The optimal Mallavin remainder problems for Beurling generalized numbers

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I Introduction: Generalized primes and integers were introduced by Beurling [3] in 1937 in order to study how independent the PNT (prime number theorem) is from the additive structure of the natural numbers.

A Beurling generalized prime number system (g-prime) is simply a non-decreasing unbounded sequence of real numbers $\mathcal{P}=\{p_k\}_{k=1}^{\infty}$ satisfying the only requirement $p_k\geq 1$, so

$$\mathcal{P}: \ 1<p_1\leq p_2\leq \cdots \leq p_k \longrightarrow \infty.$$

The associated generalized integers (g-integers) are 1 and all possible products of powers of g-primes counted according to multiplicities; we arrange them in a non-decreasing fashion:

$$\mathcal{N}=\{n_k\}_{k=0}^{\infty}: \quad n_0=1<n_1\leq n_2\leq \cdots \leq n_k \longrightarrow \infty.$$

One then considers their counting functions

$$\pi(x) = \sum_{p_k \le x} 1 \quad \text{and} \quad N(x) = \sum_{n_k \le x} 1 ,$$

and the Beurling zeta function

$$\zeta(s) = \sum_{k=0}^{\infty} \frac{1}{n_k^s} = \prod_{k=1}^{\infty} \frac{1}{1 - \frac{1}{p_k^s}} .$$

Here and below we give our Beurling functions the same names used in classical number theory. For instance, ψ and Π stand for the Chebyshev and Riemann weighted counting functions of the g-number system.

Example 1 Let $\mathcal{I}$ be the set of ideals of the ring of algebraic integers $\mathcal{O}_K$ of a number field K. Denote by $|I|$ the norm of an ideal of $\mathcal{O}_K$, that is, $|I| = \#\mathcal{O}_K/I$. Then, upon ordering, $\mathcal{P} = \{|I| : I \text{ is prime}\}$ and $\mathcal{N} = \{|I| : I \in \mathcal{I}\}$ can be regarded as a g-number system. Its Beurling zeta function is given by ζ_K, the Dedekind zeta function of the algebraic number field. ///

A central question in Beurling g-number theory is to determine conditions as minimal as possible, on one of the functions $N(x)$ or $\pi(x)$ such that the other one

becomes close to its classical counterpart.

Landau [9] antedated Beurling and essentially showed

Theorem 1 (Landau's PNT, 1903). If

$$(1)\quad N(x) = \rho x + O(x^{\theta}) \qquad (\rho > 0,\ 0 < \theta < 1),$$

then $\pi(x) \sim \dfrac{x}{\log x}$. ///

Beurling was able to weaken Landau's requirement on $N(x)$:

Theorem 2 (Beurling's PNT, 1937). The condition

$$N(x) = \rho x + O\left(\frac{x}{\log^{\gamma} x}\right) \qquad (\rho > 0)$$

for some $\gamma > \frac{3}{2}$ still ensures the PNT in the form $\pi(x) \sim \dfrac{x}{\log x}$.

If $\gamma = \frac{3}{2}$, the PNT might not hold. ///

2 The Mollifier error terms: Several authors have investigated the PNT with error term. Actually, Landau also antedated everyone [11]. He showed the next result using local analytic lemmas:

Theorem 3 (Landau, 1924). If (1) holds, then the PNT is satisfied with de la Vallée Poussin remainder:

$$(2)\quad \pi(x) = \mathrm{Li}(x) + O\left(\exp\left(-c\,(\log x)^{1/2}\right)\right) \qquad (c > 0),$$

where $\mathrm{Li}(x)$ stands for the logarithmic integral. ///

Apart from the exponent $1/2$ in (2), Mallivin [12] generalized Landau's result Theorem 3 in several directions. Hereafter α and β stand for numbers belonging to $(0,1]$.

Theorem 4 (Mallivin, 1961) If

$$(N_\beta) \qquad N(x) = \rho x + O(\exp(-b\log^\beta x)) \quad (\rho > 0)$$

holds (for some $b>0$), then

$$(P_\alpha) \qquad \pi(x) = Li(x) + O(\exp(-a\log^\alpha x))$$

is satisfied for some α (and some $a>0$). Conversely, given $\alpha \in (0,1]$, the condition (P_α) ensures (N_β) for some β. ///

In 1969, Bateman and Diamond essentially formulated the following problems, which we refer to as the optimal Mallivin remainder problems.

Problem 1. Given $\beta \in (0,1]$, determine

$$\alpha^* = \alpha^*(\beta) = \sup\{\alpha > 0 : (N_\beta) \Rightarrow (P_\alpha)\}. ///$$

Problem 2. Given $\alpha \in (0,1]$, find

$$\beta^* = \beta^*(\alpha) = \sup\{\beta > 0 : (P_\alpha) \Rightarrow (N_\beta)\}. ///$$

Mallivin himself gave the lower bounds $\alpha^* \geq \frac{\beta}{10}$ and $\beta^* \geq \frac{\alpha}{\alpha+2}$. Note that when $\beta = 1$, Landau's Theorem 3 delivers the better value $\alpha^* \geq \frac{1}{2}$.

We will survey what is known about Problems 1 and 2 in the next two sections.

In their influential work [2], Bateman and Diamond essentially conjectured the following values:

Conjecture 1 (Bateman and Diamond, 1969). For $\alpha, \beta \in (0,1]$,

$$\alpha^*(\beta) = \frac{\beta}{1+\beta} \quad \text{and} \quad \beta^*(\alpha) = \frac{\alpha}{1+\alpha}.$$

3 Problem 1: the remainder in the PNT. The only case for which a solution is known is $\beta = 1$, that is, the optimality of Landau's Theorem 3, which was established in [8] (see also [14]).

Theorem 5 (Diamond, Montgomery, Vorhauer, 2006) Let $\frac{1}{2} < \theta < 1$ and $a > 4\sqrt{(1-\theta)/2}$. There is a g-number system such that (2) holds for its g-integer counting function, its Beurling zeta function has infinitely many zeros on

$$\sigma = 1 - \frac{a^2}{4 \log t}, \quad t \geq 2$$

and no zeros on $\sigma > 1 - \frac{a}{\log t}$, and its Chebyshev function satisfies

$$\psi(x) = x + O\left(x \exp\left(-a\sqrt{\log x}\right)\right)$$

and

$$N(x) = x + \Omega_{\pm}\left(x \exp\left(-a\sqrt{\log x}\right)\right). \quad ///$$

Therefore, as RH fails for g-numbers, the message is that one can never succeed in proving RH solely based on multiplicative properties of the integers.

If we combine Theorem 3 and 5, we obtain

$$\alpha^*(1) = \frac{1}{2}.$$

In 2021, Broucke adapted the D-M-V method to establish an upper bound for general $\alpha^*(\beta)$, which actually agrees with the conjectural value, see [4].

The best lower bound is contained in Diamond & Zhang's book [10] (who noticed a little improvement in earlier work by Hall). We summarize these results:

Theorem 6: Given $\beta \in (0,1)$,

$$\frac{\beta}{\beta+1+U} \leq \alpha^*(\beta) \leq \frac{\beta}{1+\beta},$$

where the number U is the solution to one of the Landau extremal problems for positive trigonometric polynomials.

Remark 1 (i) $U = \inf\left\{ \frac{f(0)}{a_1 - a_0} : f(x) = \sum_{k=0}^{N} a_k \cos kx \geq 0 \right\}$

$a < a_j$, $0 < a_0 < a_1$

(ii) $5.8726\ldots < U < 5.90529\ldots$ (Chakalov)

(Arestov-Kondrat'ev)

(iii) See [13] for results concerning U.

4 **Problem 2: Solution.** The solution to the second Mallivain remainder problem is known. In this case the Bateman-Diamond conjecture holds true, that is,

$$\beta^*(\alpha) = \frac{\alpha}{\alpha+1},$$

and in fact $\beta^*(\alpha)$ is a minimum. To give further particulars, we state it in two theorems.

The first result gives the lower bound and it is essentially due to Diamond [7], although we add here the value of the best constant b that was first observed in [5,6], based on Balazard's adaptation [1] of Diamond's method. We rather work here with Riemann's weighted prime counting function

$$\Pi(x) = \sum_{j=1}^{\infty} \frac{1}{j} \pi(x^{\frac{1}{j}}).$$

Theorem 7 (Diamond 1970; Balazard 1999; Broucke, Debruyne, V. 2020 & 2024). If (P_α) holds, then

$$N(x) = \mathfrak{p}x + O\left(x \exp\left(-(a(\alpha+1))^{\frac{1}{\alpha+1}} (\log x \log_2 x)^{\frac{\alpha}{\alpha+1}} (1+o(1))\right)\right)$$

On the other hand,

Theorem 8 (Broucke, Debruyne, V. 2024).

Let $0<\alpha<1$ and $q>0$, where we require $q\leq 1$ if $\alpha=1$. There is a g-number system such that

$$\Pi(x)-Li(x) \ll \begin{cases} x\exp(-a\log^{\alpha}x) & \text{if } \alpha<1 \text{ or } \alpha=1 \text{ and } q<1 \\ \log_2 x & \text{if } \alpha=q=1. \end{cases}$$

and

$$N(x)=px+\mathcal{R}_{\pm}\left(x\exp\left(-(a(\alpha+1)^{\alpha+1})^{\frac{1}{\alpha+1}}(\log x\log_2 x)^{\frac{\alpha}{\alpha+1}}(1+\gamma(1))\right)\right).$$

Remark 2:

(i) When $\alpha=1$ and $\frac{1}{2}\leq q$ in Theorem 8, the g-number system satisfies RH, but has integer with extrem oscillation.

(ii) The construction from [5,6] makes use of a widely spread idea in Beurling theory. One first attempts to construct a "antiruous" example of a number system and then proceeds to discretize it via an approximation procedure. We employed the following random g-prime approximation scheme from [15] (which improves upon the one from [8]):

Theorem 9 (Broucke and V. 2024). Let F be a right continuous non-decreasing function such that $F(1)=0$ and $F(x) << \frac{x}{\log x}$. Then there is a g-prime number system whose g-prime counting function approximates F as follows:

$$F(x) = \pi(x) + O(1)$$

and (uniformly in t)

$$\left| \sum_{p_k \le x} p_k^{-it} - \int_1^x 0^{-it} \, dF(u) \right| << \sqrt{x} + \sqrt{\frac{x \log(|t|+1)}{\log(|x|+1)}}$$

(iii) The main challenge was the construction of an "absolutely continuous number system" having the properties stated in Theorem 8.

Definition 1 A non-necessarily discrete Beurling g-number system is a pair of right-continuous non-decreasing functions Π and N such that $\Pi(1)=0$, $N(1)=1$, and $dN = \exp^*(d\Pi)$, where the exponential is taken with respect to the multiplicative convolution of measures. ///

Example 2. Let $N_0(x) = x$ for $x \geq 1$. If

$$\Pi_0(x) = \int_1^x \frac{1 - \frac{1}{u}}{\log u} du, \quad x \geq 1,$$

then Π_0 and N_0 form a g-number system. ////

The idea carried out in [5,6] is inspired by Bohr's treatment of the optimality of the convexity bounds for Dirichlet series, obtained in his thesis back in 1910. We considered perturbations of the number system from Example 2 constructed by adding an infinite number of sparse "sine-arcs". More concretely, we worked with the Chebyshev functions and considered perturbations of the form:

$$\Psi_C(x) = x - 1 - \log x + \sum_{k=0}^{\infty} (R_k(x) + S_k(x)),$$

where for carefully chosen parameters $\{\gamma_k\}_{k=1}^{\infty}$ and $A_k < B_k < C_k < A_{k+1}$

$$R_k(x) = \begin{cases} \frac{1}{2} \int_{A_k}^{B_k} (1 - u^{-1}) \cos(\gamma_k \log u) du, & A_k \leq x \leq B_k \\ 0 & \text{otherwise} \end{cases}$$

$$S_k(x) = \begin{cases} R_k(B_k) + \frac{1}{2}(B_k - 1 - \log B_k)(x - 1 - \log x), & B_k \leq x \leq C_k \\ 0 & \text{otherwise} \end{cases}$$

The (many!) technical details go beyond the scope of the talk.

(iv) Harold Diamond pointed out that our construction to prove Theorem 8 resembles the butterfly effect: it is remarkable that a small perturbation in the primes (from Example 2) can reinforce itself in such a way, analogous to the butterfly effect, to produce such a big discrepancy in the integers; nevertheless without fully destroying the integer law.

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