Some points on the integration theory for functions of one real variable. A general integration theory

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- In this talk we present the construction of a new integral for functions of one variable $f : [a, b] \to \overline{\mathbb{R}}$.
- We also present an overview of some standard integration theories.

The integration theory to be presented is a collaborative work with R. Estrada.

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Introduction

The main drawbacks of the Riemann integral are:

1. The class of Riemann integrable functions is too small.
2. $|f|$ integrable $\not\Rightarrow$ f integrable.
3. Lack of convergence theorems.
4. The fundamental theorem of calculus

$$\int_a^x f(t)\mathrm{d}t = F(x)$$

where $F'(t) = f(t)$, *for all* t, is not always valid.

5. Existence of improper integrals escaping the theory.

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- It is more general than the Lebesgue integral. Indeed, f is Lebesgue integrable iff $|f|$ is (Denjoy) integrable.
- The fundamental theorem of calculus is always valid.
- Every improper integral is proper! (Hake's theorem)

For example, Denjoy integral integrates

$$\int_0^1 \frac{1}{x} \sin\left(\frac{1}{x^2}\right) \mathrm{d}x$$

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An simple example showing a basic difference

Define the piecewise constant function

$$f(x) = \begin{cases} 0\,, & \text{if } \; x \leq 0 \; \text{ or } \; x \geq 1\,, \\ c_n\,, & \text{if } \; \frac{1}{n+1} \leq x < \frac{1}{n}\,. \end{cases} \tag{1}$$

Let $a_n = c_n \left(\frac{1}{n} - \frac{1}{n+1} \right)$, so that

$$\int_x^1 f(t)\mathrm{d}t = \sum_{n \leq x^{-1}} a_n + c_{[1/x]} \left(\frac{1}{[1/x]} - x \right), \quad x \in (0, 1).$$

- Lebesgue integrable if and only if $\sum_{n=1}^{\infty} |a_n| < \infty$.
- Denjoy integrable if and only if the series is convergent.

So: Convergence of series might be out of Lebesgue theory!

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The integral that we shall construct has the following properties:

1. It is more general than the Denjoy-Perron-Henstock integral, and in particular than the Lebesgue integral.
2. It identifies a new class of functions with Schwartz distributions.
3. It enjoys all useful properties of the standard integrals:
   - Convergence theorems.
   - Integration by parts and substitution formulas.
   - Mean value theorems.
4. It satisfies Hake's theorem.
5. If $\beta > 0$, it integrates unbounded functions such as

$$\frac{1}{|x|^{\gamma}} \sin\left(\frac{1}{|x|^{\beta}}\right) \quad \text{for all } \gamma \in \mathbb{R}$$

not Denjoy-Perron-Henstock integrable if $\beta + 1 \leq \gamma$.

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Outline

Denjoy integral

In the construction of his integral, Denjoy developed a complicated procedure that he called “totalization”. He made use of transfinite induction. It is very well explained in Hobson's book:

The theory of functions of a real variable and the theory of Fourier series, vol.1, Dover, New York, 1956.

A few months later, N. Lusin connected the new integral with the notion of generalized absolutely continuous functions in the restricted sense. See the book of Gordon:

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Perron integral
Major and minor functions

In 1914, Perron developed another approach which is equivalent to the Denjoy integral.

Definition

Let $f : [a, b] \to \overline{\mathbb{R}}$.

1. U is a (continuous) major function of f if it is continuous on $[a, b]$, $U(a) = 0$, and
$$f(x) \leq \underline{D}U(x) \text{ and } -\infty < \underline{D}U(x), \ \forall x \in [a, b].$$
2. V is a (continuous) minor function of f if it is continuous on $[a, b]$, $V(a) = 0$, and
$$\overline{D}V(x) \leq f(x) \text{ and } \overline{D}V(x) < \infty, \ \forall x \in [a, b].$$

Perron integral

Definition

A function $f : [a, b] \to \overline{\mathbb{R}}$ is said to be Perron integrable on $[a, b]$ if it has at least one major and one minor function and the numbers

$$\inf \{U(b) : U \text{ is continuous major function of } f\}$$

$$\sup \{V(b) : V \text{ is continuous minor function of } f\}$$

are equal and finite. The common value is said to be its Perron integral.

Henstock-Kurzweil integral

In the 1950's Kurzweil introduced an integral which was motivated by his study in differential equations. His integral coincides with the Denjoy-Perron integral and it was systematically studied by Henstock during the 1960's.

Interestingly, the definition of Henstock-Kurzweil integral does not differ much from that of Riemann integral. It is explained in detail in the monographs by Bartle and Gordon:

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Henstock integral

Gauges and Tagged Partitions

Definition

A function $\delta : [a, b] \to \mathbb{R}_+$ is said to be a gauge on $[a, b]$.

If $P = \{I_j\}_{j=1}^{n}$ is a partition of $[a, b]$ such that for each I_j there is assigned a point $t_j \in I_j$, then we call t_j a tag of I_j. We say that the partition is tagged and write

$$\dot{P} = \{(I_j, t_j)\}_{j=1}^{n}.$$

Definition

$\dot{P}$ is said to be δ-fine if $I_j \subseteq [t_j - \delta(t_j), t_j + \delta(t_j)]$.

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Given a tagged partition $\dot{P} := \{(I_j, t_j)\}_{j=1}^{n}$, we denote the Riemann sum of f corresponding to $\dot{P}$ as

$$S(f;\dot{P}) = \sum_{j=1}^{n} f(t_j)\ell(I_j) \quad (\ell(I_j) \text{ is the length of } I_j).$$

Definition

A function $f : [a, b] \to \overline{\mathbb{R}}$ is said to be Henstock integrable if $\exists A$ such that $\forall \varepsilon > 0$ there exists a gauge δ on $[a, b]$ such that if $\dot{P} := \{(I_j, t_j)\}_{j=1}^{n}$ is δ-fine, then

$$\left| S(f;\dot{P}) - A \right| < \varepsilon \quad (\text{we say then } A \text{ is its integral }).$$

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McShane integral

McShane gave a surprising definition of the Lebesgue integral which goes in the same lines as the previous definition:

If we do not require the tags t_j to belong to I_j, but merely to $[a, b]$, then a miracle occurs! We obtain the Lebesgue integral.

See for example the book by Gordon or the one by McShane:

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Peano differentials

In 1935 Denjoy studied the problem of integration of higher order differential coefficients.

Let F be continuous on $[a, b]$, we say that F has a Peano n^{th} derivative at $x \in (a, b)$ if there are n numbers $F_1(x), \ldots, F_n(x)$ such that

$$F(x+h) = F(x) + F_1(x)h + \cdots + F_n(x)\frac{h^n}{n!} + o(h^n) , \quad \text{as } h \to 0 .$$

We call each $F_j(x)$ its Peano j^{th} derivative at x.

If $n > 1$ and this holds at every point, then $F'(x)$ exists everywhere, but this does not even imply that $F \in C^1[a, b]$.

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Denjoy higher order integration problem

Suppose that F has a Peano n^{th} derivative $\forall x \in (a, b)$. Denjoy asked:

1. If $F_n(x) = 0$ for all $x \in [a, b]$, is F a polynomial of degree at most $n - 1$?
2. Is it possible to reconstruct F, in a constructive manner, from the values $F_n(x)$?

Denjoy solved these two problems with an extremely difficult "totalization procedure" (once again involving transfinite induction).

- In 1957, Łojasiewicz found, using distribution theory, a more transparent solution to the first problem.
- Our integral, to be defined, gives in particular another solution yet to the second Denjoy problem.

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Distributions and functions

We denote by $\mathcal{D}(\mathbb{R})$ the Schwartz space of compactly supported smooth functions. Its dual space $\mathcal{D}'(\mathbb{R})$ is the space of Schwartz distributions.

Distributions will be denoted by $\mathbf{f}, \mathbf{g}, \dots,$ while functions by $f, g, \dots$.

It is well known that if f is (Lebesgue) integrable over any compact, then it corresponds in a unique fashion to the distribution

$$\langle \mathbf{f}(x), \psi(x) \rangle = \int_{-\infty}^{\infty} f(x)\psi(x)\mathrm{d}x,$$

This also holds for the Denjoy-Perron-Henstock integral! We write $f \leftrightarrow \mathbf{f}$ whenever there is a precise association between a function and a distribution.

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Łojasiewicz point values

Schwartz definition of distributions does not consider pointwisely defined values. Inspired by Denjoy, Łojasiewicz defined the value of a distribution at a point.

Definition

A distribution $\mathbf{f} \in \mathcal{D}'(\mathbb{R})$ is said to have a value, $\mathbf{f}(x)$, distributionally, at the point $x \in \mathbb{R}$, if there exist n and a continuous function F such that $\mathbf{F}^{(n)} = \mathbf{f}$ near x, $F \leftrightarrow \mathbf{F}$, and F has Peano n^{th} derivative $F_n(x) = \mathbf{f}(x)$ at the point.

Equivalently, $\mathbf{f}(x)$ exists if and only if for every $\varphi \in \mathcal{D}(\mathbb{R})$

$$\lim_{\varepsilon \to 0} \langle \mathbf{f}(x + \varepsilon t), \varphi(t) \rangle = \mathbf{f}(x) \int_{-\infty}^{\infty} \varphi(t)\mathrm{d}t.$$

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Łojasiewicz uniqueness theorem

Łojasiewicz was able to show the following fundamental theorem:

Theorem

Let $\mathbf{f} \in \mathcal{D}'(\mathbb{R})$. If $\mathbf{f}$ has point values everywhere in (a, b) and if $\mathbf{f}(x) = 0$, $\forall x \in (a, b)$, then $\mathbf{f} = 0$ on (a, b).

Corollary (Denjoy first problem)

If a continuous function F has zero Peano n^{th} derivative everywhere on (a, b), then it is a polynomial of degree at most $n - 1$.

Proof: Define $\mathbf{f} = \mathbf{F}^{(n)} \in \mathcal{D}'(\mathbb{R})$, where $\mathbf{F} \leftrightarrow F$, then $\mathbf{f}(x) = 0$, for all point in the interval, thus, $\mathbf{F}^{(n)} = \mathbf{f} = 0$ on the interval. So, F has to be a polynomial with the right degree.

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Let $\mathbf{f} \in \mathcal{D}'(\mathbb{R})$. *If* $\mathbf{f}$ *has point values everywhere in* (a, b) *and if* $\mathbf{f}(x) = 0$, $\forall x \in (a, b)$, *then* $\mathbf{f} = 0$ *on* (a, b).

Corollary (Denjoy first problem)

If a continuous function F *has zero Peano* n^{th} *derivative everywhere on* (a, b), *then it is a polynomial of degree at most* $n - 1$.

Proof: Define $\mathbf{f} = \mathbf{F}^{(n)} \in \mathcal{D}'(\mathbb{R})$, where $\mathbf{F} \leftrightarrow F$, then $\mathbf{f}(x) = 0$, for all point in the interval, thus, $\mathbf{F}^{(n)} = \mathbf{f} = 0$ on the interval. So, F has to be a polynomial with the right degree.

Łojasiewicz functions and distributions

Łojasiewicz theorem gives a precise meaning to $\mathbf{f} \leftrightarrow f$.

Definition

Let $\mathbf{f} \in \mathcal{D}'(\mathbb{R})$. It is said to be a Łojasiewicz distribution if $\mathbf{f}(x)$ exists for all $x \in \mathbb{R}$.

Definition

Let $f : [a, b] \to \mathbb{R}$ be a function. It is said to be a Łojasiewicz function if there exists a Łojasiewicz distribution $\mathbf{f} \in \mathcal{D}'(\mathbb{R})$ such that $f(x) = \mathbf{f}(x)$ for all $x \in [a, b]$.

- Łojasiewicz functions are not continuous, in general.
- They are Baire class 1, and thus Darboux functions.
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Our motivation is the construction of a new integral so that:

- It integrates every Łojasiewicz function.
- It extends the Denjoy-Perron-Henstock integral, and in particular that of Lebesgue.
- It solves Denjoy second problem on the integration of higher order differential coefficients in a constructive way (Łojasiewic functions do not solve this problem).
- It identifies a new class of functions with distributions in a precise manner.

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Notation

$\mathcal{E}'(\mathbb{R})$ denotes the space of compactly supported distributions, the dual of $\mathcal{E}(\mathbb{R}) = C^\infty(\mathbb{R})$.

Given $\phi \in \mathcal{E}(\mathbb{R})$, we define the ϕ-transform of $\mathbf{f} \in \mathcal{E}'(\mathbb{R})$ as the smooth function of two variables:

$$F_\phi \mathbf{f}(x,y) = (\mathbf{f} * \check{\phi}_y)(x), \quad (x,y) \in \mathbb{H} = \mathbb{R} \times \mathbb{R}_+,$$

where $\check{\phi}_y(t) := \frac{1}{y}\phi\left(-\frac{t}{y}\right)$.

We will always assume that ϕ is normalized, meaning

$$\int_{-\infty}^{\infty} \phi(x)\mathrm{d}x = 1.$$

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Upper and lower values of the ϕ-transform

If $x_0 \in \mathbb{R}$, denote by $C_{x_0,\theta}$ the cone in $\mathbb{H}$ starting at x_0 of angle θ,

$$C_{x_0,\theta} = \{(x,t) \in \mathbb{H} : |x - x_0| \leq (\tan\theta)t\} .$$

If $\mathbf{f} \in \mathcal{E}'(\mathbb{R})$, then the upper and lower angular values of its $\phi-$transform at x_0 are

$$\mathbf{f}^{+}_{\phi,\theta}(x_0) = \limsup_{\substack{(x,t)\to(x_0,0) \\ (x,t)\in C_{x_0,\theta}}} F_\phi \mathbf{f}(x,t)$$

$$\mathbf{f}^{-}_{\phi,\theta}(x_0) = \liminf_{\substack{(x,t)\to(x_0,0) \\ (x,t)\in C_{x_0,\theta}}} F_\phi \mathbf{f}(x,t) .$$

For $\theta = 0$, we obtain the upper and lower radial limits of the $\phi-$transform.

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For $\theta = 0$, we obtain the upper and lower radial limits of the $\phi-$transform.

Classes of test functions

Definition

- The class $\mathcal{T}_0$ consists of all positive normalized functions $\phi \in \mathcal{E}(\mathbb{R})$ that satisfy the following condition:

$$\exists \alpha < -1 \text{ such that } \quad \phi^{(k)}(x) = O\left(|x|^{\alpha - k}\right) \quad |x| \to \infty.$$

- The class $\mathcal{T}_1$ is the subclass of $\mathcal{T}_0$ consisting of those functions that also satisfy

$$x\phi'(x) \leq 0 \quad \text{for all} \quad x \in \mathbb{R}.$$

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Definition of major distributional pairs

A pair $(\mathbf{u}, \mathbf{U})$ is called a major distributional pair for the function f if:

1. $\mathbf{u} \in \mathcal{E}'[a,b]$, $\mathbf{U} \in \mathcal{D}'(\mathbb{R})$, and
$$\mathbf{U}' = \mathbf{u}\,.$$
2. $\mathbf{U}$ is a Łojasiewicz distribution, with $\mathbf{U}(a) = 0$.
3. There exist a set E, with $|E| \leq \aleph_0$, and a set of null Lebesgue measure Z, $m(Z) = 0$, such that for all $x \in [a,b] \setminus Z$ and all $\phi \in \mathcal{T}_0$ we have
$$(\mathbf{u})^{-}_{\phi,0}(x) \geq f(x)\ ,$$
while for $x \in [a,b] \setminus E$ and all $\phi \in \mathcal{T}_1$
$$(\mathbf{u})^{-}_{\phi,0}(x) > -\infty\,.$$

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A pair $(\mathbf{v}, \mathbf{V})$ is called a minor distributional pair for the function f if:

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The distributional integral

Definition

A function $f : [a, b] \to \overline{\mathbb{R}}$ is called distributionally integrable if it has both major and minor distributional pairs and if

$$\sup_{(\mathbf{v},\mathbf{V}) \text{ minor pair}} \mathbf{V}(b) = \inf_{(\mathbf{u},\mathbf{U}) \text{ major pair}} \mathbf{U}(b) .$$

When this is the case this common value is the integral of f over $[a, b]$ and is denoted as

$$(\mathfrak{dist}) \int_a^b f(x)\, \mathrm{d}x ,$$

or just as $\int_a^b f(x)\, \mathrm{d}x$ if there is no risk of confusion.

Properties

We list some properties:

- Distributionally integrable functions are measurable and finite almost everywhere.
- Any Denjoy-Perron-Henstock integrable function is distributionally integrable, and the two integrals coincide within this class of functions.
- Any Łojasiewicz function is distributionally integrable, but not conversely.
- The distributional integral integrates higher order differential coefficients, and thus solves Denjoy's second problem in a constructive manner.

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Indefinite integrals

Theorem

Assume f is distributionally integrable on $[a,b]$ and set

$$F(x) := \int_a^x f(t)\mathrm{d}t \quad x \in [a,b].$$

Then F is a Łojasiewicz function. Moreover if $\mathbf{F} \leftrightarrow F$, then $\mathbf{F}'$ has distributional point values almost everywhere, and actually,

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Distributions and functions

The association $f \leftrightarrow \mathbf{f} = \mathbf{F}'$ is a natural one.

Theorem

Let f be distributionally integrable over $[a, b]$, let its indefinite integral be F, with associated distribution $\mathbf{F}$, $F \leftrightarrow \mathbf{F}$, and let $\mathbf{f} = \mathbf{F}' \in \mathcal{E}'(\mathbb{R})$, so that $\mathbf{f}(x) = f(x)$ almost everywhere in $[a, b]$. Then for any $\psi \in \mathcal{E}(\mathbb{R})$,

$$\langle \mathbf{f}, \psi \rangle = (\mathfrak{dist}) \int_a^b f(x) \psi(x)\, \mathrm{d}x .$$

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Given $\{c_n\}_{n=1}^{\infty}$, define the function

$$f(x) = \begin{cases} 0\,, & \text{if } \ x \leq 0 \ \text{ or } \ x \geq 1\,, \\ c_n\,, & \text{if } \ \frac{1}{n+1} \leq x < \frac{1}{n}\,. \end{cases} \tag{2}$$

Let $a_n = c_n \left(\frac{1}{n} - \frac{1}{n+1} \right)$, so that

$$\int_x^1 f(t)\mathrm{d}t = \sum_{n \leq x^{-1}} a_n + c_{[1/x]} \left(\frac{1}{[1/x]} - x \right), \quad x \in (0,1).$$

Then f is, on the interval $[0,1]$,

- Lebesgue integrable if and only if $\sum_{n=1}^{\infty} |a_n| < \infty$.
- Denjoy-Perron-Henstock integrable if and only if the series is convergent.
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(Continuation of last example)

In case $\sum_{n=1}^{\infty} a_n$ is Cesàro summable, we have

$$(\mathfrak{dist}) \int_0^1 f(x)\, \mathrm{d}x = \sum_{n=1}^{\infty} a_n \quad (\mathrm{C}).$$

For example, if $c_n = (-1)^n n(n+1)$, so that $a_n = (-1)^n$, we obtain

$$(\mathfrak{dist}) \int_0^1 f(x)\, \mathrm{d}x = -1/2$$

and this function is not Denjoy-Perron-Henstock integrable.

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Example

Consider the functions

$$s_{\alpha}(x) := |x|^{\alpha} \sin\left(\frac{1}{x}\right) \quad \text{for} \quad \alpha \in \mathbb{C}.$$

Near $x = 0$:

- If $-1 < \Re e\, \alpha$, then it is Lebesgue integrable.
- If $-2 < \Re e\, \alpha \leq -1$, then it is not Lebesgue integrable but Denjoy-Perron-Henstock integrable.
- If $\Re e\, \alpha \leq -2$, it is not Denjoy-Perron-Henstock integrable, but distributional integrable.

The family of distributions $\mathbf{s}_{\alpha}$, where $\mathbf{s}_{\alpha} \leftrightarrow s_{\alpha}$, is analytic in α.

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For further details about this new integral, I refer to my joint article with R. Estrada:

A general integral, Dissertationes Math. 483 (2012), 1-49.