

# Small maximal partial spreads in classical finite polar spaces

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## Abstract

We prove lower bounds on the size of small maximal partial spreads in  $Q^+(4n+1, q)$ ,  $W(2n+1, q)$ , and  $H(2n+1, q^2)$ . This research on the size of smallest maximal partial spreads in classical finite polar spaces is part of a detailed study on small and large maximal partial ovoids and spreads in classical finite polar spaces, performed in [2, 3].

## 1 Introduction

A polar space is a geometry that satisfies the “one or all axiom”:

- Let  $l$  be a line and let  $P$  be a point not on  $l$ . Then either  $P$  is connected to exactly one point of  $l$  by a line or  $P$  is connected to all points of  $l$ .

A polar space contains projective spaces as subspaces. A projective subspace of maximal dimension is called a generator.

A polar space is called classical if its points and lines are the totally isotropic points and lines of a projective space with respect to some non-degenerated sesquilinear form. There exist five different types of polar spaces.

- The elliptic quadric  $Q^-(2n+1, q)$  formed by all points of  $PG(2n+1, q)$  which satisfy the standard equation  $x_0x_1 + \cdots + x_{2n-2}x_{2n-1} + f(x_{2n}, x_{2n+1}) = 0$  where  $f$  is an irreducible polynomial of degree 2 over  $\mathbb{F}_q$ .
- The parabolic quadric  $Q(2n, q)$  formed by all points of  $PG(2n, q)$  which satisfy the standard equation  $x_0x_1 + \cdots + x_{2n-2}x_{2n-1} + x_{2n}^2 = 0$ .
- The hyperbolic quadric  $Q^+(2n+1, q)$  formed by all points of  $PG(2n, q)$  which satisfy the standard equation  $x_0x_1 + \cdots + x_{2n}x_{2n+1} = 0$ .
- The symplectic polar space  $W(2n+1, q)$  that consists of all points of  $PG(2n+1, q)$  together with the totally isotropic lines with respect to the symplectic form  $\theta(x, y) = x_0y_1 - x_1y_0 + \cdots + x_{2n}y_{2n+1} - x_{2n+1}y_{2n}$ .

- The hermitian polar space  $H(n, q^2)$  formed by all points of  $PG(n, q^2)$  which satisfy the standard equation  $x_0^{q+1} + \dots + x_n^{q+1} = 0$ .

The study of ovoids and spreads of a classical finite polar space has drawn a great interest in the past years (see [8] for an overview and further references). In this article, we will study only partial spreads.

**Definition 1**

Let  $P$  be a classical finite polar space.

A partial spread  $\mathcal{S}$  of  $P$  is a set of pairwise disjoint generators. A partial spread is maximal, if it is not a proper subset of an other partial spread. A partial spread  $\mathcal{S}$  is called a spread if all points of  $P$  are covered by the elements of  $\mathcal{S}$ .

It is a natural question to ask for the possible size of a maximal partial spread. Especially, we are interested in the minimal size of a maximal partial spread.

## 2 Glynn's counting technique

The technique we use is standard and goes back to Glynn [6].

Let  $P$  be some classical finite polar space. Denote by  $g$  the number of generators. Let  $a$  be the number of generators skew to a given generator  $G_1$ . For two skew generators  $G_1$  and  $G_2$ , we denote the number of generators skew to both by  $b$ , and let  $c$  be a lower bound on the number of generators skew to three pairwise skew generators  $G_1, G_2$  and  $G_3$ .

Let  $S$  be a maximal partial spread of size  $s$  of the polar space  $P$ . Denote by  $n_i$  the number of generators not in  $S$  that meet exactly  $i$  elements of  $S$ . We have the following equations:

$$\begin{aligned} \sum n_i &= g - s \\ \sum n_i i &= s(g - 1 - a) \\ \sum n_i i(i - 1) &= s(s - 1)(g - 2a + b) \\ \sum n_i i(i - 1)(i - 2) &\leq s(s - 1)(s - 2)(g - 3a + 3b - c) . \end{aligned}$$

The first equation simply expresses the fact that  $S$  has size  $s$ . The second equation is obtained from counting pairs  $(\pi, \tau)$  of generators with  $\pi \notin S, \tau \in S$  and  $\pi \cap \tau \neq \emptyset$ . For the third equation, we have to count triples  $(\pi, \tau_1, \tau_2)$ , with  $\pi \notin S, \tau_i \in S, \tau_1 \neq \tau_2$  and  $\pi \cap \tau_i \neq \emptyset$  ( $i = 1, 2$ ). Note that  $2a - b$  generators miss either  $\tau_1$  or  $\tau_2$ . The last equation follows similarly by counting quadruples  $(\pi, \tau_1, \tau_2, \tau_3)$  with  $\pi \notin S, \tau_i \in S$  and  $\pi \cap \tau_i \neq \emptyset$  ( $i = 1, 2, 3$ ),  $\tau_1, \tau_2, \tau_3$  pairwise distinct. By inclusion-exclusion, the number of generators missing at least one of three pairwise skew generators is at least  $3a - 3b + c$ .

Now look at the equation:

$$\sum n_i(i-1)(i-3)(i-4) = \sum n_i i(i-1)(i-2) - 5 \sum n_i i(i-1) + 12 \sum n_i i - 12 \sum n_i .$$

As the partial spread is maximal, we have  $n_0 = 0$  and thus the sum on the left hand side is non-negative. It follows that:

$$0 \leq s(s-1)(s-2)(g-3a+3b-c) - 5s(s-1)(g-2a+b) + 12s(g-1-a) - 12(g-s) . \quad (1)$$

Now we obtain the desired bound on  $s$  by solving this inequality. To do so, we must know  $g$ ,  $a$ ,  $b$  and  $c$ . While  $g$  and  $a$  are simple expressions,  $b$  and especially  $c$  are difficult to obtain. The major part of this paper will deal with the calculation of these numbers. Our technique to determine  $c$  uses a geometric property of polar spaces in which the rank of a generator is half of the rank of the ambient projective space, therefore our technique solves the case of the hyperbolic quadric, the symplectic variety and the hermitian variety in odd dimensions. For other classical finite polar spaces, Glynn's counting technique gives no result. The main reason is that the number of generators skew to three pairwise skew generators  $G_1$ ,  $G_2$  and  $G_3$  depends on the three selected generators in  $S$ . The different possible values vary up to a factor of order  $q$ , which gives a too weak lower bound on  $c$ .

### 3 A useful lemma

We denote by  $\theta_n$  the number  $(q^{n+1} - 1)/(q - 1)$  of points of  $PG(n, q)$ . We also use the  $q$ -analog of the binomials

$$\begin{bmatrix} n \\ k \end{bmatrix} = \prod_{i=1}^k \frac{q^{n+1-i} - 1}{q^i - 1} .$$

The number of  $k$ -dimensional subspaces of  $PG(n, q)$  is  $\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}$ .

We will need the  $q$ -binomial theorem:

**Result 1 (see for example [4] page 124)**

$$\prod_{k=0}^{n-1} (1 + q^k t) = \sum_{k=0}^n q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} t^k .$$

#### Lemma 2

For some property  $P$  of the generators of a classical finite polar space, denote by  $c_k$  the number of generators with property  $P$  intersecting a given generator  $S$  in a subspace of dimension  $k$ :

$$c_k = |\{\pi \mid P(\pi) \wedge \dim(S \cap \pi) = k\}| .$$

Let  $x_k$  be the number of pairs  $(U, \pi)$ , with  $U \subset S$  a subspace of dimension  $k$ ,  $U \subset \pi$  and where  $\pi$  satisfies property  $P$ ,

$$x_k = |\{(U, \pi) \mid P(\pi) \wedge \dim(U) = k \wedge U \subset S \wedge U \subset \pi\}|.$$

Then

$$\begin{aligned} c_k &= x_k - \sum_{l>k} c_l \begin{bmatrix} l+1 \\ k+1 \end{bmatrix} \\ &= \sum_{l \geq k} (-1)^{l-k} \begin{bmatrix} l+1 \\ k+1 \end{bmatrix} q^{\binom{l-k}{2}} x_l. \end{aligned}$$

**Proof**

For each  $l > k$ , each subspace  $\pi$  intersecting  $S$  in an  $l$ -dimensional subspace occurs in  $\begin{bmatrix} l+1 \\ k+1 \end{bmatrix}$  pairs  $(U, \pi)$ . Thus:

$$c_k = x_k - \sum_{l>k} c_l \begin{bmatrix} l+1 \\ k+1 \end{bmatrix}.$$

Now we prove the second equation.

For  $k = \dim S$ , we simply have  $c_k = x_k$  and the equation is trivial. Suppose now that the equation holds for all indices larger than  $k$ . Then

$$\begin{aligned} c_k &= x_k - \sum_{l>k} c_l \begin{bmatrix} l+1 \\ k+1 \end{bmatrix} \\ &= x_k - \sum_{l>k} \begin{bmatrix} l+1 \\ k+1 \end{bmatrix} \sum_{j \geq l} (-1)^{j-l} \begin{bmatrix} j+1 \\ l+1 \end{bmatrix} q^{\binom{j-l}{2}} x_j \\ &= x_k - \sum_{j>k} x_j \sum_{j \geq l>k} (-1)^{j-l} \begin{bmatrix} j+1 \\ l+1 \end{bmatrix} \begin{bmatrix} l+1 \\ k+1 \end{bmatrix} q^{\binom{j-l}{2}} \\ &= x_k - \sum_{j>k} x_j \begin{bmatrix} j+1 \\ k+1 \end{bmatrix} (-1)^{j-k} \sum_{j \geq l>k} (-1)^{k-l} \begin{bmatrix} j-k \\ l-k \end{bmatrix} q^{\binom{j-l}{2}}. \end{aligned}$$

The inner sum simplifies as follows:

$$\begin{aligned}
\sum_{j \geq l > k} (-1)^{k-l} \begin{bmatrix} j-k \\ l-k \end{bmatrix} q^{\binom{j-l}{2}} &= \sum_{l=1}^{j-k} (-1)^l \begin{bmatrix} j-k \\ l \end{bmatrix} q^{\binom{j-k-l}{2}} \\
&= \sum_{l=1}^{j-k} (-1)^l \begin{bmatrix} j-k \\ j-k-l \end{bmatrix} q^{\binom{j-k-l}{2}} \\
&= \sum_{l=0}^{j-k-1} (-1)^{j-k-l} \begin{bmatrix} j-k \\ l \end{bmatrix} q^{\binom{l}{2}} \\
&= - \begin{bmatrix} j-k \\ j-k \end{bmatrix} q^{\binom{j-k}{2}} + (-1)^{j-k} \sum_{l=0}^{j-k} (-1)^l \begin{bmatrix} j-k \\ l \end{bmatrix} q^{\binom{l}{2}} \\
&= -q^{\binom{j-k}{2}}.
\end{aligned}$$

The last sum is equal to 0 by the  $q$ -binomial theorem (Result 1) with  $t = -1$ . This finishes the proof.  $\square$

## 4 Small maximal partial spreads of $Q^+(4n+3, q)$

### 4.1 Skew generators

It is a known result that every partial spread of  $Q^+(4n+1, q)$  contains at most two generators. Now we determine a lower bound on the size of maximal partial spreads on  $Q^+(4n+3, q)$ . As will be indicated in Section 7, this will imply the same lower bound on the size of maximal partial spreads on the parabolic quadric  $Q(4n+2, q)$ .

#### Lemma 3

The number of generators in  $Q^+(2n+1, q)$  skew to a given generator  $G$  is  $a_n = q^{\binom{n+1}{2}}$ .

#### Proof

We prove this by induction on  $n$ . For  $n = 0$ , it is trivial.

Let  $n \geq 1$ . Fix an  $i$ -dimensional subspace  $U$  of  $G$ . In the quotient geometry to  $U$ , we see a generator  $G'$  that intersects  $G$  in  $U$  as a skew generator to the quotient of  $G$ . Thus there are  $a_{n-i-1}$  generators that intersect  $G$  in  $U$ . Thus the number of generators that intersect  $G$  in an  $i$ -dimensional subspace is  $\begin{bmatrix} n+1 \\ i+1 \end{bmatrix} a_{n-i-1}$ .

We find

$$\begin{aligned}
a_n &= \prod_{i=0}^n (q^i + 1) - \sum_{i=0}^n \begin{bmatrix} n+1 \\ i+1 \end{bmatrix} a_{n-i-1} \\
&= \prod_{i=0}^n (q^i + 1) - \sum_{i=0}^n \begin{bmatrix} n+1 \\ n-i \end{bmatrix} q^{\binom{n-i}{2}} \\
&= q^{\binom{n+1}{2}},
\end{aligned}$$

where the last equation is the  $q$ -binomial theorem for  $t = 1$ .  $\square$

Consider in  $Q^+(2n+1, q)$  two generators  $G_1$  and  $G_2$  that meet in a subspace of dimension  $v$ ,  $-1 \leq v \leq n$ . Then we denote by  $b_v^n$  the number of generators missing  $G_1$  and  $G_2$ .

For  $v = n$ , we have  $G_1 = G_2$ . We have  $b_n^n = a_n = q^{\binom{n+1}{2}}$ .

If  $v \not\equiv n \pmod{2}$ , the generators  $G_1$  and  $G_2$  belong to different classes which implies that each generator must meet either  $G_1$  or  $G_2$ , i.e.  $b_v^n = 0$  for  $v \not\equiv n \pmod{2}$ .

**Lemma 4**

We have the following recursion formula

$$b_v^n = (q^{n-v-1} - 1)q^{v+1}b_{v+1}^{n-1}.$$

**Proof**

Let  $V = G_1 \cap G_2$  be the subspace of dimension  $v$ . We count the number of pairs  $(U, G)$ , where  $G$  is a generator that is skew to  $G_1$  and  $G_2$ , and where  $U$  is a totally isotropic subspace on  $V$  such that  $\dim(U) = v + 1$  and  $U \cap G \neq \emptyset$ .

Starting with one of the  $b_v^n$  generators  $G$  skew to  $G_1$  and  $G_2$ , we have  $\dim(V^\perp \cap G) = n - v - 1$ . Thus there are  $\theta_{n-v-1}$  choices for  $U$ .

Now we start with a subspace  $U$  on  $V$  of dimension  $v + 1$ . It occurs in a pair  $(U, G)$  only if  $U \cap G_1 = U \cap G_2 = V$ . Going in the quotient geometry of  $V$  and the hyperbolic quadric  $Q^+(2n - 2v - 1, q)$  induced there,  $G_1$  and  $G_2$  become two skew generators of dimension  $n - v - 1$  and the subspaces  $U$  we are interested in correspond to the points of  $Q^+(2n - 2v - 1, q)$  outside  $G_1$  and  $G_2$ . Thus the number of these subspaces  $U$  is  $|Q^+(2n - 2v - 1, q)| - 2\theta_{n-v-1}$ .

In the ambient space  $PG(2n - 2v - 1, q)$ , there exists a unique line on  $U$  meeting  $G_1$  and  $G_2$ . This line contains at least three points of  $Q^+(2n - 2v - 1, q)$  and therefore must lie inside the hyperbolic quadric  $Q^+(2n - 2v - 1, q)$ .

Now we go back to  $PG(2n + 1, q)$ . We choose one of the  $|Q^+(2n - 2v - 1, q)| - 2\theta_{n-v-1}$  subspaces  $U$ , and inside  $U$  we choose one of the  $q^{v+1}$  points  $P$  of  $U \setminus V$ . Now count the number of generators  $G$  skew to  $G_1$  and  $G_2$  intersecting  $U$  in  $P$ . In the quotient geometry of  $P$ , we see an induced hyperbolic quadric  $Q^+(2n - 1, q)$ . Here  $G_1 \cap P^\perp$  and  $G_2 \cap P^\perp$  are generators that meet in a subspace of dimension  $v + 1$ , namely  $V$  and the special line on  $U$ . Thus there are  $b_{v+1}^{n-1}$  generators of  $Q^+(2n - 1, q)$  skew to  $G_1 \cap P^\perp$  and  $G_2 \cap P^\perp$ .

Thus the number of pairs  $(U, G)$  is

$$[|Q^+(2n - 2v - 1, q)| - 2\theta_{n-v-1}] q^{v+1} b_{v+1}^{n-1}.$$

This gives

$$b_v^n \theta_{n-v-1} = [ |Q^+(2n - 2v - 1, q)| - 2\theta_{n-v-1} ] q^{v+1} b_{v+1}^{n-1}.$$

and dividing by  $\theta_{n-v-1}$  gives the desired formula.  $\square$

**Corollary 5**

For  $-1 \leq v \leq n$  and  $v \equiv n \pmod{2}$ , we have

$$b_v^n = q^{2\binom{(n+v)/2+1}{2} - \binom{v+1}{2}} \prod_{i=1}^{(n-v)/2} (q^{2i-1} - 1).$$

**Proof**

We use induction on  $n - v$ . For  $n = v$ , we have  $b_n^n = q^{\binom{n+1}{2}}$ , which is the result given by the formula.

For  $n - v \geq 1$ , we find

$$\begin{aligned} b_v^n &= (q^{n-v-1} - 1) q^{v+1} b_{v+1}^{n-1} \\ &= (q^{n-v-1} - 1) q^{v+1} \cdot \left( q^{2\binom{(n+v)/2+1}{2} - \binom{v+2}{2}} \prod_{i=1}^{(n-v-2)/2} (q^{n-v-1-2i} - 1) \right) \\ &= q^{2\binom{(n+v)/2+1}{2} - \binom{v+1}{2}} \prod_{i=1}^{(n-v)/2} (q^{n-v+1-2i} - 1). \end{aligned}$$

Substituting  $(n - v)/2 + 1 - i$  for  $i$  gives the desired result.  $\square$

For the next step we need the following geometric property of  $Q^+(4n + 3, q)$ .

**Theorem 6**

Let  $G_1, G_2$  and  $G_3$  be three mutually skew generators of  $Q^+(4n + 3, q)$ . Then the lines of  $G_1$  that lie in a totally isotropic 3-space intersecting  $G_2, G_3$  in a line, form a symplectic space  $W(2n + 1, q)$  in  $G_1$ .

The proof of this theorem is inspired by the proof of Result 13 in Section 5.

**Proof**

We will denote the polarity defining  $Q^+(4n + 3, q)$  by  $\theta$ .

Let  $P_1$  be a point of  $G_1$ . In the ambient space there exists a unique line  $l$  through  $P_1$  that meets  $G_2$  and  $G_3$  in points  $P_2$  and  $P_3$ . This line contains three points of the quadric and hence lies on the quadric.

We define a mapping  $\theta_1$  from the points of  $G_1$  to the hyperplanes of  $G_1$  by  $\theta_1 : P_1 \mapsto G_1 \cap l^\theta$ .

Let  $P_1, P'_1$  and  $P''_1$  be three collinear points of  $G_1$ . The corresponding points  $P_2, P'_2$  and  $P''_2$  on  $G_2$  are collinear and thus the hyperplanes  $G_1 \cap l^\theta = G_1 \cap P_2^\theta$ ,  $G_1 \cap l^\theta = G_1 \cap P'_2^\theta$  and  $G_1 \cap l^\theta = G_1 \cap P''_2^\theta$  of  $G_1$  belong to a pencil of hyperplanes.

Furthermore, if  $G_1 \cap l^\theta = G_1 \cap l'^\theta$ , then  $\langle G_1 \cap l^\theta, P_2, P'_2 \rangle$  is a totally isotropic  $(2n+2)$ -space. This is a contradiction, thus  $\theta_1$  is an anti-automorphism of  $G_1$ .

Let  $Q_1 \in G_1 \cap l^\theta$ . Let  $Q_2$  and  $Q_3$  be its corresponding points in  $G_2$  and  $G_3$ . As  $Q_1, Q_3 \in P_3^\theta$ , then  $Q_2 \in P_3^\theta$  and hence  $Q_2 \in l^\theta$ . Since  $Q_1 \in l^\theta$ , we have  $Q_1 Q_2 \subseteq l^\theta$  or  $l \subseteq (Q_1 Q_2)^\theta$ . Especially,  $P_1 \in G_1 \cap (Q_1 Q_2)^\theta$ , i.e.  $\theta_1$  is a polarity of  $G_1$ . But every point of  $G_1$  is isotropic with respect to  $\theta_1$ , thus  $\theta_1$  defines a symplectic variety  $W(2n+1, q)$  in  $G_1$ .  $\square$

We also need the number  $\alpha(m, s, v)$  of subspaces of rank  $m$  in  $W(2v-1, q)$  meeting the symplectic polar space in a cone with base  $W(2s-1, q)$  and vertex of rank  $m-2s$ . (For a projective space  $PG(n, q)$  the rank is the rank of the underlying vector space, i.e.  $n+1$ .) This number was determined by Wan in [13, Theorem 1.3. with  $k=l=0$ ].

$$\alpha(m, s, v) = \frac{q^{2s(v+s-m)} \cdot \prod_{i=v+s-m+1}^v (q^{2i} - 1)}{\prod_{i=1}^s (q^{2i} - 1) \prod_{i=1}^{m-2s} (q^i - 1)}$$

**Lemma 7**

Given in  $Q^+(4n+3, q)$  three mutually skew generators  $G_1, G_2$  and  $G_3$ . Let  $x_k^n$  be the number of pairs  $(U, G)$ , where  $U \subseteq G_1$  is a  $k$ -dimensional subspace,  $G \supseteq U$ , with  $G$  a generator that misses  $G_2$  and  $G_3$ . For  $-1 \leq k \leq 2n+1$ , we have

$$x_k^n = \sum_{\max\{-2, 2(k-1-n)\} \leq 2s \leq k-1} \alpha(k+1, s+1, n+1) b_{k-2s-2}^{2n-k}.$$

**Proof**

Let  $U$  be a subspace of dimension  $k$  that intersects the variety  $W(2n+1, q)$  inside  $G_1$  in a cone over a  $W(2s+1, q)$  with vertex of dimension  $v = k-2s-2$ . If we go in the quotient geometry of  $U$ , we see  $Q^+(4n+3, q)$  as  $Q^+(2(2n-k)+1, q)$  and  $\langle U^\perp \cap G_2, U \rangle$  and  $\langle U^\perp \cap G_3, U \rangle$  are generators of this  $Q^+(2(2n-k)+1, q)$ .

Then  $U$  intersects the variety  $W(2n+1, q)$  in  $G_1$  in a cone with vertex of dimension  $v$ . Let  $V_1$  be that vertex. Let  $V_2$  and  $V_3$  be the corresponding spaces in  $G_2$  and  $G_3$ , which exist by Theorem 6. Then  $\langle V_1, V_2 \rangle = \langle V_1, V_3 \rangle$  is a totally isotropic subspace, which lies in  $\langle U^\perp \cap G_2, U \rangle$  and  $\langle U^\perp \cap G_3, U \rangle$ , i.e.  $\langle U^\perp \cap G_2, U \rangle$  and  $\langle U^\perp \cap G_3, U \rangle$  share a  $v$ -dimensional space in the quotient to  $U$ . On the other hand, if  $\langle U^\perp \cap G_2, U \rangle$  and  $\langle U^\perp \cap G_3, U \rangle$  share a  $v'$ -dimensional space in the quotient geometry, it defines a totally isotropic space which intersects  $G_1$  in a  $v'$ -dimensional space  $V'$ . This  $V'$  will be a part of the vertex in the intersection of  $U$  with the induced symplectic variety  $W(2n+1, q)$  in  $G_1$ . Thus  $\langle U^\perp \cap G_2, U \rangle$  and  $\langle U^\perp \cap G_3, U \rangle$  meet in the quotient geometry of  $U$  in a subspace of dimension  $v$ .

This implies that the number of generators of  $Q^+(2n+1, q)$  through  $U$  that miss  $G_2$  and  $G_3$  is  $b_v^{2n-k}$ .

Thus the number  $x_k$  of pairs  $(U, G)$  is

$$x_k^n = \sum_{\max\{-2, 2(k-1-n)\} \leq 2s \leq k-1} \alpha(k+1, s+1, n+1) b_{k-2s-2}^{2n-k}$$

as desired, where the bounds on  $s$  express the fact for  $k > n$  that the non-degenerate part  $W(2s+1, q)$  cannot be empty but must be at least of dimension  $2k - 2n - 1$ .  $\square$

Now we can apply Lemma 2 to obtain the following result.

**Corollary 8**

Given in  $Q^+(4n+3, q)$  three mutually skew generators  $G_1, G_2$  and  $G_3$ . Let  $c_k^n$  be the number of generators that meet  $G_1$  in a  $k$ -dimensional subspace and miss  $G_2$  and  $G_3$ . For  $-1 \leq k \leq 2n+1$ , we have

$$c_k^n = \sum_{l=k}^{2n+1} \left[ (-1)^{l-k} \begin{bmatrix} l+1 \\ k+1 \end{bmatrix} q^{\binom{l-k}{2}} \sum_{\max\{-2, 2(l-1-n)\} \leq 2s \leq l-1} \alpha(l+1, s+1, n+1) b_{l-2s-2}^{2n-l} \right].$$

**4.2 Estimating  $c_{-1}^n$**

Using the previous subsection, we find that for three mutually skew generators  $G_1, G_2$  and  $G_3$ , there exist  $c_{-1}^n$  skew generators. We now derive a lower bound

on  $c_{-1}^n$ . By Corollary 8, we have

$$\begin{aligned}
c_{-1}^n &= \sum_{l=-1}^{2n+1} \left[ (-1)^{l+1} \begin{bmatrix} l+1 \\ 0 \end{bmatrix} q^{\binom{l+1}{2}} \sum_{\max\{-2, 2(l-1-n)\} \leq 2s \leq l-1} \alpha(l+1, s+1, n+1) b_{l-2s-2}^{2n-l} \right] \\
&= \sum_{l=-1}^{2n+1} \sum_{\max\{-2, 2(l-1-n)\} \leq 2s \leq l-1} (-1)^{l+1} q^{\binom{l+1}{2}} \alpha(l+1, s+1, n+1) b_{l-2s-2}^{2n-l} \\
&= \sum_{l=-1}^{2n+1} \sum_{\max\{-2, 2(l-1-n)\} \leq 2s \leq l-1} (-1)^{l+1} q^{\binom{l+1}{2}} q^{2(s+1)(n+s-l+1)} q^{2\binom{n-s}{2} - \binom{l-2s-1}{2}} \\
&\quad \frac{\prod_{i=1}^{n-l+s+1} (q^{2i-1} - 1) \prod_{i=n+s-l+2}^{n+1} (q^{2i} - 1)}{\prod_{i=1}^{s+1} (q^{2i} - 1) \prod_{i=1}^{l-2s-1} (q^i - 1)}.
\end{aligned}$$

We substitute  $l = 2s + x$ . The bound  $s \geq l - 1 - n$  gives  $s \leq n + 1 - x$  and thus  $x$  runs from 1 to  $n + 2$ . Note that  $x \geq 1$  since  $l \geq 2s + 1$ . Furthermore,  $l - 1 - n \geq -1$ , so  $n \leq l$ . If  $n = l$ , then  $s = -1$ , so  $x = n + 2$ . We obtain

$$\begin{aligned}
c_{-1}^n &= \sum_{x=1}^{n+2} \sum_{s=-1}^{n+1-x} (-1)^{x+1} q^{\binom{2s+x+1}{2}} q^{2(s+1)(n-s-x+1)} q^{2\binom{n-s}{2} - \binom{x-1}{2}} \\
&\quad \frac{\prod_{i=1}^{n-s-x+1} (q^{2i-1} - 1) \prod_{i=n-s-x+2}^{n+1} (q^{2i} - 1)}{\prod_{i=1}^{s+1} (q^{2i} - 1) \prod_{i=1}^{x-1} (q^i - 1)} \\
&= \sum_{x=1}^{n+2} \sum_{s=-1}^{n+1-x} t(x, s, n) \\
&= \sum_{s=-1}^n \sum_{x=1}^{n+1-s} t(x, s, n).
\end{aligned}$$

The inner term  $t(x, s, n)$  of the double sum is an integer. In fact, it is a polynomial in  $q$  of degree  $2n^2 + 3n - \frac{1}{2}x(x-3)$ .

To estimate this polynomial, we use the following inequalities: For  $n_1 < \dots < n_k$ , we have

$$(q^{n_1} - 1) \prod_{j=2}^k (q^{n_j} \pm 1) \leq q^{\sum_{j=1}^k n_j}$$

and

$$(q^{n_1} + 1) \prod_{j=2}^k (q^{n_j} \pm 1) \geq q^{\sum_{j=1}^k n_j}.$$

The inequalities express, that the second largest term is  $q^{\sum_{j=2}^k n_j}$ , which dominates all other terms. The sign of the term depends on the sign in the factor  $q^{n_1} \pm 1$ .

To obtain a lower bound on  $(q^{n_1} - 1) \prod_{j=2}^k (q^{n_j} \pm 1)$ , we rewrite it as  $\frac{q^{n_1}-1}{q^{n_1+1}} (q^{n_1} + 1) \prod_{j=2}^k (q^{n_j} \pm 1)$  and apply the inequality above.

Thus we have the inequalities

$$\frac{q^{n_1} - 1}{q^{n_1 + 1}} q^{\sum_{i=1}^k n_i} \leq (q^{n_1} - 1) \prod_{i=2}^k (q^{n_i} \pm 1) \leq q^{\sum_{i=1}^k n_i}$$

and

$$q^{\sum_{i=1}^k n_i} \leq (q^{n_1} + 1) \prod_{i=2}^k (q^{n_i} \pm 1) \leq \frac{q^{n_1} + 1}{q^{n_1} - 1} q^{\sum_{i=1}^k n_i} .$$

We can use these inequalities to estimate  $t(x, s, n)$  as

$$t(x, s, n) \geq \begin{cases} \frac{q-1}{q+1} \cdot \frac{q^{2n-2s-2x+4}-1}{q^{2n-2s-2x+4}+1} q^{2n^2+3n-\frac{1}{2}x(x-3)} & \text{for } x \text{ odd,} \\ -\frac{q^2+1}{q^2-1} \cdot \frac{q+1}{q-1} q^{2n^2+3n-\frac{1}{2}x(x-3)} & \text{for } x \text{ even.} \end{cases}$$

From this, we see that  $t(x, s, n) + t(x+1, s, n) \geq 0$  for  $x \geq 5$ ,  $x$  odd.

We may bound  $\sum_{x=1}^{n-s+1} t(x, s, n) \geq \sum_{x=1}^4 t(x, s, n)$ . This is a finite sum and a direct but tedious calculation gives that this is a polynomial in  $q$  of degree  $n^2 + 4sn - 3s^2 + 3n$ , which is increasing in  $s$ .

Thus the only terms which give a contribution to the leading coefficients of the double sum are those with  $s > n - 5$ . Thus we can bound  $c_{-1}^n$  by a finite sum. Doing the calculation gives us the bound:

$$c_{-1}^n \geq q^{\binom{2n+2}{2}-5} (q^5 - 2q^4 + q^3 - 2q^2 + 4q - 4) \quad (2)$$

provided that  $n \geq 2$ . For  $n = 1$ , we have to do the precise calculation and obtain:

$$c_{-1}^1 = q^6 - 2q^5 + q^4 - 2q^3 + 3q^2 .$$

### 4.3 A lower bound on the size of maximal partial spreads of $Q^+(4n+3, q)$

Now we have computed the numbers  $g$ ,  $a$ ,  $b$  and  $c$ . Note that in Equation (1) the coefficient of  $c$  is negative, so we can use the lower bound derived in the previous section on  $c_{-1}^n$ . The coefficient of  $b$  is positive, so we use the upper bound:

$$b = b_{-1}^{2n+1} = q^{2\binom{n+1}{2}} \prod_{i=1}^{n+1} (q^{2n+3-2i} - 1) \leq q^{\binom{2n+2}{2}-9} (q-1)(q^3-1)(q^5-1) .$$

The coefficient of  $g$  is  $(s-1)(s-3)(s-4) \geq 0$ , so we can estimate  $g$  as

$$g = \prod_{i=1}^{2n+1} (q^i + 1) \leq q^{\binom{2n+2}{2}-6} (q+1)(q^2+1)(q^3+2) .$$

Note that we know that the generators of  $Q^+(4n+3, q)$  fall into two classes and generators of different classes have a common intersection. So we may restrict ourselves to generators from one class. This is the reason why we start the product with  $i = 1$ .

We can plug in all these bounds in Inequality (1). This gives a degree 3 inequality in  $s$ . The coefficients are polynomials in  $q$ . The solution of the inequality is simplified by the observation, that the expression is monotone increasing in  $s$ . For  $s = 2q$ , it is still negative and for  $s = 2q + 1$ , it becomes positive (provided that  $q \geq 7$ ). We summarise this in the following theorem.

**Theorem 9**

For  $n \geq 1$  and  $q \geq 7$ , every maximal partial spread of  $Q^+(4n+3, q)$  has at least  $2q + 1$  elements.

## 5 Small maximal partial spreads on $H(2n+1, q^2)$

The arguments for the hermitian polar space are similar to the arguments for the hyperbolic quadric. The only difference is that the recursion formula that replaces Lemma 4 now has two terms instead of only one and that the estimation of  $c_{-1}^n$  becomes simpler.

### 5.1 Skew generators

**Lemma 10**

The number of generators in  $H(2n+1, q^2)$  skew to a given generator  $G$  is  $a_n = q^{(n+1)^2}$ .

**Proof**

We prove this by induction on  $n$ . For  $n = 0$ , it is trivial.

Let  $n \geq 1$ . Fix an  $i$ -dimensional subspace  $U$  of  $G$ . In the quotient geometry of  $U$ , we see a generator  $G'$  that intersects  $G$  in  $U$  as a skew generator to the quotient of  $G$ . Thus there are  $a_{n-i-1}$  generators that intersect  $G$  in  $U$ . The number of generators that intersect  $G$  in an  $i$ -dimensional subspace is therefore  $\sum_{i=0}^{n+1} \binom{n+1}{i+1}_{q^2} a_{n-i-1}$ .

We find

$$\begin{aligned} a_n &= \sum_{i=0}^n (q^{2i+1} + 1) - \sum_{i=0}^n \binom{n+1}{i+1}_{q^2} a_{n-i-1} \\ &= \sum_{i=0}^n (q^{2i+1} + 1) - \sum_{i=0}^n \binom{n+1}{n-i}_{q^2} q^{(n-i)^2} \\ &= q^{(n+1)^2}, \end{aligned}$$

where the last equation is the  $q$ -binomial theorem for  $t = q$  (note:  $2\binom{k}{2} + k = k^2$ ).  
□

Consider in  $H(2n+1, q^2)$  two generators  $G_1$  and  $G_2$  that meet in a subspace of dimension  $v$ ,  $-1 \leq v \leq n$ . Then we denote by  $b_v^n$ ,  $-1 \leq v \leq n$ , the number of generators missing  $G_1$  and  $G_2$ ; here we put  $b_{-1}^{-1}$  as 1.

For  $v = n$ , we have  $G_1 = G_2$  and  $b_n^n = a_n = q^{(n+1)^2}$ .

We also denote by  $T_i$ ,  $i \geq -1$ , the number  $(q^{2i+2} - 1)/(q^2 - 1)$  of points of  $\text{PG}(i, q^2)$ .

**Lemma 11**

*We have the following recursion formula*

$$b_v^n = q^{2(v+1)} \left( q^{2n-2v-2} (q-1) b_v^{n-1} + (q^{2(n-v-1)} - 1) b_{v+1}^{n-1} \right).$$

**Proof**

Let  $G_1$  and  $G_2$  be generators such that  $V := G_1 \cap G_2$  has dimension  $v$ . We count the number of pairs  $(U, G)$ , where  $G$  is a generator that is skew to  $G_1$  and  $G_2$  and where  $U$  is a totally isotropic subspace on  $V$  such that  $\dim(U) = v+1$  and  $U \cap G \neq \emptyset$ . Starting with one of the  $b_v^n$  generators  $G$  skew to  $G_1$  and  $G_2$ , we have  $\dim(V^\perp \cap G) = n - v - 1$ , so there are  $T_{n-v-1}$  choices for  $U$ .

Now we start with a subspace  $U$  on  $V$  of dimension  $v+1$ . It occurs in a pair  $(U, G)$  only if  $U \cap G_1 = U \cap G_2 = V$ . Go in the quotient of  $V$  and the hermitian space  $H(2n - 2v - 1, q^2)$  induced there. In the quotient geometry of  $V$ , we see  $G_1$  and  $G_2$  as skew generators (of dimension  $n - v - 1$ ) and the subspaces  $U$  we are interested in correspond to the points of  $H(2n - 2v - 1, q^2)$  outside  $G_1 \cup G_2$ . Thus the number of these subspaces  $U$  is

$$x := |H(2n - 2v - 1, q^2)| - 2T_{n-v-1} = (q^{2n-2v-1} - 1)T_{n-v-1}.$$

In the ambient space  $\text{PG}(2n - 2v - 1, q^2)$ , there exists a unique line on  $U$  that meets  $G_1$  and  $G_2$ ; this line may be totally isotropic or not. The number of points  $U$  for which this line is totally isotropic is equal to the number of totally isotropic lines  $l$  meeting  $G_1$  and  $G_2$  times  $q^2 - 1$  for the number of points on  $l \setminus (G_1 \cup G_2)$ . This number is equal to

$$y := T_{n-v-1} T_{n-v-2} (q^2 - 1).$$

Back in  $H(2n+1, q^2)$ , this means that there exist  $x$  subspaces  $U$  of dimension  $v+1$  on  $V$  that meet  $G_1$  and  $G_2$  only in  $V$ ; also for  $x - y$  of these subspaces  $U$ , we have that  $\langle U, U^\perp \cap G_1 \rangle$  and  $\langle U, U^\perp \cap G_2 \rangle$  meet only in  $U$ , whereas for the remaining  $y$  subspaces  $U$ , we have that  $\langle U, U^\perp \cap G_1 \rangle$  and  $\langle U, U^\perp \cap G_2 \rangle$  meet in a subspace of dimension  $v+2$  on  $U$ .

Consider a subspace  $U$  of the first type, and let  $P$  be one of the  $q^{2(v+1)}$  points of  $U \setminus V$ . Going in the quotient geometry of  $P$ , we see a hermitian variety  $H(2n - 1, q^2)$  and  $G_1$  and  $G_2$  are generators meeting in the subspace  $\langle V, P \rangle = U$  which in the quotient geometry has dimension  $v$ . Hence the number of generators of this  $H(2n - 1, q^2)$  skew to  $G_1$  and  $G_2$  is  $b_v^{n-1}$ .

If we instead consider a subspace of the second type, then in the corresponding quotient geometry of a point  $P \in U \setminus V$ , we see  $G_1$  and  $G_2$  as generators

meeting in a subspace of dimension  $v + 1$  and then the number of generators of this  $H(2n - 1, q^2)$  skew to  $G_1$  and  $G_2$  is  $b_{v+1}^{n-1}$ .

As there are  $q^{2(v+1)}$  choices for a point  $P \in U \setminus V$  for each  $U$ , we find that the number of pairs  $(U, G)$  we are counting, is equal to

$$(x - y)q^{2(v+1)}b_v^{n-1} + yq^{2(v+1)}b_{v+1}^{n-1}.$$

It follows that

$$b_v^n T_{n-v-1} = (x - y)q^{2(v+1)}b_v^{n-1} + yq^{2(v+1)}b_{v+1}^{n-1}.$$

Plugging in  $x$  and  $y$  and dividing by  $T_{n-v-1}$ , we find the statement.  $\square$

**Corollary 12**

For  $-1 \leq v \leq n$ , we have

$$b_v^n = q^{(n+1)^2 - \binom{n-v+1}{2}} \prod_{i=1}^{n-v} (q^i + (-1)^i).$$

**Proof**

We use induction on  $n$ .

Let  $n = 1$  and consider two generators  $G_1$  and  $G_2$ , which are lines. The number of generators missing  $G_1$  is  $q^4$  by Lemma 10. If  $G_1 = G_2$ , then these generators also miss  $G_2$ , so  $b_1^1 = q^4$ . If  $G_1$  and  $G_2$  meet in point, then  $G_2$  meets exactly  $q^2 q$  of these  $q^4$  generators, so  $b_0^1 = q^4 - q^3$ . Finally, if  $G_1 \cap G_2 = \emptyset$ , then  $G_2$  meets  $1 + (q^2 + 1)(q - 1)$  generators that miss  $G_1$ , so  $b_{-1}^1 = q^4 - 1 - (q^2 + 1)(q - 1) = (q^2 + 1)(q - 1)q$ .

Now consider the case  $n \geq 2$ . Then we find from the recursion formula and the induction hypothesis

$$\begin{aligned}
b_v^n &= q^{2(v+1)} \left( q^{2n-2v-2}(q-1)b_v^{n-1} + (q^{2(n-v-1)} - 1)b_{v+1}^{n-1} \right) \\
&= q^{2(v+1)} \left( q^{2n-2v-2}(q-1)q^{n^2 - \binom{n-v}{2}} \prod_{i=1}^{n-v-1} (q^i + (-1)^i) \right. \\
&\quad \left. + (q^{2(n-v-1)} - 1)q^{n^2 - \binom{n-v-1}{2}} \prod_{i=1}^{n-v-2} (q^i + (-1)^i) \right) \\
&= q^{2(v+1)} q^{n^2 - \binom{n-v-1}{2}} \left( \prod_{i=1}^{n-v-2} (q^i + (-1)^i) \right) \\
&\quad \left( (q-1)q^{n-v-1}(q^{n-v-1} + (-1)^{n-v-1}) + (q^{2n-2v-2} - 1) \right) \\
&= q^{2(v+1)} q^{n^2 - \binom{n-v-1}{2}} \left( \prod_{i=1}^{n-v-1} (q^i + (-1)^i) \right) \\
&\quad \left( (q-1)q^{n-v-1} + (q^{n-v-1} - (-1)^{n-v-1}) \right) \\
&= q^{2(v+1)} q^{n^2 - \binom{n-v-1}{2}} \left( \prod_{i=1}^{n-v} (q^i + (-1)^i) \right)
\end{aligned}$$

as desired, as  $2(v+1) + n^2 - \binom{n-v-1}{2} = (n+1)^2 - \binom{n-v+1}{2}$ , which is the power of  $q$  before the product in the statement of the theorem.  $\square$

To obtain the number of generators skew to three given skew generators  $G_1$ ,  $G_2$  and  $G_3$ , we need the following geometrical property of  $H(2n+1, q^2)$ .

**Result 13 (see [12])**

Let  $G_1$ ,  $G_2$  and  $G_3$  be three mutually skew generators of  $H(2n+1, q^2)$ . Then the points of  $G_1$ , that lie on a totally isotropic line intersecting  $G_2$  and  $G_3$ , form a hermitian variety  $H(n, q^2)$  in  $G_1$ .

For a hermitian space  $H(d, q^2) \subset \text{PG}(d, q^2)$ , we denote by  $\alpha(d, v, s)$  the number of subspaces  $U$  of  $\text{PG}(d, q^2)$  with the property that  $U$  meets  $H(d, q^2)$  in a cone with vertex of dimension  $v$  and base a  $H(s, q^2)$ . Clearly,  $\alpha(d, v, s) = 0$  if  $2v + s + 2 > d$ . For  $2v + s + 2 \leq d$ , the numbers are as follows (cf. [7, Theorem 23.4.3] and [13])

$$\alpha(d, v, s) = \frac{q^{(d-s-2v-2)(s+1)} \prod_{i=s+2}^{d+1} (q^i - (-1)^i)}{\prod_{i=1}^{d-s-2v-2} (q^i - (-1)^i) \cdot \prod_{i=1}^{v+1} (q^{2i} - 1)}.$$

**Lemma 14**

Given in  $H(2n+1, q^2)$  three mutually skew generators  $G_1$ ,  $G_2$ , and  $G_3$ , let  $x_k^n$  be the number of pairs  $(U, G)$  where  $U \subseteq G_1$  is a  $k$ -dimensional subspace and

$G \supseteq U$ , with  $G$  a generator that misses  $G_2$  and  $G_3$ . For  $-1 \leq k \leq n$ , we have

$$x_k^n = \sum_{v,s} \alpha(n, v, s) b_v^{n-1-k},$$

where the sum runs over all pairs  $(v, s)$  with the property that  $v + s = k - 1$  and  $2v + 2 + s \leq n$ .

The proof is almost identical to the proof of Lemma 7.

**Proof**

Consider a subspace  $U$  of dimension  $k$  of  $G_1$ . Then  $U$  meets the variety  $H(n, q^2)$  inside  $G_1$  in a variety  $S_v H(s, q^2)$  for some integers  $s, v \geq -1$ ,  $s \neq 0$  and  $v + s + 1 = k$ . Furthermore we find  $2v + s + 2 = k + s + 1 \leq n$ . If we go in  $H(2n + 1, q^2)$  in the quotient geometry of the subspace  $U$ , we see a variety  $H(2(n - 1 - k) + 1, q^2)$  and  $\langle U^\perp \cap G_2, U \rangle$  and  $\langle U^\perp \cap G_3, U \rangle$  are generators of this  $H(2(n - 1 - k) + 1, q^2)$ . As in the case  $Q^+(4n + 3, q)$  (Lemma 7), we have that these two generators of  $H(2(n - 1 - k) + 1, q^2)$  meet in a subspace of dimension  $v$ . This implies that the number of generators of  $H(2n + 1, q^2)$  on  $U$  that miss  $G_2$  and  $G_3$  is equal to  $b_v^{n-1-k}$ .

Thus, the number of pairs  $(U, G)$ , where  $U \subseteq G_1$  has dimension  $k$  and where  $G$  is a generator on  $U$  skew to  $G_2$  and  $G_3$  is  $\sum_{v,s} \alpha(n, v, s) b_v^{n-1-k}$ , where the sum runs over all pairs  $(v, s)$  with the property that  $v + s = k - 1$  and  $2v + 2 + s \leq n$ .  $\square$

Now we apply Lemma 2 and obtain the following corollary.

**Corollary 15**

Given in  $H(2n + 1, q^2)$  three mutually skew generators  $G_1, G_2$ , and  $G_3$ , let  $c_k^n$  be the number of generators that meet  $G_1$  in a  $k$ -subspace and that miss  $G_2$  and  $G_3$ .

For  $-1 \leq k \leq n$ , we have

$$c_k^n = \sum_{l=k}^n (-1)^{l-k} \begin{bmatrix} l+1 \\ k+1 \end{bmatrix}_{q^2} q^{2\binom{l-k}{2}} \sum_{\substack{v,s \\ v+s+1=l}} \alpha(n, v, s) b_v^{n-1-l},$$

where the sum runs over all pairs  $(v, s)$  with the property that  $v + s = l - 1$  and  $2v + 2 + s \leq n$ .

**5.2 Estimating  $c_{-1}^n$**

Now we have to estimate  $c_{-1}^n$ .

In  $H(2n + 1, q^2)$ , we have

$$\begin{aligned}
c_{-1}^n &= \sum_{l=-1}^n (-1)^{l+1} q^{2\binom{l+1}{2}} \sum_{\substack{v,s \\ v+s+1=l}} \alpha(n, v, s) b_v^{n-1-l} \\
&= \sum_{l=-1}^n \sum_{v=-1}^l (-1)^{l+1} q^{2\binom{l+1}{2}} \alpha(n, v, l-1-v) b_v^{n-1-l} \\
&= \sum_{l=-1}^n \sum_{v=-1}^l (-1)^{l+1} q^{2\binom{l+1}{2}} q^{(n-l-v-1)(l-v)} q^{(n-l)^2 - \binom{n-l-v}{2}} \cdot \\
&\quad \frac{\prod_{i=1}^{n-1-l-v} (q^i + (-1)^i) \prod_{i=l+1-v}^{n+1} (q^i - (-1)^i)}{\prod_{i=1}^{n-l-v-1} (q^i - (-1)^i) \prod_{i=1}^{v+1} (q^{2i} - 1)}.
\end{aligned}$$

The inner term  $t(n, l, v)$  of the double sum is an integer and in fact it is a polynomial in  $q$  of degree  $n^2 + 2n - l - (v + 1)^2$  satisfying

$$t(n, l, v) \geq \begin{cases} \frac{(q-1)^3}{(q+1)^3} q^{n^2+2n-l-(v+1)^2} & \text{if } l \text{ is odd,} \\ -\frac{q^2+1}{(q-1)^2} q^{n^2+2n-l-(v+1)^2} & \text{if } l \text{ is even.} \end{cases}$$

It follows for  $q \geq 5$  and for odd integers  $l$  that the inner sum for  $l$  plus the inner sum for  $l + 1$  is non-negative. The same is true for  $q \leq 4$  but some ad-hoc arguments are needed, which we omit.

Hence, we find a lower bound on  $c_{-1}^n$  by restricting the outer sum to  $l = -1, \dots, 4$ . Note that it suffices then to add the terms  $t(n, l, v)$  for  $-1 \leq v \leq l \leq 4$ , as  $t(n, l, v) = 0$  when  $v > l$ . Using this information, it follows that

$$c_{-1}^n \geq (q^5 - 2q^4 + 3q^3 - 6q^2 + 9q - 14)q^{(n+1)^2-5}, \quad (3)$$

provided that  $n \geq 5$ .

### 5.3 A lower bound on the size of maximal partial spreads of $H(2n + 1, q^2)$

Now we have computed the numbers  $g$ ,  $a$ ,  $b$  and  $c$ . The remaining part of the computation is exactly the same as in the case of the hyperbolic quadric  $Q^+(4n + 3, q)$ .

The estimation of  $b$  and  $g$  gives us the bounds

$$\begin{aligned}
b_{-1}^n &= q^{(n+1)^2 - \binom{n+2}{2}} \prod_{i=1}^{n+1} (q^i + (-1)^i) \\
&\leq q^{(n+1)^2 - 21} \prod_{i=1}^6 (q^i + (-1)^i) \quad \text{for } n \geq 5
\end{aligned}$$

and

$$g = \prod_{i=0}^n (q^{2i+1} + 1) \leq q^{(n+1)^2 - 16} (q+1)(q^3+1)(q^5+1)(q^7+2).$$

Now we can plug in all these bounds in Inequality (1) to obtain an inequality of degree 3 in  $s$ . Solving this inequality gives us the following theorem.

**Theorem 16**

For  $n \geq 2$  and  $q \geq 13$ , every maximal partial spread of  $H(2n+1, q^2)$  has at least  $2q+3$  generators.

## 6 Small maximal partial spreads on $W(2n+1, q)$

The case of the symplectic polar space is the most difficult one. Here the problem is that the geometric property that replaces Theorem 6 and Result 13 is more complicated and makes differences between  $q$  odd and  $q$  even.

### 6.1 Skew generators

**Lemma 17**

The number of generators in  $W(2n+1, q)$  skew to a given generator  $G$  is  $a_n = q^{\binom{n+2}{2}}$ .

**Proof**

The proof is identical to the proof of Lemma 3. □

Consider in  $W(2n+1, q)$  two generators  $G_1$  and  $G_2$  that meet in a subspace of dimension  $v$ ,  $-1 \leq v \leq n$ . As in the previous sections, denote by  $b_v^n$  the number of generators missing  $G_1$  and  $G_2$ .

**Lemma 18**

$$b_v^n = (q-1)q^n b_v^{n-1} + (q^{n-v-1} - 1)q^{v+1} b_{v+1}^{n-1}.$$

**Proof**

Let  $V = G_1 \cap G_2$  be a subspace of dimension  $v$ . We count the number of pairs  $(U, G)$ , where  $G$  is a generator that is skew to  $G_1$  and  $G_2$  and where  $U$  is a totally isotropic subspace on  $V$  of dimension  $v+1$  with  $U \cap G \neq \emptyset$ .

Starting with one of the  $b_v^n$  generators  $G$  skew to  $G_1$  and  $G_2$ , we have  $\dim(V^\perp \cap G) = n - v - 1$ . Thus there are  $\theta_{n-v-1}$  choices for  $U$ .

Now we start with a  $(v+1)$ -dimensional subspace  $U$  on  $V$ . It occurs in a pair  $(U, G)$  only if  $U \cap G_1 = U \cap G_2 = V$ . In the quotient geometry of  $V$ , we see an induced symplectic space  $W(2n-2v-1, q)$ . There,  $G_1$  and  $G_2$  are skew generators (of dimension  $n-v-1$ ) and the subspace  $U$  is a point outside  $G_1$  and  $G_2$ . Thus, the number of these subspaces is

$$x = |W(2n-2v-1, q)| - 2\theta_{n-v-1} = \theta_{2n-2v-1} - 2\theta_{n-v-1}.$$

In the space  $W(2n - 2v - 1, q)$ , there exists a unique line on  $U$  that meets  $G_1$  and  $G_2$ . This line may be totally isotropic or not. The number of points  $U$  for which this line is totally isotropic is the number of totally isotropic lines  $l$  meeting  $G_1$  and  $G_2$  times  $q - 1$  for the number of points on  $l \setminus (G_1 \cup G_2)$ . This number is equal to

$$y = \theta_{n-v-1}\theta_{n-v-2}(q - 1) .$$

Back in  $W(2n + 1, q)$ , this means that there exist  $x$  subspaces  $U$  of dimension  $v + 1$  satisfying  $U \cap G_1 = U \cap G_2 = V$ . For  $x - y$  of these subspaces  $U$ , we have that  $\langle U, U^\perp \cap G_1 \rangle$  and  $\langle U, U^\perp \cap G_2 \rangle$  meet only in  $U$ , and for  $y$  of these subspaces,  $\langle U, U^\perp \cap G_1 \rangle$  and  $\langle U, U^\perp \cap G_2 \rangle$  meet in a subspace of dimension  $v + 2$  on  $U$ .

Consider a subspace  $U$  of the first type and let  $P$  be one of the  $q^{v+1}$  points of  $U \setminus V$ . Going into the quotient geometry of  $P$ , we see a symplectic variety  $W(2n - 1, q)$  and  $G_1$  and  $G_2$  as generators meeting in the subspace  $\langle V, P \rangle = U$  which has dimension  $v$  in the quotient geometry of  $P$ . Hence the number of generators of this  $W(2n - 1, q)$  skew to  $G_1$  and  $G_2$  is  $b_v^{n-1}$ .

If we take a subspace of the second type, we see in the corresponding quotient geometry  $G_1$  and  $G_2$  as generators meeting in a subspace of dimension  $v + 1$  and then the number of generators skew to  $G_1$  and  $G_2$  is  $b_{v+1}^{n-1}$ .

As there are  $q^{v+1}$  choices for a point  $P \in U \setminus V$ , the number of pairs  $(U, G)$  is equal to

$$(x - y)q^{v+1}b_v^{n-1} + yq^{v+1}b_{v+1}^{n-1} .$$

It follows that

$$b_v^n \theta_{n-v-1} = (x - y)q^{v+1}b_v^{n-1} + yq^{v+1}b_{v+1}^{n-1} .$$

Plugging in  $x$  and  $y$  gives

$$b_v^n \theta_{n-v-1} = [\theta_{2n-2v-1} - 2\theta_{n-v-1}]q^{v+1}b_v^{n-1} + \theta_{n-v-1}\theta_{n-v-2}(q-1)q^{v+1}(b_{v+1}^{n-1} - b_v^{n-1}) .$$

Dividing by  $\theta_{n-v-1}$  gives the desired recursion formula.  $\square$

### Corollary 19

For  $-1 \leq v \leq n$ , we have

$$b_v^n = q^{\binom{n+2}{2} - \lceil (n-v)/2 \rceil^2} \prod_{i=1}^{\lceil (n-v)/2 \rceil} (q^{2i-1} - 1) ,$$

where  $\lceil x \rceil$  denotes the smallest integer greater than or equal to  $x$ .

### Proof

For  $n = 1$ , consider two generators  $G_1$  and  $G_2$  of  $W(3, q)$  which are lines. By Lemma 17 the number of generators skew to  $G_1$  is  $q^3$ , i.e.  $b_1^1 = q^3$ . If  $G_2$  meets  $G_1$  in a point, then  $G_2$  meets exactly  $q \cdot q$  generators missing  $G_1$ , so  $b_0^1 = q^3 - q^2$ .

Finally, if  $G_1 \cap G_2 = \emptyset$ , then  $G_2$  meets  $1 + (q+1)(q-1)$  generators that miss  $G_1$ , so  $b_{-1}^1 = q^3 - 1 - (q+1)(q-1) = (q-1)q^2$ .

Now consider  $n \geq 2$ . We find from the recursion formula and the induction hypothesis:

$$\begin{aligned} b_v^n &= (q-1)q^n b_v^{n-1} + (q^{n-v-1} - 1)q^{v+1} b_{v+1}^{n-1} \\ &= (q-1)q^n q^{\binom{n+1}{2} - \lceil (n-1-v)/2 \rceil^2} \prod_{i=1}^{\lceil (n-1-v)/2 \rceil} (q^{2i-1} - 1) \\ &\quad + (q^{n-v-1} - 1)q^{v+1} q^{\binom{n+1}{2} - \lceil (n-1-v-1)/2 \rceil^2} \prod_{i=1}^{\lceil (n-1-v-1)/2 \rceil} (q^{2i-1} - 1). \end{aligned}$$

If  $n \not\equiv v \pmod{2}$ , we obtain

$$\begin{aligned} b_v^n &= (q-1)q^n q^{\binom{n+1}{2} - ((n-1-v)/2)^2} \prod_{i=1}^{(n-1-v)/2} (q^{2i-1} - 1) \\ &\quad + (q^{n-v-1} - 1)q^{v+1} q^{\binom{n+1}{2} - ((n-1-v)/2)^2} \prod_{i=1}^{(n-1-v)/2} (q^{2i-1} - 1) \\ &= (q^{n+1} - q^{v+1})q^{\binom{n+1}{2} - ((n-1-v)/2)^2} \prod_{i=1}^{(n-1-v)/2} (q^{2i-1} - 1) \\ &= [(q^{n-v} - 1)q^{n+1} \cdot q^{-(n-v)}]q^{\binom{n+1}{2} - ((n-1-v)/2)^2} \prod_{i=1}^{(n-1-v)/2} (q^{2i-1} - 1) \\ &= q^{\binom{n+2}{2} - ((n-v-1)/2+1)^2} \prod_{i=1}^{(n-v-1)/2+1} (q^{2i-1} - 1) \\ &= q^{\binom{n+2}{2} - \lceil (n-v)/2 \rceil^2} \prod_{i=1}^{\lceil (n-v)/2 \rceil} (q^{2i-1} - 1), \end{aligned}$$

and if  $n \equiv v \pmod{2}$ , we obtain

$$\begin{aligned}
b_v^n &= (q-1)q^n q^{\binom{n+1}{2} - ((n-v)/2)^2} \prod_{i=1}^{(n-v)/2} (q^{2i-1} - 1) \\
&\quad + (q^{n-v-1} - 1)q^{v+1} q^{\binom{n+1}{2} - ((n-v-2)/2)^2} \prod_{i=1}^{(n-v-2)/2} (q^{2i-1} - 1) \\
&= [(q-1)q^n (q^{-(n-v-1)})(q^{n-v-1} - 1) + (q^{n-v-1} - 1)q^{v+1}] \cdot \\
&\quad q^{\binom{n+1}{2} - ((n-v-2)/2)^2} \prod_{i=1}^{(n-v-2)/2} (q^{2i-1} - 1) \\
&= (q^{n-v-1} - 1)q^{v+2} q^{\binom{n+1}{2} - ((n-v-2)/2)^2} \prod_{i=1}^{(n-v-2)/2} (q^{2i-1} - 1) \\
&= q^{\binom{n+2}{2} - ((n-v)/2)^2} \prod_{i=1}^{(n-v)/2} (q^{2i-1} - 1) \\
&= q^{\binom{n+2}{2} - \lceil (n-v)/2 \rceil^2} \prod_{i=1}^{\lceil (n-v)/2 \rceil} (q^{2i-1} - 1).
\end{aligned}$$

This finishes the induction.  $\square$

To obtain the value  $c$  for the number of generators skew to three pairwise skew generators  $G_1$ ,  $G_2$  and  $G_3$ , we need an analogon to Theorem 6 and Result 13. We start with the 3-dimensional case.

**Lemma 20**

*Let  $G_1$ ,  $G_2$  and  $G_3$  be three pairwise skew lines of  $W(3, q)$ . If  $q$  is even, then there exist either exactly one or  $q+1$  lines of  $W(3, q)$  that intersect  $G_1$ ,  $G_2$  and  $G_3$ . If  $q$  is odd, there exist either two lines of  $W(3, q)$  intersecting  $G_1$ ,  $G_2$  and  $G_3$ , or no such line.*

**Proof**

The generalised quadrangle  $W(3, q)$  is dual to  $Q(4, q)$ . The generators  $G_1$ ,  $G_2$  and  $G_3$  correspond to points  $P_1$ ,  $P_2$  and  $P_3$  of  $Q(4, q)$  that lie in a plane  $\pi$  intersecting  $Q(4, q)$  in a conic  $Q(2, q)$ .

Let  $\perp$  be the polarity of  $Q(4, q)$ . The question is now how many points lie in  $\pi^\perp$ .

If  $q$  is even, then  $Q(4, q)$  has a nucleus and  $\pi^\perp$  is either a line or a plane through that nucleus (depending on whether the nucleus lies in  $\pi$  or not). Thus  $\pi^\perp$  is either a tangent to  $Q(4, q)$ , i.e. it contains a unique point of  $Q(4, q)$ , or it is a plane that intersects  $Q(4, q)$  in a conic and then it contains  $q+1$  points.

If  $q$  is odd,  $\pi^\perp$  is either a passant or secant line.  $\square$

**Theorem 21**

Let  $G_1, G_2$  and  $G_3$  be three pairwise skew generators of  $W(2n + 1, q)$ ,  $n \geq 2$ . Let  $\mathcal{P}$  be the set of points  $P$  in  $G_1$  such that there exists a line in  $W(2n + 1, q)$  through  $P$  intersecting  $G_2$  and  $G_3$ .

For  $q$  even and  $n$  even,  $\mathcal{P}$  forms a pseudo-polarity of  $G_1$ . For  $q$  even and  $n$  odd,  $\mathcal{P}$  is either a pseudo-polarity or a symplectic polarity (depending on the relative position of  $G_1, G_2$  and  $G_3$ ).

For  $q$  odd and  $n$  even,  $\mathcal{P}$  is a parabolic quadric in  $G_1$ . For  $q$  odd and  $n$  odd,  $\mathcal{P}$  is either an elliptic or hyperbolic quadric (depending on the relative position of  $G_1, G_2$  and  $G_3$ ).

**Proof**

Let  $P_1$  be a point of  $G_1$ . Then in  $PG(2n + 1, q)$ , there exists a unique line  $l$  through  $P_1$  that intersects  $G_2$  and  $G_3$ . As in the proof of Theorem 6, we define a polarity  $\theta_1$  on  $G_1$  by  $\theta_1 : P_1 \mapsto G_1 \cap l^\theta$ , where  $\theta$  denotes the polarity of  $W(2n + 1, q)$ . Then  $\theta_1$  defines a non-singular polar space  $\mathcal{P}$  in  $G_1$ .

Let  $l_1$  be a line of  $G_1$ , let  $l_2 = G_2 \cap \langle G_3, l_1 \rangle$  and  $l_3 = G_3 \cap \langle G_2, l_1 \rangle$ . Then  $l_1, l_2$  and  $l_3$  span a 3-space  $\pi$ .

If  $\pi$  is totally isotropic, then  $l_1$  lies in the polar space  $\mathcal{P}$ . If  $\pi$  intersects  $W(2n + 1, q)$  in a singular symplectic variety, then  $\pi$  consists of  $q + 1$  planes through a common line  $l$ . In this case,  $|l_1 \cap \mathcal{P}| = 1$ .

If  $\pi$  intersects  $W(2n + 1, q)$  in a non-singular symplectic variety  $W(3, q)$ , we can apply Lemma 20 to determine  $|l_1 \cap \mathcal{P}|$ .

It follows that for  $q$  even,  $\mathcal{P}$  intersects every line either in 1 or  $q + 1$  points. Thus  $\mathcal{P}$  is a hyperplane or the whole space. If  $n$  is even, the only possibility is that  $\theta_1$  is a pseudo-polarity. If  $n$  is odd, then  $\theta_1$  is either a pseudo-polarity or a symplectic variety  $W(n, q)$ .

If  $q$  is odd, then  $\mathcal{P}$  intersects every line in 0, 1, 2 or  $q + 1$  points and the intersections of size less than  $q + 1$  occur. Thus  $\mathcal{P}$  is a quadric.  $\square$

Now we can apply the counting techniques as for  $Q^+(4n + 3, q)$  and  $H(2n + 1, q^2)$  to determine the value  $c$ . We will need intersection numbers for different polar spaces.

The elliptic quadric has type 0, the parabolic quadric has type 1 and the hyperbolic quadric has type 2. The number of  $m$ -dimensional spaces intersecting a non-degenerate quadric  $Q_n$  in  $PG(n, q)$  of type  $w$  in a cone with vertex of dimension  $m - t - 1$  over a non-singular quadric  $Q_t$  in  $t$  dimensions of type  $v$  is

$$N_Q(m, t, v, n, w) = q^{\frac{1}{2}[(n+t-2m)(t+1+vw(2-v)(2-w))-v(2-v)(w-1)^2]} \cdot \frac{\prod_{i=\frac{1}{2}[n+t-2m+v+(1+3v-2v^2)w-v(2-v)w^2]}^{\frac{1}{2}(n+1-w)} (q^i + 1) \prod_{i=\frac{1}{2}[n+t-2m+2-v-(1-5v+2v^2)w-v(2-v)w^2]}^{\frac{1}{2}(n-1+w)} (q^i - 1)}{\prod_{i=v(2-v)}^{\frac{1}{2}(t+1-v)} (q^i + 1) \prod_{i=1}^{\frac{1}{2}(t-1+v)} (q^i - 1) \prod_{i=1}^{m-t} (q^i - 1)}$$

(The proof of this result can be found in [7, Theorem 22.8.2] or in [13]. Note that the formula assumes that quadrics with given type exist. Substituting other numbers (for example  $n$  even,  $w \neq 1$ ) will give garbage.)

We now obtain the number  $\alpha_w(m, t, n)$  of  $m$ -dimensional spaces intersecting a quadric  $Q_n$  of type  $w$  in a cone with vertex dimension  $m - t - 1$  and base of dimension  $t$  by

$$\alpha_w(m, t, n) = \begin{cases} N_Q(m, t, 1, n, w) & \text{for } t \text{ even,} \\ N_Q(m, t, 0, n, w) + N_Q(m, t, 2, n, w) & \text{for } t \geq 1 \text{ odd,} \\ N_Q(m, -1, 2, n, w) & \text{for } t = -1 \end{cases}$$

(For  $t = -1$ , the convention is that the only non-singular quadric is the hyperbolic quadric.)

In the case  $q$  even, we need the intersection numbers for the pseudo-polarity. If  $n$  is even, the pseudo-polarity forms a  $W(n-1, q)$  in the hyperplane of absolute points. If  $n$  is odd, the pseudo-polarity forms a cone over a  $W(n-2, q)$  in the hyperplane of absolute points. Thus we need intersection numbers for symplectic polar spaces. These numbers can be found in [13].

The number of subspaces of rank  $m$  which intersect  $W(2v-1, q)\Pi_{l-1}$  in a cone over an  $W(2s-1, q)$  and with a vertex intersecting  $\Pi_{l-1}$  in a subspace of rank  $k$  is

$$N_W(m, s, k, l, v) = \frac{\prod_{i=v+s-m+k+1}^v (q^{2i} - 1) \prod_{i=l-k+1}^l (q^i - 1)}{\prod_{i=1}^s (q^{2i} - 1) \prod_{i=1}^{m-k-2s} (q^i - 1) \prod_{i=1}^k (q^i - 1)} q^{2s(v+s-m+k)+(m-k)(l-k)}.$$

(Note that the formula gives garbage if you substitute impossible values, like  $k > l$ . To simplify the following formulas, let  $\hat{N}_w(m, s, k, l, v) = N_W(m, s, k, l, v)$  for all possible values of  $m, s, k, l, v$  and  $\hat{N}_W(m, s, k, l, v) = 0$  in all the other cases.)

With this, we obtain the intersection numbers for the pseudo-polarity.

The number  $\alpha_{pseudo}(m, t, n)$  of  $m$ -dimensional spaces that intersect a pseudo-polarity in  $PG(n, q)$  in a cone with vertex dimension  $m - t - 1$  is

$$\alpha_{pseudo}(m, t, n) = \begin{cases} \hat{N}_w(m+1, \frac{t+1}{2}, 0, 0, \frac{n}{2}) + \hat{N}_w(m, \frac{t-1}{2}, 0, 0, \frac{n}{2})[q^{n-m} - q^{n-2m+t-1}] & \text{for } n \text{ even, } t \text{ odd} \\ \hat{N}_w(m, \frac{t}{2}, 0, 0, \frac{n}{2})q^{n-2m+t} & \text{for } n \text{ even, } t \text{ even} \\ \hat{N}_W(m+1, \frac{t+1}{2}, 0, 1, \frac{n-1}{2}) + \hat{N}_W(m+1, \frac{t+1}{2}, 1, 1, \frac{n-1}{2}) + \hat{N}_W(m, \frac{t-1}{2}, 0, 1, \frac{n-1}{2})[q^{n-m} - q^{n-2m+t-1}] + \hat{N}_W(m, \frac{t-1}{2}, 1, 1, \frac{n-1}{2})q^{n-m} & \text{for } n \text{ odd, } t \text{ odd} \\ \hat{N}_W(m, \frac{t}{2}, 0, 1, \frac{n-1}{2})q^{n-2m+t} & \text{for } n \text{ odd, } t \text{ even} \end{cases}$$

Note that  $t$  even means that the base of the cone cannot lie completely in the hyperplane of absolute points. In this case we have to choose an  $(m-1)$ -dimensional subspace in the hyperplane of absolute points and extend it to an

$m$ -dimensional space. The extension must lie in the orthogonal space of the vertex which gives us the factor  $q^{n-2m+t}$ .

The case  $t$  odd includes the cases in which the  $m$ -space lies completely in the hyperplane  $\Pi$  of absolute points (the terms  $\hat{N}_W(\dots)$  without an accompanying  $q$  power) and the cases in which the  $m$ -dimensional space  $U$  does not lie in the hyperplane, but  $U \cap U^\perp$  differs from the vertex of the polarity in  $U \cap \Pi$ .

In the case  $q$  even and  $n$  odd, we need the intersection numbers for a symplectic variety. The number  $\alpha_{\text{symplectic}}(m, t, n)$  of  $m$ -dimensional spaces intersecting a symplectic variety  $W(n, q)$  in a cone with vertex dimension  $m - t - 1$  is

$$\alpha_{\text{symplectic}}(m, t, n) = \begin{cases} N_W(m+1, \frac{t+1}{2}, 0, 0, \frac{n+1}{2}) & \text{if } t \text{ is odd,} \\ 0 & \text{if } t \text{ is even.} \end{cases}$$

(The latter case expresses that the base must always be a non-degenerate symplectic variety  $W(t, q)$ , i.e. it must be of odd dimension  $t$ .)

Now we can proceed with the computation of  $c$  as in the previous cases.

**Lemma 22**

Given in  $W(2n+1, q)$  three mutually skew generators  $G_1, G_2$  and  $G_3$ . Let  $x_k^n$  be the number of pairs  $(U, G)$ , where  $U \subset G_1$  is a  $k$ -dimensional subspace,  $G \supset U$ , with  $G$  a generator that misses  $G_2$  and  $G_3$ . For  $-1 \leq k \leq n$ , we have

$$x_k^n = \sum_{s=-1}^k \alpha(k, s, n) b_{k-s-1}^{n-1-k}$$

where  $\alpha$  is either  $\alpha_w$ ,  $\alpha_{\text{pseudo}}$  or  $\alpha_{\text{symplectic}}$  depending on the parity of  $q$  and  $n$ , and on the position of  $G_1, G_2$  and  $G_3$ .

**Proof**

Let  $U$  be a subspace of dimension  $k$  inside  $G_1$  intersecting the polarity induced in  $G_1$  in a cone with vertex of dimension  $v = k - s - 1$ . If we go in the quotient of  $U$ , we see  $W(2n+1, q)$  as an  $W(2(n-k-1)+1, q)$  and  $\langle U^\perp \cap G_2, U \rangle$  and  $\langle U^\perp \cap G_3, U \rangle$  are generators of this  $W(2(n-k-1)+1, q)$ . As in the case  $Q^+(4n+3, q)$  (Lemma 7) we have that these generators meet in a subspace of dimension  $v$ . This implies that the number of generators of  $W(2n+1, q)$  through  $U$  that miss  $G_2$  and  $G_3$  is  $b_v^{n-k-1}$ . Thus the number of pairs  $(U, G)$  is

$$x_k^n = \sum_{s=-1}^k \alpha(k, s, n) b_{k-s-1}^{n-1-k}$$

as desired. □

Now we apply Lemma 2 and obtain:

**Corollary 23**

Given in  $W(2n+1, q)$  three mutually skew generators  $G_1, G_2$  and  $G_3$ , let  $c_k^n$  be the number of generators that meet  $G_1$  in a  $k$ -dimensional subspace and are

skew to  $G_2$  and  $G_3$ . For  $-1 \leq k \leq n$ , we have

$$c_k^n = \sum_{l=k}^n \left[ (-1)^{l-k} \begin{bmatrix} l+1 \\ k+1 \end{bmatrix} q^{\binom{l-k}{2}} \sum_{s=-1}^l \alpha(l, s, n) b_{l-s-1}^{n-1-l} \right].$$

## 6.2 Estimating $c_{-1}^n$

We obtain a lower bound on  $c_{-1}^n$  by the same techniques as in the previous section. The only difference is, that the term for  $\alpha$  is more complex which results in a longer computation. We omit some of the tedious steps in between and give only the important steps.

The inner term of the double sum is a polynomial in  $q$  and the degree of this polynomial is  $\frac{1}{2}n(n+3) - \frac{1}{2}(l-s)(l-s+1) - l$ .

We see that the degree is decreasing in  $l$ . Thus we can copy the arguments from the case  $H(2n+1, q^2)$  and find a lower bound by restricting the outer sum to  $l = -1, \dots, 4$ .

Doing this, we obtain the bound for  $q \geq 5$  odd, and  $n \geq 4$

$$c_{-1}^n \geq q^{\binom{n+2}{2}-5} (q^5 - 2q^4 + q^3 - 2q^2 + 4q - 4).$$

This bound does not depend on the type of the induced quadric in  $G_1$ . To be precise,  $c_{-1}^n$  depends on the type of the induced quadric, but the highest degree coefficients are not affected by the type.

For  $q$  even,  $q \geq 8$  and  $n \geq 4$ , we obtain

$$c_{-1}^n \geq q^{\binom{n+2}{2}-5} (q^5 - 2q^4 + q^3 - 2q^2 + 4q - 4).$$

The bound is independent from the type of the induced variety (pseudo-polarity or symplectic).

For small dimensions, we must do the exact computations and obtain for  $q$  odd:

$$c_{-1}^1 = q^3 - 2q^2 + q + 1 \quad \text{if the induced quadric in } G_1 \text{ is } Q^-(1, q),$$

$$c_{-1}^1 = q^3 - 2q^2 + q - 1 \quad \text{if the induced quadric in } G_1 \text{ is } Q^+(1, q),$$

$$c_{-1}^2 = q^6 - 2q^5 + q^4 - 2q^3 + 3q^2,$$

$$c_{-1}^3 = q^{10} - 2q^9 + q^8 - 2q^7 + 4q^6 - 2q^5 + q^4 - 2q^3 \quad \text{if the induced quadric in } G_1 \text{ is } Q^-(3, q),$$

$$c_{-1}^3 = q^{10} - 2q^9 + q^8 - 2q^7 + 4q^6 - 2q^5 - q^4 + 2q^3 \quad \text{if the induced quadric in } G_1 \text{ is } Q^+(3, q),$$

and for  $q$  even:

$$c_{-1}^1 = q^3 - 2q^2 + q \quad \text{if the induced variety in } G_1 \text{ is a pseudo-polarity},$$

$$c_{-1}^1 = q^3 - 2q^2 \quad \text{if the induced variety in } G_1 \text{ is a } W(1, q),$$

$$c_{-1}^2 = q^6 - 2q^5 + q^4 - 2q^3 + 3q^2,$$

$$c_{-1}^3 = q^{10} - 2q^9 + q^8 - 2q^7 + 4q^6 - 2q^5 \quad \text{if the induced variety in } G_1 \text{ is a pseudo-polarity},$$

$$c_{-1}^3 = q^{10} - 2q^9 + q^8 - 2q^7 + 3q^6 \quad \text{if the induced variety in } G_1 \text{ is a } W(3, q).$$

### 6.3 A lower bound on the size of maximal partial spreads of $W(2n + 1, q)$

Now we have computed the numbers  $g$ ,  $a$ ,  $b$  and  $c$  for the symplectic polar space. In the cases in which the number  $c$  depends on the induced variety in  $G_1$ , we have to use the smaller values as a lower bound on  $c$ .

Simplifying Inequality (1), we obtain:

**Theorem 24**

For  $n \geq 2$  and  $q \geq 5$ , every maximal partial spread of  $W(2n + 1, q)$  has at least the size  $2q + 1$ .

We note that in the case of  $W(3, q)$ ,  $q$  even, the trivial lower bound  $q + 1$  is sharp (see [5]).

## 7 Overview of the results

If the rank of the generators of a classical finite polar space is less than half of the rank of the ambient space, then the number of generators skew to  $G_1$ ,  $G_2$  and  $G_3$  depends on the relative position of  $G_1$ ,  $G_2$  and  $G_3$ . The different possible values varies up to a factor of order  $q$ . Thus Glynn's counting technique is not applicable in this case.

However, the following observations make it possible to extend our results to parabolic quadrics.

1.  $Q(4n+2, q)$  admits a maximal partial spread of size  $s$  if and only if  $Q^+(4n+3, q)$  admits a maximal partial spread of size  $s$  [8].
2.  $Q(2n, q)$ ,  $q$  even, is isomorphic to  $W(2n - 1, q)$ ,  $q$  even [10].

Thus we have proven that a maximal partial spread of  $Q(4n + 2, q)$ ,  $n \geq 1$ ,  $q \geq 7$ , and  $Q(2n, q)$ ,  $q$  even,  $n \geq 3$ ,  $q \geq 8$ , has at least size  $2q + 1$ .

By completely different techniques, it is possible to prove directly that every maximal partial spread of  $Q(2n, q)$ ,  $n \geq 3$ , has at least size  $q + \delta$ , where  $\delta$  is the smallest value for which a non-trivial blocking set of size  $q + \delta$  in  $PG(2, q)$  exists (see [11]).

The generalised quadrangle  $Q^-(5, q)$  is isomorphic to the dual of  $H(3, q^2)$ . It is known that the smallest maximal partial ovoid of  $H(3, q^2)$  has size  $q^2 + 1$  for  $q$  even (see [1]) or is at least of size  $q^2 + 1 + \frac{4}{9}q$  for  $q$  odd (see [9]).

For  $Q^-(2n + 1, q)$  the only known lower bound is found in [3, Theorem 8.1].

The following table summarises the known results on small maximal partial spreads in classical finite polar spaces.

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Polar space	Lower bound	
$Q^+(4n+1, q), n \geq 1$	2	sharp
$Q^+(4n+3, q), n \geq 1, q \geq 7$	$2q+1$	
$Q(4, q)$	$q+1$	sharp
$Q(4n+2, q), n \geq 1, q \geq 7$	$2q+1$	
$Q(2n, q), q$ even, $q \geq 8, n \geq 3$	$2q+1$	
$Q(2n, q), q$ odd, $n \geq 3$	$q+\delta$	
$Q^-(5, q), q$ even	$q^2+1$	sharp
$Q^-(5, q), q$ odd	$q^2+1+\frac{4}{9}q$	
$Q^-(2n+1, q)$	$q^2+1$	
$W(3, q), q$ even	$q+1$	sharp
$W(3, q), q$ odd	$1.414q$	
$W(2n+1, q), n \geq 2, q \geq 5$	$2q+1$	
$H(2n+1, q^2), n \geq 2, q \geq 13$	$2q+3$	
$H(4, q^2)$	$q^3 + q\sqrt{q} - \frac{q}{2} - \frac{3}{8}\sqrt{q} + \frac{7}{8}$	

Table 1: Lower bounds on the size of maximal partial spreads

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