

# Factorization theorems in Denjoy-Carleman classes associated to representations of $(\mathbb{R}^d, +)$

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# Introduction

Factorization theorems in modules over function algebras is an important subject with a long tradition in mathematical analysis.

A module  $\mathcal{M}$  over a non-unital algebra  $\mathcal{A}$  is said to have the **strong factorization** property if

$$\mathcal{M} = \mathcal{A} \cdot \mathcal{M} = \{a \cdot m \mid a \in \mathcal{A}, m \in \mathcal{M}\}.$$

It is said to have the **weak factorization** property if

$$\mathcal{M} = \text{span}(\mathcal{A} \cdot \mathcal{M}).$$

We will present some new results about strong factorization:

- 1 A strong factorization theorem of Dixmier-Malliavin type for ultradifferentiable vectors of representations of  $(\mathbb{R}^d, +)$ .
- 2 We have established the strong factorization property for many families of convolution modules of ultradifferentiable functions. We will give some examples.

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# Factorization in classical function algebras

- Factorization theorems on  $\mathbb{T}$  go back to Salem and Zygmund.
- Rudin showed (1957-1958):  $L^1(\mathbb{R}^d) = L^1(\mathbb{R}^d) * L^1(\mathbb{R}^d)$ .
- Cohen (1959) extended this result to the function algebra of a locally compact abelian group  $G$ ,

$$L^1(G) = L^1(G) * L^1(G).$$

- Hewitt (1964) used Cohen technique to prove a general factorization theorem for Banach modules.
- Cohen-Hewitt type factorization theorems also hold for various Fréchet modules.
- Essential hypothesis: existence of bounded approximative units on the algebra under consideration.
- Many locally convex algebras **do not** have bounded approximative units. Examples: many basic algebras of smooth functions occurring in analysis.

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Does  $\mathcal{D}(\mathbb{R}^d)$  factorize as  $\mathcal{D}(\mathbb{R}^d) = \mathcal{D}(\mathbb{R}^d) * \mathcal{D}(\mathbb{R}^d)$ ?

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- Yulmukhametov (1999): in contrast  $\mathcal{D}(\mathbb{R}) = \mathcal{D}(\mathbb{R}) * \mathcal{D}(\mathbb{R})$  holds.
- Several authors have independently shown (Miyazaki; Petzeltová and P. Vrbová; Dixmier and Malliavin; Voigt; ...)

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# Factorization on Lie groups

- Let  $G$  be a real connected Lie group.
- Dixmier and Malliavin showed (1979) that

$$\mathcal{D}(G) = \text{span}(\mathcal{D}(G) * \mathcal{D}(G))$$

and, when additionally  $G$  is nilpotent,

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(hereafter: convolution = left convolution)

- Let  $E$  be a locally convex Hausdorff (sequentially complete) space and denote as  $GL(E)$  its group of isomorphisms.
- A group homomorphism  $\pi : G \rightarrow GL(E)$  such that

$$G \times E \rightarrow E, \quad (g, e) \mapsto \pi(g)e$$

is separately continuous is a representation of  $G$  on  $E$ .

- We call  $e \in E$  a **smooth vector** if its orbit mapping

$$G \rightarrow E \quad g \mapsto \pi(g)e, \quad \text{belongs to } C^\infty(G; E).$$

- $E^\infty$  is the subspace of smooth vectors



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# The induced algebra representation $\Pi$

A representation of  $G$  on  $E$  induces a natural action on function convolution algebras.

- If  $f \in C_c(G)$ , we can define:

$$(f, e) \mapsto \Pi(f)e, \quad C_c(G) \times E \rightarrow E, \quad \text{where}$$

$$\Pi(f)e = \int_G f(g)\pi(g)e \, dg \in E$$

- Note  $\Pi(f_1 * f_2) = \Pi(f_1) \circ \Pi(f_2)$ , where  $*$  is left-convolution.
- If  $\Pi(g) = L_g$  is left-translation and  $E$  is a function space,

$$(\Pi(f)e)(x) = \int_G f(g)e(g^{-1}x)dg,$$

so that  $\Pi(f)e = f * e$ .

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# Dixmier-Malliavin factorization theorems

## Weak factorization of smooth vectors

A representation also induces an action of the convolution algebra  $\mathcal{D}(G)$  on the smooth vectors,

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So,  $E^\infty$  is module over  $\mathcal{D}(G)$ .

### Theorem

If  $E$  is a Fréchet space,  $E^\infty$  has the **weak** factorization property w.r.t.  $\mathcal{D}(G)$ , that is,  $E^\infty = \text{span}(\Pi(\mathcal{D}(G))E^\infty)$ .

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$$\Pi(f)e = \int_G f(g)\pi(g)e \, dg \in E$$

So,  $E^\infty$  is module over  $\mathcal{D}(G)$ .

### Theorem

If  $E$  is a Fréchet space,  $E^\infty$  has the **weak** factorization property w.r.t.  $\mathcal{D}(G)$ , that is,  $E^\infty = \text{span}(\Pi(\mathcal{D}(G))E^\infty)$ .

# Dixmier-Malliavin factorization theorems

## Strong factorization

### Theorem

If  $G$  is a compact Lie group, one always has  $E^\infty = (\Pi(C^\infty(G))E^\infty)$ .

Strong factorization also holds in other situations, but one needs to take into account the growth of the representation.

- Let  $\vartheta$  be the distance associated to a left-invariant Riemannian metric and  $1 \in G$  the group identity. We write  $|g| := \vartheta(1, g)$ .
- If  $E$  is Banach there is  $n$  such that  $\|\pi(g)\|_{L_b(E)} \leq e^{n|g|}$ .

- Thus,  $\Pi(f) = \int_G f(g)\pi(g) \, dg$  is well defined as long as  $f$  is exponentially rapidly decreasing on  $G$ .

### Theorem

If  $E$  is a Hilbert space, the representation is unitary, and  $G$  is nilpotent, then  $E^\infty$  has the **strong** factorization property w.r.t.  $\mathcal{S}(G)$ .

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# Analytic factorization of Lie group representations

- $e \in E$  is an **analytic vector** if  $g \mapsto \pi(g)e$  is an analytic mapping.
- $E^\omega$ : subspace of analytic vectors.
- A representation is called an  $F$ -representation if
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# Analytic factorization for $(\mathbb{R}^d, +)$

The convolution algebra  $\mathcal{A}(\mathbb{R}^d)$  consists of real analytic functions  $f$  admitting holomorphic extension to  $\mathbb{R}^d + i] - h, h]^d$  for some  $h > 0$  and satisfying

$$\sup_{|\operatorname{Im} z| \leq h} e^{n|\operatorname{Re} z|} |f(z)| < \infty, \quad \text{for each } n \in \mathbb{N}.$$

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# Some remarks

Our results hold for more general representations than  $F$ -representations:

- projective generalized proto-Banach representations;
- inductive generalized proto-Banach representations.

Also, they apply to more general classes than that of analytic vectors:

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# Projective and inductive generalized proto-Banach representations

Let  $\text{csn}(E)$  be collection of all continuous seminorms on  $E$ .

## Definition

A representation  $(\pi, E)$  is said to be a **projective generalized proto-Banach** representation if

$$\forall p \in \text{csn}(E) \exists q_p \in \text{csn}(E) \exists \kappa_p > 0 \forall x \in \mathbb{R}^d \forall e \in E : \\ p(\pi(x)e) \leq e^{\kappa_p |x|} q_p(e)$$

$\mathfrak{B}(E)$  stands for the collection of non-empty absolutely convex closed bounded subsets of  $E$  and for  $B \in \mathfrak{B}(E)$  we denote  $E_B = \text{span}(B)$ .

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# Denjoy-Carleman classes

Consider a log-convex sequence  $M = (M_p)_p$  of positive numbers and

$$\omega_M(t) = \sup_{p \in \mathbb{N}} \log \left( \frac{t^p M_0}{M_p} \right), \quad t > 0.$$

We impose the assumption:

$$1 < \liminf_{t \rightarrow \infty} \frac{\omega_M(\lambda t)}{\omega_M(t)} \leq \limsup_{t \rightarrow \infty} \frac{\omega_M(\lambda t)}{\omega_M(t)} < \infty, \quad \forall \lambda > 1.$$

Prototypical example:  $M_p = (p!)^\sigma$ , with  $\sigma > 0$ . Then,  $\omega_M(t) \asymp t^{1/\sigma}$ .

- A vector  $e \in E$  is **ultradifferentiable** of class  $[M]$  if its orbit mapping is (bornologically) ultradifferentiable of class  $[M]$ .
- $[M]$  is the common notation for both the Beurling  $(M)$  and  $\{M\}$  Roumieu cases of ultradifferentiability.
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# Denjoy-Carleman classes

Consider a log-convex sequence  $M = (M_p)_p$  of positive numbers and

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# Factorization theorem for ultradifferentiable vectors

For  $h > 0$ , we define the Fréchet space

$$\mathcal{K}^{M,h}(\mathbb{R}^d) = \{\varphi \in \mathcal{C}^\infty(\mathbb{R}^d) \mid \sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{h^{|\alpha|} |\partial^\alpha \varphi(x)| e^{n|x|}}{M_{|\alpha|}} < \infty, \quad \forall n \in \mathbb{N}\}.$$

We set

$$\mathcal{K}^{(M)}(\mathbb{R}^d) = \varprojlim_{h \rightarrow \infty} \mathcal{K}^{M,h}(\mathbb{R}^d) \quad \text{and} \quad \mathcal{K}^{\{M\}}(\mathbb{R}^d) = \varinjlim_{h \rightarrow 0^+} \mathcal{K}^{M,h}(\mathbb{R}^d).$$

If  $M_p = p!$ , then  $\mathcal{A}(\mathbb{R}^d) = \mathcal{K}^{\{M\}}(\mathbb{R}^d)$ .

Theorem (Debrouwere, Prangoski, and V. (2021))

*Let  $(\pi, E)$  be either a projective or an inductive generalized proto-Banach representation of  $(\mathbb{R}^d, +)$  on a sequentially complete lchS  $E$ . Then,  $E^{[M]}$  has the strong factorization property w.r.t.  $\mathcal{K}^{[M]}(\mathbb{R}^d)$*

$$E^{[M]} = \Pi(\mathcal{K}^{[M]}(\mathbb{R}^d))E^{[M]}.$$

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# Factorization of modules of ultradifferentiable functions

- Our factorization theorem implies the strong factorization property for many concrete families of modules of ultradifferentiable functions.

## Example:

- Let  $\omega : \mathbb{R}^d \rightarrow (0, \infty)$  be a continuous weight function satisfying

$$\sup_{x \in \mathbb{R}^d} \frac{\omega(x + \cdot)}{\omega(x)} \in L_{loc}^\infty(\mathbb{R}^d).$$

- Consider  $E = L_\omega^p = \{f \mid \omega \cdot f \in L^p(\mathbb{R}^d)\}$  if  $1 \leq p < \infty$ .
- The ultradifferentiable vectors are (w.r.t. regular representation)

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For more details, see our article:

- A. Debrouwere, B. Prangoski, J. Vindas, *Factorization in Denjoy-Carleman classes associated to representations of  $(\mathbb{R}^d, +)$* , J. Funct. Anal. **280** (2021), Article 108831.

Related works on factorization theorems for representations:

- J. Dixmier, P. Malliavin, *Factorisations de fonctions et de vecteurs indéfiniment différentiables*, Bull. Sci. Math. **102** (1978), 307–330.
- H. Gimperlein, B. Krötz, C. Lienau, *Analytic factorization of Lie group representations*, J. Funct. Anal. **262** (2012), 667–681.
- H. Glöckner, *Continuity of LF-algebra representations associated to representations of Lie groups*, Kyoto J. Math. **53** (2013), 567–595.