \rightarrow S_{t-\rho_2,s-\rho_1}^s(\mathbb{H})
\]

and

\[
\mathcal{M}_\psi : S_{t-\rho_2,s-\rho_1}^s(\mathbb{H}) \rightarrow (S_{t}^{s})_0(\mathbb{R})
\]

are continuous.
The wavelet transform in Gelfand-Shilov spaces

Let \( s, t, \tau_1, \tau_2 \). Define \( S^s_{t, \tau_1, \tau_2}(\mathbb{H}) \) as the space of smooth functions satisfying estimates

\[
\partial_a^m \partial_b^n F(b, a) \ll m B^n(n!)^s \exp \left( -c \left( a^{1/\tau_1} + a^{-1/\tau_2} + |b|^{1/t} \right) \right)
\]

We have made a thorough analysis of wavelet transforms on Gelfand-Shilov spaces. In their simplest forms, our results yield:

**Theorem**

Let \( \psi \in (S^\rho_{\rho_2})_0(\mathbb{R}) \) where \( \rho_1 \geq 0 \) and \( \rho_2 > 1 \). Set \( \sigma = \rho_1 + \rho_2 - 1 \). If \( s > \sigma \) and \( t > \sigma + 1 \), the wavelet mappings

\[
\mathcal{W}_\psi : (S^s_{t-\sigma})_0(\mathbb{R}) \rightarrow S^s_{t, t-\rho_2, s-\rho_1}(\mathbb{H})
\]

and

\[
\mathcal{M}_\psi : S^s_{t, t-\rho_2, s-\rho_1}(\mathbb{H}) \rightarrow (S^s_t)_0(\mathbb{R})
\]

are continuous.
Some references

For more details about the subject of this talk, see my joint articles with D. Rakić, S. Pilipović, and N. Teofanov:


For details on the construction of wavelets of subexponential decay, see e.g.:

Some references

For more details about the subject of this talk, see my joint articles with D. Rakić, S. Pilipović, and N. Teofanov:


For details on the construction of wavelets of subexponential decay, see e.g.:
