Exterior sets of hyperbolic quadrics

Andreas Klein
University of Giessen

Abstract

Extensive studies have been made on exterior sets to hyperbolic quadrics $Q^+(2n-1,q)$ that contain exactly $(q^n-1)/(q-1)$ points. There are only few theorems on exterior sets with less than $(q^n-1)/(q-1)$ points. In this article we will prove better upper bounds for exterior sets.

AMS Mathematics classification codes: 51E20, 51E21, 51E23, 05B25

1 Introduction and Basic Results

A set $\mathfrak{X}$ of points of a projective space $PG(d,q)$ (d odd) is called an *exterior set* with respect to the hyperbolic quadric $Q^+(d,q)$, if no line joining two distinct elements of $\mathfrak{X}$ has a point in common with $Q^+(d,q)$. For $d = 2n-1$, we have that

$$|\mathfrak{X}| \leq \frac{q^n-1}{q-1} \quad , \tag{1}$$

because there are $(q^n-1)/(q-1)$ subspaces of dimension n that contain a fixed $(n-1)$-dimensional singular subspace and each of these subspaces can contain at most one point of $\mathfrak{X}$. By a singular subspace we mean a subspace of $PG(d,q)$ contained in $Q^+(d,q)$.

Exterior sets $\mathfrak{X}$ to $Q^+(2n-1,q)$ with $(q^n-1)/(q-1)$ points are called *maximal exterior sets (MES)*. The maximal exterior sets are completely classified (see [6], [1] and [2]).

Result 1
The only MES of $Q^+(2n-1,q)$, $n \geq 2$ are

(i) the unique MES of $Q^+(5,2)$,

(ii) the linear MES of $Q^+(3,q)$,

(iii) the Thas MES of $Q^+(3,q)$, q odd,

(iv) the exceptional MES of $Q^+(3,q)$, $q = 11, 23, 59$.

(A linear MES consist of the $q+1$ points of a exterior line. The MES of Thas-type consists of $\frac{q+1}{2}$ points on a line l and $\frac{q+1}{2}$ points of a line l'. In addition l and l' are orthogonal with respect to the quadratic form defining $Q^+(3,q)$.)

As we can see, in most cases equality cannot be reached in (1). In this article, better upper bounds for $|\mathfrak{X}|$ will be proved. Define

$$M(2n-1,q) = \max\{|\mathfrak{X}|, \mathfrak{X} \text{ is an exterior set of } Q^+(2n-1,q)\} \tag{2}$$

Equation (1) says $M(2n-1,q) \leq (q^n-1)/(q-1)$.

In section 2 of this article we prove a recursion formula for $M(2n-1,q)$ that is better than (1). In section 3 and 4 we prove bounds for $M(5,q)$ as starting values of the recursion formula.

2 A recursion formula

The recursion formula of Theorem 1 is better than (1), because (3) together with $M(3,q) = q+1$ implies (1).

Theorem 1

For each $n \geq 2$ and each prime power q we have

$$M(2n+1,q) \leq \frac{q^{n+1}-1}{q^n-1} M(2n-1,q) \quad . \tag{3}$$

Proof

Let $\mathfrak{X}$ be an exterior set with respect to the hyperbolic quadric $Q^+(2n+1,q)$ with $|\mathfrak{X}| = M(2n+1,q)$. Let $\perp$ be the polarity of $PG(2n+1,q)$ related to $Q^+(2n+1,q)$. For two points P and X we have $P \in X^\perp$ if and only $X \in P^\perp$.

We will count the number m of pairs (P,X) with $P \in Q^+(2n+1,q)$ and $X \in \mathfrak{X} \cap P^\perp$.

For a point $X \in \mathfrak{X}$, the set $X^\perp \cap Q^+(2n+1,q)$ is a parabolic quadric $Q(2n,q)$. Therefore

$$m = |X| \cdot |Q(2n,q)| = M(2n+1,q)\frac{(q^n-1)(q^n+1)}{q-1} \quad . \tag{4}$$

For each point $P \in Q^+(2n+1,q)$ let n_P be the number of points in $\mathfrak{X} \cap P^\perp$. Put $\mathfrak{Y} = \{PX | X \in \mathfrak{X} \cap P^\perp\}$. Since $\mathfrak{X}$ is an exterior set, each plane spanned by two lines in $\mathfrak{Y}$ has only the point P in common with $Q^+(2n+1,q)$. Therefore $\mathfrak{Y}$ is an exterior set to the hyperbolic quadric $Q^+(2n+1,q)/P$ in $P^\perp/P$. It follows that $n_P = |\mathfrak{Y}| \leq M(2n-1,q)$. This yields

$$m = \sum_{P \in Q^+(2n+1,q)} n_p \leq \frac{(q^{n+1}-1)(q^n+1)}{q-1} M(2n-1,q) \quad . \tag{5}$$

Equations (4) and (5) together imply (3). □

Corollary 1
For each $n > 2$ and each prime power q we have

$$M(2n+1,q) \leq \frac{q^{n+1}-1}{q^3-1} M(5,q)$$

Proof
This follows immediately from Theorem 1 by induction. □

Note that the statement of Corollary 1 is weaker than the recursion formula of Theorem 1. For example in the next section we prove $M(5,4) \leq 20$. Corollary 1 yields $M(9,4) \leq 324.762$. Since $M(9,4)$ must be an integer we get $M(9,4) \leq 324$. If we use the recursion formula of Theorem 1 we get $M(7,4) \leq 80.9524$ and therefore $M(7,4) \leq 80$. Using the recursion formula a second time we obtain $M(9,4) \leq 320$.

3 Bounds for exterior sets of $Q^+(5,q)$, q even

In this section we will assume that q is even.

Theorem 2
An exterior set with respect to $Q^+(5,q)$ where $q = 2^n \geq 4$ has at most $q^2+q+1-\frac{1}{4}q$ points.

For the proof of Theorem 2, we need the following result about linear spaces. Recall that a linear space is an incidence structure consisting of points and lines such that any two points are joined by a unique line and such that every line has at least two points.

For a linear space $\mathcal{L}$, let v denote the number of points of $\mathcal{L}$ and b the number of lines of $\mathcal{L}$. For each point $P \in \mathcal{L}$, let r_P be the number of lines through P.

Result 2 (Erdős, Flower, Sós, Wilson, [3])
If $\mathcal{L}$ has more than one line that do not contain the point P, then the number $b - r_P$ of lines that do not contain P is at least $\lfloor v - \sqrt{v} \rfloor$. If equality holds then $\mathcal{L}$ is a projective plane.

Proof of Theorem 2
Let $\mathfrak{X}$ be an exterior set of $Q^+(5,q)$ and put $c := q^2+q+1-|\mathfrak{X}|$. Since $q = 2^n \geq 4$, Theorem 1 shows that $c > 0$. For the proof we may assume that $2c \leq q$ and we have to show that $4c \geq q$.

In the following we denote the polarity of $PG(5,q)$ associated to $Q^+(5,q)$ by $\perp$.

For $P \in Q^+(5,q)$, the set $\mathfrak{Y} = \{PX | X \in \mathfrak{X} \cap P^\perp\}$ is an exterior set of the hyperbolic quadric $Q^+(5,q)/P$, as was shown in the proof of Theorem 1. Equation (1) yields that there are at most $q+1$ points of $\mathfrak{X}$ in $P^\perp$. We call a point $P \in Q^+(5,q)$ *big* if there are $q+1$ points of $\mathfrak{X}$ in $P^\perp$. Otherwise we call P *small*. By s we denote the number of small points.

Now we proceed in several steps.

Step 1: *For a big point P, the $q+1$ points of $\mathfrak{X} \cap P^\perp$ lie in one plane on P.*

Result 1 yields that for a big point P the set $\mathfrak{Y} = \{PX | X \in \mathfrak{X} \cap P^\perp\}$ is a line of $P^\perp/P$. This means that all $q+1$ points of $\mathfrak{X} \cap P^\perp$ lie in one plane on P.

Step 2: *We have $s \le c(q^3+q^2+q+1)$.*

We count the number of pairs (P, X) with $P \in Q^+(5,q)$, $X \in \mathfrak{X}$ and $X \in P^\perp$. For every point $X \in \mathfrak{X}$ there are q^3+q^2+q+1 (= number of points in $Q(4,q)$) points $P \in Q^+(5,q)$ with $X \in P^\perp$. For each big point P there are $q+1$ points $X \in \mathfrak{X}$ with $X \in P^\perp$ and for each small point P there are at most q points of $\mathfrak{X}$ in $P^\perp$. This gives

$$[(q^2+q+1)(q^2+1)-s](q+1)+sq \ge (q^2+q+1-c)(q^3+q^2+q+1),$$

hence

$$s \le c(q^3+q^2+q+1) \quad . \tag{6}$$

proving Step 2.

Since $\mathfrak{X}$ is an exterior set, a line joining two points of $\mathfrak{X}$ is an exterior line of the quadric $Q^+(5,q)$. Therefore its pole $l^\perp$ meets the quadric in an $Q^-(3,q)$, which is an ovoid in the 3-space $l^\perp$.

Step 3: *If a line l meets $\mathfrak{X}$ in at least two and at most $q/2+1$ points, then the ovoid $l^\perp \cap Q^+(5,q)$ contains at least q^2+1-2q small points.*

Let l be a line that contains $d \ge 2$ points of $\mathfrak{X}$. Then $\mathcal{O}_l = l^\perp \cap Q^+(5,q)$ is an ovoid, because l is an exterior line of $Q^+(5,q)$. Let s_l be the number of small points in $\mathcal{O}_l$. For each big point $P \in \mathcal{O}_l$ there are at least d and therefore $q+1$ points of $\mathfrak{X}$ in the plane $Pl \subset P^\perp$. We count the number m of points of $\mathfrak{X}$ that lie in one of the planes Pl with $P \in \mathcal{O}_l$. If we only look at the planes Pl with big points P we obtain:

$$m \ge (q^2+1-s_l)(q+1-d)+d \quad .$$

Since $m \le |\mathfrak{X}| = q^2+q+1-c$, this yields:

$$q^2+q+1-c \ge (q^2+1-s_l)(q+1-d)+d \quad ,$$

that is

$$s_l \ge q^2+1-\frac{q^2+q+1-c-d}{q+1-d} \quad . \tag{7}$$

Since $d \le q/2+1$ and $c > 0$, it follows that $s_l > q^2-2q$ proving Step 3.

Step 4: *There exists at most one line that meets $\mathfrak{X}$ in more than $q/2+1$ points.*

Assume there exists two lines h and h' that contain more than $q/2+1$ points of $\mathfrak{X}$. First suppose that h and h' lie in a plane π. Then π contains more than $q+1$ points of $\mathfrak{X}$. Since every plane meets $Q^+(5,q)$, the plane π contains a singular point, which then lies on a line of π having two points in $\mathfrak{X}$. But $\mathfrak{X}$ is an exterior set, a contradiction.

Now suppose that h and h' are skew lines. The 3-dimensional space $\langle h, h' \rangle$ intersects $Q^+(5,q)$ in a 3–dimensional hyperbolic quadric, an Ovoid or a cone. In each case there exists a point $X \in h \cap \mathfrak{X}$ for which $h'X$ is a plane that intersects $Q^+(5,q)$ in a conic. Since $\mathfrak{X}$ is an exterior set, all lines XX' with $X' \in h' \cap \mathfrak{X}$ are exterior lines to this conic. But h' contains more than $q/2+1$ points of $\mathfrak{X}$ and no point in a plane lies on that many lines that miss a conic. This contradiction proves Step 4.

For every big point P we denote by π_P the plane on P that contains the $q+1$ points of $\mathfrak{X} \cap P^\perp$. From now on, we fix a singular plane S. Then S lies in q^2+q+1 solids. Since $\mathfrak{X}$ is an exterior set with $q^2+q+1-c$ points, exactly $q^2+q+1-c$ of these solids contain one point of $\mathfrak{X}$ and the remaining c solids $H_1, \dots, H_c$ do not contain a point of $\mathfrak{X}$. The subspaces $H_i^\perp$ are lines of S and a point of S is small iff it lies on one of these lines. We choose a line l of S different from the c lines $H_i^\perp$. Then l contains at most c small points and therefore at least $q+1-c$ big points. The subspace $l^\perp$ is a solid on S, which meets $\mathfrak{X}$ in a unique point. We denote this point by R. Then R is the unique point of $\mathfrak{X} \cap \pi_P \cap \pi_{P'}$ for any two different points P and P' of l.

By Step 4, there exists at most one line h with more than $q/2+1$ points in $\mathfrak{X}$. If such a line h exists, then h is an exterior line and $h^\perp \cap S$ is a point. In this case we choose l in such a way that l does not contain this point $h^\perp \cap S$.

Step 5: *If P is a big point of l, then there exist at least $q - \sqrt{q}$ lines that contain two points of $P^\perp \cap \mathfrak{X}$ that do not contain R, and that have at most $q/2+1$ points in $\mathfrak{X}$.*

Consider the linear space $\mathcal{L}$ induced by $PG(5,q)$ on the $q+1$ points of $P^\perp \cap \mathfrak{X}$. We have chosen l in a way that each line of $\mathcal{L}$ contains at most $q/2+1$ points. The point R is a point of the linear space $\mathcal{L}$ and the claim is that $\mathcal{L}$ has at least $q - \sqrt{q}$ lines that do not contain R.

By Theorem 2 there is either exactly one line in $\mathcal{L}$ that does not contain R or there are at least $\lfloor q+1-\sqrt{q+1} \rfloor > q - \sqrt{q}$ of these lines.

Since every line of $\mathcal{L}$ contains at most $\frac{q}{2}+1$ points, the second case must occur. Thus the claim is established.

Step 6: $s \geq \frac{1}{4}q^4 - \frac{3}{4}q^3 + q^{5/2} - 2q^2 + \frac{3}{2}q^{3/2} + q - \sqrt{q}$.

We count the number of small points that lie in ovoids $\mathcal{O}_h$ for all lines h that satisfy the following two conditions:

1. h lies in a plane π_P for a big point $P \in l$ and h does not contain R.

2. h contains at least 2 and at most $q/2+1$ points of $\mathfrak{X}$.

By Step 5 we find at least $q - \sqrt{q}$ such lines in every plane π_P for the big points P of l.

For two lines h and h' in the same plane π_P we have $h^\perp \cap h'^\perp = \pi_P^\perp$. Thus the only common point of $\mathcal{O}_h$ and $\mathcal{O}_{h'}$ is P, and P is not a small point. If h lies in π_P and h' in $\pi_{P'}$ for $P \neq P'$, then $\langle h, h' \rangle$ is a 3-dimensional space, because otherwise $h \cap h' \neq \emptyset$ but $\pi_P \cap \pi_{P'} = \{R\}$. Thus $h^\perp \cap h'^\perp$ is a line and $\mathcal{O}_h$ and $\mathcal{O}_{h'}$ have at most two points in common.

By Step 3, for each line h we get at least $q^2 - 2q + 1$ small points. Using first $q - \sqrt{q}$ lines h in π_P for the first big point P of l, then $q - \sqrt{q}$ for the second and so on for exactly $q/2 + 1$ of the at least $q + 1 - c$ big points of l, we obtain

$$s \geq (q - \sqrt{q})[q^2 - 2q + 1)] + (q - \sqrt{q})[q^2 - 2q + 1 - 2(q - \sqrt{q})] + \dots \\ + (q - \sqrt{q})[q^2 - 2q + 1 - 2\frac{q}{2}(q - \sqrt{q})] \quad . \qquad (8)$$

Using the summation formula for arithmetic sums, this establishes the claim in Step 6.

Now we can complete the proof of Theorem 2. Step 2 and Step 6 together imply

$$c(q^3 + q^2 + q + 1) \geq \frac{1}{4}q^4 - \frac{3}{4}q^3 + q^{5/2} - 2q^2 + \frac{3}{2}q^{3/2} + q - \sqrt{q}$$

Hence

$$\begin{aligned} c &\geq \frac{\frac{1}{4}q^4 - \frac{3}{4}q^3 + q^{5/2} - 2q^2 + \frac{3}{2}q^{3/2} + q - \sqrt{q}}{(q^3 + q^2 + q + 1)} \\ &\geq \frac{1}{4}q - 1 + \frac{q^{5/2} + O(q^2)}{q^3 + q^2 + q + 1}. \end{aligned} \qquad (9)$$

Since $q \geq 4$, this implies $c \geq \frac{1}{4}q$. (The $O(\dots)$ term is small enough.) Theorem 2 is thus proved. □

4 Bounds for exterior sets of $Q^+(5, q)$, q odd

If q is odd and $q \neq 11, 23, 59$, then a maximal exterior set of $Q^+(3, q)$ is either linear or an exterior set of Thas-type. In this case we can prove an upper bound for $M(5, q)$ similar to Theorem 2.

For the proof we will need the following result on linear spaces:

Result 3 (Schmidt, [5])
For a linear space $\mathcal{L}$ with v points, let n be the unique positive number with $n^2 + n + 1 = v$. If P_1 and P_2 are two distinct points of $\mathcal{L}$, then the number of lines that do not contain P_1 or P_2 is either at most one or at least $n^2 - n$.

Now we are able to prove:

Theorem 3
Let $\mathfrak{X}$ be an exterior set with respect to $Q^+(5, q)$, q odd and $q \neq 11, 23, 59$. If $|\mathfrak{X}| = q^2 + q + 1 - c$ then

$$c \geq (\sqrt{5} - 2)q + (\frac{22\sqrt{5}}{5} - 10)\sqrt{2q + 3} + (\frac{1077\sqrt{5}}{50} - \frac{101}{2}). \qquad (10)$$

Proof
By Result 1 we have $c > 0$. Furthermore we can assume $q \geq 11$, because for $q < 11$ the inequality (10) only implies $c > 0$.

We go through the same steps as in the proof of Theorem 2. We can assume that $c \leq \frac{1}{2}q$. As in the proof of Theorem 2 we denote the polarity associated with $Q^+(5,q)$ by $\perp$.

For each point P there are at most $q+1$ points of $\mathfrak{X}$ in $P^\perp$. We say P is a *big* point if there are exactly $q+1$ points of $\mathfrak{X}$ in $P^\perp$ and otherwise P is a *small* point. By s we denote the number of small points.

Step 1: *For a big point P, the $q+1$ points of $\mathfrak{X} \cap P^\perp$ lie in one plane or there are two plans through P and exactly $\frac{q+1}{2}$ of the points lie in each of these planes*

Since $q \neq 11, 23, 59$ Result 1 shows that the exterior set $\mathfrak{Y} = \{PX | X \in \mathfrak{X} \cap P^\perp\}$ in the quotient geometry at P is either linear or of Thas-type.

Step 2: We have $s \leq c(q^3 + q^2 + q + 1)$.

Step 2 is the same as in the proof of Theorem 2.

Let l be a line that contains either $d \geq 3$ points of $\mathfrak{X}$, or $d = 2$ points of $\mathfrak{X}$ which are not orthogonal with respect to $Q^+(5,q)$. We will call such lines *nice* lines.

Step 3: *If a nice line l meets $\mathfrak{X}$ in at most $q/4+1$ points, then the ovoid $l^\perp \cap Q^+(5,q)$ contains at least q^2+1-4q small points.*

As in Step 3 of the proof of Theorem 2 we put $\mathcal{O}_l := Q^+(5,q) \cap l^\perp$. Let s_l be the number of small points in $\mathcal{O}_l$. For a big point P in $\mathcal{O}_l$ the exterior set $\mathfrak{Y} = \{PX | X \in \mathfrak{X} \cap P^\perp\}$ is either linear or of Thas-type. As in the proof of Theorem 2 we obtain the bound:

$$q^2 + q + 1 - c \geq (q^2 + 1 - s_l)(\frac{q+1}{2} - d) + d \quad ,$$

that is

$$s_l \geq q^2 + 1 - \frac{q^2 + q + 1 - c - d}{\frac{q+1}{2} - d} \quad . \tag{11}$$

(In the prove of Theorem 2 we have the term $q+1-d$ instead of $\frac{q+1}{2} - d$. This difference is due to the possibility that $\mathfrak{Y}$ may be a set of Thas-type.)

Since $d \leq \frac{q+1}{4}$ and $c > 0$, it follows that $s_l > q^2 - 4q$ proving Step 3.

From now on we fix a singular plane S of $Q^+(5,q)$. For each big point $P \in S$, there is either one plane π_P through P that contains all $q+1$ points of $P^\perp \cap \mathfrak{X}$ or there are two planes $\pi_P^{(1)}$ and $\pi_P^{(2)}$ through P that contain exactly $\frac{q+1}{2}$ points of $P^\perp \cap \mathfrak{X}$. Let P be a big point of S and π be the plane π_P or one of the planes $\pi_P^{(i)}$. Let b be the number of lines in π that contain at least 3 points of $\mathfrak{X}$, or 2 points of $\mathfrak{X}$ which are not orthogonal to each other. Either all points of $\mathfrak{X} \cap \pi$ lie on one line or there is at most one point $R \in \pi \cap \mathfrak{X}$ which is incident with more than $\frac{b+1}{2}$ of these lines. In the second case $R^\perp \cap S$ is a line of S and we say P corresponds to the line $R^\perp \cap S$. Since there are at most two planes $\pi_P^{(1)}$, $\pi_P^{(2)}$ that belong to P, P corresponds to at most 2 lines of S.

Step 4: *There is a line $l \in S$ that contains at least $q+1-c$ big points and that corresponds to at most one big point.*

In S there are c lines that contain the small points of S and each point of these c lines is small. (See also Step 4 of the proof of Theorem 2.) Therefore the number of small points of S is greater that c. It follows that there are more lines which contain big points than big points in S. Since each big point corresponds to at most two lines, there is a line $l \in S$ which corresponds with at most one big point P. Let R be the unique point of $l^\perp \cap \mathfrak{X}$. (As we have shown in the proof of Theorem 2 R exists and is unique.)

In S there are c lines that contain the small points of S and therefore l has at least $q+1-c$ big points.

For a big point $P \in l$ let π be the plane in $P^\perp$ through R which contains either $v = \frac{q+1}{2}$ or $v = q+1$ points of $\mathfrak{X}$. Let $\mathcal{L}$ be the linear space defined by these v points. We call a line of $\mathcal{L}$ which contains exactly two points X_1, X_2 of $\mathfrak{X}$ with $X_1 \in X_2^\perp$ a *bad* line. (We call these lines bad, because they cause additional trouble compared with the proof of theorem 2.)

Step 5: *For at least $q-1-c$ big points of l the linear space $\mathcal{L}$ contains at least $\frac{q+1}{2} - \sqrt{2q+3}$ lines that are not bad, that contain at most $q/4+1$ points and that do not contain R.*

Let P be a point of l and define π and $\mathcal{L}$ as above. We denote the v points of $\mathfrak{X} \cap P^\perp$ by $X_1, \ldots, X_v$.

Suppose $\{X_1, X_2\}$ and $\{X_1, X_3\}$ are bad lines. It follows that $X_2X_3 = X_1^\perp \cap \pi$. Since $P \in \pi$ and $X_1 \in P^\perp$, it follows $P \in X_2X_3$. A contradiction to X is an exterior set. This yields: Two bad lines have no points of $\mathfrak{X}$ in common.

We construct a new linear space $\mathcal{L}'$. $\mathcal{L}'$ contains all points and lines of $\mathcal{L}$. Furthermore $\mathcal{L}'$ contains one special point X' which lies on every bad line. If $X_i \in \mathcal{L}$ lies not on a bad line $\{X', X_i\}$ is a line of $\mathcal{L}'$. The number of lines of $\mathcal{L}'$ which are not incident with X' is equal to the number of non bad lines of $\mathcal{L}$.

By Theorem 3 the number of lines of $\mathcal{L}'$ that do not contain X' and R is either zero or one or at least n^2-n, where n is the positive number with $n^2+n+1 = v+1$. (Note: $\mathcal{L}'$ has $v+1$ points.)

We now investigate the first two possibility's:

- **All points of $\mathcal{L}$ lie on one line (i.e. $\mathcal{L}'$ is a near pencil).**
  If $v = q+1$, $\mathfrak{X}$ contains a whole line h. Suppose $X \in \mathfrak{X} - h$. Then Xh must be an exterior plane to $Q^+(5,q)$. This is impossible.

  Now we assume $v = \frac{q+1}{2}$. We prove that for all other big points $P' \in l$ this case can not occur. Suppose the opposite. In π_P and $\pi_{P'}$ together lie q points $X_1, \ldots, X_q$ of $\mathfrak{X}$. ($X_q = R$ lies in both planes.) Since the $\frac{q+1}{2}$ points in π_P and $\pi_{P'}$ are part of an exterior set of Thas type, it follows that $\|X_i\|$ is a square for all X_i, $i = 1, \ldots, q$ or $\|X_i\|$ is a non-square for all X_i. ($\|X_i\| = b(X_i, X_i)$ for the bilinear form b that belongs to $Q^+(5,q)$. That $\|X_i\|$ is always a square or always a non-square is part of the construction of exterior sets of Thas type (see [4]).) Let τ be the plane that contains the points $X_1, \dots, X_q$. Without loss of generality we assume that $\|X_i\|$ is always a non-square.

Suppose τ contains only one point Q of $Q^+(5,q)$. All points X of τ with $\|X\|$ is a non-square lie on $\frac{q+1}{2}$ lines through Q. It follows Q lies at least on one line X_iX_j, a contradiction to $\mathfrak{X}$ is an exterior set.

Now suppose τ intersects $Q^+(5,q)$ in a conic. We can assume that the conic has the equation $x_1x_2 + x_3^2 = 0$. Investigate the hyperbolic quadric with equation $x_1x_2 + x_3^2 - x_4^2$. The points X_i, $i = 1, \dots, q$ and $\bar{X} := \langle(0,0,0,1)\rangle$ form a maximal exterior set with respect two this hyperbolic quadric. ($X_i * \bar{X} = b(X_i, X')^2 - \|X_i\|\|\bar{X}\|$ is a non-square, so $X_i\bar{X}$ is an exterior line (see [4])). This set is neither linear nor of Thas type, a contradiction to the assumption $q \neq 11, 23, 59$.

- **There is only one line in $\mathcal{L}'$ that contains neither R nor X'.**
  In this case $\mathcal{L}$ is a near-pencil and all but one line of $\mathcal{L}$ contain the point R. We have chosen l so that this can occur for at most one point of l

Since each of the above cases can occur only once we have shown:
For at least $q-c-1$ of the $q+1-c$ big points of l there are at least n^2-n lines in $\mathcal{L}$ that are not bad and do not contain R. If $v = \frac{q+1}{2}$ the number n^2-n is equal to $\frac{q+3}{2} - \sqrt{2q+3}$ and if $v = q+1$ the number n^2-n is equal to $q+2-\sqrt{4q+5}$.

In addition we have shown that at least $\frac{q+1}{2} - \sqrt{2q+3}$ lines in $\mathcal{L}$ are not bad and contain $\leq \frac{q+1}{4}$ points. If $v = q+1$ this clear, because at most 4 lines contain more than $\frac{q+1}{4}$ points and in this case $n^2-n-4 > \frac{q+1}{2} - \sqrt{2q+3}$ If $v = \frac{q+1}{2}$ (i.e. $n^2 - n = \frac{q+3}{2} - \sqrt{2q+3}$) then at most one line contains more than $\frac{q+1}{4}$ points. Thus the claim of Step 5 follows.

In the following calculations we put $z = 2q+3$.

Step 6:

$$s \geq \frac{z^4}{64} - \frac{21}{32}z^3 - \frac{c-3}{4}z^{5/2} - \frac{c^2-23c-96}{16}z^2 + \frac{c^2-15}{2}z^{3/2} - \frac{28c^2+268c+531}{32}z - \frac{(c-15)(2c+5)}{4}\sqrt{z} - \frac{(2c-67)(2c+5)}{64}$$

As proven in Step 5 there are at least $q+1-c$ big points in l and in at least $q-1-c$ planes π_P (P is a big point of l) there are at least $\frac{q+1}{2} - \sqrt{2q+3}$ nice lines that do not contain the point R.

As in the proof of Theorem 2 we count the number of small points in ovoids of type $\mathcal{O}_h$ where h is a line in one of the planes π_P for a big point $P \in l$. In the each plane we have at least $\frac{q+1}{2} - \sqrt{2q+3}$ nice lines that do not contain R. Continuing as in the proof of Theorem 2 using Step 3 we obtain the bound:

$$s \geq (\frac{q+1}{2} - \sqrt{2q+3})\left[q^2+1-4q\right]$$
$$+(\frac{q+1}{2} - \sqrt{2q+3})\left[(q^2+1-4q) - 2(\frac{q+1}{2} - \sqrt{2q+3})\right] +$$
$$(\frac{q+1}{2} - \sqrt{2q+3})\left[(q^2+1-4q) - 2\cdot 2(\frac{q+1}{2} - \sqrt{2q+3})\right] + \cdots +$$
$$(\frac{q+1}{2} - \sqrt{2q+3})\left[(q^2+1-4q) - (q-c-2)\cdot 2(\frac{q+1}{2} - \sqrt{2q+3})\right] \quad (12)$$

Using the formula for arithmetic sums, this establishes the claim of Step 6.

Now we can complete the proof of Theorem 3. Step 2 and Step 6 together imply a quadric inequality for c. Solving this inequality we obtain:

$$c \geq \frac{\sqrt{5z^6 + 8z^{11/2} - 160z^5 + O(z^{9/2})} - 2z^3 - 4z^{5/2} + 37z^2 + O(z^{3/2})}{2(z - 4\sqrt{z} - 1)^2} \quad (13)$$

Using polynomial division this simplifies to:

$$c \geq (\frac{\sqrt{5}}{2} - 1)z + (\frac{22\sqrt{5}}{5} - 10)\sqrt{z} + (\frac{501\sqrt{5}}{25} - \frac{95}{2}) + \epsilon_q \quad (14)$$

With $\epsilon_q > 0$ and $\epsilon_q \to 0$ for $q \to \infty$.
Replacing z by $2q+3$ we obtain the inequality (10). □

Remark 1
Since the inequality (10) weaker than inequality (13), we can sometimes (expecially for small values q) improve our result, if we use the exact solution of the quadratic inequality. In the following table we list the fist values of q in which we can achieve an improvement:

q	3	5	7	9	11	13	17	23	25	27	31	37	41	49
$c \geq$	1	1	1	1	1	1	2	3	4	4	5	6	7	9

q	59	73	81	109	...
$c \geq$	11	14	16	22	...

Of course we can now use Theorem 1 or Corollary 1 to derivate bounds for exterior sets with respect to $Q^+(2n-1,q)$, $n > 3$. For example for q even, $q > 2$ and $n > 2$ we have $M(2n+1,q) \leq \frac{q^{n+1}-1}{q^3-1}(q^2 + \frac{3}{4}q + 1)$.

Acknowledgements
I would like to thank Prof. Dr. K. Metsch for detailed comments on the first draft.

References

[1] Laura Bader and Guglielmo Lunardon. On the Flocks of $Q^+(3, q)$. *Geom. Deticata*, 29(2):177–183, 1989.

[2] Frank de Clerck and Joseph A. Thas. Exterior Sets with Respect to the Hyperbolic Quadric in $PG(2n-1, q)$. In *Finite Geometries*, volume 103 of *Lectures and Notes in Pure and Applied Mathematics*, 1985.

[3] P. Erdős, J. C. Fowler, V.T. Sós, and R. M. Wilson. On 2-Designs. *J. Comb. Theory*, Ser. A 38:131–142, 1985.

[4] J. Chris Fisher and Joseph A. Thas. Flocks in $PG(3, q)$. *Math. Z.*, 169:1–11, 1979.

[5] Birgit Schmidt. Externe Geraden in Linearen Räumen. Master's thesis, Justus-Liebig-Universität Gießen, 1997.

[6] Joseph A. Thas. Maximal Exterior Sets of Hyperbolic Quadrics: The Complete Classification. *J. Comb. Theory*, Ser. A 56(2):303–308, 1991.