New developments in the non-linear theory of generalized functions: optimal embeddings of ultradistributions and hyperfunctions

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In this talk we present various new developments in the non-linear theory of generalized functions.

We will discuss non-linear theories for ultradistributions and hyperfunctions. This includes:

- Construction of new differential algebras and embeddings.
- Optimality with respect to preservation of multiplication of functions.
- We also study Hörmander's sheaves of infrahyperfunctions (quasianalytic distributions).
- Connection to the Cousin problem for quasianalytic functions.

The talk is based on collaborative works with Andreas Debrouwere and Hans Vernaeve.

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Introduction

- The non-linear theory of generalized functions was initiated by Colombeau, who gave a framework for non-linear operations with distributions.
- Schwartz 'impossibility' result (roughly): There is no differential algebra containing $\mathcal{D}'$ as a differential linear subspace and simultaneously C^k as a subalgebra ($k < \infty$).
- Colombeau showed that the construction of such an algebra is possible if C^k is replaced by C^∞.
- In 1992, T. Gramchev initiated the non-linear theory of ultradistributions. However, there were always unsolved problems concerning optimality of embeddings ...
- The same year, the corresponding question for hyperfunctions was posed by M. Oberguggenberger.

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Some very basic notions from sheaf theory

Let $\mathcal{F}$ be a sheaf (always of vector spaces, always on $\mathbb{R}^d$). Notation:

- Sections on a open set Ω will be indistinctly denoted as $\mathcal{F}(\Omega) = \Gamma(\Omega, \mathcal{F})$.
- $\Gamma_K(\Omega, \mathcal{F})$ is the space of sections on Ω with supports in $K \Subset \Omega$.
- We write $\mathcal{F}_c(\Omega) = \Gamma_c(\Omega, \mathcal{F}) = \bigcup_{K \Subset \Omega} \Gamma_K(\Omega, \mathcal{F})$.
- For S closed, the space of germs is $\mathcal{F}[S] = \Gamma[S, \mathcal{F}]$.
- $\mathcal{F}$ is soft if sections over a closed set can be extended globally.

Lemma (Extension principle)

Let X be second countable and let $\mathcal{F}$ and $\mathcal{G}$ be soft sheaves on X. Let $\rho_c : \mathcal{F}_c(X) \to \mathcal{G}_c(X)$ be a linear mapping such that

$$\operatorname{supp} \rho_c(T) \subseteq \operatorname{supp} T, \qquad T \in \mathcal{F}_c(X). \quad \textit{(a local operator!)}$$

Then, there is a unique sheaf morphism $\rho : \mathcal{F} \to \mathcal{G}$ such that, for every open set U in X, we have $\rho_U(T) = \rho_c(T)$ for all $T \in \mathcal{F}_c(U)$. If, moreover,

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The central problem: Setting

Most linear spaces of generalize functions (distributions, hyperfunctions, ...) arise as sheaves having the following properties:

Let $\mathcal{F}$ be a sheaf of vector spaces (generalized functions) on $\mathbb{R}^d$ and let $\mathcal{R}$ be a subsheaf (regular elements). Assume:

1. Every $\mathcal{R}(\Omega) \subseteq C^\infty(\Omega)$ is a topological algebra with continuous action of partial derivatives.
2. The sections of $\mathcal{F}$ with support in a given compact set $K \Subset \Omega$ are given as follows:
$$\Gamma_K(\Omega, \mathcal{F}) = \mathcal{R}'[K],$$
with $\mathcal{R}[K]$ the space of germs on K.
3. $\mathcal{F}$ is an $\mathcal{R}$-module and $\mathcal{F}$ has a "natural" action of linear PDOs with coefficients in $\mathcal{R}$.
4. Sometimes $\mathcal{R}$ and $\mathcal{F}$ come with additional intrinsic differential structures (actions of infinite order differential operators).

Remark: Often, the third and fourth properties follow immediately from the first and second one. (For example, if $\mathcal{F}$ is soft).

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The central problem: Formulation

Suppose that $\mathcal{F}$ and $\mathcal{R}$ are a above. The central problem of the non-linear theory of generalized functions is:

Problem (Differential algebra embedding)

Find a sheaf of differential algebras $\mathcal{G}$ and a linear sheaf embedding $\iota : \mathcal{F} \to \mathcal{G}$ such that

1. *ι commutes with all partial derivatives.*
2. *ι preserves the multiplication on $\mathcal{R}$, namely, for all open set*

$$\iota_\Omega(f \cdot g) = \iota_\Omega(f) \cdot \iota_\Omega(g), \quad \forall f, g \in \mathcal{R}(\Omega).$$

I refer to property 2 above and the next one as optimality:

Problem (Preservation of natural structures)

If $\mathcal{R}$ and $\mathcal{F}$ have an additional "differential structure", find $\mathcal{G}$ with the same structure in a embedding preserving fashion.

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A general strategy

Here is a recipe one may try to follow.

- Suppose additionally that $\mathcal{F}$ is soft.
- One can try to construct a suitable soft sheaf of differential algebras $\mathcal{G}$ and a linear embedding at the level of compact sections:

$$\iota_c : \mathcal{F}_c(\mathbb{R}^d) \to \mathcal{G}_c(\mathbb{R}^d),$$

  commuting with partial derivatives (and possibly, preserving additional differential structures of $\mathcal{F}$).
- If ι_c is support preserving, the extension principle takes care of the existence of

$$\iota : \mathcal{F} \to \mathcal{G},$$

  usually with all desired properties, except perhaps preservation of multiplication on $\mathcal{R}$.
- Usually, ι_c is realized via a "regularization procedure". The regularization procedure should be good enough to encode multiplication of the "regular sheaf of functions" $\mathcal{R}$.

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The Colombeau algebra: Distribution case

The distribution case $\mathcal{F} = \mathcal{D}'$ and $\mathcal{R} = C^{\infty}$ was solved by Colombeau. We review here the construction of the so-called special algebra.

Consider the Fréchet space s of rapidly decreasing sequences,

$$s = \{(a_n)_n \in \mathbb{C}^{\mathbb{N}} : a_n = O(1/n^{\alpha}),\ \forall \alpha > 0\},$$

and its dual

$$s' = \{(a_n)_n \in \mathbb{C}^{\mathbb{N}} : a_n = O(n^{\alpha}),\ \text{for some } \alpha > 0\}.$$

The Colombeau algebra on Ω is then

$$\mathcal{G}(\Omega) = C^{\infty}(\Omega; s')/C^{\infty}(\Omega; s).$$

- The embedding $\iota_c : \mathcal{D}'_c(\mathbb{R}^d) = \mathcal{E}'(\mathbb{R}^d) \to \mathcal{G}_c(\mathbb{R}^d)$ is realized as $f \mapsto [(f * \phi_n)_n]$, where $\phi_n(x) = n^d \phi(nx)$, $\phi \in \mathcal{S}(\mathbb{R}^d)$ is such that

$$\int_{\mathbb{R}^d} \phi(x)dx = 1, \quad \int_{\mathbb{R}^d} x^{\alpha}\phi(x)dx = 0, \quad \forall \alpha \neq 0.$$

- Sheaf theory can be avoided: glue with partitions of the unity.

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Denjoy-Carleman classes of ultradifferentiable functions: Roumieu type

Let $(M_p)_{p\in\mathbb{N}}$ be a weight sequence, that is, a positive increasing sequence of real numbers with $M_0 = 1$.

- $\mathcal{E}^{\{M_p\}}(\Omega)$ consists of $\varphi \in C^\infty(\Omega)$ such that: for each $K \Subset \Omega$ there is $h > 0$ such that

$$\sup_{x\in K} \frac{|\partial^\alpha \varphi(x)|}{h^{|\alpha|} M_{|\alpha|}} < \infty.$$

- Topology of $\mathcal{E}^{\{M_p\}}(\Omega)$: take first inductive limit with respect to h and then projective limit with respect to K.
- These are highly non-metrizable spaces!
- If $M_p = p!^\sigma$, $\sigma > 0$, one recovers the Gevrey classes.
- The case $M_p = p!$ is the space of real analytic functions. It deserves a special notation (and attention!)

$$\mathcal{A}(\Omega) = \mathcal{E}^{\{M_p\}}(\Omega)$$

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Basic assumptions on the weight sequences

We shall impose the following three conditions on M_p:

$(M.1)$ $M_p^2 \leq M_{p-1}M_{p+1}$. (logarithmic convexity $\Rightarrow \mathcal{E}^{\{M_p\}}(\Omega)$ is an algebra)

$(M.2)$ $M_p \leq AH^p \min_{0\leq q\leq p} M_q M_{p-q}$, (stability under ultradifferential operators)

Ultradifferential operators: $P(D) = \sum_{\alpha\in\mathbb{N}^d} c_\alpha D^\alpha$, where

$$| c_\alpha | \leq C_L \frac{L^{|\alpha|}}{M_{|\alpha|}}, \quad (\forall L > 0.)$$

(NE) $p! \subset M_p$ (which translates in the dense inclusion $\mathcal{A}(\Omega) \subseteq \mathcal{E}^{\{M_p\}}(\Omega)$)

The associated function of M_p is defined as: $M(t) = \sup_{p\in\mathbb{N}} \log_+ \frac{t^p}{M_p}$.

One then splits into two cases (Denjoy-Carleman theorem)

$(M.3)'$ $\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty$ (non-quasianalyticity: $\mathcal{D}^{\{M_p\}}(\Omega) = \mathcal{D}(\Omega) \cap \mathcal{E}^{\{M_p\}}(\Omega)$)

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One then splits into two cases (Denjoy-Carleman theorem)

$(M.3)'$ $\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty$ (non-quasianalyticity: $\mathcal{D}^{\{M_p\}}(\Omega) = \mathcal{D}(\Omega) \cap \mathcal{E}^{\{M_p\}}(\Omega)$)

(QA) $\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} = \infty$ (quasianalyticity: $\mathcal{D}^{\{M_p\}}(\Omega) = \{0\}$)

Basic assumptions on the weight sequences

We shall impose the following three conditions on M_p:

$(M.1)$ $M_p^2 \leq M_{p-1}M_{p+1}$. (logarithmic convexity $\Rightarrow \mathcal{E}^{\{M_p\}}(\Omega)$ is an algebra)

$(M.2)$ $M_p \leq AH^p \min_{0\leq q\leq p} M_q M_{p-q}$, (stability under ultradifferential operators)

Ultradifferential operators: $P(D) = \sum_{\alpha\in\mathbb{N}^d} c_\alpha D^\alpha$, where

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Sheaves of linear generalized functions

We are interested in the sheaves of generalized functions corresponding to the sheaf of regular functions $\mathcal{R} = \mathcal{E}^{\{M_p\}}$.

- Non-quasianalytic case: Here it is easy $\mathcal{F} = \mathcal{D}'^{\{M_p\}}$, the sheaf of non-quasianalytic ultradistributions.
- Analytic case: $\mathcal{R} = \mathcal{A}$ is the sheaf of real analytic functions, and $\mathcal{F} = \mathcal{B}$ is the sheaf of Sato hyperfunctions. ($\Gamma_K(\mathbb{R}^d, \mathcal{B}) = \mathcal{A}'[K]$ is the Martineau-Harvey duality theorem).
- General quasianalytic case: $\mathcal{F} = \mathcal{B}^{\{M_p\}}$ is the sheaf of infrahyperfunctions (also called quasianalytic ultradistributions), constructed first by Hörmander in his seminal paper “Between distributions and hyperfunctions”.

Hörmander’s construction relies on a “hard analysis" approach to quasianalytic functionals, that is, the dual spaces $\mathcal{E}'^{\{M_p\}}(\Omega)$.

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Assume (QA).

- The space of germs $\mathcal{E}^{\{M_p\}}[K]$ is a (DFN)-space.
- $\mathcal{E}^{\{M_p\}}(\Omega) \cong \varprojlim_{K \Subset \Omega} \mathcal{E}^{\{M_p\}}[K]$. Modern terminology: a (PLN)-space.
- Consequently, $\mathcal{E}'^{\{M_p\}}(\Omega) \cong \varinjlim_{K \Subset \Omega} \mathcal{E}'^{\{M_p\}}[K] = \bigcup_{K \Subset \Omega} \mathcal{E}'^{\{M_p\}}[K]$.
- We say that $K \Subset \Omega$ is a $\{M_p\}$-carrier of $f \in \mathcal{E}'^{\{M_p\}}(\Omega)$ if $f \in \mathcal{E}'^{\{M_p\}}[K]$.
- For $f \in \mathcal{A}'(\Omega)$, there is a smallest $\{p!\}$-carrier of f, denoted by $\operatorname{supp}_{\mathcal{A}'} f$.

Theorem (Hörmander's support theorem)

In the quasianalytic case: For every quasianalytic functional $f \in \mathcal{E}'^{\{M_p\}}(\Omega)$ there is a smallest compact set among its $\{M_p\}$-carriers and it coincides with $\operatorname{supp}_{\mathcal{A}'} f$. We simply denote this set by $\operatorname{supp} f$ and call it its support.

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Hörmander's support theorem is the key to show:

Theorem (Hörmander)

Assume (QA) *holds. There exists an (up to isomorphism) unique flabby sheaf* $\mathcal{B}^{\{M_p\}}$ *such that*

$$\Gamma_K(\mathbb{R}^d, \mathcal{B}^{\{M_p\}}) = \mathcal{E}'^{\{M_p\}}[K], \qquad K \Subset \mathbb{R}^d.$$

Moreover, for any relatively compact open subset Ω *of* $\mathbb{R}^d$*,*

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Let us supplement the previous result:

Theorem

(*i*) *Let N_p be non-quasianalytic. We have the support preserving sheaf inclusions*

$$\mathcal{D}' \to \mathcal{D}'^{\{N_p\}} \to \mathcal{B}^{\{M_p\}} \to \mathcal{B}$$

(*ii*) *For every ultradifferential operator $P(D)$ of class $\{M_p\}$ there is a unique sheaf morphism*

$$P(D) : \mathcal{B}^{\{M_p\}} \to \mathcal{B}^{\{M_p\}}$$

that coincides on $\Gamma_c(\mathbb{R}^d, \mathcal{B}^{\{M_p\}}) = \mathcal{E}'^{\{M_p\}}(\mathbb{R}^d)$ with the usual action of $P(D)$ on quasianalytic functionals.

Non-linear theory of ultradistributions and infrahyperfunctions: What was known?

For non-quasianalytic ultradistributions:

- T. Gramchev constructed a differential algebra containing Gevrey ultradistributions ($M_p = p!^{\sigma}$). Drawbacks:
  1. Only works for $\sigma > 2$.
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We have introduced the following new algebra.

First a sequence space. Consider the (*DFS*)-space

$$s^{\{M_p\}} = \left\{ (a_n) \in \mathbb{C}^{\mathbb{N}} : |a_n| = O(e^{-M(\lambda n)}), \text{ for some } \lambda > 0 \right\},$$

its strong dual is the (*FS*)-space

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- We clearly have the embedding $\sigma : \mathcal{E}^{\{M_p\}} \to \mathcal{G}^{\{M_p\}}$ mapping f into the (equivalence class) of a constant sequence $f_n = f$.
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We use a mapping $\iota_c : \mathcal{E}'^{\{M_p\}}(\mathbb{R}^d) \to \mathcal{G}_c^{\{M_p\}}(\mathbb{R}^d)$

$$f \to [(f * \phi_n)_n].$$

The key point is to improve the properties of ϕ. We choose:

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Non-quasianalytic case: Embedding $\mathcal{D}'^{\{M_p\}}$ into $\mathcal{G}^{\{M_p\}}$

Theorem

Suppose M_p satisfies $(M.1)$, $(M.2)$, and $(M.3)'$. There is a sheaf monomorphism $\iota : \mathcal{D}'^{\{M_p\}} \to \mathcal{G}^{\{M_p\}}$ having the following properties on any open subset Ω of $\mathbb{R}^d$

(i) $\iota_{|\mathcal{E}'^{\{M_p\}}(\Omega)} = \iota_c$.

(ii) ι *commutes with $\{M_p\}$-ultradifferential operators $P(D)$,*

$$P(D)\iota(f) = \iota(P(D)f), \quad f \in \mathcal{D}'^{\{M_p\}}(\Omega).$$

(iii) $\iota_{|\mathcal{E}^{\{M_p\}}(\Omega)}$ *coincides with the constant embedding σ. In particular,*

$$\iota(fg) = \iota(f)\iota(g), \qquad f, g \in \mathcal{E}^{\{M_p\}}(\Omega).$$

Quasianalytic case: Is $\mathcal{G}^{\{M_p\}}$ even a sheaf?

- Quasianalytic case: $\mathcal{G}^{\{M_p\}}$ is a presheaf but not obvious at all whether it is a sheaf (no partitions of the unity available).
- We realized that showing $\mathcal{G}^{\{M^p\}}$ is a sheaf reduces to solve the Cousin problem for $\mathcal{E}^{\{M_p\}}(\Omega; s^{\{M_p\}})$.
- We needed the following assumption: Set $m_p = M_p/M_{p-1}$

$(M.2)^*$ $2m_p \leq Cm_{pQ}$, for some $Q \in \mathbb{N}$, $C > 0$.

- $(M.2)^*$ is intrinsically related to the topology of $s^{\{M_p\}}$: characterizes Vogt's (DN) property for its dual space.

Theorem

$\mathcal{G}^{\{M_p\}}$ is a soft sheaf under $(M.1)$, $(M.2)$, $(N.E)$, and $(M.2)^*$.

- Sheaf property: We showed the solvability of Cousin problem for vector-valued quasianalytic functions.
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The (first) Cousin problem for holomorphic functions

Theorem (Oka-Cartan)

Let $\Omega \subseteq \mathbb{C}^d$ be a Stein open and let $\{\Omega_i : i \in I\}$ be an open covering of Ω consisting of Stein open sets. Suppose $\varphi_{i,j} \in \mathcal{O}(\Omega_i \cap \Omega_j)$, $i, j \in I$, are given such that

$$\varphi_{i,j} + \varphi_{j,k} + \varphi_{k,i} = 0 \qquad \text{on } \Omega_i \cap \Omega_j \cap \Omega_k.$$

Then, there are $\varphi_i \in \mathcal{O}(\Omega_i)$, $i \in I$, such that

$$\varphi_{i,j} = \varphi_i - \varphi_j \qquad \text{on } \Omega_i \cap \Omega_j.$$

- Since every open set in $\mathbb{R}^d$ has a fundamental system of complex neighborhoods consisting of open sets, the Cousin problem is solvable for the sheaf of real analytic functions on $\mathbb{R}^d$ (now for arbitrary open sets and coverings)
- Is the Cousin problem solvable in general spaces of quasianalytic functions?

The (first) Cousin problem for holomorphic functions

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The Cousin problem for quasianalytic functions

We solved the vector-valued Cousin problem in the following case:

Theorem

Assume $(M.1)$, $(M.2)'$, (QA), and (NE) and let F be a (DFS)-space such that its strong dual F'_β has the property $(\underline{DN})$. Let $\Omega \subseteq \mathbb{R}^d$ be open and $\mathcal{M} = \{\Omega_i : i \in I\}$ be an open covering of Ω. Suppose $\varphi_{i,j} \in \mathcal{E}^{\{M_p\}}(\Omega_i \cap \Omega_j; F)$, $i, j \in I$, are given F-valued functions such that

$$\varphi_{i,j} + \varphi_{j,k} + \varphi_{k,i} = 0 \qquad \text{on } \Omega_i \cap \Omega_j \cap \Omega_k,$$

for all $i, j, k \in I$.Then, there are $\varphi_i \in \mathcal{E}^{\{M_p\}}(\Omega_i; F)$, $i \in I$, such that

$$\varphi_{i,j} = \varphi_j - \varphi_i \qquad \text{on } \Omega_j \cap \Omega_i,$$

for all $i, j \in I$.

A Féchet space E with a generating system of semi-norms $\{\| \, \|_k : k \in \mathbb{N}\}$ has the Vogt property $(\underline{DN})$ if

$$(\exists m \in \mathbb{N})(\forall k \in \mathbb{N})(\exists j \in \mathbb{N})(\exists \tau \in (0,1))(\exists C > 0)$$

$$\|x\|_k \leq C \|x\|_m^{1-\tau} \|x\|_j^{\tau}, \qquad x \in E.$$

Quasianalytic case: Carrying out the regularization procedure

Now we know that $\mathcal{B}^{\{M_p\}}$ and $\mathcal{G}^{\{M_p\}}$ are soft, the next step is to construct a regularization procedure for compact sections,

$$\iota_c : \mathcal{E}'^{\{M_p\}}(\mathbb{R}^d) \to \mathcal{G}_c^{\{M_p\}}(\mathbb{R}^d) \qquad f \to [(f * \theta_n)_n].$$

We constructed θ_n as follows ("analytic mollifier sequence"):

- Take a Hörmander analytic cut-off sequence $0 \leq \chi_n \leq 1$ for the unit ball:
  - (a) $\chi_n \equiv 1$ on $B(0,1)$,
  - (b) $(\chi_n)_n$ is a bounded sequence in $\mathcal{D}(B(0,2))$,
  - (c) there is $L \geq 1$ such that $\|\chi_n^{(\alpha)}\|_{L^\infty(\mathbb{R}^d)} \leq L(Ln)^{|\alpha|}, \ |\alpha| \leq n$.
- We define θ_n via Fourier transform $\theta_n(x) = n^d \mathcal{F}^{-1}(\chi_n)(nx)$ and ask: for every $c > 0$ there are $C, \delta, \gamma > 0$ such that

$$\sup_{|x|\geq c} |\theta_n^{(\alpha)}(x)| \leq C e^{-\delta n} \gamma^{|\alpha|} \alpha!, \qquad \alpha \in \mathbb{N}^d, n \in \mathbb{N}.$$

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$$\sup_{|x|\geq c} |\theta_n^{(\alpha)}(x)| \leq Ce^{-\delta n}\gamma^{|\alpha|}\alpha!, \qquad \alpha \in \mathbb{N}^d, n \in \mathbb{N}.$$

The above regularization procedure works to obtain:

Theorem

Suppose M_p satisfies $(M.1)$, $(M.2)$, (QA), (NE), and $(M.2)^*$. There is a sheaf monomorphism $\iota : \mathcal{B}^{\{M_p\}} \to \mathcal{G}^{\{M_p\}}$ having the following properties on any open subset Ω of $\mathbb{R}^d$

(i) $\iota_{|\mathcal{E}'^{\{M_p\}}(\Omega)} = \iota_c$.

(ii) ι *commutes with $\{M_p\}$-ultradifferential operators $P(D)$.*

(iii) $\iota_{|\mathcal{E}^{\{M_p\}}(\Omega)}$ *coincides with the constant embedding* σ
$\Longrightarrow$ ι *preserves the multiplication of $\mathcal{E}^{\{M_p\}}$-functions .*

For further details on the topic of this talk, see our recent articles:

- A. Debrouwere, H. Vernaeve, J. Vindas, Optimal embeddings of ultradistributions into differential algebras, Monatsh. Math., to appear, doi:10.1007/s00605-017-1066-6.
- A. Debrouwere, H. Vernaeve, J. Vindas, A non-linear theory of infrahyperfunctions, Kyoto J. Math., to appear (preprint: arXiv:1701.06996).
- A. Debrouwere, J. Vindas, Solution to the first Cousin problem for vector-valued quasianalytic functions, Ann. Mat. Pura Appl., doi:10.1007/s10231-017-0649-0.