Harmonic representations of generalized functions

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In this talk we present an "elementary" approach to the construction of all major sheaves of (linear!) generalized functions on $\mathbb{R}^n$, namely,

$$\mathcal{D}' \to \text{ultradistributions} \to \text{infrahyperfunctions} \to \mathfrak{B}$$

A few words on their classical constructions:

- Distributions are easy: they arise as a dual space.
- Ultradistributions: same as distributions.
- Hyperfunctions: more involved, a lot of prerequisites (several complex variables, homological algebra)
- Infrahyperfunctions: also very involved. First construction is due Hörmander (1985).

We will discuss how to define these sheaves and obtain their properties via harmonic functions.

The talk is based on collaborative works (in progress) with Andreas Debrouwere and Ricardo Estrada.

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Analogies between distributions and hyperfunctions

Distributions satisfy:

1. $\mathcal{D}'$ is a sheaf on $\mathbb{R}^n$.
2. For compact sets $K \subset \Omega$

$$\mathcal{D}'_K(\Omega) = \{f \in \mathcal{D}'(\Omega) : \operatorname{supp} f \subset K\} = \mathcal{E}'(K),$$

   where $\mathcal{E}(K) = C^\infty(K)$.
3. $\mathcal{D}'$ is a fine sheaf (existence of partitions of the unity).

Hyperfunctions satisfy

1. $\mathfrak{B}$ is a sheaf on $\mathbb{R}^n$.
2. For compact sets $K \subset \Omega$

$$\mathfrak{B}_K(\Omega) = \mathcal{A}'[K] \qquad \text{(Martineau-Harvey duality theorem)}$$

   where $\mathcal{A}[K]$ is the space of (germs of) real analytic functions.
3. $\mathfrak{B}$ is a flabby sheaf.

- The 3rd properties are different, but contained in being soft
- Properties 2 uniquely determine these soft sheaves on $\mathbb{R}^n$.

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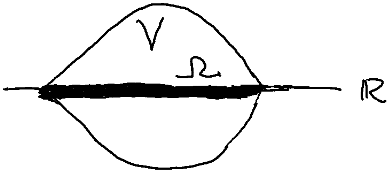
Hyperfunctions in one variable

Let $\Omega \subseteq \mathbb{R}$ be open and $V \subset \mathbb{C}$ be a complex neighborhood containing Ω as closed set. Let $\mathcal{O}$ be the sheaf of holomorphic functions.



The space of hyperfunctions:

$$\mathfrak{B}(\Omega) = \mathcal{O}(V \setminus \Omega)/\mathcal{O}(V).$$

Every $f \in \mathfrak{B}(\Omega)$ is the "boundary value" of some $F \in \mathcal{O}(V \setminus \Omega)$

$$f(x) = F(x + i0) - F(x - i0).$$

The desired three properties of $\mathfrak{B}$ follow from:

- The Mittag-Leffler theorem: $H^1(V, \mathcal{O}) = 0$, for any open $V \subseteq \mathbb{C}$.
- The Köthe-Silva duality theorem $\mathcal{A}'[K] \cong \mathcal{O}(V \setminus K)/\mathcal{O}(V)$.

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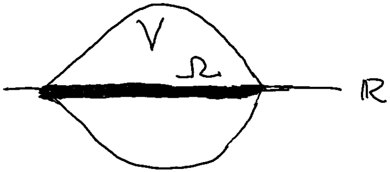

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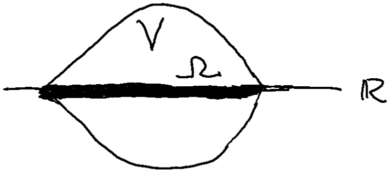

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Hyperfunctions in several variables

In several variables the situation is more complicated.

- Sato's original definition is

$$\mathfrak{B}(\Omega) = H^n_\Omega(V, \mathcal{O}),$$

  where the right-hand side is the nth relative cohomology group with support in Ω, a concept introduced by himself and, independently, by Grothendieck.
- Martineau developed a functional analysis approach. For a bounded open set, he defines

$$\mathfrak{B}(\Omega) = \mathcal{A}'(\overline{\Omega})/\mathcal{A}'(\partial\Omega).$$

  Extension to unbounded Ω: Mittag-Leffler procedure.
- Martineau's method requires showing the existence of the support of an analytic functional (= minimal carrier).
- Martineau's support theorem is shown via harmonic functions in Shapira's (1969) and Hörmander's (1991) books.
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Let $(M_p)_{p\in\mathbb{N}}$ be a weight sequence, that is, a positive increasing sequence of real numbers with $M_0 = 1$.

- $\mathcal{E}^{\{M_p\}}(\Omega)$ consists of $\varphi \in C^\infty(\Omega)$ such that: for each $K \Subset \Omega$ there is $h > 0$ such that
$$\sup_{x\in K} \frac{|\partial^\alpha \varphi(x)|}{h^{|\alpha|} M_{|\alpha|}} < \infty.$$
- The case $M_p = p!$ is the space of real analytic functions.

We shall impose:

$(M.1)^*$ $M_p^2/p \leq M_{p-1}M_{p+1}/(p+1)$ (strong logarithmic convexity)

$(M.2)$ $M_{p+q} \leq AH^{p+q}M_pM_q$ (stability under ultradifferential operators)

(QA) $\sum_{p=1}^\infty M_{p-1}/M_p = \infty$ (quasianalyticity: $\mathcal{D}^{\{M_p\}}(\Omega) = \{0\}$)

The associated function of $M_p/p!$ is: $M^*(t) = \sup_{p\in\mathbb{N}} \log_+ \dfrac{p!t^p}{M_p}$.

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Sheaves of infrahyperfunctions

- Inspired by Martineau functional analysis scheme, Hörmander constructed for the first time the sheaf of infrahyperfunctions $\mathfrak{B}^{\{M_p\}}$ in his seminal paper "Between distributions and hyperfunctions".
- Hörmander's construction relies on a "hard analysis" approach to quasianalytic functionals, that is, the dual spaces $\mathcal{E}'^{\{M_p\}}(\Omega)$.
- In particular, this requires establishing the so-called support theorem for quasianalytic functionals.

Very important fascinating open problem

Is it possible to construct a sheaf $\mathfrak{B}^{(M_p)}$? That is, a sheaf of infrahyperfunctions of Beurling type. So far it seems that no one has been able to overcome topological obstructions ...

Quasianalytic functionals

Assume $(M.1)^*$, $(M.2)$, and (QA).

- $\mathcal{E}^{\{M_p\}}[K]$ denotes space of germs of quasianalytic functions .
- $\mathcal{E}^{\{M_p\}}(\Omega) \cong \varprojlim_{K \Subset \Omega} \mathcal{E}^{\{M_p\}}[K]$.
- Consequently, $\mathcal{E}'^{\{M_p\}}(\Omega) \cong \varinjlim_{K \Subset \Omega} \mathcal{E}'^{\{M_p\}}[K] = \bigcup_{K \Subset \Omega} \mathcal{E}'^{\{M_p\}}[K]$.
- We say that $K \Subset \Omega$ is a $\{M_p\}$-carrier of $f \in \mathcal{E}'^{\{M_p\}}(\Omega)$ if $f \in \mathcal{E}'^{\{M_p\}}[K]$.
- For $f \in \mathcal{A}'(\Omega)$, Martineau's theorem states: there is a smallest $\{p!\}$-carrier of f, denoted by $\operatorname{supp} f$.

Theorem (Hörmander's support theorem)

For every quasianalytic functional $f \in \mathcal{E}'^{\{M_p\}}(\Omega)$ there is a smallest compact set among its $\{M_p\}$-carriers and it coincides with $\operatorname{supp} f$.

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Hörmander's support theorem is the key to show:

Theorem (Hörmander)

There exists an (up to isomorphism) unique flabby sheaf $\mathcal{B}^{\{M_p\}}$ such that

$$\mathcal{B}_K^{\{M_p\}}(\mathbb{R}^n) = \mathcal{E}'^{\{M_p\}}[K], \qquad K \Subset \mathbb{R}^n.$$

The harmonic function method we now proceed to sketch leads to a new approach to

- Hörmander's support theorem as well as
- an explicit construction of Hörmander's infrahyperfunctions.

Similar considerations lead to represent the (space of sections of the) sheaves $\mathcal{D}'$ and (non-quasinalytic ultradistributions) $\mathcal{D}'^*$ as quotients of spaces of harmonic functions.

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Infrahyperfunctions

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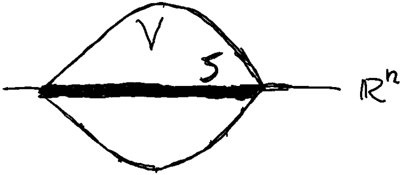
Some spaces of harmonic functions

- $\mathcal{H}(W) = \{$harmonic functions on W$\}$.
- We write $(x, y) \in \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$.
- We always consider an open neighborhood $V \subseteq \mathbb{R}^{n+1}$ of $S \subseteq \mathbb{R}^n$ such that S is closed in V and V is symmetric with respect to $\mathbb{R}^n$:

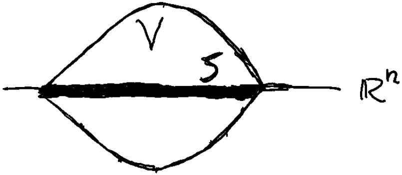


- We write $\mathcal{H}_o(V \setminus S) = \{U \in \mathcal{H}(V \setminus S) : \ U(x, -y) = -U(x, y)\}$.
- In particular $\mathcal{H}_o(V) = \{U \in \mathcal{H}(V) : \ U(x, 0) = 0\}$.
- We define $\mathcal{H}_o^{\{M_p\}}(V \setminus S)$ as those $U \in \mathcal{H}_0(V \setminus S)$ such that

$$U(x, y) \ll e^{M^*\left(\frac{h}{d(K;(x,y))}\right)}, \qquad \forall h > 0 \text{ and in compacts of } V \setminus S.$$

Some spaces of harmonic functions

- $\mathcal{H}(W) = \{$harmonic functions on W$\}$.
- We write $(x, y) \in \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$.
- We always consider an open neighborhood $V \subseteq \mathbb{R}^{n+1}$ of $S \subseteq \mathbb{R}^n$ such that S is closed in V and V is symmetric with respect to $\mathbb{R}^n$:



- We write $\mathcal{H}_o(V \setminus S) = \{U \in \mathcal{H}(V \setminus S) : \ U(x, -y) = -U(x, y)\}$.
- In particular $\mathcal{H}_o(V) = \{U \in \mathcal{H}(V) : \ U(x, 0) = 0\}$.
- We define $\mathcal{H}_o^{\{M_p\}}(V \setminus S)$ as those $U \in \mathcal{H}_0(V \setminus S)$ such that

$$U(x, y) \ll e^{M^*\left(\frac{h}{d(K;(x,y))}\right)}, \qquad \forall h > 0 \text{ and in compacts of } V \setminus S.$$

Harmonic representation of infrahyperfunctions

Theorem (Debrouwere, Estrada, and V., 2019)

On any open set $\Omega \subseteq \mathbb{R}^n$, the flabby sheaf of infrahyperfunctions can also be defined as

$$\mathfrak{B}^{\{M_p\}}(\Omega) = \mathcal{H}_o^{\{M_p\}}(V \setminus \Omega)/\mathcal{H}_o(V). \tag{1}$$

Its compact sections are

$$\mathfrak{B}_K^{\{M_p\}}(\Omega) \cong \mathcal{H}_o^{\{M_p\}}(V \setminus K)/\mathcal{H}_o(V) \cong \mathcal{E}'^{\{M_p\}}[K].$$

- Our ideas directly prove that (1) is a flabby sheaf, without relying on Hörmander's approach to quasianalytic functionals.
- The isomorphism $\mathcal{E}'^{\{M_p\}}[K] \cong \mathcal{H}_o^{\{M_p\}}(V \setminus K)/\mathcal{H}_o(V)$ is explicit, as we now explain. This also yields a new proof of Hörmander's support theorem.

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Harmonic representations of functionals

We assume $(M.1)^*$ and $(M.2)$ and let $K \Subset \mathbb{R}^n$.

- Let P be the Poisson kernel of the upper half-space. The Poisson transform of $f \in \mathcal{A}'[K]$ is

$$P[f](x,y) := \langle f(t), P(x-t,y)\rangle, \qquad (x,y) \in \mathbb{R}^{n+1}\backslash K.$$

One can show: $P[\cdot\,] : \mathcal{E}'^{\{M_p\}}[K] \to \mathcal{H}_o^{\{M_p\}}(\mathbb{R}^{n+1}\backslash K)$.

- Let $U \in \mathcal{H}_o^{\{M_p\}}(V\backslash K)$ and fix an open $K \subset \Omega \subset V \cap \mathbb{R}^n$ and a cut-off $\chi \in \mathcal{D}(\Omega)$ being 1 on K. The boundary value mapping is

$$\langle \mathrm{bv}(U), \varphi\rangle = \lim_{y\to 0^+} \int_{\mathbb{R}^n} U(x,y)\chi(x)\varphi(x)dx, \qquad \varphi \in \mathcal{E}^*(\Omega).$$

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$0 \longrightarrow \mathcal{H}_o(V) \longrightarrow \mathcal{H}_o^{\{M_p\}}(V\backslash K) \xrightarrow{\mathrm{bv}} \mathcal{E}'^{\{M_p\}}[K] \longrightarrow 0$ *is exact and the Poisson transform is a linear right inverse of* bv.

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Important tool: almost harmonic extensions

Let $V \subseteq \mathbb{R}^{n+1}$ be an open neighborhood of $K \Subset \mathbb{R}^n$ and let $\varphi \in \mathcal{A}[K]$. By the Cauchy-Kowalevski theorem, there is an open $K \subset W \subseteq V$ with and a solution $\Phi \in \mathcal{H}_o(W)$ to the Cauchy problem

$$\begin{cases} \Delta\Phi(x,y) = 0 & (x,y) \in W, \\ \Phi(x,0) = 0 & x \in \Omega' \\ \partial_y \Phi(x,0) = \varphi(x) & x \in \Omega'. \end{cases} \tag{2}$$

Taking $\rho \in \mathcal{D}(W)$ being equal to 1 in an $\mathbb{R}^{n+1}$-neighborhood of K ,

$$\langle \mathrm{bv}(U), \varphi \rangle = -\int_V U(x,y)\Delta(\rho\Phi)(x,y)dxdy, \qquad U \in \mathcal{H}_0(V \setminus K)$$

- This formula also holds for $U \in \mathcal{H}_o^{\{M_p\}}(V \setminus K)$ and $\varphi \in \mathcal{E}^{\{M_p\}}[K]$, but a “harmonic extension” Φ as in (2) won't exist in general.
- We then introduced so-called almost harmonic extensions of ultradifferentiable functions.

Almost harmonic extensions of ultradifferentiable functions

Theorem (Debrouwere and V., 2019)

Assume $(M.1)^*$ and $(M.2)$ and let $\Omega \subseteq \mathbb{R}^n$ be open. For $\varphi \in C^2(\Omega)$ the following statements are equivalent:

(*i*) $\varphi \in \mathcal{E}^{\{M_p\}}(\Omega)$.

(*ii*) *For every $\Omega' \Subset \Omega$ there is $\Phi \in C^2(\Omega' \times \mathbb{R})$ such that*

- (*a*) *Φ is odd with respect to y. In particular, $\Phi(x,0) = 0$.*
- (*b*) *$\partial_y \Phi(x,0) = \varphi(x)$ for all $x \in \Omega'$.*
- (*c*) *For every $h > 0$,*

$$\sup_{(x,y)\in\Omega'\times\mathbb{R}} |\Delta\Phi(x,y)| e^{M^*(h/|y|)} < \infty.$$

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