Mental Arithmetic Tricks

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The normal tuxmath questions are simple. With the options TRICK_<trickname>_ALLOWED you can enable some tricky problems which need explanation.

The first five tricks are suitable for disciples at 5th or 6th grade. I add a short explanation for each trick. Teachers should work out longer explanations with more examples before presenting them to the class.

1. **The trick with the 37 (TRICK_37_ALLOWED)**
   
   \[37 \times 3 = 111\]. This gives us \[37 \times (3n) = nnn\].

   **Example:**
   
   \[37 \times 12 = 444\]
   \[12 \div 3 = 4\]

2. **The trick with the 25 (TRICK_37_ALLOWED)**
   
   \[25 \times 4 = 100\]. This makes it easy to multiply 25 with any other two digit number.

   **Example:**
   
   \[25 \times 35 = 875\]
   \[35 \div 4 = 8 \text{ remainder } 3, \ 3 \times 25 = 75\]

3. **The mirror at 5 (TRICK_MIRROR_5_ALLOWED)**
   
   For the next trick we need two numbers with the property that the first digits are the same and the sum of the last digits is 10.

   Then it easy to calculate the product
   
   \[(ab)_{10} \times (ac)_{10} = (xy \ zw)_{10}\]
   \[a \times (a+1) = (xy)_{10}\]
   \[b \times c = (zw)_{10}\]

   **Example:**
62 × 68 = 4216

6 × 7 = 42

2 × 8 = 16

(Important: if the last two digits are 1 and 9, you must write \(1 \times 9 = 09\)
don’t forget the leading 0!)
The proof of the trick is just a little exercise in calculus.

4. The mirror at 10 (TRICK_MIRROR_10_ALLOWED)
The third binomial theorem \((x + y)(x - y) = x^2 - y^2\) give us a nice trick for mental arithmetic, if we specialise \(x\) to a multiple of 10 and \(y\) to a number less than 10.

Example:

\[
\begin{array}{l}
76 \times 64 = 4864 \\
7 \times 7 - 1 = 48 \\
6 \times 6 = 36, 100 - 36 = 64
\end{array}
\]

5. Squares (TRICK_SQUARE20_ALLOWED)
This is not really a trick, but it is useful to know the squares up to 20.
The next tricks are designed to make a show out of your mental arithmetic skills.

1. Exercising third powers (TRICK_CUBES_ALLOWED)
You must learn the third power for the numbers from 1 to 10.

<table>
<thead>
<tr>
<th>(x)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x^3)</td>
<td>1</td>
<td>8</td>
<td>27</td>
<td>64</td>
<td>125</td>
<td>216</td>
<td>343</td>
<td>512</td>
<td>729</td>
<td>1000</td>
</tr>
</tbody>
</table>

2. Cube roots (TRICK_CUBEROOT_ALLOWED)
Ask one in the audience to choose a two digit number \(n\) and compute \(n^3\). When you hear \(n^3\) you can tell immediately \(n\).

The trick is quite easy \(x \mapsto x^3 \mod 10\) is a bijection on \(\{0, \ldots, 9\}\) (see table above). You know all third powers, so you get directly the last digit.
The first digit you get from the size of the number.

Example
You hear 658503. Remove the last three digits. You get 658, since \(8^3 \leq 658 < 9^3\) the result must be between 80 and 89.
Since last digit is 3 and \(7^3 = 343\) the last digit of the result must be 7.
So you have the answer 87.

3. **5th roots** (TRICK_FIFTH_ROOT_ALLOWED)

Taking the 5th root is even easier than the cuberoot. You have \(x^5 \equiv x \mod 10\), so the last digit is simple.

For the first digit you learn the following table:

<table>
<thead>
<tr>
<th>(x)</th>
<th>Size of (x^5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100 thousand</td>
</tr>
<tr>
<td>2</td>
<td>3 million</td>
</tr>
<tr>
<td>3</td>
<td>24 million</td>
</tr>
<tr>
<td>4</td>
<td>100 million</td>
</tr>
<tr>
<td>5</td>
<td>300 million</td>
</tr>
<tr>
<td>6</td>
<td>777 million</td>
</tr>
<tr>
<td>7</td>
<td>1.6 milliards</td>
</tr>
<tr>
<td>8</td>
<td>3 milliards</td>
</tr>
<tr>
<td>9</td>
<td>6 milliards</td>
</tr>
<tr>
<td>10</td>
<td>10 milliards</td>
</tr>
</tbody>
</table>

(Our American friends must replace the word milliards by billions.)

You hear 8587340257. As soon as you hear 8 milliards you know by the table that the first digit must be 9. You ignore the rest and wait for the last digit which is 7, so the result is 97.

4. **Square roots** (TRICK_SQUAREROOT_ALLOWED)

You can also take fast square roots, if you know that the solution must be an integer. Consider the problem \(\sqrt{7569}\).

Since \(8^2 \leq 75 < 9^2\) you know that the first digit must be an 8.

\(x^2 \equiv 9 \mod 10\) has two solutions \(x = 3\) and \(x = 7\), so the last digit is either 3 or 7. To decide which one is correct you calculate \(85^2 = 7225\) (use the trick mirror at 5). \(7569 > 7225\), so the last digit must be 7.

Solution \(\sqrt{7569} = 87\).

5. **Cube roots of large numbers** (TRICK_LARGE_CUBEROOT_ALLOWED)

Now it gets really hard. Ask one of the audience to compute the cube of a three digit number. You will tell the cube root.

I explain two ways to solve the problem

**First method:**

Learn the following table.

<table>
<thead>
<tr>
<th>(x)</th>
<th>(x^3) mod 11</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
</table>

When you hear 582182875 you know \(8^3 \leq 582 < 9^3\) so the first digit must be 8. You see the last digit 5 and as in the case of the two digit result
you know that the last digit of the result must be 5. So the answer is 800 + 10b + 5. To determine b you compute 582182875 mod 11. Of course you don’t dived by 11 you use the alternating sum of digits

\[ 582182875 \equiv 5 - 8 + 2 - 1 + 8 - 2 + 8 - 7 + 5 = 10 \quad \text{mod} \ 11 \]

(Warning: Make sure that use start with the right sign. The sign of the last digit must be positive).

By the table you know that the result leaves the remainder 10 modulo 11. So you have to solve 800 + 10b + 5 ≡ 10 mod 11. Again you use the alternating sum of digits and get b = 8 + 5 − 10 = 3.

So the answer is 835.

**Second method:**

It is helpful to memorize the following table:

<table>
<thead>
<tr>
<th>x</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>3x^2 mod 10</td>
<td>3</td>
<td>2</td>
<td>7</td>
<td>8</td>
<td>5</td>
<td>8</td>
<td>7</td>
<td>2</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>3x^2</td>
<td>3</td>
<td>12</td>
<td>27</td>
<td>48</td>
<td>75</td>
<td>108</td>
<td>147</td>
<td>192</td>
<td>243</td>
<td>300</td>
</tr>
</tbody>
</table>

As in the case of to digit numbers you easily find the first and the last digit. Now you must determine the middle digit.

The easy case is that last digit is an odd number different from 5.

**Example:**

You hear 258474853. From the 3 at the end you now that the last digit is 7 and form 6^3 ≤ 258 < 7^3 you know that the first digit is 6. Thus the solution is 600 + b * 10 + 7.

What is b. Compute the result modulo 100. You get b × 3 × 7^2 + 7^3.

Now you subtract 7^3 from 258474853 you are only interested in the tens place. The result is 5 − 4 = 1. So you must solve b × 3 × 7^2 ≡ 1 mod 10 or b × 7 ≡ 1 mod 10. (That the reason why you learned the table.) This equation has only one solution, b = 3.

So you get the result 637.

The case that the last digit is even (and different from 0) is similar.

**Example:**

You hear 191102976. Thus the fist digit must be 5 and last digit must be 6.

Subtract 6^3 you need only the digit at the tens place, which is 7 − 1 = 6.

So must solve b × 3 × 6 ≡ 6 mod 10. This has two solutions b = 2 and b = 7. To decide which correct you again at the front. 191 close to 6^3 = 216 than to 5^3 = 125, so b must be 7. (You can also compare 191 with 5.5^3 ≈ 166.)

<table>
<thead>
<tr>
<th>x</th>
<th>1.5</th>
<th>2.5</th>
<th>3.5</th>
<th>4.5</th>
<th>5.5</th>
<th>6.5</th>
<th>7.5</th>
<th>8.5</th>
<th>9.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>x^3</td>
<td>3.375</td>
<td>15.625</td>
<td>42.875</td>
<td>91.125</td>
<td>166.375</td>
<td>274.625</td>
<td>421.875</td>
<td>614.125</td>
<td>857.375</td>
</tr>
</tbody>
</table>
Thus the result is 576.

5 as last digit is very nasty.

**Example:**

You hear 256047875. As always the first digit 6 and the last digit 5 is easy. From the 7 at the tens place you learn that the middle digit of the result must be odd. (Other wise the power would end with 25.)

Since the first digit even you have \((600 + 10b + 5)^3 \equiv (10b + 5)^3 \mod 1000\).

You know:

<table>
<thead>
<tr>
<th>(b)</th>
<th>((10b + 5)^3 \mod 1000)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>375</td>
</tr>
<tr>
<td>1</td>
<td>375</td>
</tr>
<tr>
<td>2</td>
<td>875</td>
</tr>
<tr>
<td>3</td>
<td>375</td>
</tr>
<tr>
<td>4</td>
<td>375</td>
</tr>
<tr>
<td>5</td>
<td>875</td>
</tr>
<tr>
<td>6</td>
<td>375</td>
</tr>
<tr>
<td>7</td>
<td>375</td>
</tr>
<tr>
<td>8</td>
<td>875</td>
</tr>
<tr>
<td>9</td>
<td>375</td>
</tr>
</tbody>
</table>

So \(b = 3, 5\) or 7.

Look again at the front compare 256 is between 6\(^3\) and 6.5\(^3\), so \(b\) has to be 3.

The result 635.

When presenting this trick in a show, you can make your live much easier, if you ensure that no one choose a number divisible by 5.

You can combine both method to compute cube roots even if the result has 4 digits.

It needs some exercise to present this tricks fast, but I guarantee you that people will be very impressed if you use them in a show.

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