

On the Number of Membranes in Unary P Systems

Rudolf Freund^(A,B) Andreas Klein^(C) Martin Kutrib^(D)

^(A)Fakultät für Informatik, Technische Universität Wien,
Favoritenstraße 9–11, A-1040 Wien, Austria
rudi@emcc.at

^(C)Department of Pure Mathematics and Computer Algebra, Ghent University,
Krijgslaan 281-S22, B-9000 Ghent, Belgium
klein@cage.ugent.be

^(D)Institut für Informatik, Universität Giessen,
Arndtstraße 2, D-35392 Giessen, Germany
kutrib@informatik.uni-giessen.de

Abstract. We consider P systems with a linear membrane structure working on objects over a unary alphabet using sets of rules resembling homomorphisms. Such a restricted variant of P systems allows for a unique minimal representation of the generated unary language and in that way for an effective solution of the equivalence problem. Moreover, we examine the descriptive complexity of unary P systems with respect to the number of membranes.

Keywords: descriptive complexity, equivalence problem, number of membranes, P systems, unique representation

1 Introduction

One decade ago, inspired by biological systems to be found in nature, Gheorghe Păun introduced membrane systems (e.g., see [4]) as a class of distributed parallel computing models, originally working on multisets of objects. The main feature of membrane systems (soon called P systems in the literature) are the hierarchically structured membranes embedded in the outermost *skin* membrane. Every membrane encloses a *region* possibly containing other membranes as well as multisets of specific objects that evolve according to *evolution rules* associated with the regions. A *computation* is a sequence of consecutive *configurations* of the system evolving by applying the evolution rules in parallel to the objects in every region in a maximal manner. The vector of the multiplicities of objects present in the specified *output membrane* in the final configuration of a halting computation is considered to be the *result* of such a computation; in that way, a membrane system *computes* a set of vectors of natural numbers.

^(B)corresponding author

Many variants of P systems are able to characterize the family of recursively enumerable sets of vectors of natural numbers (the Parikh sets associated with recursively enumerable languages). The first detailed definitions and results can be found in [4], and an overview of many variants is given in [6]. The first monograph on membrane systems [5] shows the great variety and potentials of P systems; for the current state of the art in membrane systems we refer the interested reader to the P page [9].

In [7] it is shown that special variants of P systems with the underlying membrane structure being only a linear tree can be computationally complete when using adequate evolution rules. In [3], several quite simple variants of such P systems were investigated that allowed for establishing an infinite hierarchy with respect to the number of membranes, which usually cannot be the case for computationally complete variants of P systems. These special variants of P systems were inspired by the tree systems of morphisms (introduced in [2]) and the compound Lindenmayer systems (investigated in [1]). In this paper, we consider those variants as introduced in [3], yet especially over a unary alphabet, and investigate the descriptonal complexity of the unary languages generated by such systems.

The rest of the paper is organized as follows: In the second section we give some preliminary definitions of notions needed in this paper and then define the special variant of self-reproducing P systems to be investigated in the succeeding sections; moreover, we also construct a suitable example and prove some first results. In the third section, we show how to construct a unique minimal representation of the unary language generated by a unary P system, which also allows us to show the decidability of the equivalence problem for unary P systems in the succeeding section. The descriptonal complexity of unary P systems with respect to the number of membranes is investigated in the fifth section.

2 A Specific Variant of P Systems

After having specified some notions, in this section we introduce the main variant of P systems investigated in this paper, i.e., P systems with a linear membrane structure and rules mapping each object to powers of itself. Moreover, we give an illustrative example and prove two useful lemmas valid for this specific variant of P systems.

2.1 Preliminaries

For the basic notions of formal language theory we refer the reader to monographs as [8]. We just mention some specific notions used throughout the paper: The empty string is denoted by λ , and for the length of a string w we write $|w|$. The set of mappings from a set M to a set N is denoted by N^M . We denote the set of positive integers by \mathbb{N}_+ and the set of non-negative integers by \mathbb{N} . \mathbb{N}^n denotes the set of vectors (x_1, \dots, x_n) with $x_i \in \mathbb{N}$, $1 \leq i \leq n$, and for $(x_1, \dots, x_n) \in \mathbb{N}^n$,

$(x_1, \dots, x_n)^T$ denotes the corresponding transposed vector. Moreover, we write \mathbb{P} for the set of prime numbers, and throughout the paper p_i denotes the i -th prime number, i.e., $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, \dots

2.2 Self-reproducing P Systems

Now let us consider the following very special variant of P systems, where

- the underlying membrane structure is a linear tree $[_0[1\dots[n]_n\dots]_1]_0$;
- the only axiom is in the innermost membrane;
- except for the skin membrane (labeled by 0), each region $i \geq 1$ contains the membrane dissolving operation δ as well as for each object a exactly one (noncooperative) rule of the special form $a \rightarrow a^{m(i,a)}$ for some $m(i,a) > 0$;
- there are no rules in the skin membrane;
- the result of a computation is the multiset finally appearing in the skin membrane (which constitutes a halting computation).

There are no additional features used when applying the (noncooperative) rules of the special form $a \rightarrow a^{m(i,a)}$ in a maximally parallel manner; for example, there are no priority relations among the rules.

The movement of objects from the innermost membrane to the skin membrane only happens by dissolving one membrane after the other one using the membrane dissolving operation δ which may be chosen at any time for a membrane containing some objects.

These special P systems as described above were introduced in [3] and had been inspired by the tree systems of morphisms (introduced in [2]) and the compound Lindenmayer systems (investigated in [1]). In the case of linear trees, these tree systems with morphisms intuitively correspond with the self-reproducing deterministic linear P systems as defined above. The main difference is the (additional) skin membrane, which only serves for terminating the computation and collecting its result.

Due to this correspondence with tree systems with morphisms, we can easily describe a P system of the special form as defined above in the following way:

$$\Pi = (V, h_1, \dots, h_n, w)$$

where V is an alphabet, n is the height of the linear tree describing the membrane structure, w is the axiom over V , and the h_i are the homomorphisms defined by the rules of the form $a \rightarrow a^{m(i,a)}$ in region i . As each object is mapped to powers of itself only, we call these P systems *self-reproducing*.

As it is easy to see, the language generated by a self-reproducing P system

$$\Pi = (V, h_1, \dots, h_n, w)$$

can be written as

$$h_1^*(h_2^* \cdots (h_n^*(w)) \cdots),$$

in other words,

$$L_*(\Pi) = \{h_1^{m_1}(h_2^{m_2} \cdots (h_n^{m_n}(w)) \cdots) \mid m_i \geq 0, 1 \leq i \leq n\}.$$

We may also demand that every homomorphism has to be applied at least once before the corresponding membrane may be dissolved; in that case, the corresponding language generated by $\Pi = (V, h_1, \dots, h_n, w)$ can be written as

$$h_1^+(h_2^+ \cdots (h_n^+(w)) \cdots),$$

in other words,

$$L_+(\Pi) = \{h_1^{m_1}(h_2^{m_2} \cdots (h_n^{m_n}(w)) \cdots) \mid m_i \geq 1, 1 \leq i \leq n\}.$$

2.3 An Illustrative Example and First Results

Example 1. For each $n \geq 1$, consider the language

$$L(n) = \{a_1^{p_1^{m_1}} \cdots a_n^{p_n^{m_n}} \mid m_i \geq 0, 1 \leq i \leq n\}$$

where p_i denotes the i -th prime number.

$L(n)$ is generated by the P system

$$\Pi(n) = (V, h_1, \dots, h_n, w)$$

with $V = \{a_1, \dots, a_n\}$, $w = a_1 \dots a_n$, $h_i(a_i) = a_i^{p_i}$ and $h_i(a_k) = a_k$ for $k \neq i$, $1 \leq k \leq n$, $1 \leq i \leq n$.

Obviously, $a_1^{p_1^{m_1}} \cdots a_n^{p_n^{m_n}}$ is obtained as $h_1^{m_1}(\cdots (h_n^{m_n}(a_1 \dots a_n)) \cdots)$; moreover, only elements from $L(n)$ can be generated by $\Pi(n)$, hence, $L(n) = L_*(\Pi(n))$. Finally, we observe that

$$L_+(\Pi(n)) = \{a_1^{p_1^{m_1}} \cdots a_n^{p_n^{m_n}} \mid m_i \geq 1, 1 \leq i \leq n\}.$$

As it was shown in [3], $L(n)$ cannot be generated by a self-reproducing P system with less than n homomorphisms (or equivalently, less than $n+1$ membranes), i.e., these variants of P systems establish an infinite hierarchy with respect to the number of membranes.

We now establish two simple but nevertheless useful lemmas. The first observation is that the ordering of the homomorphisms is irrelevant (which directly follows from the commutativity of multiplication):

Lemma 2. *Let $\Pi = (V, h_1, \dots, h_n, w)$ be a self-reproducing P system, u be a permutation of $1, \dots, n$, and $\Pi' = (V, h_{u(1)}, \dots, h_{u(n)}, w)$. Then $L_*(\Pi) = L_*(\Pi')$ as well as $L_+(\Pi) = L_+(\Pi')$.*

Proof: Let $a \in V$ and let $|w|_a$ denote the number of objects a in the axiom w of Π ; then obviously

$$h_{u(1)}^{m_{u(1)}}(h_{u(2)}^{m_{u(2)}} \cdots (h_{u(n)}^{m_{u(n)}}(a^{|w|_a})) \cdots) = h_1^{m_1}(h_2^{m_2} \cdots (h_n^{m_n}(a^{|w|_a})) \cdots),$$

as, due to the commutativity of multiplication, $(a^m)^k = a^{mk} = a^{km} = (a^k)^m$ for all $k, m \geq 1$. \square

Using Lemma 2, next we show that we may move the first application of every homomorphism into the axiom, i.e., for every self-reproducing P system Π there is an equivalent self-reproducing P system Π' such that $L_+(\Pi) = L_*(\Pi')$.

Lemma 3. *Let Π be a self-reproducing P system. Then we can effectively construct a self-reproducing P system Π' such that $L_+(\Pi) = L_*(\Pi')$.*

Proof: For $\Pi = (V, h_1, \dots, h_n, w)$ we obtain

$$\begin{aligned} L_+(\Pi) &= \{h_1^{m_1}(h_2^{m_2} \cdots (h_n^{m_n}(w)) \cdots) \mid m_i \geq 1, 1 \leq i \leq n\} \\ &= \{h_1^{m_1-1}(h_2^{m_2-1} \cdots (h_n^{m_n-1}(h_1(\cdots (h_n(w)) \cdots))) \cdots) \\ &\quad \mid m_i \geq 1, 1 \leq i \leq n\} \\ &= \{h_1^{m'_1}(h_2^{m'_2} \cdots (h_n^{m'_n}(w')) \cdots) \mid m'_i \geq 0, 1 \leq i \leq n\} \\ &= L_*(\Pi'), \end{aligned}$$

where $\Pi' = (V, h_1, \dots, h_n, w')$ with $w' = h_1(\cdots (h_n(w)) \cdots)$. \square

The following example shows that the converse of Lemma 3 does not hold:

Consider self-reproducing P systems over a one letter alphabet, i.e., *unary P systems* of the form $\Pi = (\{a\}, h_1, \dots, h_n, w)$; without loss of generality, let us assume that none of the homomorphisms h_i with $1 \leq i \leq n$ equals the identity $h_i(a) = a$, because it would have no effect on the results. Clearly, the axiom w belongs to the language $L_*(\Pi)$. Moreover, there cannot be any other string in $L_*(\Pi)$ that is shorter than the axiom. So, choosing $w = a^{p_i}$ and a sole homomorphism h_1 such that $h_1(a) = a^{p_j}$, where p_i and p_j are different prime numbers, we obtain a system $\Pi = (\{a\}, h_1, a^{p_i})$ whose language $L_*(\Pi)$ is $\{a^{p_i p_j^m} \mid m \geq 0\}$. On the other hand, it is easy to see that for any other unary system Π' the string a^{p_i} cannot belong to $L_+(\Pi')$ unless the axiom is a and in Π' there must be a homomorphism h' with $h'(a) = a^{p_i}$. But in this case, $a^{p_i^2}$ belongs to $L_+(\Pi')$, too. This is a contradiction, since p_i and p_j are different prime numbers and, thus, $a^{p_i^2}$ cannot belong to $L_*(\Pi)$.

3 Representation of Unary P Systems

We now consider *unary P systems*, i.e., systems of the form

$$\Pi = (\{a\}, h_1, \dots, h_n, w)$$

where $n \geq 1$, $w \in \{a\}^*$ is the axiom, and $h_i : \{a\} \rightarrow \{a\}^*$, $1 \leq i \leq n$, are homomorphisms. Depending on whether or not each homomorphism has to be applied at least once, as above we distinguish two languages generated by Π :

$$\begin{aligned} L_*(\Pi) &= \{h_1^{m_1}(h_2^{m_2} \cdots (h_n^{m_n}(w)) \cdots) \mid m_i \geq 0, 1 \leq i \leq n\} \\ L_+(\Pi) &= \{h_1^{m_1}(h_2^{m_2} \cdots (h_n^{m_n}(w)) \cdots) \mid m_i \geq 1, 1 \leq i \leq n\} \end{aligned}$$

Now we turn to a language representation which is advantageous for our purposes. To this end, we consider the decomposition of positive integers into prime factors. We denote by F the set of mappings from \mathbb{P} to \mathbb{N} with finite support, i.e., only finitely many primes are mapped to non-zero values. Then the decompositions are represented by the mapping $pf : \mathbb{N}_+ \rightarrow F$. For example, let $n = q_1^{e_1} \cdots q_k^{e_k}$, where $k \geq 1$, q_i prime, $q_i \neq q_j$ for $i \neq j$, $e_i \geq 1$, $1 \leq i, j \leq k$. Then $(pf(n))(q_i) = e_i$, for $1 \leq i \leq k$, and $(pf(n))(q) = 0$, for $q \notin \{q_1, \dots, q_k\}$. For convenience, we write pf_n for the mapping $pf(n)$. Clearly, it holds $pf_{mn} = pf_m + pf_n$. Next, let $gpf(m)$ be the index of the greatest prime factor with non-zero exponent in the decomposition of m , if $m > 1$, and set $gpf(1) = 1$.

Since pf is injective, every unary language L is uniquely represented by the set $P(L) = \{pf_m \mid a^m \in L\}$. Conversely, since pf is surjective every subset $V \subseteq F$ uniquely represents a unary language

$$P^{-1}(V) = \{a^m \mid \text{there exists } f \in V \text{ such that } m = p_1^{f(p_1)} p_2^{f(p_2)} p_3^{f(p_3)} \cdots\}.$$

(Recall that p_i always denotes the i -th prime number.) Alternatively, V may be a set of vectors of natural numbers, e.g., $(m_1, \dots, m_n)^T \in V$ implies $a^m \in P^{-1}(V)$, where $m = p_1^{m_1} \cdots p_n^{m_n}$. From this point of view, a vector of natural numbers may be seen as a mapping from F and vice versa. In the sequel we will use both notions in a synonymous way.

Let $\Pi = (\{a\}, w, h_1, \dots, h_n)$ be a system, where $h_i(a) = a^{c_i}$, for $c_i \geq 1$, $1 \leq i \leq n$. We set $k = \max\{gpf(|w|), gpf(c_1), \dots, gpf(c_n)\}$ to be the largest prime factor with non-zero exponent appearing in $|w|$ or one of the exponents c_i , and define the matrix:

$$M_\Pi = \begin{pmatrix} pf_{c_1}(p_1) & pf_{c_2}(p_1) & \cdots & pf_{c_n}(p_1) \\ pf_{c_1}(p_2) & pf_{c_2}(p_2) & \cdots & pf_{c_n}(p_2) \\ \vdots & \vdots & \ddots & \vdots \\ pf_{c_1}(p_k) & pf_{c_2}(p_k) & \cdots & pf_{c_n}(p_k) \end{pmatrix}$$

Lemma 4. *Let $\Pi = (\{a\}, h_1, \dots, h_n, w)$ be a unary P system. Then*

$$L_+(\Pi) = P^{-1}(M_\Pi(\mathbb{N}_+^n)^T + b) \quad \text{and} \quad L_*(\Pi) = P^{-1}(M_\Pi(\mathbb{N}^n)^T + b),$$

where $b = (pf_{|w|}(p_1), pf_{|w|}(p_2), \dots, pf_{|w|}(p_k))^T$.

Proof: Let $k = \max\{gpf(|w|), gpf(c_1), \dots, gpf(c_n)\}$ and $h_i(a) = a^{c_i}$, for $c_i \geq 1$, $1 \leq i \leq n$, be defined as above. There exist $(m_1, \dots, m_n) \in \mathbb{N}_+^n$ and $(e_1, \dots, e_k)^T \in M_\Pi(\mathbb{N}_+^n)^T + b$ such that the following is true.

$$\begin{aligned}
 & a^m \in L_+(\Pi) \\
 \iff & a^m = h_1^{m_1}(h_2^{m_2} \dots (h_n^{m_n}(w)) \dots) \\
 \iff & m = |w|c_n^{m_n} c_{n-1}^{m_{n-1}} \dots c_1^{m_1} \\
 \iff & p f_m(p_i) = p f_{|w|}(p_i) + m_n p f_{c_n}(p_i) + \dots + m_1 p f_{c_1}(p_i), \quad 1 \leq i \leq k \\
 \iff & p f_m(p_i) = (M_\Pi(m_1, m_2, \dots, m_n)^T + b)[i], \quad 1 \leq i \leq k \\
 \iff & p f_m(p_i) = e_i, \quad 1 \leq i \leq k, \quad p f_m(q) = 0, \quad q \notin \{p_1, \dots, p_k\} \\
 \iff & m = p_1^{e_1} \dots p_k^{e_k} \\
 \iff & a^m \in P^{-1}(M_\Pi(\mathbb{N}_+^n)^T + b)
 \end{aligned}$$

The assertion for $L_*(\Pi)$ follows in an analogous way. \square

Due to the pumping lemma for regular languages, we immediately observe the following result:

Corollary 5. *The only context-free languages that can be generated by a unary P system are of the form $\{a^m\}$ for some $m \geq 1$.*

4 Equivalence and Minimality of Unary P Systems

We now may take advantage of the chosen language representation in order to show the decidability of the equivalence problem for unary P systems. Moreover, we obtain a unique minimal representation.

Consider the partial ordering

$$f \leq g \iff f(p) \leq g(p) \text{ for all } p \in \mathbb{P}$$

defined on the set of mappings $\mathbb{N}^{\mathbb{P}}$. As usual, we call an element f minimal if $g \leq f$ implies the identity $g = f$. Let $\Pi = (\{a\}, h_1, \dots, h_n, w)$ be a unary P system and $V = M_\Pi(\mathbb{N}^n)^T + b$ be such that $L_*(\Pi) = P^{-1}(V)$. In any other system $\Pi' = (\{a\}, h'_1, \dots, h'_n, w')$ which generates the same language, i.e., $V' = M_{\Pi'}(\mathbb{N}^{n'})^T + b'$ such that $P^{-1}(V'^{-1}) = P^{-1}(V)$, the same prime factors as in Π must occur. Hence, M_Π and $M_{\Pi'}$ must have the same number of rows. Moreover, a decomposition $b = b_1 + b_2$ implies $b \geq b_1$. Therefore, b is the unique minimal element in V and, thus, the unique minimal element in V' , which implies $b = b'$. For this reason, in order to deal with equivalence and minimality, for what follows we may assume $b = 0$ and consequently may minimize and compare sets of the form $M_\Pi(\mathbb{N}^n)^T$.

An element $x \in M_\Pi(\mathbb{N}^n)^T \setminus \{0_k\}$ – where 0_k denotes the zero-vector from $(\mathbb{N}^n)^T$ – is called irreducible, if there is no decomposition of the form $x = x_1 + x_2$, where $x_1, x_2 \in M_\Pi(\mathbb{N}^n)^T \setminus \{0_k\}$. The next theorem leads to a unique minimal representation.

Theorem 6. *Let Π be a unary P system with n homomorphisms. If M' is the matrix whose columns are the n' irreducible elements of $M_{\Pi}(\mathbb{N}^n)^T \setminus \{0_k\}$, then (i) all columns of M' are columns of M_{Π} and (ii) $M'(\mathbb{N}^{n'})^T = M_{\Pi}(\mathbb{N}^n)^T$.*

Proof: Since an irreducible element of $M_{\Pi}(\mathbb{N}^n)^T \setminus \{0\}$ cannot be decomposed into summands, it has a representation of the form $M_{\Pi}(0, \dots, 0, 1, 0, \dots, 0)^T$. Therefore, it is a column of M_{Π} , and assertion (i) follows.

By (i), the inclusion $M'(\mathbb{N}^{n'})^T \subseteq M_{\Pi}(\mathbb{N}^n)^T$ follows immediately. Now assume that in contrast to assertion (ii), the inclusion is a proper one. Then there is a minimal element $x \in M_{\Pi}(\mathbb{N}^n)^T \setminus M'(\mathbb{N}^{n'})^T$. Since $0_k \in M_{\Pi}(\mathbb{N}^n)^T \cap M'(\mathbb{N}^{n'})^T$, we have $x \neq 0_k$. Moreover, x is not irreducible, since otherwise it would be a column of M' and, thus, would belong to $M'(\mathbb{N}^{n'})^T$. So, there is a decomposition $x = x_1 + x_2$, where $x_1, x_2 \in M_{\Pi}(\mathbb{N}^n)^T \setminus \{0\}$. This implies $x_1 \leq x$ as well as $x_2 \leq x$. Since x is minimal in $M_{\Pi}(\mathbb{N}^n)^T \setminus M'(\mathbb{N}^{n'})^T$, both elements x_1 and x_2 belong to $M'(\mathbb{N}^{n'})^T$. Obviously, $M'(\mathbb{N}^{n'})^T$ is closed under addition. Therefore, x belongs to $M'(\mathbb{N}^{n'})^T$, too. From this contradiction we infer assertion (ii). \square

The preceding theorem shows that $M'(\mathbb{N}^{n'})^T + b$ is the unique minimal representation of the language generated by Π , i.e., $L_*(\Pi) = P^{-1}(M'(\mathbb{N}^{n'})^T + b)$. It is easy to determine the irreducible columns out of the n columns of M_{Π} . Hence, matrix M' can effectively be constructed from M_{Π} . Conversely, given some M' with n' columns and b , the unary P system Π' whose language is $P^{-1}(M'(\mathbb{N}^{n'})^T + b)$ can effectively be constructed, too.

Corollary 7. *There exists an effective algorithm which minimizes the number of homomorphisms (membranes) of a unary P system.*

Corollary 8. *The equivalence of unary P systems is decidable.*

5 Descriptive Complexity of Unary P Systems

This section is devoted to descriptive complexity issues of unary P systems with a different number of homomorphisms (membranes). The key question is how succinct a language, given by some unary P system with n homomorphisms, can be represented by some unary P system with at least n homomorphisms.

In order to talk about the economy of descriptions we first have to define what is meant by the size of a system. In general, we are interested to measure the length of the string that defines a system. In particular, we use more convenient size measures, such that there is a recursive upper bound for the length of the defining string depending on the chosen size measure. For example, the size of a finite automaton equals the product of the number of its states and the number of its input symbols.

The *size* $|\Pi|$ of a unary P system $\Pi = (\{a\}, h_1, \dots, h_n, w)$ is defined to be

$$|\Pi| = |w| + \sum_{1 \leq i \leq n} |h_i(a)|.$$

Denote the family of unary P systems by \mathcal{F} . Clearly, the considered measure implies a total, recursive function mapping a unary P system to its size, such that \mathcal{F} is recursively enumerable in order of increasing size, and does not contain infinitely many members of the same size.

Let \mathcal{F}_1 and \mathcal{F}_2 be two subfamilies of \mathcal{F} . A function $f : \mathbb{N}_+ \rightarrow \mathbb{N}_+$, with $f(n) \geq n$, is said to be an upper bound for the increase in size when changing from a minimal description in \mathcal{F}_1 to an equivalent minimal description in \mathcal{F}_2 , if

$$\min\{|\Pi| \mid \Pi \in \mathcal{F}_2 \text{ generates } L\} \leq f(\min\{|\Pi| \mid \Pi \in \mathcal{F}_1 \text{ generates } L\})$$

for all languages L generated by some system in \mathcal{F}_1 as well as by some system in \mathcal{F}_2 .

The following theorem is an immediate consequence of Theorem 6. Since any matrix of some representation contains at least the columns of irreducible elements, any equivalent system has at least the homomorphisms associated with these columns, respectively.

Theorem 9. *Let Π be a unary P system with n homomorphisms where $L_*(\Pi) = P^{-1}(M_\Pi(\mathbb{N}^n)^T + b)$. If the columns of M_Π are different irreducible elements of $M_\Pi(\mathbb{N}^n)^T \setminus \{0_k\}$, then every unary system Π' with $L_*(\Pi) = L_*(\Pi')$ is at most of size $|\Pi|$.*

So, in case of $L_*(\Pi)$, the identity is an upper bound for the increase of size when possibly changing from a description with n homomorphisms to an equivalent description with $n' < n$ homomorphisms. The situation is different in case of $L_+(\Pi)$:

Theorem 10. *Let Π be a unary P system with n homomorphisms. If there is a unary P system Π' with $n - 1$ homomorphisms such that $L_+(\Pi) = L_+(\Pi')$, then $|\Pi'| \in O(|\Pi|^2)$.*

Proof: If a language $L = L_+(\Pi)$ is generated by some unary P system

$$\Pi = (\{a\}, h_1, \dots, h_n, w)$$

whose number of homomorphisms can be reduced, then there is at least one column of M_Π which is not an irreducible element of $M_\Pi(\mathbb{N}^n)^T \setminus \{0_k\}$. Let h_n be the associated homomorphism, then

$$\Pi' = (\{a\}, h_1, \dots, h_{n-1}, h_n(w))$$

generates L , too, i.e., $L = L_+(\Pi')$. Therefore, we can approximate the size of Π' as follows:

Let $m = \max\{|h_i(a)| \mid 1 \leq i \leq n\}$, then

$$|\Pi'| = |\Pi| - |h_n(a)| - |w| + m|w| \leq m|w| \leq \frac{|\Pi|}{2} \frac{|\Pi|}{2} \in O(|\Pi|^2).$$

□

The next lemma shows that the upper bound of Theorem 10 can be reached.

Lemma 11. *Let $\Pi = (\{a\}, h_1, h_2, w)$ where $h_1(a) = h_2(a) = a^m$ and $w = a^m$, for some $m \geq 2$, then any unary P system Π' with $L_+(\Pi) = L_+(\Pi')$ has size $\Omega(|\Pi|^2)$.*

Proof: Obviously, $h_1(a) = a^m$ is associated with a necessary column in M_Π . So, the sole equivalent unary P system with one homomorphism is

$$\Pi' = (\{a\}, h_1, h_2(w)).$$

For the sizes we obtain $|\Pi| = 3m$ and $|\Pi'| = m + m^2 \in \Omega(|\Pi|^2)$. \square

If we reduce the number of homomorphisms by more than one, i.e., to iterate the construction x times, then we do not obtain sizes of the form $O(|\Pi|^{2^x})$. The reason is that we have to multiply the sizes by at most a factor $|\Pi|$ in each step.

Corollary 12. *Let Π be a unary P system with n homomorphisms. If there is a unary P system Π' with $n - x$ homomorphisms such that $L_+(\Pi) = L_+(\Pi')$, then $|\Pi'| \in O(|\Pi|^{x+1})$.*

By an immediate generalization of the proof of Lemma 11 we obtain a matching lower bound (in the order of magnitude) in the worst case.

Lemma 13. *Let $\Pi = (\{a\}, h_1, \dots, h_n, w)$ where $h_1(a) = \dots = h_n(a) = a^m$ and $w = a^m$, for some $m \geq 2$, then any unary P system Π' with $n - x$ homomorphisms, $1 \leq x \leq n - 1$, and $L_+(\Pi) = L_+(\Pi')$ has size $\Omega(|\Pi|^{x+1})$.*

Finally, if we fix the number m and consider arbitrarily large n , then the trade-off in the size may become exponential with respect to the base m .

Example 14. Let $\Pi = (\{a\}, h_1, \dots, h_n, w)$ where $h_1(a) = \dots = h_n(a) = a^m$ and $w = a^m$, for some $m \geq 2$, then any unary P system Π' with one homomorphism and $L_+(\Pi) = L_+(\Pi')$ has size $\Omega(|\Pi|^n)$.

The representation and the descriptive complexity of languages generated by less restricted variants of P systems remains as a challenging task for future research; for example, we may ask to which extent the results proved above for unary P systems can also be formulated for self-reproducing P systems.

References

- [1] Jürgen Dassow. On Compound Lindenmayer Systems. In Grzegorz Rozenberg, Arto Salomaa (eds.), *The Book of L*. Springer-Verlag, Berlin, 1986, 75–86.
- [2] Jürgen Dassow, Gheorghe Păun, Gabriel Thierrin, Sheng Yu. Tree-Systems of Morphisms. *Acta Informatica* **38** (2001), 131–153.
- [3] Rudolf Freund. Special Variants of P Systems Inducing an Infinite Hierarchy with Respect to the Number of Membranes. *Bulletin of the EATCS* **75** (2001), 209–219.
- [4] Gheorghe Păun. Computing with membranes. *J. of Computer and System Sciences* **61**, 1 (2000), 108–143, and *TUCS Research Report* **208** (1998) (<http://www.tucs.fi>).
- [5] Gheorghe Păun. *Membrane Computing. An Introduction*. Springer-Verlag, Berlin, 2002.
- [6] Gheorghe Păun, Grzegorz Rozenberg. A Guide to Membrane Computing. *Theoretical Computer Science*, **287**, 1 (2002), 73–100.
- [7] Gheorghe Păun, Yasubumi Sakakibara, Takashi Yokomori. P Systems on Graphs of Restricted Forms. *Publicationes Mathematicae Debrecen* **60** (2002), 635–660.
- [8] G. Rozenberg, A. Salomaa (eds.). *Handbook of Formal Languages* (3 volumes). Springer-Verlag, Berlin, 1997.
- [9] <http://ppage.psystems.eu>. *The P Systems Web Page*.