On partitions of the q-ary Hamming Space into few spheres

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Abstract

In this article we present a generalization of a result due to Hollmann, Körner and Litsyn [9]. They prove that each partition of the n-dimensional binary Hamming space into spheres consists of either one or two or at least $n+2$ spheres. We prove a q-ary version of that gap theorem and consider the problem of the next gaps.

1 Introduction

The geometry of the Hamming space is of special interest in combinatorics. In particular the partitions of the Hamming space into spheres have been examined by several authors. A partition into spheres with equal radii is known as a perfect code (see [11] for an overview). For a prime power q all perfect codes are known [14, 18]. If q is not a prime power, a perfect e-error-correcting code with $e \geq 3$ does not exist [2, 10]. Partitions into spheres with different radii are called *generalized perfect codes* [1].

In this paper we extend the classification of generalized perfect codes with few spheres given in [9].

Hollmann, Körner and Litsyn prove the following "gap theorem": For each partition of the n-dimensional binary Hamming space into k spheres we have either $k \leq 2$ or $k \geq n+2$. In each partition with $n+2$ spheres one sphere must have radius $n-2$ and all other spheres have radius 0.

An alternate proof of this theorem can be found in [8].

For $q > 2$ the situation is more difficult. For small dimensions n there are sporadic examples. But we can prove: For large n the first gap is between 1 and $(q-1)^n+1$ (Theorem 9). As a generalization of the binary gap theorem we prove that the second gap lies between $(q-1)^n+1$ and $1+(q-1)^{n-1}(n+q-1)-A_{q-1}(n,1)n(q-1)$ where $A_{q-1}(n,1)$ denotes the maximal number of words in a $(q-1)$-ary n-dimensional 1 error-correcting code (Theorem 10). (For $q=2$ this reduces to gap theorem proven in [9], if we formally set $A_1(n,1)=0$.)

Further we investigate the next gaps for the case $q > 4$ (Theorem 11) and $q=2$ (Theorem 12).

For $q = 2$ we can completely solve the problem of the second gap and classify the sporadic examples (Theorem 13).

The key idea of the proofs is simple: Either the average size of the spheres is small, i.e. the number of spheres is large, or a very large sphere must exist. In the latter case we count the spheres adjacent to that large sphere. However, we have to work hard to get these counting arguments right (see the lemmata in Section 3). With these lemmata we can quickly derive the main theorems in Section 4. The classification of all sporadic examples in the binary case (Theorem 13) involves the same ideas, but requires investigation of many cases.

2 Preliminaries

We consider the n-dimensional q-ary Hamming space.

A sphere of radius r with center C is set of all points with distance $\leq r$ from C. We call the set of all points with distance from c exactly r the surface of the sphere. A partition of the Hamming space into spheres is a family of disjoint spheres that covers all points.

We denote the volume of a sphere with radius r by

$$V_{n,q}(r) = \sum_{j=0}^{r} \binom{n}{j} (q-1)^j$$

and the number of points of weight r by

$$S_{n,q}(r) = \binom{n}{r} (q-1)^r \ .$$

Note $V_{n,q}(r) = \sum_{j=0}^{r} S_{n,q}(j)$.

It is well known that the asymptotic behavior of $V_{n,q}(r)$ and $S_{n,q}(r)$ can be described by the q-ary entropy function

$$H_q(x) = \begin{cases} 0 & \text{for } x = 0 \\ x \log_q(q-1) - x \log_q x - (1-x) \log_q(1-x) & \text{for } 0 < x \leq 1 \end{cases} .$$

We remember that the size of the spheres in the q-ary Hamming space can be bounded as the following lemma shows. Here and in the following asymptotic calculations terms like λn always mean $[\lambda n]$.

Lemma 1
For $0 \leq \lambda \leq 1 - \frac{1}{q}$ we have

$$e^{-\frac{1}{6}} \frac{1}{\sqrt{2\pi n \lambda (1-\lambda)}} q^{H_q(\lambda) n} \leq S_{n,q}(\lambda n) \leq V_{n,q}(\lambda n) \leq q^{H_q(\lambda) n} \ .$$

Similarly, for $1 - \frac{1}{q} \leq \lambda \leq 1$ we have

$$e^{-\frac{1}{6}} \frac{1}{\sqrt{2\pi n \lambda (1-\lambda)}} q^{H_q(\lambda) n} \leq S_{n,q}(\lambda n) \leq \sum_{n \geq i \geq \lambda n} S_{n,q}(i) \leq q^{H_q(\lambda) n} \ .$$

(See for example [17] Theorem 1.4.5.)

With

$$S_{n,q}(r+1) = \binom{n}{r+1}(q-1)^{r+1} = \frac{n-r}{r+1}(q-1)S_{n,q}(r)$$

we obtain:

Lemma 2

$S_{n,q}(r)$ is increasing for $r \in [0, (1-\frac{1}{q})n]$ and decreasing for $r \in [(1-\frac{1}{q})n, n]$.

3 Lemmata about partitions into few spheres

The following lemma ensures that there are "only few large spheres".

Lemma 3

In each partition of the q-ary Hamming space into spheres there are at most q spheres with radius greater than $\left(\frac{1}{2} - \frac{1}{2\binom{q+1}{2}}\right)n$.

Proof

Assume that there are $q+1$ spheres of this kind. The center points of two such spheres differ in more than

$$2\left(\frac{1}{2} - \frac{1}{2\binom{q+1}{2}}\right)n = n - \frac{n}{\binom{q+1}{2}}$$

coordinates. Thus the center points of all $q+1$ spheres differ in more than $n - \binom{q+1}{2}\frac{n}{\binom{q+1}{2}} = 0$ coordinates. Hence there is at least one coordinate in which all $q+1$ center points differ. □

Now we prove that partitions without a "large sphere" must contain "many" spheres.

Lemma 4

Let $q \geq 3$. Then for sufficiently large n each partition of the n-dimensional q-ary Hamming space into spheres of radius $\leq (1-\frac{1}{q})n$ contains at least

$$\frac{e^{-\frac{1}{6}}}{2\sqrt{2\pi n\frac{1}{q}(1-\frac{1}{q})}}(q-\frac{1}{2})^n$$

spheres.

Proof

We distinguish two cases:

Case 1: At least one sphere has radius r with

$$\left(1-\frac{1}{q}-\frac{H_q^{-1}\left(1-\log_q(q-\frac{1}{2})\right)}{4}\right)n \le r \le (1-\frac{1}{q})n\ .$$

Without loss of generality we assume that the large sphere has its center in origin.

Let $\varepsilon = \frac{H_q^{-1}(1-\log_q(q-\frac{1}{2}))}{4}n$. As a first step we derive a bound for the number of points of weight between $r+1$ and $(1-\frac{1}{q})n+\varepsilon$ covered by spheres of radius $\ge 4\varepsilon$.

We consider a sphere S with a center of weight $r+x \ge r+4\varepsilon$ and radius $r' < x$. We denote the number of points of weight $r+x'$ in S by $N(x')$. Then

$$N(x') = \sum_{\substack{a+b+c\le r'\\ x-x'=a-c}} \binom{r+x}{a}\binom{r+x-a}{b}(q-2)^b\binom{n-r-x}{c}(q-1)^c\ .$$

(To see this, start with the center of S and change a nonzero coordinates to zero, b nonzero coordinates to another nonzero value and c zero coordinates to nonzero.)

We prove that $N(x')$ is strictly increasing for $x' \in [r+1, (1-\frac{1}{q})n+\varepsilon]$.

$$\begin{aligned} N(x'+1) &= \sum_{\substack{a+b+c\le r'\\ x-x'-1=a-c}} \binom{r+x}{a}\binom{r+x-a}{b}(q-2)^b\binom{n-r-x}{c}(q-1)^c \\ &\ge \sum_{\substack{a+b+c\le r'\\ x-x'=a-c}} \binom{r+x}{a-1}\binom{r+x-(a-1)}{b+1}(q-2)^{b+1}\binom{n-r-x}{c}(q-1)^c \end{aligned}$$

We observe that

$$\begin{aligned} &\binom{r+x}{a-1}\binom{r+x-(a-1)}{b+1}(q-2)^{b+1} \\ &\quad= \frac{a(r+x-a+1)}{(r+x-a+1)(b+1)}(q-2)\left[\binom{r+x}{a}\binom{r+x-a}{b}(q-2)^b\right] \\ &\quad\ge \binom{r+x}{a}\binom{r+x-a}{b}(q-2)^b \qquad \text{, since } a \ge x-x' \ge 2\varepsilon \text{ and } b \le x' \le 2\varepsilon \end{aligned}$$

and therefore

$$N(x'+1) \ge N(x')\ .$$

Thus there are at least $S_{n,q}((1-\frac{1}{q})n) - S_{n,q}((1-\frac{1}{q})n+\varepsilon)$ points of weight $(1-\frac{1}{q})n$ which are not covered by spheres of radius $\ge 4\varepsilon$. We need at least

$$\frac{S_{n,q}((1-\frac{1}{q})n) - S_{n,q}((1-\frac{1}{q})n+\varepsilon)}{V_{n,q}(4\varepsilon)}$$

spheres of radius $\leq 4\varepsilon$ to cover these points. By Lemma 1 we find

$$\frac{S_{n,q}((1-\frac{1}{q})n) - S_{n,q}((1-\frac{1}{q})n+\varepsilon)}{V_{n,q}(4\varepsilon)}$$

$$\geq \frac{e^{-\frac{1}{6}}\frac{1}{\sqrt{2\pi n\frac{1}{q}(1-\frac{1}{q})}}q^n - q^{H_q(1-\frac{1}{q}+\frac{\varepsilon}{n})n}}{q^{H_q(\frac{4\varepsilon}{n})n}}$$

$$\geq \frac{e^{-\frac{1}{6}}}{2\sqrt{2\pi n\frac{1}{q}(1-\frac{1}{q})}}\frac{q^n}{q^{H_q(\frac{4\varepsilon}{n})n}} \qquad \text{for } n \text{ sufficiently large}$$

$$= \frac{e^{-\frac{1}{6}}}{2\sqrt{2\pi n\frac{1}{q}(1-\frac{1}{q})}}(q-\frac{1}{2})^n \qquad \text{, since } \frac{4\varepsilon}{n} = H_q^{-1}(1-\log_q(q-\frac{1}{2})).$$

Case 2: All spheres have radii smaller than $\left(1-\frac{1}{q}-\frac{H_q^{-1}\left(1-\log_q(q-\frac{1}{2})\right)}{4}\right)n$.

Before we deal with the general case, we study the situation in which all spheres have radius

$$r < r_s = H_q^{-1}\left(1-\log_q\left(q-\frac{1}{2}\right)\right)n\ .$$

In this case Lemma 1 yields that the volume of each sphere is bounded by

$$q^{H_q(r_s/n)n} < q^{[1-\log_q(q-\frac{1}{2})]n}\ ,$$

and thus the number of spheres is at least

$$\frac{q^n}{q^{[1-\log_q(q-\frac{1}{2})]n}} = \left(q-\frac{1}{2}\right)^n\ .$$

Now we are ready to deal with the general case. We prove that in this case the average size of the spheres is too small, i.e. there are more than $(q-\frac{1}{2})^n$ spheres.

First we note that by Lemma 3 there are at most q spheres of radius greater or equal to $\left(\frac{1}{2}-\frac{1}{2\binom{q+1}{2}}\right)n$. These spheres cover at most

$$qV_{n,q}\left(\left(\frac{1}{2}-\frac{1}{2\binom{q+1}{2}}\right)n\right) \leq q^{H_q\left(\frac{1}{2}-\frac{1}{2\binom{q+1}{2}}\right)n+1} \leq \bar{q}^n$$

points with $\bar{q} < q$, since $H_q\left(\frac{1}{2}-\frac{1}{2\binom{q+1}{2}}\right) < 1$. Thus the surface of these spheres is at most $\bar{q}^n$, too. We restrict our focus to the remaining $q^n - \bar{q}^n$ points. By Lemma 1 we find that the surface of the remaining spheres must be at least

$$\frac{\sqrt{2}}{\sqrt{\pi n}}(q^n-\bar{q}^n)\ .$$

Thus at least

$$\frac{\sqrt{2}}{\sqrt{\pi n}}q^n - \left(\frac{\sqrt{2}}{\sqrt{\pi n}} + 1\right)\bar{q}^n$$

points on the surface of small spheres (i.e. spheres with radius $< (\frac{1}{2} - \frac{1}{2\binom{q+1}{2}})n$) are covered by small spheres. Since $\bar{q} < q$ we find for n sufficiently large:

$$\frac{\sqrt{2}}{\sqrt{\pi n}}q^n - \left(\frac{\sqrt{2}}{\sqrt{\pi n}} + 1\right)\bar{q}^n \geq \frac{1}{\sqrt{\pi n}}q^n \ .$$

We investigate the spheres adjacent to a sphere S of size

$$\left(\frac{1}{2} - \frac{1}{2\binom{q+1}{2}}\right)n > r \geq r_s \ .$$

Without loss of generality we assume that the center of S is the zero vector. There are $A_1 = \binom{n}{r+1}(q-1)^{r+1}$ points of weight $r+1$ and $A_2 = \binom{n}{r+3}(q-1)^{r+3}$ points of weight $r+3$. We find

$$A_2 = A_1(q-1)^2\frac{(n-r-1)(n-r-2)}{(r+2)(r+3)} \leq cA_1$$

with $c = (q-1)^2\left(\frac{n}{r_s}\right)^2$. (Note that $\frac{n}{r_s}$ is independent of n.)

We remember that r_s was chosen in such a way that

$$q^n S_{n,q}(r_s)^{-1} > q^n V_{n,q}(r_s)^{-1} > (q-1)^n + 1 \ .$$

Now we choose δ such that

$$\frac{1}{2\sqrt{\pi n}}q^n S_{n,q}(\delta r_s)^{-1} > \left(q - \frac{1}{2}\right)^n \ . \tag{1}$$

Observe that $\delta \to 1$ for $n \to \infty$.

Now we investigate a sphere S' with radius $\left(\frac{1}{2} - \frac{1}{2\binom{q+1}{2}}\right)n > r' \geq \delta r_s$ adjacent to S. The number of points of weight $r+1$ in S' is

$$B_1 = \binom{r+r'+1}{r'} \ .$$

On the other hand, the number B_2 of points of weight $r+3$ in S' is bounded by

$$B_2 \geq \binom{r+r'+1}{r'-1}\binom{n-(r+r'+1)}{1}(q-1) \ .$$

(To see this, start with the center of S' and change $r-1$ nonzero coordinates to zero and one zero to a nonzero. This point has weight $r+3$ and distance r' from the center of S'.)

We find

$$B_2 \geq B_1 \frac{r'}{r+1}(n-r-r'-1)(q-1) \geq c'nB_1$$

with

$$c' = \frac{\delta r_s}{\left(\frac{1}{2} - \frac{1}{2\binom{q+1}{2}}\right)n} \cdot \frac{1}{\binom{q+1}{2}} \cdot (q-1) \ .$$

For

$$n \geq \frac{2c}{c'} = \frac{1}{\delta}(q-1)\left(\frac{n}{r_s}\right)^3 \binom{q+1}{2}\left(\frac{1}{2} - \frac{1}{2\binom{q+1}{2}}\right) \tag{2}$$

we have $2c \leq c'n$, i.e. at least $\frac{A_1}{2}$ points of weight $r+1$ are covered by spheres of radius $\leq \delta r_s$.

We have proved that at least half of the $\frac{1}{\sqrt{\pi n}}q^n$ points on the surface of small spheres are covered by spheres of radius at most δr_s. Thus the number of these spheres must be greater than

$$\frac{1}{2\sqrt{\pi n}} q^n S_{n,q}(\delta r_s)^{-1} > (q - \frac{1}{2})^n \ .$$

□

Lemma 4 says that partitions with "few" spheres must contain "large" spheres. From now we study partitions with one sphere of radius $> (1-\frac{1}{q})n$.

Lemma 5

Let $q \geq 4$. We consider a partition of the n-dimensional q-ary Hamming space into spheres which contain a sphere S of radius $(1-\frac{1}{q})n < r = n-d < n$. The number N of spheres satisfy

$$(cn^{d-1} + O(n^{d-2}))(q-1)^n \leq N \leq (c'n^{d-1} + O(n^{d-2}))(q-1)^n \ ,$$

with $c = \frac{1}{2(q-1)^{d-1}(d-1)!}$ and $c' = 2c = \frac{1}{(q-1)^{d-1}(d-1)!}$.

Proof

Without loss of generality we may assume that the center of the sphere S is the zero vector. We investigate the $\binom{n}{r+1}(q-1)^{r+1}$ points of weight $r+1$ adjacent to S and the $\binom{n}{r+2}(q-1)^{r+2}$ points of weight $r+3$. By Lemma 2 we find

$$\binom{n}{r+1}(q-1)^{r+1} > \binom{n}{r+2}(q-1)^{r+2} \ ,$$

since $r > 1 - \frac{1}{q}$.

Now we count the spheres of radius 0 tangent to S. The center of a sphere S' with radius r' tangent to S has weight $r+r'+1$. Thus S' contains $\binom{r+r'+1}{r'}$ points of weight $r+1$. The sphere S' contains

$$\binom{r+r'+1}{r'-1} + \binom{r+r'+1}{r'}\binom{r'}{1}(q-2) \tag{3}$$

points of weight $r+2$. (To see this, start with the center of S' and either choose $r'-1$ nonzero coordinates and change them to zero or choose r' nonzero coordinates, set all but one to zero and change the value of the remaining one.)

For $r \geq 1$ we find $\binom{r+r'+1}{r'}\binom{r'}{1}(q-2) + \binom{r+r'+1}{r'-1} \geq 2\binom{r+r'+1}{r'}$. This proves that each sphere with radius ≥ 1 adjacent to S contains twice as many points of weight $r+2$ than points of weight $r+1$. Thus at least

$$\binom{n}{r+1}(q-1)^{r+1} - \frac{1}{2}\binom{n}{r+2}(q-1)^{r+2}$$

points of weight $r+1$ are not covered by a sphere with radius ≥ 1.

For $(1-\frac{1}{q})n \leq r \leq n-2$ we obtain by Lemma 2:

$$\begin{aligned} \binom{n}{r+1}(q-1)^{r+1} - \frac{1}{2}\binom{n}{r+2}(q-1)^{r+2} &\geq \frac{1}{2}\binom{n}{r+1}(q-1)^{r+1} \; . \\ &= \frac{1}{2(q-1)^{d-1}}\binom{n}{d-1}(q-1)^n \end{aligned} \tag{4}$$

We obtain an upper bound on the number of spheres if we assume that one sphere has radius $r = n-d$ and all other spheres have radius 0. In this case the number of spheres is

$$1 + (q-1)^n + \binom{n}{1}(q-1)^{n-2} + \cdots + \binom{n}{d-1}(q-1)^{n-d+1} \; . \tag{5}$$

The cases $q=3$ and $q=2$ need specials arguments.

For $q=3$ we can use the same arguments as in Lemma 5, but since $q-2 = 1 \not\geq 2$ the resulting bounds are a bit weaker.

Lemma 6

Consider a partition of the n-dimensional ternary Hamming space into spheres which contain a sphere S of radius $(1-\frac{1}{3})n < r = n-d < n$. The number N of spheres satisfies

$$(cn^{d-2} + O(n^{d-3}))2^n \leq N \leq (c'n^{d-1} + O(n^{d-2}))2^n \; ,$$

with $c = \frac{1}{2^d(d-1)!}$ and $c' = \frac{1}{2^{d-1}(d-1)!}$.

Proof

For $q=3$ we need a more careful estimation than in the proof of Lemma 6, since $q-2=1$.

We proceed as in the proof of Lemma 5, but instead of counting the points of weight $r+1$ not covered by spheres of radius ≥ 1, we count the points of weight $r+1$ not covered by spheres of radius ≥ 2. Now we find that in equation (3) we have $\binom{r'}{1} \geq 2$ and once again we conclude that

$$\binom{n}{r+1}2^{r+1} - \frac{1}{2}\binom{n}{r+2}2^{r+2}$$

points of weight $r+1$ are not covered by spheres of radius ≥ 2.

Each sphere of radius 1 with a center of weight $r+2$ covers exactly $r+2$ points of weight $r+1$. Thus we can estimate the number of spheres by:

$$\begin{aligned}
\frac{1}{r+2}&\left[\binom{n}{r+1}2^{r+1} - \frac{1}{2}\binom{n}{r+2}2^{r+2}\right] \\
&\geq \frac{1}{2(r+2)}\binom{n}{r+1}2^{r+1} \qquad \text{by Lemma 2} \\
&\geq \frac{1}{2(r+2)}\left[\frac{1}{(d-1)!}n^{d-1} + O(n^{d-2})\right]\frac{2^n}{2^{d-1}} \\
&\geq \frac{1}{2^d(d-1)!}n^{d-2} + O(n^{d-3}) \ .
\end{aligned}$$

This proves the lower bound. The upper bound is the same as in Lemma 5.

For $r = n-2$ we can improve this result.

Lemma 7
Each partition of the n-dimensional ternary Hamming space into spheres where one sphere has radius $n-2$ contains more than $(n-2)2^{n-1}$ spheres.

Proof
In this case all the other spheres have either radius 0 or radius 1. The center of a sphere with radius 1 must have weight n. Thus the spheres of radius 1 form a 1-error correcting code in the binary Hamming space $\{1,2\}^n$. By the well known sphere packing bound we conclude that the number of spheres of radius 1 is at most $\frac{1}{n+1}2^n$. Thus the number of points of weight $n-1$ not covered by spheres with radius 1 is at least

$$\binom{n}{n-1}2^{n-1} - \frac{n}{n+1}2^n > (n-2)2^{n-1} \ .$$

For $q = 2$ we use a technique developed in [9].

We study a partition with a sphere S of radius r with center 0. We investigate all spheres adjacent to a sphere S with radius r and center in $(0\dots0)$. By Proposition 1 of [9] the center points of the spheres adjacent to S form a $(r+1)-(n,*,1)$ generalized Steiner system.

(A $(r+1)-(n,*,1)$ generalize Steiner System is a subset $\mathcal{S}$ of $\mathcal{P}(\{1,2,\dots,n\})$ such that each $(r+1)$ subset of $\{1,\dots,n\}$ is a subset of exactly one set in $\mathcal{S}$. The elements of $\mathcal{S}$ are called Blocks and the elements of $\{1,\dots,n\}$ are called points. A $2-(n,*,1)$ generalize Steiner System is also called a linear space. If all Block have the same size b we say it is a $(r+1)-(n,b,1)$ Steiner System or $(r+1)-(n,b,1)$ design.)

The number of blocks in such a Steiner system can be bounded by the following lemma.

Lemma 8
A $t-(n,*,1)$ *generalized Steiner system contains at least*

$$\binom{n}{t} - \binom{n}{t+1}\frac{t+1}{t+2}$$

blocks of size t*. Equality holds if and only if an* $(n-t-2)-(n,n-t-1,1)$ *Steiner system exists.*

Proof
By Proposition 1 of [9] the spheres of radius k around the blocks of size $t+k$ are disjoint. (Each Steiner system arises from a sphere packing.)

For a sphere S of radius $k \geq 1$ with a word w of weight $t+k$ as center we estimate the ratio

$$r(k) = \frac{\text{number of words of weight } t \text{ in } S}{\text{number of words of weight } t+2 \text{ in } S} \ .$$

For $k = 1$ we find

$$r(1) = \frac{\binom{t+1}{t}}{\binom{n-(t+1)}{1}} = \frac{t+1}{n-t-1} \ .$$

For $k \geq 2$ we find

$$\text{number of words of weight } t+2 \text{ in } S > \binom{t+k}{t+1}(n-(t+k)) \ .$$

(We only count the words of weight $t+2$ which share $t+1$ entries equal to 1 with w.)

Thus

$$r(k) < \frac{\binom{t+k}{t}}{\binom{t+k}{t+1}(n-t-k)} = \frac{t+1}{k(n-t-k)} \ .$$

For $2 \leq k \leq n-t-1$ we find

$$r(k) < \frac{t+1}{n-t-1} = r(1) \ .$$

We can now estimate the number N of words of weight t which are covered by spheres of radius k $(1 \leq k \leq n-t-1)$.

$$N \leq \binom{n}{t+2} r(1)$$

This proves that the generalized Steiner System contains at least

$$\binom{n}{t} - \binom{n}{t+2}\frac{t+1}{n-t-1} = \binom{n}{t} - \binom{n}{t+1}\frac{t+1}{t+2} \ .$$

blocks of size t.

Equality holds if and only if all blocks are of size $t+1$ or t and the spheres of radius 1 around the blocks of size $t+1$ cover all words with weight $t+2$. In this case the complements of the $t+1$ blocks form an $(n-t-2)-(n, n-t-1, 1)$ Steiner system. □

(Remark: At present the following $(n-t-2)-(n, n-t-1, 1)$ designs are known:

A $2-(n, 3, 1)$ designs exists if and only if $n \equiv 1, 3 \mod 6$ (see [12]).

A $3-(n, 4, 1)$ designs exists if and only if $n \equiv 2, 4 \mod 6$ (see [7]).

Further more a $4-(11, 5, 1)$, a $4-(15, 5, 1)$, $4-(17, 5, 1)$, a $4-(23, 5, 1)$, a $4-(27, 5, 1)$, a $5-(16, 6, 1)$, a $5-(18, 6, 1)$, a $5-(24, 6, 1)$, a $5-(28, 6, 1)$, a $6-(17, 7, 1)$, a $7-(18, 8, 1)$, an $8-(19, 9, 1)$ and a $9-(20, 10, 1)$ design is known (see [13] and the references given there).)

4 Proof of the main theorems

We are now ready to prove the main theorems.

We start by characterizing the first gap in the q-ary case. (Corresponding to the "gap" between 1 and 2 in the binary case.)

Theorem 9
Let $q > 2$. For n sufficiently large, each partition of the n-dimensional q-ary Hamming space in more than one sphere requires at least $(q-1)^n + 1$ spheres with equality if and only if one sphere has radius $n-1$ and the other $(q-1)^n$ spheres have radius 0.

Proof
By Lemma 4 we find that for large n a partition into spheres of size $\leq (1-\frac{1}{q})n$ contains more than

$$\frac{e^{-\frac{1}{6}}}{2\sqrt{2\pi n \frac{1}{q}(1-\frac{1}{q})}}(q-\frac{1}{2})^n > (q-1)^n + 1$$

spheres.

Now we consider a partition in which one sphere has radius $r = n - d > (1-\frac{1}{q})n$. For $q \geq 4$ Lemma 5 applies and we find that for $d \geq 2$ the number of spheres is greater than (see also Equation (4))

$$\begin{aligned} \frac{1}{2(q-1)^{d-1}}\binom{n}{d-1}(q-1)^n &\geq \frac{1}{2}\binom{n}{n-1}(q-1)^{n-1} && \text{by Lemma 2} \\ &\geq (q-1)^n + 1 && \text{for } n > 2q-2. \end{aligned}$$

Similarly for $q = 3$ and $n \geq 16$, Lemma 6 proves that for $n-2 > r > (1-\frac{1}{q})n$ each sphere packing contains more than $(q-1)^n + 1$ points. The case $q = 3$ and $r = n-2$ is dealt by Lemma 7.

Thus the partition with the fewest spheres contain one sphere of radius $n-1$ and $(q-1)^n$ spheres of radius 0. □

We remark that inequality (2) gives the precise meaning of "sufficiently large". (All other bounds used in the proof are much weaker.) To give the reader an impression of the growth of the bound for n we give a table that lists for small values of q the number N_0 with the property that Theorem 1 holds for all $n > N_0$.

q	3	4	5	10
N_0	$1.8 \cdot 10^5$	$4.1 \cdot 10^5$	$1.4 \cdot 10^6$	$1.3 \cdot 10^8$

(The table entries are computed as follows. First we substitute the asymptotic formula for $S_{n,q}(\delta r_s)$ stated in Lemma 1 in equation (1) to compute δ approximately. We substitute this and $r_s = H_q^{-1}\left(1 - \log_q\left(q - \frac{1}{2}\right)\right) n$ in equation 2 which finally leads us to the calculation

```
1*(q-1) * 1/Hinv(q,1-log[q](q-1)-(q-1)^(-2)/ln(q))^3 *
     binomial(q+1,2)*(0.5-0.5/binomial(q+1,2))
```

(Maple notation).)

The table shows us that a change of Theorem 9 into a full classification can not be done by simple investigating all small cases but would need additional ideas to improve the bounds for n.

In general we have the bound $N_0 \leq c\left(\frac{q}{H^{-1}(1-\log_q(q-1))}\right)^3$ for some constant c, where H is the q-ary entropy function.

The lemmata of the preceding section can also be used to characterize the second smallest partition.

Theorem 10

Let $q \geq 3$. Then for sufficiently large n each partition of the n-dimensional q-ary Hamming space into more than $(q-1)^n + 1$ spheres contains at least

$$1 + (q-1)^{n-1}(n + q - 1) - A_{q-1}(n,1)n(q-1)$$

spheres, where $A_{q-1}(n,1)$ denotes the maximal size of a $(q-1)$-ary one error correction code of length n.

Proof

By Lemma 4 we find that a partition with less than $1 + (n+q-1)(q-1)^{n-1}$ spheres must contain a sphere of radius $\geq (1 - \frac{1}{q})n$.

For the moment let us assume that $q \geq 4$. Then we can apply Lemma 5 and conclude that each partition with less than $1 + (n+q-1)(q-1)^{n-1}$ spheres must contain a sphere of radius $n-1$ or $n-2$. The only possible partition with a sphere of size $n-1$ contains $1 + (q-1)^n$ spheres. Thus we are left with the case that one sphere has radius $n-2$ and center in 0. Let a be the number of spheres in that partition with radius 1.

The number of spheres of radius 0 is $q^n - (V_{n,q}(n-2) + aV_{n,q}(1))$ and thus the total number of spheres in this partition is $1 + (q-1)^{n-1}(n+q-1) - an(q-1)$.

The center points of the spheres with radius 1 have weight n and thus they form a one-error-correcting code in the set of all words of weight n, i.e. $a \le A_{q-1}(n, 1)$.

Now we deal with the case $q = 3$. In this case we must apply the weaker Lemma 6 and conclude that each partition with less than $1+(n+q-1)(q-1)^{n-1}$ spheres must contain a sphere of radius $n-1$, $n-2$ or $n-3$. The cases $n-1$ and $n-2$ are the same as in the case $q \ge 4$. We now prove that the case $n-3$ cannot occur, i.e. each partition of the ternary Hamming space in which one sphere has radius $n-3$ contains more than $1 + (n + q - 1)(q - 1)^{n-1}$ spheres.

Consider a partition of the ternary Hamming space where one sphere has radius $n-3$ and center 0. Let a be the number of spheres of radius 1. Then the number of spheres of radius 0 is $3^n - (V_{n,3}(n-3) + aV_{n,3}(1))$ and therefore the total number of spheres is $2+n2^{n-1}+\binom{n}{2}2^{n-2}-2an$. But each sphere of radius 1 contains at least 2 points of weight n. (Either the center has weight n and the sphere contains $1+n$ points of weight n or the center has weight $n-1$ and the sphere contains 2 points of weight n.) Thus $a \le \frac{1}{2}2^n$. This proves that the number of spheres is at least $1+(n+2)2^{n-1}+\binom{n}{2}2^{n-2}-2^n n = 1+O(n^2 2^n)$. $\Box$

Little is know about the function $A_q(n, 1)$ especially if q is not a prime power. From the sphere packing bound we know $A_q(n, 1) \le \frac{q^n}{n(q-1)+1}$. Equality is achieved by the hamming codes. For q equal 2, 3 or 4 table with good bounds for small n exists [3, 5, 4] but even for that small values we have only bounds like $2720 \le A_2(16, 3) \le 3276$.

Note that Lemma 5 can be used to characterize even the next gaps.

Theorem 11

Let $q \ge 4$ and $c_i = \frac{1}{(q-1)^i i!}$ for $i \in \mathbb{N}_0$. Further let m be the number of spheres in a partition of the n-dimensional q-ary Hamming space.

There exists an increasing function $t_q(n)$ with $t_q(n) \to \infty$ for $n \to \infty$ and the following property:

If

$$m \le c_{t_q(n)} n^{t_q(n)} (q-1)^n$$

then there exists a $0 \le d \le t_q(n)$ with

$$\frac{1}{2^{d+1}} c_d n^d (q-1)^n \le m \le c_d n^d (q-1)^n \ ,$$

or in other words there are gaps *between $c_d n^d (q-1)^n$ and $\frac{1}{2^{d+2}} c_{d+1} n^{d+1} (q-1)^n$ for $d \in \{0, \ldots, t_q(n)\}$.*

Proof

By Lemma 4 we know that there exists an increasing function $t_q(n)$, with $t_q(n) \to \infty$ for $n \to \infty$ and the property:

If $m \le c_{t_q(n)} n^{t_q(n)} (q-1)^n$, then one sphere of the partition has a radius $\ge (1 - \frac{1}{q})n$.

But now Lemma 5 can be applied and the Theorem follows. (Note that the O-terms in Lemma 5 can be avoided if we use the bound $\frac{n^d}{2^d d!} \le \binom{n}{d} \le \frac{n^d}{d!}$ instead of $\binom{n}{d} = \frac{n^d}{d!} + O(n^{d-1})$ in the equations (4) and (5).)

For $q = 2$ we can use Lemma 8 to obtain a similar result.

Theorem 12
Consider a partition of the n-dimensional binary Hamming space $\{0,1\}^n$ into $N < n^{\sqrt{n/32}}$ spheres then there exists a number $t \le \sqrt{n/32}$ such that

$$\binom{n}{t} - \binom{n}{t+1}\frac{t+1}{t+2} < N \le \binom{n}{t} + \binom{n}{t-1} + \cdots + \binom{n}{1} + 2$$

and the partition contains a sphere of radius $n-t-1$, i.e. there are gaps *between $\binom{n}{t} + \binom{n}{t-1} + \cdots + \binom{n}{1} + 2$ and $\binom{n}{t+1} - \binom{n}{t+2}\frac{t+2}{t+3}$.*

Proof
Suppose the largest sphere S in the partition has radius r. If $r \le \sqrt{n/2}$ then by Lemma 1 the number of spheres must be larger than $n^{\sqrt{n/32}}$.

Now suppose that $\sqrt{n/2} \le r \le n - \sqrt{n/2}$. By Proposition 1 of [9] the center points of the spheres adjacent to S form a $(r+1)-(n,*,1)$ generalized Steiner system. By [16] the number of blocks b in a $(r+1)-(n,*,1)$ generalized Steiner system satisfies $b(b-1) \ge (r+1)\binom{n}{r+1}$. Thus in this case the number of spheres is at least $\sqrt{\binom{n}{t}}$ with $t = \lfloor \sqrt{n/2} \rfloor$. But

$$\sqrt{\binom{n}{t}} \ge \frac{(n/2)^{t/2}}{t^{t/2}} \ge n^{t/4} \ ,$$

since $n > 4t^2$, i.e. in this case there are more than $n^{\sqrt{n/32}}$ spheres.

Thus we are left with the case that the radius r of S is greater than $n - \sqrt{n/2}$. In this case we apply Lemma 8 and see that the number of spheres is larger than

$$\binom{n}{r+1} - \binom{n}{r+2}\frac{r+2}{r+3} \ .$$

This is the lower bound in Theorem 12. The upper bound follows if we assume that all spheres but S have radius 0. □

This characterizes the next gaps in the binary case.

Note that the left hand side of the inequality in Theorem 12 is sharp (one sphere has radius $n-t-1$ and all other spheres have radius 0). We conjecture that the inequality on the right hand side can be improved to $\binom{n}{t} + 2 \le N$ with equality if one sphere has radius $n-t-1$, an other sphere has radius t and all other spheres have radius 0.

5 Small Dimensions

As remarked Theorem 9 holds only for large n. A bound for N_0 such that Theorem 1 holds for every $n > N_0$ is given by (2). Even if this bound is certainly not optimal, Theorem 9 is definitely false for small dimensions. In this section we list the known exceptions in small dimensions.

The $[q+1, q-1, 3]_q$ Hamming-Code has q^{q-1} codewords and the spheres of radius 1 with these codewords as center form a partition of the Hamming space. (The Hamming-Code is a perfect code.) Since $q^{q-1} < (q-1)^{q+1} + 1$ for $q \geq 3$, this example shows that Theorem 9 is not valid for all n.

For $n = 3$ we can construct a sphere packing with q spheres of radius 1 (centers $(0,0,0), \ldots, (q-1, q-1, q-1)$) and $q^3 - q(3(q-1)+1)$ spheres of radius 0. Thus the packing with one sphere of radius $n-1 = 2$ is not the unique packing with $(q-1)^3 + 1$ spheres.

For $q = 3$ we know one other exception. The $[11, 6, 5]_3$ Golay-Code is a perfect code, i.e. the 3^6 spheres with radius 2 around the codewords form a sphere packing. Since $3^6 < 2^{11} + 1$, Theorem 9 does not hold for this situation.

Now we investigate the gap after $n+2$ spheres in the binary cases. By Theorem 12 for large n the next partition must contain a sphere of radius $n-3$. Each sphere packing with a sphere of radius $n-3$ contains either $\frac{n(n-1)}{2} + 2$ or $\frac{n(n-1)}{2} + n + 2$ spheres (either one sphere has radius 1 or all other spheres have radius 0). Thus the next gap is between $n+2$ and $\frac{n(n-1)}{2} + 2$. Using the bound given in Theorem 12 we see that this is true for $n > 128$ but if we go through the proof of Theorem 12 and use more careful estimations for the special case $t = 2$ we see that this is even true for $n > 32$. For small n there are the following exceptions:

Theorem 13
Each partition of the binary n-dimensional Hamming space ($n \geq 9$) contains either $1, 2, n+2$ or at least $\frac{n(n-1)}{2} + 2$ spheres (exactly $\frac{n(n-1)}{2} + 2$ spheres if and only if one sphere has radius $n-3$, one sphere has radius 1 and all other spheres have radius 0). For $n \leq 8$ we have the following sporadic examples:

$n \leq 2$ *In this case no partition can contain more than $n+2$ spheres.*

$n = 3$ *The only partition with more than $5 = n+2$ spheres consists of 8 spheres of radius 0.*

$n = 4$ *The partition with two spheres of radius 1 contains 8 spheres. (Note: This is the case described above, since $n-3 = 1$.)*

$n = 5$ *Each partition with more than $n+2 = 7$ spheres must contain 12 spheres. But in addition to the packing with one sphere of radius $n-3 = 2$ there exists a packing in which 4 spheres have radius 1.*

$n = 6$ *The packing with two spheres of radius 2 and two spheres of radius 1 (center points (000000), (111100), (110011) and (001111)) contains 10 spheres.*

$n = 7$ *The* $(7,4)$*-Hamming code contains* 16 *spheres of radius* 1 *but* $\frac{7(7-1)}{2} + 2 = 23$.

$n = 8$ *There exists a packing with* 2 *spheres of radius* 2 *and* 12 *spheres of radius* 1.

Proof
The proof is organized as follows:

1. First we rule out all cases with a big sphere.
2. We show that the is now conter example with a sphere of size 3 or 4.
3. We prove that result for $n > 12$
4. We check all $n \leq 12$ individually. The cases $n = 9$ and $n = 10$ are the most difficult.

Part I: (big spheres)

First we can check the cases in which one sphere has radius $\geq n - 3$. We find that in these cases the number of spheres is either 1, 2, $n+2$, $\binom{n}{2}+2$ or $\binom{n}{2}+n+2$ as we have seen above.

If one sphere has radius r with $\frac{n}{2} \leq r < n - 3$ we can apply Lemma 8 to see that the number of spheres is greater than $\binom{n}{2} + 2$.

If one sphere has radius $r \geq 3$, the adjacent spheres induce an $(r+1)-(n,*,1)$ generalized Steiner system. By a result due to Wilson and Petrenjuk (see for example [15]) we know that a $t-(n,*,1)$ Steiner system contains at least $\binom{n}{s}$ blocks if $t \geq 2s$. Therefore the $(r+1)-(n,*,1)$ Steiner system contains at least $\binom{n}{2}$ blocks if $r = 3, 4$ and at least $\binom{n}{3}$ blocks if $r \geq 5$.

Part II: (spheres of radius $r = 3$ or $r = 4$) In a hypothetical counter example with a sphere S of radius $r = 3$ or 4 all but one spheres must be adjacent to S. We now prove that this is impossible.

In a first step we prove that two spheres S_1 and S_2 with radii r_1 and r_2 adjacent to S are not adjacent to each other. Otherwise, let C, C_1, C_2 be the center points of S, S_1, S_2. We have $d(C, C_1) = r + r_1 + 1$, $d(C, C_2) = r + r_2 + 1$ and $d(C_1, C_2) = r_1 + r_2 + 1$. But on the other hand we have

$$d(C_1, C_2) \equiv d(C_1, C) + d(C, C_2) \mod 2$$

which is a contradiction.

Next we note that in each sphere packing with more than two spheres, every sphere has at least two neighbors. (Assume S_1 with radius r_1 and center point 0 has only the neighbor S_2 with radius r_2. Then the center point of S_2 has weight $r_1 + r_2 + 1$ and S_2 covers all points of weight $r_1 + 1$. This implies $n = r_1 + r_2 + 1$, i.e. there are only two spheres.)

Thus if a packing with a sphere S of radius $r = 3$ or 4 contains at most $\binom{n}{2} + 2$ spheres, there must be an other sphere S' not adjacent to S and exactly $\binom{n}{2}$ spheres adjacent to S and S'.

Each of the remaining $\binom{n}{2}$ spheres have only to neighbors S and S'. Using the above argument with the Steiner system we can conclude that these spheres must have radius 0, since otherwise there must be at least n adjacent spheres.

But the cases in with all but two spheres have radius 0 is exactly the case described in the Theorem.

Part III: (prove for $n \geq 12$)
At this point we have proven that in a hypothetical counter example all spheres must have radius ≤ 2. We now investigate the remaining cases.

Since all spheres have radius ≤ 2 the number of spheres is at least $\frac{2^n}{\binom{n}{2}+n+1}$. For $n > 12$ this is $> \binom{n}{2} + 2$. Thus we must only check the cases in which $n \leq 12$.

It is easy to check that the counter examples listed in the Lemma above exist. For $n = 8$ the 2 spheres of radius 2 have center points (00000000) and (11111111). The center points of 6 spheres of radius 1 have the form $(xxyyzzww)$ where exactly two of variables have the value 1. The center points of the remaining 6 spheres of radius 1 have the form $(xXyYzZwW)$, where $X = x + 1, \ldots, W = w + 1$ and exactly two of the four variables x, y, z, w have the value 1.

Now we check that the list is complete.

Part IV: (the small cases) For $n \leq 5$ this clear.

For $n = 6$ a possible further counter example would contain one sphere of radius 2 and 6 spheres of radius 1. But this is impossible as we can check by investigating the spheres adjacent to the spheres of radius 2.

Similarly, we can check in the case $n = 7$ that no packing with a sphere of radius 2 can contain 23 spheres.

For $n = 8$ the only possible counter example is that 2 spheres has radius 2 and 12 spheres have radius 1. This is in the list.

We now come to the difficult cases $n = 9$ and $n = 10$.

For $n = 9$ the only hypothetical counter example would consist of 6 spheres of radius 2, 23 spheres of radius 1 and 6 single points. We call the 12 spheres of radius 2 or 0 the special spheres. We say that a sphere is even if its center point has even weight, similarly we call a sphere odd if its center point has odd weight.

The center of all even spheres of radius 2 form a code with minimal distance 6. We use the table of bounds on nonlinear binary block codes published by Best et. al. [3] to see that the number of even spheres of radius 2 is at most 4. Similarly the number of odd spheres of radius 2 is at most 4, i.e. we find an even sphere S of radius 2 and an odd sphere S' of radius 2.

We investigate the spheres adjacent to S. Without loss of generality we assume the center of S is 0. Look at the 7 words of weight 3 with one in the first two coordinates. A sphere of radius 0 covers none or 1 of these points. Similarly, a sphere of radius 1 [2] covers none or 2 [3] of these points. Since 7 is odd, we find that there must be an odd number of spheres of radius 0 or 2 adjacent to S whose center points have 1 in the first two coordinates. We can apply this argument to any pair of coordinates to see

$$\binom{9}{2} + 2n = x\binom{5}{2} + y\binom{3}{2},$$

where x denotes the number of spheres of radius 2 adjacent to S and y denotes the number of spheres of radius 0 adjacent to S. (The center point of a sphere of radius 2 adjacent to S has weight 5 and therefore $\binom{5}{2}$ pairs of coordinates are 1. Similarly, the y spheres of radius 0 correspond to $y\binom{3}{2}$ pairs of coordinates. Since we need an odd number of special spheres for each pairs of coordinates we find that $x\binom{5}{2} + y\binom{3}{2}$ is at least $\binom{9}{2}$ and even.)

This proves either $x \geq 4$ or $x = 3$ and $y = 2, 4, 6$. Note that all special spheres adjacent to S are odd.

We can apply the same argument to S' and find for the corresponding numbers x' and y' of special spheres adjacent to S' the same relations. But all special spheres adjacent to S' are even. Thus $x + x' \leq 6$ and $y + y' \leq 6$. The only solution is $x = x' = 3$, $y = 4$ and $y' = 2$ or vice versa.

The 3 spheres of radius 2 and 2 spheres of radius 0 cover $34 = 3\binom{5}{3} + 4\binom{3}{3}$ words of weight 3. The remaining $\binom{9}{3} - 34 = 50$ points of weight 3 must be covered by spheres of radius 1. But each such sphere covers $\binom{4}{3}$ words of weight 3. But $4 \nmid 50$, i.e. it is impossible to cover these words.

For $n = 10$ a potential counter example must have 12 spheres of radius 2 and 32 spheres of radius 1. We now prove that this is impossible.

Consider a sphere S of radius two with center point (0000000000). There exists another sphere of radius 2 adjacent to S. (Otherwise the center points of the spheres of radius two have pairwise distance ≥ 6. But this is impossible [3].) Let us assume that (1111100000) is the center point of a sphere S' of radius 2 adjacent to S. We now investigate the 8 words of weight 3 of the form $(11********)$. Each sphere of radius 2 covers either 0 or 3 of these words and each sphere of radius 1 covers either 0 or 2 of these words. Since S' covers 3 of these 8 words we know that there must be another sphere S'' of radius 2 that covers 3 more of these words. The center of S'' must have the form $(11000*****)$ and weight 5. Instead of the first two places we can choose any two of the first five places in the argument above. This proves that the center point of the 10 spheres of radius 2 different from S and S' have weight 5. Thus the center points of these spheres have pairwise distance ≥ 6. But this is impossible [3].

For $n = 11$ we know by [3] that at most 24 spheres have radius 2. But $\frac{2^{11} - 24V_{11,2}(2)}{V_{11,2}(1)} \geq 36$, i.e. there are at least $24 + 37 = 61 \geq \binom{n}{2} + 2$ spheres.

For $n = 12$ we know by [3] that the number of spheres of radius 2 is at most 32 and as in the case $n = 11$ we find that the number of spheres is larger than $\binom{n}{2} + 2$. □

6 Open Problems

The bounds in Lemma 4 are not strict. However, we see no significant improvement of these bounds. It would be interesting to improve the bounds in such a way that for $n \geq cq$ and some $c > 0$ we can apply Lemma 4 to see that the number of spheres is $\geq n(q-1)^n$.

An interesting but probably very hard problem is the classification of all "exceptions" to Theorem 9 in small dimensions.

Another open problem is the following. The gap theorem for the binary case proves that the only number k for which no n-dimensional binary Hamming space can be partitioned into k spheres is $k = 3$. This was previously proved by Fachini and Körner [6]. What are the numbers k for which a partition of a q-ary Hamming space into k spheres exists?

The q-ary gap theorem is not sufficient to answer this question. As a simple necessary condition we observe that a partition into k spheres can only exist for $k \equiv 1 \mod (q-1)$. (Take the sizes of the spheres mod $q-1$.)

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