Partial ovoids in classical finite polar spaces

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Abstract

Ovoids in finite polar spaces are related to many other objects in Finite Geometries. In this article we prove some new upper bounds for the size of partial ovoids in $Q^-(2n+1,q)$ and $W(2n+1,q)$. Further we give a combinatorial proof for the non-existence of ovoids of $H(2n+1,q^2)$ for $n > q^3$.

1 Introduction

Let P be a finite polar space of rank ≥ 2. A set O of points is called a partial ovoid, if every maximal isotropic subspace contains at most one point of O. O is called an ovoid of P if every maximal isotropic subspace contains exactly one point of O. A good survey on ovoids can be found in [7].

For a polar space P we define:

$$O(P) = \max\{|O| \mid O \text{ is a partial ovoid of } P\}$$

In the following we shall improve the counting arguments from [7] to prove inductive bounds for the size of partial ovoids in polar spaces of type $Q^-(2n+1,q)$ and $W(2n+1,q)$. Combining these inductive bounds with known upper bounds in small dimensions we obtain improved upper bounds for the size of partial ovoids. In the last section we use a similar counting argument to prove the non-existence of ovoids of $H(2n+1,q^2)$ for $n > q^3$.

Other inductive bounds on the size of partial ovoids can be found in [2], where the authors prove

$$(q^{n+1}+1) - O(P) < (q-1)((q^n+1) - O(P'))$$

for $P = Q^-(2n+1,q)$ and $P' = Q^-(2n-1,q)$ or $P = W(2n+1,q)$ and $P' = W(2n-1,q)$. This formula is weaker than the formulas in Theorem 1 and Theorem 2, but it has a natural generalization to polar spaces of other types.

2 Partial ovoids in $Q^-(2n+1,q)$

It is known that no ovoid exists in $Q^-(2n+1,q)$ for $n \geq 2$ (see [5]). We first prove:

Theorem 1
For $n \geq 3$, we have

$$O(Q^-(2n+1,q)) \leq \frac{q^n+1}{q^{n-1}+1}(O(Q^-(2n-1,q))-2)+2 \qquad (1)$$

Proof
The proof of this theorem is inspired by the proof of Theorem 22 in [7].

Let O be a partial ovoid of $Q^-(2n+1,q)$ with $|O| = O(Q^-(2n+1,q))$. Let θ be the polarity of $PG(2n+1,q)$ corresponding to $Q^-(2n+1,q)$. Let $x, y \in O$ with $x \neq y$. Define $\pi = (xy)^\theta$. π is a $(2n-1)$-dimensional subspace and $\pi \cap Q^-(2n+1,q)$ is an elliptic quadric $Q^-(2n-1,q) = Q$. Since $x, y \in \pi^\theta$ we conclude $\pi \cap O = \emptyset$.

Now we count the number N of pairs (u,v) with $u \in Q$, $v \in O \backslash \{x,y\}$ such that uv is a line of $Q^-(2n+1,q)$. For each possible v the point u must lie in the non singular quadric $(xyv)^\theta \cap Q^-(2n+1,q)$. This quadric contains $\frac{q^{2n-2}-1}{q-1}$ points, i.e. $N = (O(Q^-(2n+1,q))-2)\frac{q^{2n-2}-1}{q-1}$.

On the other hand for each $u \in Q$ the points of $O \cap u^\theta$ define a partial ovoid in u^θ/u. Since $x, y \in O \cap u^\theta$, for each possible $u \in Q$ there exist at most $O(Q^-(2n-1,q))-2$ corresponding v's. Therefore we have
$N \leq \frac{(q^{n-1}-1)(q^n+1)}{q-1}(O(Q^-(2n-1,q))-2)$.

All together we get:

$$(O(Q^-(2n+1,q))-2)\frac{q^{2n-2}-1}{q-1} \leq \frac{(q^{n-1}-1)(q^n+1)}{q-1}(O(Q^-(2n-1,q))-2)$$

Simplifying we get (1). □

The next corollary improves the bound $O(Q^-(2n+1,q)) \leq q^{n+1}-q^2+2$ of [7].

Corollary 1
$O(Q^-(2n+1,q)) \leq q^{n+1}-q^n+q^{n-1}-q^{n-2}+2$, *if* $n \geq 2$.

Proof
Thas proves in [6] that $O(Q^-(5,q)) \leq q^3+1-q(q-1)$. The corollary follows from Theorem 1 since $\frac{q^n+1}{q^{n-1}+1} \leq q$. □

Remark 1
Blokhuis and Moorhouse have proven:

$$O(Q^-(2n+1,q)) \leq [\binom{p+2n}{5} - \binom{p+2}{2n-2}]^e + 1$$

for a prime number p and $q = p^e$ (see [1]). If p is small in comparison with n this bound is better than the one of Corollary 1.

3 Partial ovoids in $W(2n+1,q)$

Ovoids of $W(3,q)$ exist only if q is even, and there are no ovoids in $W(2n+1,q)$ for $n>1$ (see [4] and [5]). In this section we prove some upper bounds for $O(W(2n+1,q))$.

Theorem 2
Let O be a partial ovoid of $W(2n+1,q)$, $n>1$. Let g be a line of $PG(2n+1,q)$ which contains m, with $2 \leq m \leq q+1$, points of O. Let M be the maximal size of a partial ovoid of $W(2n-1,q)$ which contains a line with at least m points.

Then the number of points in O is bounded above by:

$$|O| \leq \frac{q^{2n}-1}{q^{2n-1}-1}(M-m)+m \tag{2}$$

In particular:

$$O(W(2n+1,q)) \leq \frac{q^{2n}-1}{q^{2n-1}-1}(O(W(2n-1,q))-2)+2 \tag{3}$$

Proof
The proof of this theorem is inspired by the proof of Theorem 23 in [7].

Let θ be the polarity of $PG(2n+1,q)$ corresponding to $W(2n+1,q)$ and let $\pi = g^\theta$. Then π is a $(2n-1)$-dimensional subspace of $PG(2n+1,q)$ which contains no point of O.

We count the number N of pairs (u,v) with $u \in \pi$, $v \in O \backslash g$ such that uv is a singular line of $W(2n+1,q)$.

For $v \in O\backslash g$ the point u must lie in $v^\theta \cap \pi$. This is a $(2n-2)$-dimensional subspace of $PG(2n+1,q)$ which contains $\frac{q^{2n-1}-1}{q-1}$ points. Thus $N = (|O|-m)\frac{q^{2n-1}-1}{q-1}$.

For each point $u \in \pi$ the points of $O \cap u^\theta$ define a partial ovoid in u^θ/u. The points of $g \cap O$ lie in u^θ/u on the line gu/u. By assumption every partial ovoid in $W(2n-1,q)$ with at least m points on a line contains at most M points, i.e. $|u^\theta \cap O| \leq M$ and $|u^\theta \cap (O\backslash g)| \leq M-m$. Since π contains exactly $\frac{q^{2n}-1}{q-1}$ points, we conclude $N \leq \frac{q^{2n}-1}{q-1}(M-m)$.

Together we have:

$$(|O|-m)\frac{q^{2n-1}-1}{q-1} \leq \frac{q^{2n}-1}{q-1}(M-m)$$

This proves (2). Since every partial ovoid contains at least one line with 2 points, (3) follows from (2) if we set $m=2$. □

In [7] Thas proves $O(W(2n+1,q)) \leq q^{n+1}-q+2$. Now we can use Theorem 2 to improve this result.

Corollary 2
For $n \geq 2$ we have $O(W(2n+1,q)) \leq q^{n+1}-q^{n-1}+2$. If q is odd and $n \geq 1$ the bound can be improved to $O(W(2n+1,q)) \leq q^{n+1}-q^n-q^{n-1}+2$.

Proof
As a special case of the result in [7] we obtain $O(W(5,q)) \leq q^3 - q + 2$. Now the corollary follows from Theorem 2 together with $\lfloor \frac{q^{2n}-1}{q^{2n-1}-1}(q^n - q^{n-2}) \rfloor \leq q^{n+1} - q^{n-1}$.

Tallini proves in [3] that every partial spread of $Q(4,q)$ with q odd contains at most $q^2 - q + 1$ lines. Since $W(3,q)$ is isomorphic to the dual of $Q(4,q)$ this implies $O(W(3,q)) \leq q^2 + 1 - q$ if q is odd. If we apply Theorem 2 to that upper bound we obtain $O(W(2n+1,q)) \leq q^{n+1} - q^n - q^{n-1} + 2$. □

Another corollary of Theorem 2 is:

Corollary 3
Every ovoid of $W(3,q)$, q even, is an ovoid of $PG(3,q)$.

Proof
Suppose O is an ovoid of $W(3,q)$ but not an ovoid of $PG(3,q)$. In this case there is a line which contains $m > 2$ points of O. Applying Theorem 2 we get:

$$|O| \leq \frac{q^2-1}{q-1}(q+1-m) + m \leq q^2 + 2q - mq + 1 < q^2 + 1 \quad .$$

This is a contradiction since $|O| = q^2 + 1$. □

4 Ovoids in $H(d,q^2)$

The situation in $H(d,q^2)$ is more complicated than the situation in $Q^-(2n+1,q)$ and $W(2n+1,q)$. The main reason is:

Let θ be the polarity of $PG(d,q^2)$ corresponding to $H(d,q^2)$. If a plane π contains at least 3 points of an ovoid, π^θ can intersect the polar space $H(d,q^2)$ either in a polar space of type $H(d-3,q^2)$ or in a cone over a polar space of type $H(d-4,q^2)$.

It is obvious that there exist ovoids in $H(3,q^2)$. In [5] Thas proves that there are no ovoids in $H(2n,q^2)$ for $n > 1$. In this section we prove:

Theorem 3
For $n > q^3$ there exist no ovoids in $H(2n+1,q^2)$.

To prove this we need the following lemma:

Lemma 1
Let O be an ovoid of $H(2n+1,q^2)$. Let π be a plane which intersects $H(2n+1,q^2)$ in a polar space of type $H(2,q^2)$. Let $m = |\pi \cap O| \geq 2$.

Then there exists an ovoid O' of $H(2n-1,q^2)$ and a plane π', which intersect $H(2n-1,q^2)$ in a polar space of type $H(2,q^2)$ and with $|\pi' \cap O'| > m$.

Proof
Let θ be the polarity of $PG(2n+1,q^2)$ with respect to $H(2n+1,q^2)$. Then π^θ intersects $H(2n+1,q^2)$ in a polar space of type $H(2n-2,q^2)$. We distinguish two cases:

1. **There exists a point $X \in O\backslash\pi$ such that $X\pi \cap H(2n+1,q^2)$ is a cone with basis $H(2,q^2)$.**
   Let P be the vertex of the cone. Then $H(2n+1,q^2)$ induces a polar space of type $H(2n-1,q^2)$ in P^θ/P and O induces an ovoid O' in this polar space. The plane $\pi' = (\pi P)/P$ contains at least $m+1$ points of O' (the m points in π and the point X).

2. **For all points $X \in O\backslash\pi$ the space $X\pi$ intersects $H(2n+1,q^2)$ in a polar space of type $H(3,q^2)$.**
   We count the number N of pairs (u,v) with $u \in \pi^\theta \cap H(2n+1,q^2)$, $v \in O\backslash\pi$ such that uv is a singular line of $H(2n+1,q^2)$.

   For every point $u \in \pi^\theta \cap H(2n+1,q^2)$ the ovoid O induces an ovoid in the polar space u^θ/u. Therefore for every u we have exactly $q^{2n-1}+1-m$ pairs (u,v). Thus:

   $$N = \frac{(q^{2n-1}+1)(q^{2n-2}-1)}{q^2-1}(q^{2n-1}+1-m)$$

   For every point $v \in O\backslash\pi$ the space $(\pi v)^\theta$ intersects $H(2n+1,q^2)$ in a polar space of type $H(2n-3,q^2)$ (assumption of this case). Since u must lie in $(\pi v)^\theta$ we have

   $$N = \frac{(q^{2n-2}-1)(q^{2n-3}+1)}{q^2-1}(q^{2n+1}+1-m) \quad .$$

   Comparing the two equalities we find:

   $$\begin{aligned}\frac{(q^{2n-1}+1)(q^{2n-2}-1)}{q^2-1}(q^{2n-1}+1-m) &= \frac{(q^{2n-2}-1)(q^{2n-3}+1)}{q^2-1}(q^{2n+1}+1-m)\\ q^{2n+1}+1-m &= \frac{q^{2n-1}+1}{q^{2n-3}+1}(q^{2n-1}+1-m)\\ q^{2n+1}+1-m &< q^2(q^{2n-1}+1-m)\\ m &< 1\end{aligned}$$

   A contradiction, since $m \geq 2$. Thus case 2 never occurs. □

Proof of Theorem 3
Suppose that an ovoid of $H(2n+1,q^2)$ exists for $n > q^3$. At least 2 points of this ovoid lie in a plane π which intersects $H(2n+1,q^2)$ in a polar space of type $H(2,q^2)$.

Applying Lemma 1 repeatedly we conclude that there exists an ovoid of $H(2n+1-2k, q^2)$ with at least $2+k$ points in a plane. In particular for $k = q^3$ there must be an ovoid with more than q^3+1 points in a plane. Since $H(2, q^2)$ contains only q^3+1 points, this is a contradiction. $\square$

In [1] Blokhuis and Moorhouse use algebraic methods to obtain a bound for the non-existence of ovoids in $H(2n+1, q^2)$. They prove that if $q = p^e$ for some prime number p and $H(2n-1, q^2)$ contains an ovoid then $p^{2n-1} \leq \binom{p+2m-2}{2m-1}^2$. This bound is better than the one of Theorem 3 but the combinatorial arguments used in the proof of Theorem 3 describe in addition the structure of possible ovoids in small dimensions.

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