New results on covers and partial spreads of polar spaces

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Abstract

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1 Introduction

Let $H(3,q^2)$ be a hermitian surface of $\mathrm{PG}(3,q)$. The lines it contains are called its *generators*. An *ovoid* of $H(3,q^2)$ is a set of points of $H(3,q^2)$ meeting every generator exactly once, and a *partial ovoid* is a set of points meeting every generator in at most one point. It is known that $H(3,q^2)$ has ovoids, for example a hermitian curve $H(2,q^2)$ that is obtained by intersecting $H(3,q^2)$ with a non-tangent hyperplane. A blocking set of $H(3,q^2)$ is a set of points that meets generator in at least one point. The same notation is used for all other polar spaces of rank two, that is polar spaces that contain lines but no planes.

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The origin of the paper was the problem of finding the largest partial ovoid of $H(3,q^2)$ that is not an ovoid. This problem we learned from Gary Ebert [4]. It is a simple calculation to see that an ovoid of $H(3,q^2)$ has exactly q^3+1 points, and it is not difficult to construct a maximal partial ovoid of $H(3,q^2)$ of size q^3+1-q; we introduce this example in its dual form inside $Q^-(5,q)$ in Section 3. As a first result we show that this is best possible:

In this section we shall prove a result on blocking sets in cones over quadrics. This result is then applied to obtain various results on partial spreads and covers of quadrics.

Theorem 1.1 *A partial ovoid of $H(3,q^2)$ either is contained in an ovoid or has at most q^3+1-q lines.*

For the proof we work in the dual setting, that is in the elliptic quadric $Q^-(5,q)$. Here the result can be formulated as follows. Recall that a *partial spread* of a polar space is a set of mutually skew generators of the polar space; it is called a *spread* if its lines partition the point set of the polar space.

Theorem 1.2 *Suppose that S is a maximal partial spread of $Q^-(5,q)$. Then either S is a spread or $|S| \leq q^3-q+1$.*

We obtain this result by a careful analyzation of the projective space $\mathrm{PG}(4,q)$ when the blocking sets lives in a degenerate quadric. Counting arguments shows that such a blocking set must have many collinear points unless it is quite large. This will be done in Section 2. Our method also shows that a partial spread of $Q(4,q)$ either is contained in a spread (with q^2+1 lines) or has at most q^2-q+1 lines. For odd q this was proved many years ago by Tallini [9]. However, when q is even, it was only proved very recently by Brown, De Beule and Storme [2] by using the representation of $Q(4,q)$ as a $\mathcal{T}_2(\mathcal{O})$. Our proof is the first one that works for q even and odd.

The method we develop together with an algebraic trick that generalizes a result of Bichara and Korchmaros [3] will enable us to also prove the following theorem.

Theorem 1.3 *Let q be odd. Then a blocking set of $W(3,q)$ contains at least*

$$q^2-q-\frac{3}{2}+\frac{\sqrt{8q^2+20q+25}}{2}$$

points.

The bound in Theorem 1.3 is of size $q^2 + (\sqrt{2} - 1)q$, which is a significant improvement in comparison with the bound of size $q^2 + \frac{1}{3}q$ proven in [6]. We recall that $W(3, q)$, q even, has an ovoid, which is a blocking set of size q^2+1. The smallest known blocking set of $W(3, q)$ when q is odd has size q^2+q-1 and was found by Govaerts in [8].

2 Blocking sets contained in quadrics

If one studies partial spreads or covers of quadrics, then the set of points of the quadric that are not covered (for spreads) respectively the set of points of the quadric that are covered more than once (for covers) have similar properties. It is extremely useful to study the intersection of these sets with tangent hyperplanes. We shall do this in the case $Q^-(5, q)$. The crucial observation is in the following lemma.

Lemma 2.1 *Consider in* $\mathrm{PG}(4, q)$ *a quadric that is a cone with vertex a point* P *over a non-degenerate elliptic quadric* $Q^-(3, q)$*. Suppose that* B *is a set of at most* $2q$ *points contained in the quadric. If every solids of* $\mathrm{PG}(4, q)$ *meets* B*, then one of the following occurs:*

(a) Some line of the quadric is contained in B*.*

(b) $|B| > \frac{9}{5}q + 1$*,* $P \in B$*, and there exists a unique line* l *of the quadric that meets* B *in at least* $1 + \frac{1}{3}|B|$ *points. This line has at most* $|B| - 1 - q$ *points in* B*.*

Proof. Denote by l_i, $i = 1, \dots, q^2 + 1$, the lines of the quadric on P. If P is not in B, then we use that each line l_i lies on a solid meeting the quadric only in l_i to deduce that $|B| \geq q^2 + 1$; but $|B| \leq 2q$, so $P \in B$. Put $b := |B| - 1$ and $b_i := |l_i \cap B| - 1$. Let $\mathcal{S}$ be the set consisting of the q^4 solids that do not contain P, and put $b_S := |B \cap S|$ for $S \in \mathcal{S}$. Since all solids S meet B we have $b_S \geq 1$ for all $S \in \mathcal{S}$.

If one of the lines l_i is contained in B, there is nothing to show. We may thus assume that this is not the case. Consider different lines l_i and l_j and choose points $R_i \in l_i$ and $R_j \in l_j$ such that $R_i, R_j \notin B$. Then there are q^2

solids in $\mathcal{S}$ containing the line R_iR_j, and every point of B not lying on $l_i \cup l_j$ appears in exactly q of these. Thus, if we sum up b_s for the q^2 solids S of $\mathcal{S}$ on the line R_iR_j, then we obtain $(b - b_i - b_j)q$. As $b_S \geq 1$ for all $S \in \mathcal{S}$, it follows that $b - b_i - b_j \geq q$, that is

$$b_i + b_j \leq b - q \quad \text{for } i \neq j. \tag{1}$$

We use for integers $x \geq 1$ the identity

$$x - \binom{x}{2} + \binom{x}{3} - \binom{x-1}{3} = 1 \tag{2}$$

to obtain

$$q^4 = |\mathcal{S}| = \sum_{S\in\mathcal{S}} \left(b_S - \binom{b_S}{2} + \binom{b_S}{3} - \binom{b_S - 1}{3} \right). \tag{3}$$

For the first three terms on the right hand sind, standard counting arguments give

$$\begin{aligned}
\sum_{S\in\mathcal{S}} b_S &= \sum_i b_i q^3 = bq^3 \ , \\
\sum_{S\in\mathcal{S}} \binom{b_s}{2} &= \sum_{i<j} b_i b_j q^2 = \frac{1}{2} q^2 b^2 - \frac{1}{2} q^2 \sum_i b_i^2 \ , \ \text{and} \\
\sum_{S\in\mathcal{S}} \binom{b_s}{3} &= \sum_{i<j<k} b_i b_j b_k q = \frac{1}{6} b^3 q - \frac{1}{2} bq \sum_i b_i^2 + \frac{1}{3} q \sum_i b_i^3 \ .
\end{aligned}$$

The last term is handled differently. Firstly, with $\mathcal{S}_4 := \{S \in \mathcal{S} : b_S \geq 4\}$, we have

$$\sum_{S\in\mathcal{S}} \binom{b_s - 1}{3} \ \geq \ \frac{1}{12} \sum_{S\in\mathcal{S}_4} b_s(b_S - 1)(b_S - 3) \ . \tag{4}$$

Now $b_S(b_S - 1)$ is the number of ordered pairs of different points $R, R' \in B \cap S$. If we first choose different points $R, R' \in B \setminus \{P\}$ such that RP and $R'P$ are different lines l_i and l_j (otherwise R and R' do not occur together in an element of $\mathcal{S}$) and then add up all terms $b_S - 3$ for the $S \in \mathcal{S}_4$ containing R and R', then we obtain at least $(b - q - b_i - b_j)q$; this can be seen as follows: There are $b - b_i - b_j$ points in B outside $l_i \cup l_j$. Each such point occurs together with R and R' in q solids of S. Thus, if we sum $b_S - 2$ for the q^2 solids $S \in \mathcal{S}$ on RR', we obtain $(b - b_i - b_j)q$; of course the same is true if we just consider solids S satisfying $b_S \geq 3$. Since there are at most q^2 such solids, the sum of the terms $b_S - 3$ taken over all solids S on RR' with $|S \cap B| \geq 3$ is at least $(b - b_i - b_j)q - q^2$. As the solids S with $b_S = 3$ give no contribution, the claim is established. Hence from (4) we deduce that

$$\begin{aligned}
\sum_{S\in\mathcal{S}} \binom{b_s - 1}{3} &\geq \frac{1}{12} \sum_{i \neq j} b_i b_j (b - q - b_i - b_j) q \\
&= \frac{1}{12}(b-q)q \sum_{i\neq j} b_i b_j - \frac{1}{6} \sum_{i\neq j} b_i^2 b_j q \\
&= \frac{1}{12}(b-q)q \sum_i b_i (b - b_i) - \frac{1}{6} \sum_i b_i^2 (b - b_i) q \\
&= \frac{1}{12}(b-q)qb^2 - \frac{1}{12} \sum_i b_i^2 q (3b - q - 2b_i) \ .
\end{aligned}$$

Using all this, we deduce from (3) the following (note that all terms are divisible by q)

$$q^3 \leq bq^2 - \frac{1}{2}b^2 q + \frac{1}{6}b^3 - \frac{1}{12}(b-q)b^2 + \frac{1}{12}\sum_i b_i^2 (5q - 3b + 2b_i). \tag{5}$$

CASE 1: $b_i \leq \frac{1}{2}(b-q)$ for all i. Then

$$\sum_i b_i^2 (5q - 3b + 2b_i) \leq \sum_i b_i \frac{1}{2}(b-q)(4q - 2b) = b(b-q)(2q-b) \ .$$

Combining this with (5) gives $0 \leq \frac{1}{6}q(2q-b)(b-3q)$. As $b = |B|-1 \leq 2q-1$, this is a contradiction.

CASE 2: $\max\{b_i\} > \frac{1}{2}(b-q)$. We may assume that $b_1 = \max\{b_i\}$. From (1) we obtain $b_i \leq b - q - b_1$ for $i \geq 2$. As $\sum b_i = b$, it follows that

$$\begin{aligned}
\sum_i b_i^2 (5q - 3b + 2b_i) &\leq b_1^2 (5q - 3b + 2b_1) + \sum_{i\geq 2} b_i (b - q - b_1)(3q - b - 2b_1) \\
&= b_1^2 (5q - 3b + 2b_1) + (b - b_1)(b - q - b_1)(3q - b - 2b_1) \ .
\end{aligned}$$

Combining this with (5) multiplied by 12, we find that $0 \leq f(b_1)$, where $f \in \mathbb{Z}[x]$ is defined by

$$f := q(9bq - 12q^2 - b^2 + 6x^2 - 5bx + 3xq) \ . \tag{6}$$

As f is a polynomial of degree two in x and since

$$f\left(\frac{b}{3}\right) = f\left(\frac{b-q}{2}\right) = q(2q-b)(b-3q)/6 < 0$$

it follows that $f(x) < 0$ for x between $\frac{1}{2}(b-q)$ and $\frac{1}{3}b$; note that $b < 2q$ implies that $\frac{1}{3}b > \frac{1}{2}(b-q)$. As $b_1 > \frac{1}{2}(b-q)$ and $f(b_1) \geq 0$, it follows that $b_1 > \frac{b}{3}$. Thus $|l_1 \cap B| > 1 + \frac{b}{3}$ and hence $|l_1 \cap B| \geq 1 + \frac{1}{3}|B|$. From (1) we also have $b_1 < b - q$, that is $|l_1 \cap B| < |B| - q$. Using again (1) we see that $b_i \leq b - q - b_1$ for $i \geq 2$, which implies that see that $|l_i \cap B| < 1 + \frac{1}{3}|B|$ for $i \geq 2$. Finally, using $\frac{b}{3} < b_1 < b - q$ (see (1)) we find that

$$0 \leq f(b_1) < f(b-q) = q(5b - 9q) \ .$$

Hence $b > \frac{9}{5}q$. ■

3 Partial spreads of $Q^-(5,q)$

Suppose that S is a partial spread of $Q^-(5,q)$, that is a set of mutually disjoint planes of $Q^-(5,q)$. As $Q^-(5,q)$ has $(q+1)(q^3+1)$ points, then $|S| = q^3 + 1 - \delta$ for some $\delta \geq 0$. If $\delta = 0$, then S is a spread.

We use that $Q^-(5,q)$ and $H(3,q^2)$ are dual (Klein-Correspondence). A spread of $Q^-(5,q)$ translates under this duality to an ovoid of $H(3,q^2)$, that is to a set B of points that meets every line of $H(3,q)$ in a unique point. The most natural candidate for an ovoid in $H(3,q^2)$ is a hermitian curve $H(2,q^2)$. However there are many others; for example every chord in such a $H(2,q^2)$ can be replaced by its perp, and this can be done several times. A hermitian spread of $Q^-(5,q)$ is a spread dual to an ovoid $H(2,q^2)$ of $H(3,q^2)$. As chords of $H(2,q^2)$ are Baer-sublines and thus translate to reguli of $Q^-(5,q)$ (a property of the Klein-Correspondence), we see that a hermitian spread S has the property that any two lines of S lie in a unique regulus R with $R \subseteq S$.

Example. Let S be a hermitian spread of $Q^-(5,q)$, let l be a line of S and let R_1 and R_2 be two reguli on l with $R_i \subseteq S$. Let R_i^{op} be the regulus oppsite to R_i. Replace the $2q+1$ lines of S in $R_1 \cup R_2$ by $q+1$ lines of $R_1^{\mathrm{op}} \cup R_2^{\mathrm{op}}$ such that every point of l is covered exactly once. This gives a partial spread S' with $|S'| = q^3+1-q$. If one chooses at least one line of R_1^{op} and one from R_2^{op}, then the partial spread is maximal.

This example and generalizations occur also in [5]. The following theorem shows that this example is best possible.

Theorem 3.1 *Suppose that S is a maximal partial spread of $Q^-(5,q)$. Then either S is a spread or $|S| \leq q^3 - q + 1$.*

Proof. Put $\delta := q^3 + 1 - |S|$. We assume that $0 < \delta < q$ and derive a contradiction. Let H be the set consisting of the $\delta(q+1)$ points of $Q^-(5,q)$ that are not covered by S. The points of H will be called *holes*. As $\mathcal{S}$ is maximal, then H does not contain a line.

Embed $Q^-(5,q)$ in the natural way in $\mathrm{PG}(5,q)$. Every hyperplane of $\mathrm{PG}(5,q)$ meets $Q^-(5,q)$ in one modulo q points. As $|S| = q^3+1-\delta$, it follows that every hyperplane meets H in δ modulo q points. As $\delta < q$, this implies that every hyperplane contains at least δ holes.

Consider a hole P. The tangent hyperplane $P^\perp$ on P meets $Q^-(5,q)$ in a cone with vertex P over a $Q^-(3,q)$. Every line of S meets $P^\perp$ in a unique point. As $P^\perp$ contains q^3+q+1 points of the quadric, then $P^\perp$ contains $q+\delta$ holes. If S is a solid of $P^\perp$, then each of the q hyperplanes on S other than X contain at least δ holes. As the number of holes is $(q+1)\delta$ and as $P^\perp$ contains more than δ holes, it follows that S must contain a hole. Hence $P^\perp \cap H$ meets every solid of $P^\perp$. Lemma 2.1 shows that there exists a unique line l of the quadric $Q^-(5,q)$ such that $P \in l$ and $|l \cap H| \geq 1 + \frac{1}{3}(q+\delta)$. The lemma also gives $|l \cap H| \leq \delta - 1$. As this holds for every hole P, we find lines $l_1, \ldots, l_s$ in $Q^-(5,q)$ such that $\frac{1}{3}(q+\delta)+1 \leq |l_i \cap H| \leq \delta - 1$ for all l_i, and every hole is contained in exactly one of the lines l_i.

A point of the quadric that is not a hole lies on a unique line of the spread. This implies that $P^\perp$ contains exactly δ holes. Therefore, P can be contained in at most one of the lines l_i. This shows that the lines l_i are mutually skew. We have verified the hypotheses of Proposition 3.2. As $0 < \delta < q$, this proposition gives a contradiction. ■

Proposition 3.2 *Consider the elliptic quadric $Q^-(5,q)$ and its ambient space* $\mathrm{PG}(5,q)$. *Suppose that H is a set of $\delta(q+1)$ points of $Q^-(5,q)$ with the following properties.*

(a) Every hyperplane of $\mathcal{P}$ meets H in δ modulo q points.

(b) There exist s mutually skew lines $l_1,\dots,l_s$ of $Q^-(5,q)$ such that H is contained in the union of the l_i and such that $\frac{1}{3}(q+\delta)+1 \le |l_i \cap H| \le \delta - 1$ for $i=1,\dots,s$.

Then $\delta = 0$ or $\delta \ge q$.

Proof. Assume on the contrary that $1 \le \delta \le q-1$. We shall derive a contradiction. The points of H will be called *holes*. As $\delta < q$, hypothesis (a) implies that every hyperplane has at least δ holes.

PART 1. Suppose that a hyperplane X has $rq+\delta$ holes. If S is a solid of X and u its number of holes, then the other q hyperplanes on S have each at least $\delta - u$ further holes. Hence $rq+\delta$ plus $q(\delta-u)$ is at most the total number $\delta(q+1)$ of holes. This gives $u \ge r$. Hence every solid of X has at least r holes.

PART 2. As the lines l_i are mutually skew, any two lines l_i span a hyperbolic solid, that is a solid meeting the quadric in a $Q^+(3,q)$. The number s of lines l_i is bounded by

$$s \le \frac{|H|}{\frac{1}{3}(q+\delta)} = \frac{3\delta(q+1)}{(q+\delta)} < \frac{3}{2}(q+1).$$

Also, as $|l_i \cap H| < \delta$ and $|H| = \delta(q+1)$, then $s \ge q+2$. Finally we remark that $\delta \le q-1$ and the hypothesis $1+\frac{1}{3}(q+\delta) \le |l_i \cap H| \le \delta - 1$ for all i imply that $q \ge 8$.

PART 3. We shall show in this part that every hyperplane that contains two of the lines l_i contains at least $\frac{1}{2}(q+1)$ of the lines l_i. For this, suppose that X is a hyperplane that contains exactly $c \ge 2$ lines l_i, say $l_1,\dots,l_c$. We may assume that $|l_1 \cap H| \ge |l_i \cap H|$ for $i=1,\dots,c$.

Consider the hyperbolic solid $\langle l_1, l_2\rangle$, and let R be a point of l_1 that is not a hole. As at least two points of l_2 are not holes, we find a non-hole R' on l_2 such that RR' is a secant line to the quadric. Then the line RR' has no hole. Since RR' lies on q^2+q+1 planes of X, we find a plane π of X on RR' that has no hole.

Put $|X \cap H| = rq + \delta$. By Part 1, every solid of X meets H in at least r points. Considering the $q+1$ solids of X on π taking into account that two of these contain l_1 resp. l_2, we find that

$$\begin{aligned} rq+\delta &\geq (q-1)r + |l_1 \cap H| + |l_2 \cap H|. \\ \Rightarrow \quad r+\delta &\geq |l_1 \cap H| + |l_2 \cap H|. \end{aligned}$$

Each of the lines l_i with $i > c$ meets X in a unique point, which might be in H. This implies that

$$rq + \delta = |X \cap H| \leq \sum_{i \leq c} |l_i \cap H| + s - c \leq c|l_1 \cap H| + s - c.$$

Writing $rq + \delta = q(r+\delta) - \delta q + \delta$, we find

$$|l_1 \cap H|q + |l_2 \cap H|q - \delta q + \delta \leq c|l_1 \cap H| + s - c.$$

Assume that $c \leq \frac{q}{2}$. Using $|l_i \cap H| \geq \frac{1}{3}(q+\delta)$ and $s < \frac{3}{2}(q+1)$, it follows that

$$\frac{3}{2}q \cdot \frac{q+\delta}{3} < \delta q - \delta + q + \frac{3}{2}.$$

As $\delta \leq q-1$, this leads to $q < 5$. But we have seen in Part 2 that $q \geq 8$. This contradiction shows that $c \geq \frac{1}{2}(q+1)$.

PART 4: Here we study the case that every solid that contains two of the lines l_i contains at least $\frac{1}{2}(q+1)$ of the lines l_i. Consider a solid S containing $u \geq 2$ of the lines l_i. Since there are $s \geq q+2$ lines l_i and since the l_i are mutually skew, then not all lines l_i lie in S. We may assume that l_1 is not contained in S. Then l_1 spans a solid with every line l_i in S. This gives at least u solids on l_1 that all contain at least $\frac{1}{2}(q+1)$ of the s lines l_i. Hence

$$1 + u \cdot \frac{q-1}{2} \leq s < \frac{3}{2}(q+1).$$

As $u \geq \frac{1}{2}(q+1)$ and $q \geq 8$ (Step 2), this is a contradiction.

PART 5: Now we consider the case that some solid S contains u of the lines l_i with $2 \leq u < \frac{1}{2}(q+1)$. By Part 3, every hyperplane on S contains at least $\frac{1}{2}(q+1)$ lines l_i and thus at least one line l_i that is not contained in S. As $s < 2(q+1)$, it is not possible that each of the $q+1$ hyperplanes on S contains two lines l_i that do not lie in S. Hence S must contain at least

$\frac{1}{2}(q-1)$ lines l_i. Thus $u = \lfloor \frac{q}{2} \rfloor$ and similarly every solid with two lines l_i contains exactly u lines l_i. As each hyperplane on S contains a line l_i that is not contained in S; it follows that each such hyperplane contains at least $1 + u(u-1) = u^2 - u + 1$ lines l_i. Considering the $q+1$ hyperplanes on S, we find $s \geq u + (q+1)(u^2 - 2u + 1)$. But $u \geq \lfloor \frac{q}{2} \rfloor \geq 4$ and $s < \frac{3}{2}(q+1)$, a contradiction. ■

The same technique also works for partial spreads of $Q(4,q)$. As already mentioned in the introduction, this was shown in [9] when q is odd and [2] when q is even. The following new proof works for all q.

Theorem 3.3 *Suppose that S is a maximal partial spread of $Q(4,q)$. Then either S is a spread or $|S| \leq q^2 - q + 1$.*

Proof. Put $\delta := q^2 + 1 - |S|$. We assume that $0 < \delta < q$ and derive a contradiction. As in the case of $Q^-(5,q)$, the points of the quadric not covered by S form a set H consisting of $\delta(q+1)$ holes. Also H contains no line and every hyperplane meets H in δ modulo q points.

For a hole P, the tangent hyperplane $P^\perp$ contains $q+\delta$ holes and hence each plane of $P^\perp$ contains a hole. The structure of $P^\perp$ is a cone with vertex P over a conic $Q(2,q)$; such a structure can be embedded in a cone with vertex P over a $Q^-(3,q)$ and then Lemma 2.1 can be applied. Thus, as for $Q^-(5,q)$ we find lines $l_1, \ldots, l_s$ of $Q(4,q)$ such that $1 + \frac{1}{3}(q+\delta) \leq |l_i \cap H| \leq \delta - 1$ for all l_i, and every hole is contained in exactly one of the lines l_i.

Embedding $Q(4,q)$ now in a $Q^-(5,q)$, the proposition can again be applied, leading to a contradiction as before. ■

4 Covers of $Q^-(5,q)$ and $Q(4,q)$

The technique of the previous section can be slightly modified to be applicable to covers. We shall demonstrate this for $Q^-(5,q)$. However, we start more generally with weighted line sets that *cover* all points.

Lemma 4.1 *Suppose w is a function from the set $\mathcal{L}$ of lines of $Q^-(5,q)$ to $\mathbb{Z}$. For every point P denote by $w_P + 1$ the sum of the values $w(l)$ running over all lines l on P. Suppose that $w_P \geq 0$ for all P.*

If $\delta := \sum_{l\in\mathcal{L}} w(l) - (q^3+1) \le 1+\frac{4}{5}q$, then there exist lines $l_1,\dots,l_\delta$ of $Q(4,q)$ with the following property: For every point P, the number w_P is equal to the number of lines l_i that pass through P.

Proof. We have $\sum_{l\in\mathcal{L}} w(l) = q^3+1+\delta$ and thus

$$\sum_{P\in Q^-(5,q))} (w_P+1) = (q^3+1+\delta)(q+1) \quad \Rightarrow \quad \sum_{P\in Q^-(5,q))} w_P = \delta(q+1).$$

Hence $\delta \ge 0$ with equality if and only if $w_P = 0$ for all points P of $Q(4,q)$. Thus, the theorem is correct in the case $\delta = 0$. Suppose now that $0 < \delta \le \frac{4}{5}q+1$. Embed $Q^-(5,q)$ in a natural way in $\mathrm{PG}(5,q)$. For every subset A of $\mathrm{PG}(5,q)$, denote by $w(A)$ the sum of the w_P for $P \in A \cap Q^-(5,q)$. Notice that $w(\mathrm{PG}(5,q)) = \delta(q+1)$.

As every hyperplane of $\mathrm{PG}(5,q)$ meets $Q^-(5,q)$ in one modulo q points and since $\sum w(l) = q^3+1+\delta$, then $w(H)$ is congruent δ modulo q for every hyperplane H of $\mathrm{PG}(5,q)$. As a matter of fact, when S is a solid with $w(S) = 0$, then $w(\mathrm{PG}(5,q)) = \delta(q+1)$ implies that $w(H) = \delta$ for every hyperplane H on S. In other words:

(*) $w(H) > \delta$ for a hyperplane H implies $w(S) > 0$ for all solids S of H.

Put $c := \min\{w_P \mid w_P > 0\}$, and denote by P a point satisfying $w_P = c$. Then the sum of the $w(l)$ for the lines l of $Q^-(5,q)$ not on P is $q^3+1+\delta-c$. As the tangent hyperplane $P^\perp$ has $(q^2+1)q+1$ points in $Q^-(5,q)$, it follows that

$$w(P^\perp) = (c+1)(q+1) + q^3+1+\delta-c-(q^2+1)q-1 = cq+\delta.$$

Put $B := \{X \in P^\perp \mid w_X > 0\}$. Then $w(B) = cq+\delta$. As $w(X) \ge c$ for all $X \in B$, this implies that $|B| \le (cq+\delta)/c \le q+\delta$. From (*) we see that all solids of $P^\perp$ meet B. As $\delta < 1+\frac{4}{5}q$, then Lemma 2.1 implies that B contains a line l_0. Define a new function w' from the lines of $Q^-(5,q)$ to $\mathbb{Z}$ with $w'(l) = w(l)$ for $l \ne l_0$, and $w'(l_0) := w(l_0) - 1$. As $w(P) \ge 1$ for all $P \in l_0$, we see that w' fulfills the hypothesis of the lemma. As $\sum w'(l)$ is one less than $\sum w(l)$, an inductive argument completes the proof. ■

Corollary 4.2 *Suppose that S is a cover of $Q^-(5,q)$. For every point P of $Q^-(5,q)$ denote by w_P+1 the number of lines of S on P. Suppose that*

$\delta := |S| - q^3 - 1 < \frac{4}{5} + 1$. *Then there exists lines* $l_1, \ldots, l_\delta$ *of* $Q^-(5,q)$ *with the following property: For every point* P*, the number* w_P *is equal to the number of lines* l_i *that pass through* P*.*

Remark. (1) Consider a hermitian spread S of $Q^-(5,q)$ that is a spread that translates by the duality to $H(3,q^2)$ to a hermitian curve. Then the spread contains two reguli R_1 and R_2 that share precisely one line l. Let R_i^{op} be the regulus opposite to R_i, and put $S' := (S \cup R_1^{op} \cup R_2^{op}) \setminus (R_1 \cup R_2)$. Then S' is a minimal with $q^3 + 2$ lines. This shows that there does not exist a gap-theorem for covers.

(2) Consider again the spread S and its two reguli sharing the line l. Remove l von S and add $q+1$ lines of $R_1^{op} \cup R_2^{op}$ such that each point of l is covered exactly one. This gives a cover with $q^3 + q + 1$ lines. If one uses at least one line of R_1^{op} and one of R_2^{op}, the cover is minimal. However, the multiple covered points can not be written as a sum of lines as in corollary 4.2. We conjecture that there is no smaller example with this property.

An analogous result to Lemma 4.1 can be proved for $Q(4,q)$. The proof is almost identical. We therefore omit the lemma and the proof and give only the corollary.

Corollary 4.3 *Suppose that* S *is a cover of* $Q(4,q)$*. For every point* P *of* $Q(4,q)$ *denote by* $w_P + 1$ *the number of lines of* S *on* P*. Suppose that* $|S| = q^2 + 1 + \delta$ *with* $\delta < \frac{4}{5} + 1$*. Then there exists lines* $l_1, \ldots, l_\delta$ *of* $Q(4,q)$ *such that for every point* P *the number* w_P *is equal to the number of lines* l_i *that pass through* P*.*

5 An algebraic tool

We will need the following algebraic tool in the next section. We remark that the lemma reduces in the case when $w(P) \in \{0,1\}$ for all points P to a result due to Bichara and Korchmaros [3].

Lemma 5.1 *Consider a weight function* w *from the points of* $PG(2,q)$ *to* $\mathbb{Z}$ *with* $\sum_{P \in PG(2,q)} w(P) = q + 2$*.*

A point P of weight 1 is called internal nucleus if for each line l through P we find $\sum_{Q\in l} w(Q) = 2$.

If we find three internal nuclei P_1, P_2, P_3 with $w(Q) = 0$ for all $Q \in P_iP_j \backslash \{P_iP_j\}$ then q is even.

Proof. Define $P_1 = (1, 0, 0)$, $P_2 = (0, 1, 0)$, $P_3 = (0, 0, 1)$. Consider an other point P of weight $\neq 0$. By Ceva's theorem the lines PP_i intersect in $(0, \lambda_1^P, 1)$, $(1, 0, \lambda_2^P)$, $(\lambda_3^P, 1, 0)$ with $\lambda_1^P\lambda_2^P\lambda_3^P = 1$. This implies that

$$\prod_{P\in PG(2,q)\backslash\{P_1,P_2,P_3\}} (\lambda_1^P\lambda_2^P\lambda_3^P)^{w(P)} = 1\ . \tag{7}$$

As P_i is a internal nucleus, then

$$\prod_{P\in PG(2,q)\backslash\{P_1,P_2,P_3\}} \lambda_i^{P\,w(P)} = \prod_{q\in\mathbb{F}_q^*} q$$

for $i = 1, 2, 3$. Thus (7) is also equal to

$$\prod_{q\in\mathbb{F}_q^*} q^3 = -1\ .$$

This shows that q is even. ■

6 Blocking sets of $W(3, q)$

In this section we study $W(3, q)$. We represent it as the set of absolute points and lines with respect to a symplectic polarity in $\mathrm{PG}(3, q)$. The absolute lines are also called *symplectic lines*. A *blocking set* of $W(3, q)$ is a set of points of $W(3, q)$ that meets every symplectic lines. Clearly, a blocking set has at least $q^2 + 1$ points with equality iff it is an ovoid, that is, if it meets every symplectic line is a unique point. If q is odd, then $W(3, q)$ does not have an ovoid, in fact, it is known that a blocking set of $W(3, q)$, q odd, has at least $q^2 + 1 + \frac{1}{3}(q - 1)$ points, see [6]. We shall improve this result in this section. Notice that $W(3, q)$ has a blocking set with $q^2 + q$ points, since the points in the tangent-hyperplane $P^\perp$ of a point P provide such a blocking set. If one replaces in this blocking set the $q + 1$ points on a line l with $P \notin l \subseteq P^\perp$

by the q points $\neq P$ of $l^{\perp}$, a blocking set of size q^2+q-1 is obtained. This example appears first in [8] and is the smallest known blocking set for $W(3,q)$.

From now on suppose that B is a blocking set of $W(3,q)$ such that

$$|B| = q^2 + 1 + \delta \quad \text{and} \quad \delta \leq \frac{4q}{5}.$$

First we use the representation of $W(3,q)$ as the dual of $Q(4,q)$. From Corollary 4.3, we know that for a cover of $Q(4,q)$ with $q^2+1+\delta$ lines, the multiple covered points can be represented as the sum of δ lines. Translating this to $W(3,q)$ proves the following.

Lemma 6.1 *There exist δ points $N_1,\dots,N_\delta$ (a point can appear more than once) outside B with the following property. If l is a symplectic line, then $|l \cap B| - 1$ is the number of indices i with $N_i \in l$.*

From now on we also suppose that the blocking set B is minimal, that is no proper subset of B is a blocking set of $W(3,q)$. Then the points $N_1,\dots,N_\delta$ do not belong to B. Define a function w from the point set to $\mathbb{Z}$ such that $w(P)=1$ for $P \in B$, and otherwise $w(P)$ is the number of indices i with $P = N_i$. The lemma implies states that $\sum_{p\in l} w(P) = 1$ for every symplectic line l. Thus, we may view the function w as a generalized ovoid of $W(3,q)$. By construction, the sum of $w(P)$ for all points P with $w(P) < 0$ is $-\delta$. Thus Theorem 1.3 follows from the following more general statement.

Theorem 6.2 *Suppose that w is a function from the point set of $W(3,q)$ to $\mathbb{Z}$. Suppose that $w(P) \leq 1$ for every point and $\sum_{P\in l} w(P) = 1$ for every symplectic line of $W(3,q)$. Then*

$$\sum_{P, w(P)<0} -w(p) \geq -\frac{5}{2} - q + \frac{\sqrt{25+20q+8q^2}}{2}.$$

In the rest of this section we prove Theorem 6.2. This will be done in three steps.

Lemma 6.3 *If l is a line, then*

$$\sum_{P\in l} w(P) + \sum_{Q\in l^\perp} w(Q) = 2 \tag{8}$$

Proof. Through a point P in $W(3,q)$ we find $q+1$ symplectic lines, which lie in $P^\perp$. Since the weight of each such line is 1, we have

$$\sum_{Q\in P^\perp} w(Q) = q+1-qw(P)\ .$$

For any line of l of $PG(3,q)$, we conclude that

$$\sum_{P\in l}\sum_{Q\in P^\perp} w(Q) = (q+1)^2 - q\sum_{P\in l} w(P)\ .$$

Since the planes $P^\perp$ with $P\in l$ cover $PG(3,q)$ and intersect in $l^\perp$, we also find

$$\sum_{P\in l}\sum_{Q\in P^\perp} w(Q) = \sum_{Q\in PG(3,q)} w(Q) + q\sum_{Q\in l^\perp} w(Q) = q^2+1+q\sum_{Q\in l^\perp} w(Q)\ .$$

Both equations together reveal the assertion for the line l. ■

Notation. For every line l (symplectic or not) we call $w(l) := \sum_{P\in l} w(P)$ the weight of l. For any point P, let $a_k(P)$ be the number of lines in the plane $P^\perp$ that have weight k. By $b_k(P)$ we denote the number of lines through P that have weight k. The above lemma implies that

$$a_k(P) = b_{2-k}(P)$$

for every point P and all $k\in\mathbb{Z}$.

Lemma 6.4 *Suppose that $w(P_0)=0$ and put*

$$n_0 := -\sum_{\substack{Q\in P^\perp:\\ w(Q)<0}} w(Q).$$

Then $\sum_{k>2} ka_k(P) \geq q-3n_0$.

Proof. This proof works only in the plane $P_0^\perp$. We know that $q+1 = \sum_{P\in\pi} w(P)$, so the number of points of weight one in π is $q+1+n_0$. The assertion is that at least $q-3n_0$ of these lie on a line of weight at least three. Assume this is not true, that is $P_0^\perp$ contains at least $4n_0+2$ points P of weight one such that every line of $P_0^\perp$ on P has weight at most two. For such a point P, the line PP_0 has weight one, and every other line of $P_0^\perp$ on P has weight exactly two. Define

$$\hat{w}(P) = \begin{cases} 1 & \text{for } P = P_0 \\ w(P) & \text{for } P \in P_0^\perp \backslash \{P_0\} \end{cases},$$

Then $\hat{w}$ satisfies the requirements of Lemma 5.1. Also, P_0 is an internal nucleus. The hypothesis just made says that at least $4n_0+2$ other points of π are internal nuclei, so we have $4n_0+3$ internal nuclei. Consider one internal nucleus N_1. Consider a line g on N_1. If r is the sum of the numbers $-w(Q)$ for points $Q \in g$ with $w(Q) < 0$, then g contains $r+2$ points P with $w(P) = 1$, so apart from N_1 at most $r+1 \le 2r$ points P with $w(P) = 1$. Hence, the number of internal nuclei $N \neq N_1$ such that the line NN_1 contains a negative point is at most $2n_0$. Let N_2 be a nucleus such that the line N_1N_2 does not contain negative points. As before, at most $2n_0$ nuclei $N \neq N_2$ are joined to N_2 by a line with negative points. As there are $4n_0+3$ nuclei, we find a nucleus N_3 that is joined to N_1 and N_2 by a line without negative points. Lemma 5.1 gives a contradiction. ■

Lemma 6.5 *We have* $q^3 - 3\delta q \le 2(q^2+q)\delta + q\delta^2$.

Proof. Let δ be the number of negative points in $W(3,q)$ and for every point P with $w(P) = 1$ let $\delta(P)$ be the number of negative points in $P^\perp$, all counted with multiplicity. Thus

$$\delta := -\sum_{P,\ w(P)<0} w(P) \quad \text{and} \quad \sum_{P,\ w(P)\ge 0} \delta(P) \le \delta q^2.$$

The inequality follows from the fact that every negative point is perpendicular to q^2 points of $W(3,q)$. For the proof of the theorem, we may assume that $\delta \le \frac{q}{2}$. The hypotheses of the Theorem 6.2 imply that $\sum_P w(P) = q^2+1$. Hence, the number of points P with $w(P) = 1$ is $q^2+1+\delta$ and the number of points P with $w(P) = 0$ is at least $q^3 + q - 2\delta \geq q^3$. The preceding lemma shows

$$\sum_{w(P)=0} \sum_{k>2} k a_k(P) \geq \sum_{w(P)=0} (q - 3\delta(P)) \geq q^4 - 3 \sum_{w(P)=0} \delta(P) \geq q^4 - 3\delta q^2.$$

The last sign follows from the fact that every negative point is perpendicular to at most q^2 points of weight 0. Using $a_k(P) = b_{2-k}(P)$ we get

$$\sum_{P \in PG(3,q)} \sum_{k<0} (2-k) b_k \geq q^4 - 3\delta q^2$$

$$\Rightarrow \sum_{l,\ w(l)<0} (2 - w(l)) \geq q^3 - 3\delta q.$$

Here the second sum is over all symplectic and non-symplectic lines l with $w(l) < 0$. Since every line with negative weight contains at least one point with negative weight, we find a point P satisfying $w(P) < 0$ and

$$\sum_{k<0} (2-k) b_k(P) \geq \frac{-w(P)}{\delta}(q^3 - 3\delta q),$$

or in other words

$$\sum_{k>2} k a_k(P) \geq \frac{-w(P)}{\delta}(q^3 - 3\delta q). \tag{9}$$

The sum of the weights of all points but P of the plane $P^\perp$ is $(1-w(P))(q+1)$. Since the number of negative points in $P^\perp$ is at most δ we find that

$$\sum_{\substack{Q \in P^\perp \\ w(Q)>0}} 1 \leq (1 - w(P))(q+1) + \delta.$$

Every point of weight 1 lies on exactly q lines of $P^\perp \backslash \{P\}$. On the other hand each line of weight $k > 0$ in $P^\perp \backslash \{P\}$ contains at least k points of weight 1. Thus

$$\sum_{k>2} k a_k(P) \leq \sum_{k>0} k a_k(P) \leq q \sum_{\substack{Q \in P^\perp \\ w(Q)>0}} 1 \leq (1 - w(P))(q^2 + q) + q\delta. \tag{10}$$

Now we put (9) and (10) together and obtain

$$\frac{-w(P)}{\delta}(q^3 - 3\delta q) \leq (1 - w(P))(q^2 + q) + q\delta.$$

As $2\delta \leq q$, then $q^3 - 3\delta q \geq (q^2 + q)\delta$. Since $-w(P) \geq 1$, this gives the inequality in the statement. ■

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