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Parameters for which the Griesmer bound is not sharp

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Abstract

We prove for a large class of parameters t and r that a multiset of at most $t\theta_{d-k} + r\theta_{d-k-2}$ points in $\mathrm{PG}(d, q)$ that blocks every k-dimensional subspace at least t times must contain a sum of t subspaces of codimension k.

We use our results to identify a class of parameters for linear codes for which the Griesmer bound is not sharp. Our theorem generalizes the non-existence results from Maruta [On the achievement of the Griesmer bound, Des. Codes Cryptogr. 12 (1997) 83–87] and Klein [On codes meeting the Griesmer bound, Discrete Math. 274 (2004) 289–297].

Keywords: Codes; Griesmer bound; Blocking sets

1. Introduction

The famous Griesmer bound states that the minimal length $n_q(k, d)$ of a k-dimensional linear code over $\mathbb{F}_q$ with minimal distance d satisfies:

Result 1 (*Griesmer bound, Griesmer [5] for* $q = 2$ *and Solomon and Stiffler [12] for* $q > 2$).

$$n_q(k, d) \geqslant g_q(k, d) := \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil. \tag{1}$$

It is a natural question to ask for which parameters we can achieve equality. In this article, we use finite geometry to characterize a large set of parameters for which equality cannot be achieved in (1). In order to state our result, we need the notation of a blocking set.

A *blocking set* of $\mathrm{PG}(2, q)$ is a set of points such that every line of $\mathrm{PG}(2, q)$ has at least one point in this set and at least one point that is not in this set.

The smallest number of points in a blocking set of $\mathrm{PG}(2, q)$ plays an important role in our main result. We remark that this smallest number lies in between $q + 1 + \sqrt{q}$ and $\frac{3}{2}(q + 1)$, depending on q; for odd primes q the smallest blocking set has size $\frac{3}{2}(q + 1)$ by a result of Blokhuis [1]. For information on blocking sets and their sizes, we refer to [4]. For a recent article on linear codes meeting the Griesmer bound we refer to a paper of Storme [13]. Our main result is the following.

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Theorem 2. *Let* $d = sq^{k-1} - \sum_{i=1}^{k-1} t_i q^{k-1-i}$ *with* $0 \leqslant t_i < q$ *and let* i_0 *be the smallest index with* $t_{i_0} \neq 0$. *Suppose that* $t_{i_0+1} = 0$ *and*

$$\sum_{i=i_0+2}^{k-1} t_i q^{k-1-i} \leqslant r q^{k-3-i_0}.$$

Then $n_q(k,d) > g_q(k,d)$, *i.e. the Griesmer bound is not sharp provided the following conditions hold*:

- (a) $s < \min\{t_{i_0}, (k-i_0)/i_0\}$.
- (b) $t_{i_0} \leqslant \frac{1}{2}(q+1)$.
- (c) $t_{i_0} + r \leqslant q$ *and* r *is a non-negative integer.*
- (d) *if* $i_0 > 1$, *then the projective plane* PG$(2,q)$ *has no blocking set of size* $\leqslant q + t_{i_0}$.

As special cases we find that there is no $[g_q(k,d), k, d]_q$-code with $d = (k-2)q^{k-1} - (k-1)q^{k-2}$ (see [9,10]). The non-existence results proven in [8] are also special cases of Theorem 2.

We explain the background of this result. Let C be an $[n,k]$-code. Represent the columns of a generator matrix for C by points of the projective space PG$(k-1,q)$. In this way, the code gives a multiset of points in PG$(k-1,q)$. That the code meets the Griesmer bound translates to intersection properties of the multiset with the subspaces of PG$(k-1,q)$. For some sets of parameters, these properties are sufficient to recover a large part of the multiset. Then, in a final step, one can analyze whether or not such a large part can occur in the multiset of a code.

2. Notation

For integers $k \geqslant -1$ we denote by $\theta_k = (q^{k+1}-1)/q - 1$ the number of one-dimensional subspaces of $\mathbb{F}_q^{k+1}$ or, equivalently, the number of points in the finite projective Galois space PG(k,q). An $[n,k,d]_q$-code is a linear subspace C of dimension $k \geqslant 1$ of F_q^n such that d is the minimum Hamming distance between different *words* (elements) of C. For information on linear codes we refer to the standard literature. If q is clear from the context, we also speak of $[n,k,d]$-codes. If we do not want to refer to d, then we speak of an $[n,k]$-code.

The *simplex code* S_k is a $[\theta_{k-1}, k, q^{k-1}]_q$-code whose generator matrix G is formed by θ_{k-1} pairwise linear independent column-vectors in $\mathbb{F}_q^k$. Taking $t \geqslant 1$ copies of G, we obtain the generator matrix $(G\ G \cdots G)$ of a $[t\theta_{k-1}, k, tq^{k-1}]_q$-code, which we call a *t-fold simplex code*. All t-fold simplex codes satisfy the Griesmer bound.

An excellent way to construct good linear codes is to start with a copy of t simplex codes and delete columns of the generator matrix. The columns to be deleted form the generator matrix of what is called an anticode (this notation is due to Farrell [2,3], see also [11]). Distance 0 between codewords is allowed, i.e. an anticode may contain repeated codewords.

Definition 3. Let A be a $k \times m$ matrix with entries in $\mathbb{F}_q$. Then the q^k combinations of its rows form the codewords of an anticode of length m.

Notice that if $\operatorname{rank}(A) = r$, each codeword occurs q^{k-r} times. If one starts with a code of minimum distance d and deletes m columns that form an anticode with maximum distance δ, then one obtains a code with minimum distance at least $d - \delta$.

3. Geometric description of codes

Let C be an $[n,k]$-code with generator matrix G. Each column of the generator matrix describes a point of PG$(k-1,q)$. We represent C by the multiset M of these n points. For example, the t-fold simplex code is represented by the multiset, in which every point of PG$(k-1,q)$ occurs t times.

For each subset S of PG$(k-1,q)$ we denote by $c(S)$ the number of points of M in S counted with multiplicity. To be more precise, we define $c(P)$ for each point P as the number of columns in G representing P, and put $c(S) = \sum_{P \in S} c(P)$.

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Let

$$\gamma_i := \max\{c(S) \mid S \text{ is a subspace of dimension } i\}.$$

Then γ_0 is the largest integer i such that a point is represented by i columns of the generator matrix. The minimal distance of C is the minimal number of points lying in the complement of a hyperplane, i.e.

$$d = n - \gamma_{k-2}.$$

For a code meeting the Griesmer bound one can compute the γ_i from its parameters. At the moment we need only the following assertion, which is well known; for the convenience of the reader we give the short proof.

Lemma 4 (*see also Maruta [10]*). *Let* $(s-1)q^{k-1} < d \leqslant sq^{k-1}$ *and C be an* $[n,k,d]_q$ *code meeting the Griesmer bound. Then* $\gamma_0 = \max\{c(P) \mid P \in \mathrm{PG}(k-1,q)\} = s$.

Proof. By the pigeonhole principle we get $\gamma_0 \geqslant n/\theta_{k-1} > s-1$.

Assume $\gamma_0 > s$. There exists a point $P = (p_0, \dots, p_{k-1})$ described by at least $s+1$ columns of the generator matrix. Let G be the generator matrix of C and consider the subcode C' of C defined by

$$C' = \left\{ xG \mid x = (x_0, \dots, x_{k-1}) \in F_q^k, \sum x_i p_i = 0 \right\}.$$

The codewords of C' have entry 0 at the columns corresponding to P. Puncturing C' at these columns yields an $[n', k', d']_q$-code with $n' \leqslant n - s - 1$, $k' \geqslant k-1$ and $d' \geqslant d$. But the Griesmer bound says

$$n - s - 1 \geqslant n' \geqslant \sum_{i=0}^{k'} \left\lceil \frac{d'}{q^i} \right\rceil \geqslant \sum_{i=0}^{k-2} \left\lceil \frac{d}{q^i} \right\rceil = \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil - \left\lceil \frac{d}{q^{k-1}} \right\rceil = n - s,$$

a contradiction. □

Suppose now that the code C meets the Griesmer bound, and put

$$d = sq^{k-1} - \sum_{i=1}^{k-1} t_i q^{k-1-i}$$

with $0 \leqslant t_i \leqslant q-1$. Then $\gamma_0 = s$ and $n = s\theta_{k-1} - \sum_{i=1}^{k-1} t_i \theta_{k-1-i}$. For the study of those codes, a second multiset M' plays an important role. It is so to say the difference of γ_0 copies of $\mathrm{PG}(k-1,q)$ and the multiset M. To be precise, it is the multiset, in which each point P occurs $w(P) := \gamma_0 - c(P)$ times. We call $w(P)$ the *weight* of the point P, and for each subspace S we define its weight $w(S)$ to be $w(S) := \sum_{P \in S} w(P)$. We denote by w_i the minimal weight of all i-dimensional subspaces. From the definitions we have $\gamma_i + w_i = s\theta_i$. The multiset M' represents a generator matrix of an anticode. We can calculate

$$w_{k-1} = s\theta_{k-1} - \gamma_{k-1} = s\theta_{k-1} - n = \sum_{i=1}^{k-1} t_i \theta_{k-1-i}$$

and

$$w_{k-2} = s\theta_{k-2} - \gamma_{k-2} = s\theta_{k-2} - (n-d) = \sum_{i=1}^{k-2} t_i \theta_{k-2-i}.$$

Remark. The parameters give the impression that M' is a sum of subspaces of $\mathrm{PG}(k-1,q)$, namely of t_i subspaces of dimension $k-1-i$ for each i. The examples in Section 4 show that this is not necessarily the case. In fact the counterpart of Griesmer codes in projective spaces are so called minihypers, see [13] for more details. In this paper, we mainly give a criterion that ensures that M' contains certain subspaces.

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Lemma 5 (*see also Hamada* [6]). *Let w be a weight function with values in $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ on the point set* $\mathrm{PG}(k-1, q)$, *and let w_i for $0 \leqslant i \leqslant k-1$, be the minimum weight of the i-subspaces. Suppose that*

$$w_{k-1} = \sum_{i=1}^{k-1} t_i \theta_{k-1-i} \quad \text{and} \quad w_{k-2} = \sum_{i=1}^{k-2} t_i \theta_{k-2-i}.$$

Then $w_j = \sum_{i=1}^{j} t_i \theta_{j-i}$.

Proof. In a first step we prove $w_j \geqslant \sum_{i=1}^{j} t_i \theta_{j-i}$ be descending induction. For $j = k-1$ and $j = k-2$ this holds by assumption. Now suppose it holds for j where $j \leqslant k-2$. Look at a $(j-1)$-dimensional subspace of weight w_{j-1} and consider all j-dimensional subspaces through this one. We find

$$w_{j-1} + \theta_{k-1-j}(w_j - w_{j-1}) \leqslant w_{k-1}.$$

From this we conclude

$$\begin{aligned} w_{j-1} &\geqslant w_j - \frac{w_{k-1} - w_j}{\theta_{k-j-1} - 1} \\ &\geqslant \sum_{i=1}^{j} t_i \theta_{j-i} - \frac{\sum_{i=1}^{k-1} t_i \theta_{k-1-i} - \sum_{i=1}^{j} t_i \theta_{j-i}}{\theta_{k-j-1} - 1} \quad \text{induction hyp.} \\ &= \sum_{i=1}^{j-1} t_i \theta_{j-1-i} - \frac{\sum_{i=j+1}^{k-1} t_i \theta_{k-1-i}}{\theta_{k-j-1} - 1} \\ &> \sum_{i=1}^{j-1} t_i \theta_{j-1-i} - 1 \quad \text{since } t_i \leqslant q-1 \text{ and } j \leqslant k-2. \end{aligned}$$

This finishes the descending induction. Thus $w_j \geqslant \sum_{i=1}^{j} t_i \theta_{j-i}$ for all j. Assume i_0 is the smallest index with $t_{i_0} \neq 0$. Then the total weight is less than θ_{k-i_0}. Therefore some subspace of dimension $i_0 - 1$ must have weight 0, which proves $w_i = 0$ for all $i < i_0$. Now suppose that we can improve the bound for w_i for some i in the range $i_0 \leqslant i \leqslant k-3$. The induction argument shows that in this case we can improve the bound for all indices $i-1, i-2, \ldots, i_0, i_0-1$. A contradiction to $w_{i_0-1} = 0$. □

Especially if i_0 is the smallest value for which $t_{i_0} \neq 0$, this says M' meets every i_0-dimensional subspace at least t_{i_0} times. We now forget that M' comes from a code and study only multisets with $\sum_{i=i_0}^{k} t_i \theta_{k-1-i}$ points that meets every i_0-dimensional subspace at least t_{i_0} times. We give criteria on such a set that imply that M' contains the sum of t_{i_0} subspaces of codimension i_0.

4. Line blocking sets

In this section, we study multisets of points in $\mathrm{PG}(d, q)$ that meet every line at least t times. We start in dimension two. Usually we identify a multiset S with a weight function on the points of $\mathrm{PG}(d, q)$, where the weight $w(P)$ of a point is the number of times P occurs in S. The weight of a subset of $\mathrm{PG}(d, q)$ is the sum of the weights of the points in this subset. An algebraic proof of the following lemma will also occur in [4].

Lemma 6. *Let S be a multiset of $t(q+1)$ points in* $\mathrm{PG}(2, q)$ *such that every line contains at least t points of S. If $t \leqslant \frac{1}{2}(q+1)$, then S consists of t lines.*

Proof. We use induction on t, the case $t = 0$ being trivial. Let $t \geqslant 1$. Every line on a point of weight zero has weight exactly t. Also, a point of positive weight does not lie only on lines of weight t. Both statements follow from counting

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the total weight of the plane using the lines on the point. Clearly, both statements together imply that there exists a line h with only points of positive weight.

Assume that the weight $w(h)$ of h is at most $q+t-1$. Then h contains a point P of weight one. If w_i, $i=1,\ldots,q$, are the weights of the remaining q lines on P, then $\sum(w_i-1)$ is the total weight $t(q+1)$ minus the weight of h and hence

$$\sum(w_i-t)=t(q+1)-w(h)-q(t-1)=q+t-w(h).$$

As $w_i \geqslant t$ for all i and $q+1 \leqslant w(h) \leqslant q+t-1$, it follows that there exists an index i with $t<w_i\leqslant 2t-1$. As $w_i \neq t$, then all points of the corresponding line have positive weight. Hence $w_i \geqslant q+1$. Then $q+1\leqslant 2t-1$, a contradiction.

Hence all points of h have positive weight and the total weight of h is at least $q+t$. If we remove each point of h once from the multiset, we remain with a multiset meeting every line at least $t-1$ times. Now apply the induction hypothesis. ☐

Remark. The bound on t is this lemma is best possible in general. In fact, in PG$(2,q)$, q even, a dual hyperoval consists of $q+2$ lines, no three of which are collinear. These lines cover $\frac{1}{2}(q+2)(q+1)$ points. Let S be the set consisting of these $\frac{1}{2}(q+2)(q+1)$ points, where each point is given weight one. Then every line is incident with at least $\frac{1}{2}(q+2)$ points of S, but S cannot be written as the union of $\frac{1}{2}(q+2)$ lines.

Theorem 7. *Let S be a multiset of at most $t\theta_{d-1}+r\theta_{d-3}$ points in* PG(d,q) *that meets every line at least t times, where $t\geqslant 1$ and $r\geqslant 0$ are integers satisfying $2t\leqslant q+1$ and $r+t\leqslant q$. Then S contains t hyperplanes of* PG(d,q).

Proof. We prove this by induction on d. The case $d=2$ was handled in the above lemma. Now suppose that $d\geqslant 3$ and suppose the statement holds for smaller d. We prove that it holds for d and for this we use induction on t. The proof will be obtained in two steps. In Step 1 we show that S contains a hyperplane, hereby proving the induction hypothesis $t=1$. In Step 2 we show that every line of H meets S in at least $q+t$ points. For $t\geqslant 2$, this is sufficient to prove the induction step from $t-1$ to t by removing each point of H once from S to obtain a multiset S' intersecting every line still in at least $t-1$ points.

Step 1: The average number of points of S in the hyperplanes is

$$\frac{|S|\theta_{d-1}}{\theta_d}<\frac{|S|}{q}\leqslant\frac{t\theta_{d-1}+r\theta_{d-3}}{q}=t\theta_{d-2}+r\theta_{d-4}+\frac{t+r}{q}.$$

As $t+r\leqslant q$, it follows that PG(d,q) contains a hyperplane H that meets S in at most $t\theta_{d-2}+r\theta_{d-4}$ points. The induction hypothesis applied to this hyperplane shows that S contains the sum of t subspaces that have codimension one in H. This proves that H contains a subspace π of dimension $d-2$ with $|S\cap\pi|\geqslant\theta_{d-2}+(t-1)\theta_{d-3}$. We claim that some hyperplane on π lies entirely in S. Assume by way of contradiction that this is not true.

If H' is any hyperplane on π, then we find a point $P\in H'$ that does not lie in S. As each line on P meets S in at least t points, this implies that H' meets S in at most $|S|-tq^{d-1}$ points. Considering all hyperplanes on π, we find the following bound on $|S|$:

$$\begin{aligned}&|S|\leqslant|\pi\cap S|+(q+1)(|S|-tq^{d-1}-|\pi\cap S|)\\&\quad\Rightarrow|S|\geqslant|\pi\cap S|+tq^{d-2}(q+1)\\&\quad\Rightarrow|S|\geqslant q^{d-2}+t\theta_{d-1}.\end{aligned}$$

But $|S|\leqslant t\theta_{d-1}+r\theta_{d-3}$ with $r\leqslant q-t\leqslant q-1$, a contradiction.

Step 2: We consider a hyperplane H with the property that all points of H lie in S. We have to show that every line of H meets S in at least $q+t$ points. For this we consider a line l of H with the smallest number $q+1+x$ of points in S. If every point of l occurs at least twice in S, we are done. Therefore we may assume that l contains a point P that occurs only once in S. Then H contains at least $\theta_{d-1}+x\theta_{d-2}$ points from S (count the number of points of S on the lines of H through P). Now we count the average number N of points of S in a plane through l not in H.

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This average number is at most

$$\frac{1}{q^{d-2}}(t\theta_{d-1} + r\theta_{d-3} - \theta_{d-1} - x\theta_{d-2}) + (q + 1 + x).$$

This number is less than $t(q+1)+1$ due to the hypothesis $t + r \leqslant q$. Hence some plane π through l contains at most $t(q+1)$ points of S. To π we apply Lemma 6. It follows that $\pi \cap S$ is the sum of t lines. As l is contained in S, it follows that l meets S in precisely $q+t$ points. □

5. Sets meeting subspaces of higher dimension

Now we want to study multisets that meet all subspaces of a higher dimension k at least t times. In this case, we cannot hope to get as strong results as in previous section for $k = 1$ due to the following examples.

Example 8.

(a) Let q be a square. Every k-subspace of $\mathrm{PG}(2k-1, q)$ contains at least one line of the Baer subspace $\mathrm{PG}(2k-1, \sqrt{q})$. Thus $S = \mathrm{PG}(2k-1, \sqrt{q})$ is a set of $(\sqrt{q}+1)\theta_{k-1}$ points in $\mathrm{PG}(2k-1, q)$ that meets every k-dimensional subspace at least $\sqrt{q}+1$ times. But S contains no subspace of codimension k.

(b) In the previous example we project $\mathrm{PG}(2k-1, \sqrt{q})$ from an outside point to a hyperplane of $\mathrm{PG}(2k-1, q)$. This yields a multiset of $(\sqrt{q}+1)\theta_{k-1}$ points in a $\mathrm{PG}(2k-2, q)$ that meets every $(k-1)$-subspace at least $\sqrt{q}+1$ times. But this set contains no subspace of codimension $k-1$ in $\mathrm{PG}(2k-2, q)$.

The analogue of Theorem 7 for higher dimensions needs therefore additional restrictions on t. The following result on blocking sets will be used to generalize Theorem 7.

Result 9 (*Heim [7], Theorem 3.1.3*). *Let $q+1+\delta$ be the minimum number of points of a blocking set in* $\mathrm{PG}(2, q)$. *Then for $1 \leqslant t < d$, the minimum cardinality of a set of points of* $\mathrm{PG}(d, q)$ *that meets every t-dimensional subspace and that contains no subspace of codimension t is $\theta_{d-t} + q^{d-1-t}\delta$ with equality if and only if the set is a cone with a vertex a subspace of dimension $d-t-2$ and base a blocking set of size $q+1+\delta$ in a plane.*

By a *cone* over a blocking set B in a pl[illegible] π, we mean the union of the subspaces $\langle V, P\rangle$ with $P \in B$, where the vertex V of the cone is a subspace of dimension $d-3$ skew to the plane in which B lives.

By a characteristic function of a subspace U of a projective space we mean the function 1_U defined by $1_U(P) = 1$ for points P in U and $1_U(P) = 0$ for points P outside U. For two functions f and g from the point set of a projective space to the integers, we write $f \leqslant g$ if $f(P) \leqslant g(P)$ for all points.

Lemma 10. *Suppose that w is a function from the point set of* $\mathrm{PG}(k-1, q)$ *to $\mathbb{N} \cup \{0\}$ such that the total weight of all points is at most $t\theta_{d-k} + r\theta_{d-k-2}$ for integers $t, r \geqslant 0$ with $t + r \leqslant q$. Suppose furthermore that there exist t subspaces $T_1, \ldots, T_t$ of dimension $d-k$ such that $1_{T_1} + \cdots + 1_{T_t} \leqslant w$. If $S_1, \ldots, S_{t'}$ are $(d-k)$-subspaces with $1_{S_1} + \cdots + 1_{S_{t'}} \leqslant w$, then $t' \leqslant t$ and with a suitable enumeration we have $S_i = T_i$ for $i = 1, \ldots, t'$.*

Proof. Using induction on t', one sees that it is sufficient to prove the assertion for $t' = 1$. Thus, we have $1_S \leqslant w$ for some $(d-k)$-subspace S and we have to show that S is one of the subspaces T_i. Assume that this is not true. Then all subspaces $S \cap T_i$ have dimension at most $d-k-1$, so

$$\begin{aligned}\theta_{d-k} = 1_S(S) \leqslant w(S) &= \sum_i 1_{T_i}(U \cap T_i) + \Big(w - \sum 1_{T_i}\Big)(S)\\ &\leqslant t\theta_{d-k-1} + r\theta_{d-k-2} \leqslant (r+t)\theta_{d-k-1} \leqslant q\theta_{d-k-1}\end{aligned}$$

and this is a contradiction. □

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Theorem 11. *Let S be a multiset of at most $t\theta_{d-k}+r\theta_{d-k-2}$ points in* PG(d,q) *that meets every k-dimensional subspace at least t times. Suppose*

(a) $1\leqslant t\leqslant (q+1)/2$.
(b) $t+r\leqslant q$. *The number r is a non-negative integer.*
(c) *A projective plane of order q contains no blocking set of size $\leqslant q+t$.*

Then S contains t subspaces of codimension k of PG(d,q).

Proof. *Part* 1: We represent S also by a weight function w on the point set of PG(d,q), where $w(P)$ for each point P is the number of times P occurs in S. We may assume that S is a minimal multiset meeting every k-subspace at least t times. This means that every point of P of S lies on a k-subspace π satisfying $w(\pi)=t$.

We prove the theorem by induction on k. For $k=1$ apply Theorem 7. Now let $k>1$ and assume that the theorem holds for $k-1$. We assume that S contains $t'<t$ subspaces of codimension k and show that S contains at least one more subspace of codimension k. Let $S_1,\ldots,S_{t'}$ be any t' subspaces of codimension k contained in S. Then $1_{S_1}+\cdots+1_{S_{t'}}\leqslant w$, that is for each point P the weight $w(P)$ of P is at least the number of indices $i\leqslant t'$ with $P\in S_i$. Let S' be the multiset obtained from S by removing $S_1,\ldots,S_{t'}$. The corresponding weight function w' satisfies for every point P that $w'(P)$ is equal to $w(P)$ minus the number of subspaces $S_1,\ldots,S_{t'}$ containing P. We have

$$\begin{aligned} w'(\mathrm{PG}(k-1,q)) &= w(\mathrm{PG}(k-1,q))-t'\theta_{d-k}\\ &\leqslant (t-t')\theta_{d-k}+r\theta_{d-k-2}. \end{aligned}$$

We remark that the naive approach to just remove one subspace S_i of codimension k from S and try to apply the induction theorem to the obtained multiset leads to difficulties; it is not clear why the obtained multiset meets every k-subspace in at least $t-1$ points.

Part 2: Points of low multiplicity. In this part we consider a point Q not in S and look at the factor space of Q. There we see S as a multiset S/Q of at most $t\theta_{d-k}+r\theta_{d-k-2}$ points in $(d-1)$-dimensional projective space which meets every $(k-1)$-dimensional subspace at least t times. By the induction hypothesis we must see t subspaces of codimension $k-1$ in this factor space whose sum is contained in S/Q. From Lemma 10 we deduce that these t subspaces are uniquely determined; hence t' of these are the subspaces $\langle Q,S_i\rangle$, $i=1,\ldots,t'$, in the factor space of Q.

Back in PG(d,q) we see t subspaces of dimension $d-k+1$ on Q such that the weight of a line on Q is at least as big as the number of these t subspaces that contain that line. Also t' of these subspaces are the subspaces $\langle S_i,Q\rangle$, $1\leqslant i\leqslant t'$. Consider the remaining $t-t'$ subspaces. We denote by x_Q the smallest multiplicity of one of these subspaces. That is, every subspace that occurs in the list of these $t-t'$ subspaces occurs at least x_Q times, and at least one such subspace occurs exactly x_Q times.

We shall also need the following. If T is a $(d-k+1)$-dimensional subspace on Q, then its w'-weight $w'(T)$ determines the multiplicity of T in the list of these $t-t'$ subspaces. In fact, if y is the multiplicity of T, then clearly $w'(T)\geqslant y\theta_{d-k}$. On the other hand, since $t-t'-y$ of the subspaces do not contain T and since the w'-weight of the full space is at most $(t-t')\theta_{d-k}+r\theta_{d-k-2}$, we have

$$\begin{aligned} w'(T) &\leqslant (t-t')\theta_{d-k}+r\theta_{d-k-2}-(t-t'-y)q^{d-k}\\ &= y\theta_{d-k}+(t-t'-y)\theta_{d-k-1}+r\theta_{d-k-2} \end{aligned} \qquad (2)$$

and this is less than $(y+1)\theta_{d-k}$ since $t+r\leqslant q$. So indeed $w'(T)$ determines the multiplicity y of T uniquely.

Notation: Let x be the minimum of the x_Q for all points $Q\notin S$:

$$x := \min\{x_Q \mid Q\notin S\}.$$

Part 3: In this part we shall show that $w'(P)=0$ or $w'(P)\geqslant x$ for every point $P\in S$.

By the minimality of S, we find a k-subspace π on P satisfying $w(\pi)=t$. As $t<q+1$, then each subspace S_i, $i=1,\ldots,t'$, meets π in a single point and thus $w'(\pi)=w(\pi)-t'=t-t'$. As $k\geqslant 2$, we then find a line h in π on P such that $w(h)=w(P)$. Let Q be a point of $h\backslash\{P\}$, so $w(Q)=0$. As in Part 2 we see on Q the t' subspaces $\langle S_i,Q\rangle$ and $t-t'$ further subspaces of dimension $d-k+1$. Also a subspace occurring in the list of these $t-t'$ subspaces occurs in that list at least x_Q times. Each of these $t-t'$ subspaces meets π in at least a line and thus contributes at

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least one to the weight $w'(\pi)$ (recall that we have shown in Part 3 for a line h on Q that $w'(h)$ is at least as big as the number of the $t-t'$ subspaces that contain h). If none of the $t-t'$ subspaces contains P, then $w'(\pi)=t-t'$ implies that $w'(P)=0$. If on the other hand, some of these $t-t'$ subspaces contains P (and hence the line $h=PQ$), then $w'(P)=w'(h)\geqslant x_Q\geqslant x$.

Part 4: Again we consider a point $Q\notin S$ and the t subspaces of dimension $d+1-k$ on Q obtained from the induction hypothesis in the factor space. But this time we choose Q in such a way that $x_Q=x$. Let T be a $(d-k+1)$-subspace on Q of multiplicity x. Then $w'(h)\geqslant x$ for every line h of T on Q. Also $w'(T)\geqslant x\theta_{d-k}$.

If R is any point of T not in S, the last argument of Part 2 shows that we see T in the factor space of R also with multiplicity x, that is every line of T on R meets S' in at least x points.

Let B' be the set of all points in $T\cap S'$ counted without multiplicities, that is

$$B'=\{P\in T \mid P\in S',\ w'(P)>0\}.$$

From Part 3 we know that every point of S' has multiplicity at least x. Applying inequality (2) with $y=x$ thus gives

$$\begin{aligned}|B'| &\leqslant \frac{1}{x}\cdot w'(T)\\ &\leqslant \theta_{d-k}+\frac{t-t'-x}{x}\theta_{d-k-1}+\frac{r}{x}\theta_{d-k-2}\\ &\leqslant \theta_{d-k}+(t-t'-1)\theta_{d-k-1}+r\theta_{d-k-2}.\end{aligned}\tag{3}$$

Our arguments show that B' meets every line of T that contains a point that does not lie in S. As S' is obtained from S by removing the subspaces $S_1,\dots,S_{t'}$, it follows that every line of T that does not meet B' must be a line that lies entirely in one of the subspaces $S_1,\dots,S_{t'}$. For each $i\leqslant t'$, let U_i be a hyperplane of $S_i\cap T$. Then U_i meets every line contained in $S_i\cap T$ and hence

$$B := B'\cup S_1\cup\cdots\cup S\ \text{[illegible]}$$

meets every line contained in T. We have $\dim(U_i)\leqslant \dim(S_i)-1=d-k-1$ and hence

$$\begin{aligned}|B| &\leqslant |B'|+t'\theta_{d-k-1}\leqslant\theta_{d-k}+(t-1)\theta_{d-k-1}+r\theta_{d-k-2}\\ &=\theta_{d-k}+(t-1)q^{d-k-1}+(t+r-1)\theta_{d-k-2}\\ &\leqslant\theta_{d-k}+(t-1)q^{d-k-1}+(q-1)\theta_{d-k-2}\\ &<\theta_{d-k}+tq^{d-k-1}.\end{aligned}$$

Since $\mathrm{PG}(2,q)$ possesses by hypothesis no blocking set of size $q+t$ or smaller, Result 9 of Heim shows that B contains a subspace U of dimension $d-k$.

We know that U is contained in the union of B' and the subspaces $U_1,\dots,U_{t'}$, but we want to show that U is entirely contained in B'. Assume that this is not the case, that is some point A of U does not lie in B'. Then $t'\geqslant 1$.

Also there are at least $q^{d-k}-t'q^{d-k-1}$ lines in T on A that do not lie in U or any subspace S_i. Such a line has at least $q+1-t'$ points that do not belong [illegible]ny subspace S_i. Each such line meets B' (in fact, some point X of such a line does not lie in any of the subspaces U_i; if $X\in S$, then $X\in S'$ and hence $X\in B'$, and if $X\notin S$, then every line of T on X meets B' as seen above). It follows that at least $q^{d-k}-t'q^{d-k-1}$ points of B' do not lie in U. Since all points of U that do not lie in the subspaces U_i belong to B', we have

$$|U\cap B'|\geqslant q^{d-k}-(t'-1)q^{d-k-1}.$$

Hence

$$|B'|\geqslant q^{d-k}-(t'-1)q^{d-k-1}+q^{d-k}-t'q^{d-k-1}.$$

Comparing this lower bound with the upper bound (3) using $r\leqslant q-t$, we find

$$q^{d-k}+1+(t'-1)\theta_{d-k-2}\leqslant(t+t')q^{d-k-1}.$$

A[illegible] $\geqslant 1$, we find $t+t'>q$. As $t'\leqslant t-1$, this implies $t>(q+1)/2$, a contradiction. Hence U is entirely contained in B', so we have reached our goal to show that B contains beside the t' subspaces $U_1,\dots,U_{t'}$ at least one more subspace of codimension k. ☐

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6. Back to Coding Theory

In this section, we will translate our results back to Coding Theory and prove Theorem 2.

Let $d = sq^{k-1} - \sum_{i=1}^{k-1} t_i q^{k-1-i}$ with $0 \leqslant t_i < q$ and let C be an $[g_q(k,d),k,d]_q$-code. Lemma 4 shows that the anticode of C is represented by a multiset S such that the corresponding weight function w satisfies

1. $0 \leqslant w(P) \leqslant s$.
2. $w_{k-1} = w(\mathrm{PG}(k-1,q)) = \sum_{i=1}^{k-1} t_j \theta_{k-1-j}$.
3. The minimal weight of a hyperplane is $w_{k-2} = \sum_{i=1}^{k-2} t_i \theta_{k-2-i}$.

Let i_0 be the smallest value with $t_{i_0} \neq 0$. By Lemma 5 the multiset S blocks every i_0-dimensional subspace at least t_{i_0} times.

Assume that $t_{i_0+1} = 0$ and $\sum_{i=i_0+2}^{k-1} t_i q^{k-1-i} \leqslant rq^{k-3-i_0}$ for an integer $r \leqslant q-t$. The conditions (b), (c) and (d) of Theorem 2 guarantee that Theorem 7 or 11 can be applied. Thus S contains t_{i_0} subspaces of codimension i_0.

The condition (a) of Theorem 2 implies that $t_{i_0} \geqslant s+1$ and that the intersection of $s+1$ subspaces of codimension i_0 is non-empty. Hence, there exists a point P that lies in at least $s+1$ of the t_{i_0} subspaces contained in S. But then $w(P) \geqslant s+1$, a contradiction. This proves Theorem 2.

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