Recent developments on complex Tauberian theorems for Laplace transforms

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Complex Tauberian theorems have been strikingly useful tools in diverse areas such as:

- Analytic number theory.
- Spectral theory for (pseudo-)differential operators.
- Last three decades: operator theory and semigroups.

We will discuss some recent developments on complex Tauberians for Laplace transforms and power series. We will be concerned with two groups of statements:

- Wiener-Ikehara theorems.
- Ingham-Fatou-Riesz theorems.

Main questions:

1. Relax boundary requirements to a minimum.
2. Mild Tauberian hypotheses (one-sided conditions).
3. Optimal Tauberian constants: Sharp versions.

This talk is based on collaborative works with G. Debruyne.

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The classical Wiener-Ikehara theorem

Theorem (Wiener-Ikehara, Laplace-Stieltjes transforms)

Let S be a non-decreasing function (Tauberian hypothesis) such that $\mathcal{L}\{\mathrm{d}S; z\} = \int_{0^-}^{\infty} e^{-zt}\mathrm{d}S(t)$ converges for $\Re e\, z > 1$. If

$$\mathcal{L}\{\mathrm{d}S; z\} - \frac{A}{z-1}$$

has analytic continuation through $\Re e\, z = 1$, then $S(x) \sim Ae^{x}$.

Theorem (Wiener-Ikehara, version for Dirichlet series)

Let $a_n \geq 0$ and $\lambda_n \nearrow \infty$. Suppose $\sum_{n=1}^{\infty} a_n \lambda_n^{-z}$ converges for $\Re e\, z > 1$. If

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From the Wiener-Ikehara theorem to the PNT:

The Prime Number Theorem (PNT) asserts that

$$\pi(x) = \sum_{p \leq x} 1 \sim \frac{x}{\log x}$$

- PNT is equivalent to $\psi(x) = \sum_{p^{\alpha} \leq x} \log p = \sum_{n \leq x} \Lambda(n) \sim x$.
- $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ has analytic continuation to $\Re e\, z > 0$ except for simple pole with residue 1 at $z = 1$.
- Logarithmic differentiation of $\zeta(z) = \prod_p (1 - p^{-z})^{-1}$ leads to

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^z} = -\frac{\zeta'(z)}{\zeta(z)}, \quad \Re e\, z > 1.$$

- $(z-1)\zeta(z)$ has no zeros on $\Re e\, z = 1$, so

$$-\frac{d}{dz}(\log((z-1)\zeta(z))) = -\frac{\zeta'(z)}{\zeta(z)} - \frac{1}{z-1}$$

is analytic in a region containing $\Re e\, z \geq 1$. The rest follows from the Wiener-Ikehara theorem.

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Remarks on the Wiener-Ikehara theorem

- Another typical application: Weyl type spectral asymptotics for (pseudo-)differential operators.
- Historically, the Wiener-Ikehara theorem improved a Tauberian theorem of Landau (1908) by eliminating the unnecessary hypothesis $G(z) = O(|z|^N)$ on

$$G(z) = \mathcal{L}\{\mathrm{d}S; z\} - \frac{A}{z-1}$$

- The hypothesis $G(z)$ has analytic continuation to $\Re e\, z = 1$ can be significantly relaxed to "good boundary behavior":
  1. G(z) has continuous extension to $\Re e\, z = 1$.
  2. L^1_{loc}-boundary behavior: $\lim_{x \to 1^+} G(x + iy) \in L^1(I)$ for every finite interval I.
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In his very influential 1906 paper

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Fatou proved the following theorem on analytic continuation of power series.

Theorem (Fatou-Riesz theorem)

Suppose that $F(z) = \sum_{n=0}^{\infty} c_n z^n$ *converges for* $|z| < 1$ *and* $c_n = o(1)$ *(this is the Tauberian condition). If* $F(z)$ *has analytic continuation to a neigborhood of* $z = 1$*, then* $\sum_{n=0}^{\infty} c_n$ *converges and*

$$\sum_{n=0}^{\infty} c_n = F(1).$$

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Ingham theorem for Laplace transforms

In 1935 Ingham obtained a Fatou-Riesz type Tauberian theorem for Laplace transforms. The result makes use of *slow decrease*.

A function τ is called slowly decreasing if for each $\varepsilon > 0$ there is $\delta > 0$ such that

$$\liminf_{x\to\infty} \inf_{h\in[0,\delta]} (\tau(x+h) - \tau(x)) > -\varepsilon.$$

that is, $\tau(x+h) - \tau(x) > -\varepsilon$ for $x > X_{\varepsilon}$ and $0 \leq h < \delta_{\varepsilon}$.

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In 1980 Newman gave a simple contour integration proof of the next Tauberian theorem.

Theorem

Let $a_n = O(1)$ (Tauberian hypothesis). If $F(z) = \sum_{n=1}^{\infty} \frac{a_n}{n^z}$ has analytic continuation beyond $\Re e\, z = 1$, then

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Newman's Tauberian theorem from above provides a relatively simple way to prove the PNT.

- One works here with the Möbius

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^r & \text{if } n \text{ has } r \text{ distinct prime factors,} \\ 0 & \text{otherwise.} \end{cases}$$

- Property: μ is the Dirichlet convolution inverse of 1. So,

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^z} = \frac{1}{\zeta(z)} \quad (\zeta \text{ is the Riemann zeta function})$$

- Applying the previous theorem, $\displaystyle\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = \frac{1}{\zeta(1)} = 0.$
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Tauberians motivated by applications in semigroups

Newman's contour integration method was adapted to a variety of Tauberian problems in numerous articles.

Its importance was recognized by the semigroup community. Here is a sample (extending a result of Korevaar and Zagier):

Theorem (Arendt and Batty, 1988)

Let $\rho \in L^{\infty}(\mathbb{R})$ (Tauberian hypothesis) vanish on $(-\infty, 0)$. Suppose that $\mathcal{L}\{\rho; z\}$ has analytic continuation at every point of the complement of iE where $E \subset \mathbb{R}$ is a closed null set. If $0 \notin iE$ and

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If $E = \emptyset$, the result is due to Korevaar and Zagier (independently), who also obtained it via Newman's contour integration technique.

In this case, the result is contained in Ingham's Fatou-Riesz type theorem:

- Set $\tau(x) = \int_0^x \rho(u)\mathrm{d}u - b \Rightarrow \mathcal{L}\{\tau; z\} = \dfrac{\mathcal{L}\{\rho; z\} - b}{z}$ with $b = \mathcal{L}\{\rho; 0\}$.
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If $E = \emptyset$, the result is due to Korevaar and Zagier (independently), who also obtained it via Newman's contour integration technique.

In this case, the result is contained in Ingham's Fatou-Riesz type theorem:

- Set $\tau(x) = \int_0^x \rho(u)\mathrm{d}u - b \Rightarrow \mathcal{L}\{\tau; z\} = \dfrac{\mathcal{L}\{\rho; z\} - b}{z}$ with $b = \mathcal{L}\{\rho; 0\}$.
- $\mathcal{L}\{\rho; z\}$ has analytic continuation beyond $\Re e\, z = 0$ if and only if

$$\mathcal{L}\{\tau; z\} = \frac{\mathcal{L}\{\rho; z\} - b}{z}$$

does.

The Arendt-Batty Tauberian theorem readily extends to functions with values on a Banach space. Here is a sample application of the vector-valued version:

Theorem (Arendt and Batty)

Let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup on a reflexive Banach space X. Denote the spectrum of its infinitesimal generator A as $\sigma(A)$. If $\sigma(A)\cap i\mathbb{R}$ is countable and no eigenvalue of A lies on the imaginary axis, then

$$\lim_{t\to\infty} T(t)x = 0, \quad \forall x \in X.$$

In recent times, Tauberian methods have been revisited to study rates of converge that can be a applied to PDE, e.g. decay estimates for damped wave equations.

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Theorem (Katznelson and Tzafriri)

Suppose that $F(z) = \sum_{n=0}^{\infty} c_n z^n$ *converges for* $|z| < 1$ *and* $S_n = \sum_{k=0}^{n} c_k = O(1)$ *(Tauberian condition). If* $F(z)$ *has analytic continuation to every point* $\partial\mathbb{D} \setminus \{1\}$, *then* $c_n = o(1)$.

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Application in operator theory

The above theorem has also a Banach space valued counterpart, which Katznelson and Tzafriri used to prove:

Theorem (Katznelson and Tzafriri, 1986)

Let T be a power-bounded operator on a Banach space ($\sup_{n\in\mathbb{N}}\|T^n\|<\infty$). Then,

$$\lim_{n\to\infty}\|T^{n+1}-T^n\|=0$$

if and only if $\sigma(T)\cap\partial\mathbb{D}\subseteq\{1\}$.

Proof: The contraposition of the direct implication follows by standard functional calculus. For the converse, if $\lambda I-T$ is invertible for all $|\lambda|\geq 1$, $\lambda\neq 1$, then $g(z)=\sum_{n=0}^{\infty}T^nz^n$ is analytic on $\partial\mathbb{D}\setminus\{1\}$, the same is true for

$$F(z)=(I-z)g(z)=\sum_{n=0}^{\infty}(T^n-T^{n+1})z^n\Rightarrow\|T^{n+1}-T^n\|\to 0.$$

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Theorem (Katznelson and Tzafriri, 1986)

Let T be a power-bounded operator on a Banach space ($\sup_{n\in\mathbb{N}} \|T^n\| < \infty$). Then,

$$\lim_{n\to\infty} \|T^{n+1} - T^n\| = 0$$

if and only if $\sigma(T) \cap \partial\mathbb{D} \subseteq \{1\}$.

Proof: The contraposition of the direct implication follows by standard functional calculus. For the converse, if $\lambda I - T$ is invertible for all $|\lambda| \geq 1$, $\lambda \neq 1$, then $g(z) = \sum_{n=0}^{\infty} T^n z^n$ is analytic on $\partial\mathbb{D} \setminus \{1\}$, the same is true for

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Pseudofunctions and pseudomeasures are notions that naturally arise in harmonic analysis.

- Pseudomeasures: $PM(\mathbb{R}) = \{g \in \mathcal{S}'(\mathbb{R}) : \widehat{g} \in L^{\infty}(\mathbb{R})\}$
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Given an open set $U \subseteq \mathbb{R}$, we define the local spaces:

- $PM_{loc}(U) = \{g \in \mathcal{D}'(U) : \varphi g \in PM(\mathbb{R}), \forall \varphi \in \mathcal{D}(U)\}$.
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- Every Radon measure on U is a local pseudomeasure.

Let G be analytic on $\Re e\, z > \alpha$ and $U \subset \mathbb{R}$ be open.

We say that G has local pseudofunction boundary behavior on $\alpha + iU$ if it has distributional boundary values there, i.e.

$$\lim_{x\to\alpha^+} G(x + iy) = g(y) \text{ in } \mathcal{D}'(U)$$

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Analogously, local pseudomeasure boundary behavior.

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Extension of the Ingham-Fatou-Riesz theorem

Theorem (Debruyne and Vindas, 2016)

Let $\tau \in L^1_{loc}(\mathbb{R})$ be slowly decreasing, vanish on $(-\infty, 0)$, and have convergent Laplace transform on $\Re e\, z > 0$. Suppose that there is a closed null set $E \subset \mathbb{R}$ such that:

(I) *The analytic function $\mathcal{L}\{\tau; z\} - \sum_{n=1}^{N} \frac{b_n}{z - it_n}$, where $t_n \in \mathbb{R}$, has local pseudofunction boundary behavior on $i(\mathbb{R} \setminus E)$,*

(II) *for each $t \in E$ there is $M_t > 0$ such that*

$$\sup_{x>0} \left| \int_0^x \tau(u) e^{-itu} \mathrm{d}u \right| < M_t,$$

(III) $E \cap \{t_1, \dots, t_N\} = \emptyset$.

Then $\tau(x) = \sum_{n=1}^{N} e^{it_n x} + o(1)$.

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Conversely, the relation

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implies that

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has local pseudofunction boundary behavior in the whole imaginary axis $i\mathbb{R}$.

Remark: This shows that there are actually no singular points for the local pseudofunction boundary behavior of $\mathcal{L}\{\tau; z\}$ in the above theorem.

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Second version of the Ingham-Fatou-Riesz theorem

Theorem (Debruyne and Vindas, 2016)

Let $\tau \in L^1_{loc}(\mathbb{R})$ be slowly decreasing, vanish on $(-\infty, 0)$, and have convergent Laplace transform on $\Re e\, z > 0$. Let $\beta_1 \leq \cdots \leq \beta_m \in [0, 1)$ and $k_1, \dots, k_m \in \mathbb{Z}_+$. The analytic function

$$\mathcal{L}\{\tau; z\} - \frac{a}{z^2} - \sum_{n=1}^{N} \frac{b_n}{z - it_n} - \sum_{n=1}^{m} \frac{c_n + d_n \log^{k_n}(1/z)}{z^{\beta_n + 1}} \qquad (t_n \in \mathbb{R})$$

has local pseudofunction boundary behavior on $\Re e\, z = 0$ if and only if

$$\tau(x) = ax + \sum_{n=1}^{N} b_n e^{it_n x} + \sum_{n=1}^{m} \frac{c_n x^{\beta_n}}{\Gamma(\beta_n + 1)} + \sum_{n=1}^{m} d_n x^{\beta_n} \sum_{j=0}^{k_n} \binom{k_n}{j} D_j(\beta_n + 1) \log^{k_n - j} x + o(1),$$

where $D_j(\omega) = \frac{d^j}{dy^j}\left(\frac{1}{\Gamma(y)}\right)\Big|_{y=\omega}$.

Second version of the Ingham-Fatou-Riesz theorem

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Theorem (Debruyne and Vindas, 2016)

Let S be a non-decreasing function and supported in $[0,\infty)$ such that $\mathcal{L}\{\mathrm{d}S;z\} = \int_{0^-}^{\infty} e^{-zt}\mathrm{d}S(t)$ converges for $\Re e\, z > \alpha > 0$. Suppose that there are a closed null set E, constants $r_0, r_1, \dots, r_N \in \mathbb{R}$, $\theta_1, \dots, \theta_N \in \mathbb{R}$, and $t_1, \dots, t_N > 0$ such that:

(I) $$\mathcal{L}\{\mathrm{d}S;z\} - \frac{r_0}{z-\alpha} - \sum_{n=1}^{N} r_n \left(\frac{e^{i\theta_n}}{z-\alpha-it_n} + \frac{e^{-i\theta_n}}{z-\alpha+it_n} \right)$$ *admits local pseudofunction boundary behavior on $\alpha + i(\mathbb{R}\setminus E)$,*

(II) $E \cap \{0, t_1, \dots, t_N\} = \emptyset$, *and*

(III) *for every $t \in E$, $\int_0^x e^{-\alpha u - itu}\mathrm{d}S(u) = O_t(1)$.*

Then

$$S(x) = e^{\alpha x}\left(\frac{r_0}{\alpha} + 2\sum_{n=1}^{N} \frac{r_n \cos(t_n x + \theta_n - \arctan(t_n/\alpha))}{\sqrt{\alpha^2 + t_n^2}} + o(1) \right).$$

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Conversely, if S has asymptotic behavior

$$S(x) = e^{\alpha x}\left(\frac{r_0}{\alpha} + 2\sum_{n=1}^{N}\frac{r_n\cos(t_n x + \theta_n - \arctan(t_n/\alpha))}{\sqrt{\alpha^2 + t_n^2}} + o(1)\right).$$

then

$$\mathcal{L}\{\mathrm{d}S; z\} - \frac{r_0}{z-\alpha} - \sum_{n=1}^{N} r_n\left(\frac{e^{i\theta_n}}{z-\alpha-it_n} + \frac{e^{-i\theta_n}}{z-\alpha+it_n}\right)$$

has local pseudofunction boundary behavior on the whole line $\Re e\, z = \alpha$.

Extension of the Katznelson-Tzafriri theorem

Theorem (Debruyne and Vindas, 2016)

Let $F(z) = \sum_{n=0}^{\infty} c_n z^n$ be analytic in $\mathbb{D}$. Suppose that there is a closed null subset $E \subset \partial\mathbb{D}$ such that F has local pseudofunction boundary behavior on $\partial\mathbb{D} \setminus E$, whereas for each $e^{i\theta} \in E$

$$\sum_{n=0}^{N} c_n e^{in\theta} = O_\theta(1)$$

Then, $c_n = o(1)$. Moreover, the series $\sum_{n=0}^{\infty} c_n e^{in\theta_0}$ converges at every point where there is a constant $F(e^{i\theta_0})$ such that

$$\frac{F(z) - F(e^{i\theta_0})}{z - e^{i\theta_0}}$$

has pseudofunction boundary behavior at $z = e^{i\theta_0} \in \partial\mathbb{D}$, and

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$$\sum_{n=0}^{N} c_n e^{in\theta} = O_\theta(1)$$

Then, $c_n = o(1)$. Moreover, the series $\sum_{n=0}^{\infty} c_n e^{in\theta_0}$ converges at every point where there is a constant $F(e^{i\theta_0})$ such that

$$\frac{F(z) - F(e^{i\theta_0})}{z - e^{i\theta_0}}$$

has pseudofunction boundary behavior at $z = e^{i\theta_0} \in \partial\mathbb{D}$, and

$$\sum_{n=0}^{\infty} c_n e^{in\theta_0} = F(e^{i\theta_0}).$$

An important particular case

Showing all of the above four theorems may be reduced to:

Theorem

Let $\tau \in L^1_{loc}(\mathbb{R})$ be slowly decreasing, vanish on $(-\infty, 0)$, and have convergent Laplace transform on $\Re e\, z > 0$. Suppose there is a closed null set $E \subset \mathbb{R}$ such that:

(I) *$\mathcal{L}\{\tau; z\}$ has local pseudofunction boundary behavior on $i(\mathbb{R} \setminus E)$,*

(II) *for each $t \in E$ there is $M_t > 0$ such that*

$$\sup_{x>0} \left| \int_0^x \tau(u) e^{-itu} \mathrm{d}u \right| < M_t,$$

(III) *$0 \notin E$.*

Then

$$\tau(x) = o(1).$$

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Some tools

Our proof of the previous theorem is based on:

1. Boundedness theorems for Laplace transforms with local pseudo-measure behavior.
2. Characterizations of local pseudofunctions.
3. The following further simplified version of the theorem:

Theorem

$\tau \in L^1_{loc}(\mathbb{R})$ slowly decreasing with convergent $\mathcal{L}\{\tau; z\}$ on $\Re e\, z > 0$. Then,

$$\lim_{x\to\infty} \tau(x) = 0$$

if and only if $\mathcal{L}\{\tau, z\}$ has local pseudofunction boundary behavior on the imaginary axis $i\mathbb{R}$.

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Finite forms

Lip(I; M) denotes the class of Lipschitz continuous functions on I with Lipschitz constant M.

Known result: Suppose that

1. $\tau \in L^1_{loc}[0, \infty)$,
2. $\mathcal{L}\{\tau; z\}$ has local pseudofunction boundary behavior on $(-i\lambda, i\lambda)$,
3. $\tau \in \mathrm{Lip}([X, \infty); M)$ for some X.

There is an absolute contant $\mathfrak{C} > 0$ such that

$$\limsup_{x\to\infty} |\tau(x)| \leq \frac{\mathfrak{C}M}{\lambda}.$$

Some values of $\mathfrak{C}$:

$$\mathfrak{C} = 6, \quad \text{Ingham (1935)}$$
$$\mathfrak{C} = 2, \quad \text{Korevaar, Zagier, and other people...}$$

Problem: Find the optimal value of $\mathfrak{C}$.

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Sharp finite forms

Theorem (Debruyne and Vindas, 2017)

Suppose that

1. $\tau \in L^1_{loc}[0, \infty)$
2. $\mathcal{L}\{\tau; z\}$ *has local pseudofunction boundary behavior on* $(-i\lambda, i\lambda)$
3. $\tau \in \mathrm{Lip}([X, \infty); M)$ *for some large* $X > 0$

Then

$$\limsup_{x \to \infty} |\tau(x)| \leq \frac{\pi M}{2\lambda}$$

and the value of $\pi/2$ *in this inequality cannot be improved.*

Define the 'oscillation' and 'decrease' moduli at ∞ as:

$$\Psi(\delta) = \limsup_{x\to\infty} \sup_{h\in[0,\delta]} |\tau(x+h) - \tau(x)|.$$

and

$$\Psi_{-}(\delta) = -\liminf_{x\to\infty} \inf_{h\in[0,\delta]} \tau(x+h) - \tau(x).$$

Theorem (Debruyne and Vindas, 2017)

Let $\tau \in L^1_{loc}(\mathbb{R})$ have at most polynomial growth. Suppose that $\hat{\tau} \in PF_{loc}(-\lambda, \lambda)$ (in particular if continuous there). Then,

$$\limsup_{x\to\infty} |\tau(x)| \leq \inf_{\delta>0} \left(1 + \frac{\pi}{2\delta\lambda}\right) \Psi(\delta)$$

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Inequalities for functions with regular Fourier transform

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Some references

The last part of this talk is based on our recent work:

- G. Debruyne, J. Vindas, Complex Tauberian theorems for Laplace transforms with local pseudofunction boundary behavior, J. Anal. Math., to appear (preprint: arXiv:1604.05069).
- G. Debruyne, J. Vindas, Optimal Tauberian constant in the Fatou-Riesz theorem for Laplace transforms, preprint: arXiv:1705.00667.

For applications of these results in analytic number theory, see:

- G. Debruyne, J. Vindas, On PNT equivalences for Beurling numbers, Monatsh. Math., in press, doi:10.1007/s00605-016-0979-9.
- G. Debruyne, J. Vindas, On Diamond's L^1 criterion for asymptotic density of Beurling generalized integers, preprint: arXiv:1704.03771.

Important book references on Tauberians

- W. Arendt, C. J. K. Batty, M. Hieber, F. Neubrander, Vector-valued Laplace transforms and Cauchy problems, Birkhäuser/Springer Basel, 2011.
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