

Generalised dual arcs and Veronesean surfaces, with applications to cryptography

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Abstract

We start by defining generalised dual arcs, the motivation for defining them comes from cryptography, since they can serve as a tool to construct authentication codes and secret sharing schemes. We extend the characterisation of the tangent planes of the Veronesean surface V_2^4 in $PG(5, q)$, q odd, described in [4], as a set of $q^2 + q + 1$ planes in $PG(5, q)$, such that every two intersect in a point and every three are skew. We show that a set of $q^2 + q$ planes generating $PG(5, q)$ and satisfying the above properties can be extended to a set of $q^2 + q + 1$ planes still satisfying all conditions. This result is a natural generalisation of the fact that a q -arc in $PG(2, q)$, q odd, can always be extended to a $(q + 1)$ -arc. This extension result is then used to study a regular generalised dual arc with parameters $(9, 5, 2, 0)$ in $PG(9, q)$, q odd, where we obtain an algebraic characterization of such object as being the image of a *cubic* Veronesean.

1 Introduction

The quadratic Veronesean V_2^4 is one of the most important substructures in $PG(5, q)$. It is the image of the plane $PG(2, q)$ under the mapping

$$\eta : PG(2, q) \rightarrow PG(5, q) : (x_0, x_1, x_2) \mapsto (x_0^2, x_1^2, x_2^2, x_0x_1, x_0x_2, x_1x_2).$$

This quadratic Veronesean V_2^4 has been studied in great detail, and characterized in different ways.

A particular characterization uses the tangent planes to the Veronesean V_2^4 in $PG(5, q)$, q odd.

Theorem 1

(Tallini [7]) *Let \mathcal{F} be a set of $q^2 + q + 1$ planes in $PG(5, q)$, q odd, satisfying the following properties:*

1. the elements of \mathcal{F} generate $PG(5, q)$,
2. two distinct elements of \mathcal{F} intersect in a point,
3. three distinct elements of \mathcal{F} have an empty intersection.

Then \mathcal{F} consists of the set of $q^2 + q + 1$ tangent planes to a Veronesean surface V_2^4 .

We extend this result by proving that a set of $q^2 + q$ planes of $PG(5, q)$, $q > 3$ odd, spanning $PG(5, q)$ and satisfying the properties above, is equal to a set of $q^2 + q$ tangent planes of a quadratic Veronesean V_2^4 .

Our motivation for solving this problem is to characterize regular $(9, 5, 2, 0)$ -dimensional dual arcs, for q odd, $q > 3$, containing $q^2 + q + 1$ 5-spaces. This is a set of $q^2 + q + 1$ distinct 5-spaces of $PG(9, q)$, generating $PG(9, q)$, two distinct 5-spaces intersecting in a plane, three distinct 5-spaces intersecting in a point, and such that every four distinct 5-spaces have an empty intersection.

It follows from the preceding definition that in every 5-space, the intersections with the other 5-spaces form $q^2 + q$ planes satisfying the properties above, thus, by the extension result presented in Theorem 15, they are tangent planes to a Veronesean variety V_2^4 in this 5-space.

This information on the planes forming the intersections of a given 5-space with the other $q^2 + q$ distinct 5-spaces enables us to characterize the regular $(9, 5, 2, 0)$ -dimensional dual arcs, for q odd, $q > 3$, having $q^2 + q + 1$ distinct 5-spaces in an unique way.

Our characterization result also will imply that any regular $(9, 5, 2, 0)$ -dimensional dual arc, for q odd, $q > 3$, contains at most $q^2 + q + 1$ 5-spaces.

Our motivation for characterizing these $(9, 5, 2, 0)$ -dimensional dual arcs follows from the fact that they can be used to define message authentication codes (MAC). We present this link in the final section of the paper.

2 Generalised dual arcs

Definition 2

A generalised dual arc \mathcal{D} of order l with dimensions $d_1 > d_2 > \dots > d_{l+1}$ of $PG(n, q)$ is a set of subspaces of dimension d_1 such that:

1. each j of these subspaces intersect in a subspace of dimension d_j , $1 \leq j \leq l + 1$,
2. each $l + 2$ of these subspaces have no common intersection.

We call (n, d_1, \dots, d_{l+1}) the parameters of the dual arc.

Example 1

- Take a dual arc in a plane π . Embed π in a 3-dimensional space. Now we have a generalised dual arc with parameters $(3, 1, 0)$. But the 3-space is not really used.
- Take a dual arc with k elements in a plane π . Embed π in a space of dimension $k + 2$ and choose planes through the k lines of the dual arc that span $PG(2 + k, q)$. This is a generalised dual arc with parameters $(k + 2, 2, 0)$. Even if the planes span $PG(2 + k, q)$, the interesting part of the construction is contained only in the plane π .
- The following planes of $PG(4, q)$ form a generalised dual arc with parameters $(4, 2, 0)$:

$$\begin{aligned}\pi_1 &= \{[a, b, c, 0, 0] \mid a, b, c \in \mathbb{F}_q\}, \\ \pi_2 &= \{[a, 0, b, b, c] \mid a, b, c \in \mathbb{F}_q\}, \\ \pi_3 &= \{[0, a, b, c, b] \mid a, b, c \in \mathbb{F}_q\}, \\ \pi_4 &= \{[a, a, 0, b, c] \mid a, b, c \in \mathbb{F}_q\}.\end{aligned}$$

The intersection points of π_1 with the other planes lie on the line $X_2 = X_3 = X_4 = 0$. So only that line of π_1 is a real part of the generalised dual arc.

These examples motivate the notion of a regular generalised dual arc.

Definition 3

A generalised dual arc \mathcal{D} of order l with parameters $(n = d_0, \dots, d_{l+1})$ is regular if, in addition, the d_1 -dimensional spaces span $PG(n, q)$ and it satisfies the property that if π is the intersection of j elements of \mathcal{D} , $j \leq l$, then π is spanned by the subspaces of dimension d_{j+1} which are the intersections of π with the remaining elements of \mathcal{D} .

A normal d -dimensional dual arc in $PG(n, q)$ has parameters $(n, d, 0)$. A generalised dual arc of order 0 is a partial d_1 -spread.

In particular, a regular generalised dual arc with parameters $(9, 5, 2, 0)$ is a set of 5-spaces in $PG(9, q)$, generating $PG(9, q)$, such that each two intersect in a plane, each three in a point, and each four are skew. This is the particular regular generalised dual arc we will characterize later on for q odd, $q > 3$.

Construction 1

Let $PG(V)$ be a d -dimensional space and let e_i ($0 \leq i \leq d$) be a basis of V .

Let $PG(W)$ be a $\left(\binom{d+l+1}{l+1} - 1\right)$ -dimensional space and let e_{i_0, \dots, i_l} ($0 \leq i_0 \leq i_1 \leq \dots \leq i_l \leq d$) be a basis of W .

Below, we define a map which is a generalisation of the well-known quadratic Veronesean map (see [4]).

We define $\zeta : PG(V) \rightarrow PG(W)$ by

$$\zeta : \left[\sum_{i=0}^d x_i e_i \right] \mapsto \left[\sum_{0 \leq i_0 \leq \dots \leq i_l \leq d} x_{i_0} \cdots x_{i_l} e_{i_0, \dots, i_l} \right].$$

With b and B respectively, we denote the standard scalar product of V and W , i.e.,

$$b\left(\sum_{i=0}^d x_i e_i, \sum_{i=0}^d y_i e_i\right) = \sum_{i=0}^d x_i y_i,$$

and

$$B\left(\sum_{0 \leq i_0 \leq \dots \leq i_l \leq d} x_{i_0, \dots, i_l} e_{i_0, \dots, i_l}, \sum_{0 \leq i_0 \leq \dots \leq i_l \leq d} y_{i_0, \dots, i_l} e_{i_0, \dots, i_l}\right) = \sum_{0 \leq i_0 \leq \dots \leq i_l \leq d} x_{i_0, \dots, i_l} y_{i_0, \dots, i_l}.$$

For each $x \in V$, we denote by x^\perp the subspace of V perpendicular to x with respect to b . So

$$x^\perp = \{y \in V \mid b(x, y) = 0\}.$$

For each point $P = [x]$ of $PG(V)$, we define a subspace $D(P)$ of $PG(W)$ by

$$D(P) = \{[z] \in W \mid B(z, \zeta(y)) = 0 \text{ for all } y \in x^\perp\}. \quad (1)$$

Before we prove that this construction indeed gives a generalised dual arc, we show two examples.

Example 2

Starting with $PG(2, q)$, the mapping $\zeta : PG(2, q) \rightarrow PG(5, q)$ with

$$\zeta([x_0, x_1, x_2]) = [x_0^2, x_1^2, x_2^2, x_0 x_1, x_0 x_2, x_1 x_2]$$

defines the quadratic Veronesean V_2^4 .

If $P = [a, b, c]$, the planes $D(P)$ defined above, have the equation

$$D(P) = \{[ax_0, bx_1, cx_2, ax_1 + bx_0, ax_2 + cx_0, bx_2 + cx_1] \mid x_0, x_1, x_2 \in \mathbb{F}_q\}.$$

These planes form a generalised dual arc of $q^2 + q + 1$ planes with parameters $(5, 2, 0)$.

Example 3

The map $\zeta : PG(2, q) \rightarrow PG(9, q)$ with

$$\zeta([x_0, x_1, x_2]) = [x_0^3, x_1^3, x_2^3, x_0^2x_1, x_0^2x_2, x_1^2x_0, x_1^2x_2, x_2^2x_0, x_2^2x_1, x_0x_1x_2]$$

defines a configuration of $q^2 + q + 1$ 5-dimensional spaces in $PG(9, q)$. Each two of these 5-spaces intersect in a plane. Each three 5-spaces share a common point and each four 5-spaces are skew.

Three of the $q^2 + q + 1$ 5-spaces are:

$$\begin{aligned}\pi_0 &:= D([1, 0, 0]) = \{[e_0, 0, 0, e_1, e_2, e_3, 0, e_4, 0, e_5] \mid e_i \in \mathbb{F}_q\}, \\ \pi_1 &:= D([0, 1, 0]) = \{[0, e_0, 0, e_1, 0, e_2, e_3, 0, e_4, e_5] \mid e_i \in \mathbb{F}_q\}, \\ \pi_2 &:= D([0, 0, 1]) = \{[0, 0, e_0, 0, e_1, 0, e_2, e_3, e_4, e_5] \mid e_i \in \mathbb{F}_q\}.\end{aligned}$$

In each 5-space, the other $q^2 + q$ 5-spaces intersect in a configuration of $q^2 + q$ planes. These planes are a part of the Veronesean described in Example 2.

For π_0 , the corresponding Veronesean has the equation

$$\mathcal{V}_0 := [x_0^2, 0, 0, x_0x_1, x_0x_2, x_1^2, 0, x_2^2, 0, x_1x_2].$$

This Veronesean \mathcal{V}_0 has $q^2 + q + 1$ tangent planes; where $q^2 + q$ of the tangent planes are intersections of π_0 with the other 5-spaces. The extra plane has the equation

$$E_0 := \{[a_0, 0, 0, a_1, a_2, 0, 0, 0, 0, 0] \mid a_0, a_1, a_2 \in \mathbb{F}_q\}.$$

Similarly, we see in π_1 the Veronesean

$$\mathcal{V}_1 := [0, x_1^2, 0, x_0^2, 0, x_0x_1, x_1x_2, 0, x_2^2, x_0x_2]$$

and the extra plane

$$E_1 := \{[0, a_0, 0, 0, 0, a_1, a_2, 0, 0, 0] \mid a_0, a_1, a_2 \in \mathbb{F}_q\},$$

and in π_2 , we have the Veronesean

$$\mathcal{V}_2 := [0, 0, x_2^2, 0, x_0^2, 0, x_1^2, x_0x_2, x_1x_2, x_0x_1]$$

and the extra plane

$$E_2 := \{[0, 0, a_0, 0, 0, 0, 0, a_1, a_2, 0] \mid a_0, a_1, a_2 \in \mathbb{F}_q\}.$$

Let q be odd. Assume that the generalised dual arc of $q^2 + q + 1$ 5-spaces can be extended to a generalised dual arc of size $q^2 + q + 2$. The additional 5-space must intersect each of the three 5-spaces π_0, π_1, π_2 in the extra planes

E_0, E_1, E_2 . But E_0, E_1, E_2 span an 8-space. Thus our example with $q^2 + q + 1$ 5-spaces is a maximal dual arc.

For q even, the situation is a bit more complicated. Now the dual arc of Example 2 can be extended to a dual arc of size $q^2 + q + 2$, see [4, Theorem 25.1.17]. Thus π_0, π_1, π_2 contain two extra planes. But we can check that no three extra planes lie in a common 5-space. Thus the example with $q^2 + q + 1$ 5-spaces is a maximal generalised dual arc.

Theorem 4

The set $\mathcal{D} = \{D(P) \mid P \in PG(V)\}$ is a regular generalised dual arc with dimensions $d_i = \binom{d+l+1-i}{l+1-i} - 1, i = 0, \dots, l+1$.

Proof. STEP 1: The dimension of $D(P)$.

Let $P = [x]$, then x^\perp is a d -dimensional subspace of V . By ζ , this d -dimensional subspace is mapped to a $\binom{d+l}{l+1}$ -dimensional subspace W' of W . (Here we need $q > l$ since otherwise the points of the generalised Veronesean do not span the whole space.)

Since the bilinear form B is non-degenerate, the space

$$\{z \in W \mid B(z, y) = 0 \text{ for } y \in W'\}$$

has dimension $\binom{d+l+1}{l+1} - \binom{d+l}{l+1} = \binom{d+l}{l}$.

Thus $D(P)$ has projective dimension $\binom{d+l}{l} - 1$.

STEP 2: The intersection of two spaces.

We now give an alternative description of $D(P)$.

For each permutation σ , let $e_{i_{\sigma(0)}, \dots, i_{\sigma(l)}}$ be equal to e_{i_0, \dots, i_l} .

Let $\theta : V^{l+1} \rightarrow W$ be the multilinear mapping

$$\theta : \left(\sum_{i=0}^d x_i^{(0)} e_i, \dots, \sum_{i=0}^d x_i^{(l)} e_i \right) \mapsto \sum_{0 \leq i_0, \dots, i_l \leq d} x_{i_0}^{(0)} \cdot \dots \cdot x_{i_l}^{(l)} e_{i_0, \dots, i_l}. \quad (2)$$

A simple check shows us that for $b(x, y) = 0$, we have

$$B(\theta(x, v_1, \dots, v_l), \zeta(y)) = 0$$

for all possible vectors v_1, \dots, v_l of V .

Thus for $P = [x]$, we have

$$\langle \theta(x, v_1, \dots, v_l) \mid v_1, \dots, v_l \in V \rangle \subseteq D(P).$$

Since the vector space $\langle \theta(x, v_1, \dots, v_l) \mid v_1, \dots, v_l \in V \rangle$ has dimension $\binom{d+l}{l}$ (choose v_1, \dots, v_l as basis vectors), we find

$$\langle \theta(x, v_1, \dots, v_l) \mid v_1, \dots, v_l \in V \rangle = D(P). \quad (3)$$

Since $PGL(V)$ acts doubly transitively on V , it is sufficient to check the dimension of $D(P) \cap D(P')$ for two fixed points. We choose $P = [1, 0, \dots, 0]$ and $P' = [0, 1, 0, 0, \dots, 0]$. From (3), we see directly that in this case

$$D(P) \cap D(P') = \{[e_{0,1,i_2,\dots,i_l}] \mid 0 \leq i_j \leq d\}.$$

Thus $D(P) \cap D(P')$ has projective dimension $\binom{d+l-1}{l-1} - 1$.

STEP 3: The intersection of more than two spaces.

If $P = [x]$ and $P' = [y]$, we see from (3) that

$$\langle \theta(x, y, v_2, \dots, v_l) \mid v_2, \dots, v_l \in V \rangle \subseteq D(P) \cap D(P').$$

Since we now know the dimension of the intersection $D(P) \cap D(P')$, we even see that

$$\langle \theta(x, y, v_2, \dots, v_l) \mid v_2, \dots, v_l \in V \rangle = D(P) \cap D(P').$$

Thus for a fixed point P , the set $\mathcal{D}' = \{D(P) \cap D(P') \mid P' \neq P\}$ is a set of subspaces of dimension $\binom{d+l-1}{l-1} - 1$ which is defined by a generalised dual arc of order $l - 1$. By induction, we know the dimension of the intersections.

This proves that the intersection of $D(P)$ with two or more other spaces has the correct dimension. \square

Remark: Even if Equation (1) is only defined for $q > l$, a generalised dual arc defined by Equation (3) works for any q and l .

3 An extension result for $q^2 + q$ planes satisfying the properties of the tangent planes of the Veronesean surface V_2^4 in $PG(5, q)$, q odd

In Example 3, we have constructed a regular generalised dual arc with parameters $(9, 5, 2, 0)$ consisting of $q^2 + q + 1$ distinct 5-spaces of $PG(9, q)$. In each of these 5-spaces, we have a set of $q^2 + q$ planes forming a regular generalised dual arc with parameters $(5, 2, 0)$. For q odd, $q > 3$, we will show that we can always find an extra plane in such a 5-space, in order to obtain a set of $q^2 + q + 1$ planes forming a regular generalised dual arc with parameters $(5, 2, 0)$. Such a set of $q^2 + q + 1$ planes in $PG(5, q)$, q odd, is the set of tangent planes to a Veronesean variety V_2^4 [4, Theorem 25.2.12].

Therefore, in $PG(5, q)$, consider a set \mathcal{F} of $q^2 + q$ planes generating $PG(5, q)$, such that each two of them intersect in a point, and each three of them are skew.

Let q be odd.

Each plane π of \mathcal{F} contains 2 points that are covered only once. We call these points *contact points* of this plane π of \mathcal{F} . The lemmas below are paraphrases of the lemmas of the characterisation of V_2^4 , q odd, which can be found in [4].

Lemma 5

Every hyperplane Π of $PG(5, q)$ contains at most $q + 1$ planes of \mathcal{F} .

Proof. A plane of \mathcal{F} not contained in Π intersects Π in a line ℓ . Each plane of \mathcal{F} contained in Π must share a point with that line ℓ . Furthermore, no two planes of \mathcal{F} in Π intersect ℓ in the same point, so Π contains at most $q + 1$ planes of \mathcal{F} . \square

Lemma 6

Every hyperplane Π of $PG(5, q)$ contains 0, 1, $q - 1$, q , or $q + 1$ planes of \mathcal{F} .

Proof. Let Π be a hyperplane that contains k planes of \mathcal{F} , where $2 \leq k < q - 1$.

Let π' be any plane of \mathcal{F} not contained in Π . This plane π' intersects Π in a line l' . At least $q - 1$ points of l' must be covered by a second plane of \mathcal{F} . Since $q + 1 - k - 2 > 0$, there must be a second plane π'' of \mathcal{F} not contained in Π which intersects l' . Let $\pi'' \cap \Pi = l''$.

The lines l' and l'' span a plane π . Since every one of the k planes of \mathcal{F} in Π must intersect π' and π'' , these k planes intersect π' and π'' in a point on l' , respectively on l'' , hence, they intersect π in lines.

Assume that π''' is another plane of \mathcal{F} not contained in Π that intersects Π in l''' . We prove that if l''' has a point in common with l' , then it has also a point in common with l'' .

Suppose that l''' intersects l' . If l''' does not intersect l'' , then every plane of \mathcal{F} contained in Π must share a line with the plane spanned by l' and l'' , and have a point in common with l''' . Thus these planes lie in the 3-dimensional space spanned by l' , l'' and l''' . Especially they must share a line, a contradiction.

This proves that the planes of \mathcal{F} not contained in Π can be partitioned into *groups*. The planes from one group intersect each other in Π and planes from different groups intersect each other outside Π . Each group defines a plane inside Π and the k planes of \mathcal{F} contained in Π must intersect such a plane in lines.

Let π_1 and π_2 be two planes inside Π defined by such groups.

If π_1 and π_2 intersect in a line, then at most one plane of \mathcal{F} contained in Π contains the line $\pi_1 \cap \pi_2$. So at least $k - 1$ planes of \mathcal{F} contained in Π must lie in the 3-dimensional space spanned by π_1 and π_2 . Thus each two of the planes must share a line, a contradiction for $k > 2$. We now eliminate the case $k = 2$, where one of the two planes of \mathcal{F} in Π , for instance π , passes through the line $\ell = \pi_1 \cap \pi_2$.

For $k = 2$, all groups have size at least $q - 2$. For, consider a first plane π' of \mathcal{F} not in Π , then consider the line $\ell' = \pi' \cap \Pi$. This line has at most two contact points, so it is intersected by at least $q - 3$ planes of \mathcal{F} , not lying in Π , in a point. This shows that a group of planes of \mathcal{F} , not lying in Π , has at least size $q - 2$.

But now consider the line $\ell = \pi_1 \cap \pi_2$, lying in the plane π of \mathcal{F} , also lying in Π , and in the two planes π_1 and π_2 containing at least $q - 2$ lines lying in planes of \mathcal{F} , not contained in Π . Since no point lies in three planes of \mathcal{F} , and every point of ℓ already lies in the plane π of \mathcal{F} , we must have $q + 1 \geq 2(q - 2) + 1$, where the $+1$ arises from the second plane of \mathcal{F} in Π . This implies $q \leq 3$.

Thus π_1 and π_2 intersect in a point Q . But then the only possibility for a plane of \mathcal{F} contained in Π to intersect π_1 and π_2 in lines is that Q is a point of that plane. Thus all planes of \mathcal{F} contained in Π contain Q . Since every three planes of \mathcal{F} are skew, this means that $k = 2$. Since there are $q^2 + q - 2$ planes of \mathcal{F} not contained in Π , and each group can contain at most $q - 1$ planes, there are at least $q + 2$ different groups.

Each group defines a plane through Q which intersects a plane of \mathcal{F} contained in Π in a line. Since a plane contains only $q + 1$ lines through Q , there must exist two groups which define planes π_1 and π_2 intersecting in a line. But this is impossible, as we already proved. \square

Lemma 7

Let Π be a hyperplane of $PG(5, q)$ that contains $q + 1$ planes π_0, \dots, π_q of \mathcal{F} . Then $q + 1$ of the $2q + 2$ contact points in Π form a conic.

Proof. First of all, Π contains exactly $2q + 2$ contact points. Namely, every plane π of \mathcal{F} not in Π intersects Π in a line l . Every plane π_0, \dots, π_q contains one point of l , so l has no contact points. So Π only contains the contact points of π_0, \dots, π_q . Let $Q_2 = \pi_0 \cap \pi_1$, $Q_1 = \pi_0 \cap \pi_2$ and $Q_0 = \pi_1 \cap \pi_2$.

The points Q_0, Q_1, Q_2 generate a plane, since otherwise, π_0, π_1, π_2 share a line. Assume that the line Q_1Q_2 contains no contact point of π_0 . Then Q_1Q_2 contains at least one point Q that lies on a plane π of $\mathcal{F} \setminus \{\pi_0, \dots, \pi_q\}$. More precisely, $\{Q\} = \pi_0 \cap \pi$. Let $l = \pi \cap \Pi$.

Suppose that l is not a line of the plane $Q_0Q_1Q_2$. Each of the planes π_1 and π_2 has one point in common with l . This point differs from Q , i.e. it does not lie in $Q_0Q_1Q_2$. This proves that π_1 and π_2 are contained in the solid spanned by $Q_0Q_1Q_2$ and l . It follows that π_1 and π_2 have a common line, a contradiction. So l is a line of $Q_0Q_1Q_2$.

Every plane π_1, \dots, π_q which has a point in common with Q_1Q_2 must also share a point with l , i.e. all these planes intersect $Q_0Q_1Q_2$ in a line. Together with the lines in $Q_0Q_1Q_2$ coming from planes in $\mathcal{F} \setminus \{\pi_0, \dots, \pi_q\}$, these lines form a dual $(q + 2)$ -arc. A contradiction to q odd, see [3].

Let Q' be the intersection point $\pi_3 \cap \pi_0$. The argument above proves that Q_1Q_2, Q_1Q' and Q_2Q' must contain a contact point of π_0 . Since π_0 has only two contact points, this proves that $Q' \in Q_1Q_2$.

The same argument proves that $\pi_3 \cap \pi_1 \in Q_0Q_2$. Thus each of the $q + 1$ planes π_0, \dots, π_q shares a line with $Q_0Q_1Q_2$. These lines form a dual $(q + 1)$ -arc. Each of the lines contains a contact point and these contact points form

a conic, since every dual $(q + 1)$ -arc in $PG(2, q)$, q odd, consists of the tangent lines to a conic in $PG(2, q)$, q odd, see [3]. \square

Lemma 8

Let Π be a hyperplane of $PG(5, q)$ that contains q planes π_0, \dots, π_{q-1} of \mathcal{F} . Then Π contains a plane $\bar{\Pi}$ which intersects the q planes of \mathcal{F} in Π in a line.

The plane $\bar{\Pi}$ contains at least q contact points which lie on a conic.

Furthermore, every plane of \mathcal{F} not contained in Π intersects $\bar{\Pi}$ in a line. This line contains a contact point and is either skew to $\bar{\Pi}$, or lies completely in $\bar{\Pi}$. The latter case can occur only once.

Proof. The same arguments as in the previous proof show that every plane of \mathcal{F} , not contained in Π but containing a point of Q_0Q_1 , intersects $\bar{\Pi} = Q_0Q_1Q_2$ in a line; equivalently, such a plane does not intersect π_0, \dots, π_{q-1} in a point of the plane $\bar{\Pi}$, or it intersects all planes π_0, \dots, π_{q-1} in a point of $\bar{\Pi}$.

We investigate three planes π_0, π_1 , and π_2 of \mathcal{F} contained in Π . Let $Q_2 = \pi_0 \cap \pi_1$, $Q_1 = \pi_0 \cap \pi_2$ and $Q_0 = \pi_1 \cap \pi_2$. As in the previous proof, we find that Q_1Q_2 contains a contact point of π_0 . The same arguments as in the previous proof show that every plane of \mathcal{F} contained in Π intersects $\bar{\Pi} = Q_0Q_1Q_2$ in a line.

The only difference is that this time we can not exclude the case that a plane π' of \mathcal{F} , not in Π , intersects $\bar{\Pi}$ in a line l' contained in $\bar{\Pi}$. Thus we see in $\bar{\Pi}$ either a dual q -arc or a dual $(q + 1)$ -arc of lines lying in planes of \mathcal{F} . But in any case, there are contact points that lie on a conic. \square

Definition 9

A Q-hyperplane of \mathcal{F} is a hyperplane Π of $PG(5, q)$ containing q planes of \mathcal{F} , such that the plane $\bar{\Pi}$ of Π intersecting the q planes of \mathcal{F} in Π in a line contains exactly q lines lying in a plane of \mathcal{F} .

Equivalently, the line in $\bar{\Pi}$ extending the dual q -arc consisting of the intersection lines of the planes of \mathcal{F} in Π with $\bar{\Pi}$ consists of q contact points lying in the planes of \mathcal{F} in Π .

Lemma 10

Let Π be a hyperplane of $PG(5, q)$ that contains $q - 1$ planes of \mathcal{F} . Then Π contains a plane $\bar{\Pi}$ which intersects the $q - 1$ planes of \mathcal{F} in Π in a line, and which contains a conic of contact points.

Proof. The arguments of the previous lemma are still valid. The only difference is that $\bar{\Pi}$ must contain one line l' that is induced from a plane of \mathcal{F} not in Π . Namely, if no such line exists, then the intersection lines of the planes of \mathcal{F} in Π with $\bar{\Pi}$ form a dual $(q - 1)$ -arc. But then every such line contains three contact points, which is impossible.

Furthermore, $\bar{\Pi}$ may contain at most two such lines. Thus we see in $\bar{\Pi}$ either a dual q -arc or a dual $(q + 1)$ -arc of intersection lines of planes of \mathcal{F} . \square

Lemma 11

Every plane of \mathcal{F} is contained in exactly one Q -hyperplane of $PG(5, q)$. Hence, there are $q + 1$ such Q -hyperplanes.

Proof. Let π be any plane of \mathcal{F} , then the other elements of \mathcal{F} lie in hyperplanes Π_1, \dots, Π_k through π which contain either $q - 1$, q or $q + 1$ planes of \mathcal{F} . To each hyperplane Π_i , there corresponds a plane $\bar{\Pi}_i$ which contains a conic of contact points and all intersections $\pi \cap \pi'$, where π' is a plane of \mathcal{F} contained in Π_i .

The planes $\bar{\Pi}_i$ intersect π in lines l_i . These lines always go through a contact point and cover π . Thus there exists a unique hyperplane $\bar{\Pi}$ through π for which $\bar{\Pi}$ contains both contact points. The hyperplane $\bar{\Pi}$ then must contain $q - 1$ elements of $\mathcal{F} \setminus \{\pi\}$ which intersect π in the $q - 1$ non-contact points of $\bar{\Pi} \cap \pi$.

By Lemma 8, since $\bar{\Pi}$ contains q planes of \mathcal{F} , and since $\bar{\Pi} \cap \pi$ contains two contact points, it is impossible that some plane of \mathcal{F} not contained in $\bar{\Pi}$ intersects $\bar{\Pi}$ in a line, so, again by Lemma 8, every plane of \mathcal{F} not contained in $\bar{\Pi}$ is skew to $\bar{\Pi}$. The intersections of the planes of \mathcal{F} contained in $\bar{\Pi}$ with $\bar{\Pi}$ form a dual q -arc. We know by a famous theorem of Segre, see [6], that a q -arc in $PG(2, q)$, q odd, can always be extended to a $(q + 1)$ -arc. This proves that the $2q$ contact points contained in $\bar{\Pi}$ lie on a conic and a line.

We have shown that $\bar{\Pi}$ is a Q -hyperplane. This Q -hyperplane $\bar{\Pi}$ through π must be unique since the corresponding plane $\bar{\Pi}$ must contain the line of π through the two contact points in π . Hence, by Lemma 8, this Q -hyperplane contains π and the $q - 1$ planes of \mathcal{F} intersecting π in a point of the line of π through the two contact points of π .

We have proved that every plane of \mathcal{F} is contained in exactly one Q -hyperplane. Thus there are $(q^2 + q)/q = q + 1$ Q -hyperplanes. \square

From here on, assume $q > 3$.

Lemma 12

Let Π_1 and Π_2 be Q -hyperplanes. Let $\bar{\Pi}_1$ and $\bar{\Pi}_2$ be the planes in Π_1 and Π_2 intersected by the planes of \mathcal{F} in Π_1 and Π_2 in lines, and containing the "conic" of contact points. Let \tilde{l}_1 be the line extending the dual q -arc in $\bar{\Pi}_1$ consisting of the intersection lines of the planes of \mathcal{F} in Π_1 with $\bar{\Pi}_1$. Then \tilde{l}_1 consists of q tangent points and \tilde{l}_1 is completely contained in Π_2 .

Proof. First of all, it is impossible that the plane $\bar{\Pi}_1$ is contained in Π_2 . For assume the contrary, we obtain a contradiction in the following way. Every plane π of \mathcal{F} in Π_1 intersects Π_2 in a line. If $\bar{\Pi}_1$ lies completely in Π_2 , then the intersection line $\ell = \Pi_2 \cap \pi$ equals the line $\bar{\Pi}_1 \cap \pi$. This line contains at least $q - 1$ points lying in two planes of \mathcal{F} in Π_1 . But the q planes of \mathcal{F} in Π_2 must intersect π in a point. So at least q points of ℓ lie still in a plane of \mathcal{F} in Π_2 . Then there are points of ℓ lying in three planes of \mathcal{F} . This is false.

So $\bar{\Pi}_1$ intersects Π_2 in a line.

Consider again the intersection line $\ell = \Pi_2 \cap \pi$ of a plane π of \mathcal{F} in Π_1 with Π_2 . This line contains q points lying on a plane of \mathcal{F} in Π_2 . So these points do not lie in another plane of \mathcal{F} in Π_1 . The remaining point on ℓ is a contact point by Lemma 8.

Now ℓ and $\pi \cap \bar{\Pi}_1$ intersect in a point. This point must be a contact point, for else, it lies in a second plane of \mathcal{F} in Π_1 , but this was excluded in the preceding paragraph.

So π shares a contact point with Π_2 , which also lies on the intersection line of Π_2 with $\bar{\Pi}_1$.

The preceding arguments show that the plane $\bar{\Pi}_1$ contains a line having at least q contact points.

But then this line is the line \tilde{l}_1 in $\bar{\Pi}_1$ extending the dual q -arc consisting of the intersection lines of the planes of \mathcal{F} in Π_1 with $\bar{\Pi}_1$.

The preceding arguments already show that this line \tilde{l}_1 lies in Π_2 . \square

Corollary 13

Let Π_1, \dots, Π_{q+1} be the Q -hyperplanes and $\bar{\Pi}_1, \dots, \bar{\Pi}_{q+1}$ the planes containing the lines M_1, \dots, M_{q+1} containing q contact points. Then the lines $M_1, \dots, M_{q+1} \subset \Pi_1 \cap \dots \cap \Pi_{q+1}$.

Lemma 14

No three Q -hyperplanes Π_1, Π_2, Π_3 intersect in a 3-space.

Proof. Suppose that three Q -hyperplanes intersect in a 3-space Δ . Let $\pi \in \mathcal{F}$, $\pi \in \Pi_2$. Let $\pi \cap \Delta$ be a line ℓ . Then the q planes of $\mathcal{F} \cap \Pi_1$ intersect π in a point; this point lies in $\Delta = \Pi_1 \cap \Pi_2$, so this point lies on ℓ . Similarly, the q planes of $\mathcal{F} \cap \Pi_3$ intersect π in a point, this point lies again on ℓ . Then there are points on ℓ lying on 3 planes of \mathcal{F} , a contradiction. \square

Conclusion: The lines M_1, \dots, M_{q+1} lie in the intersection of all $q + 1$ different Q -hyperplanes, and by the preceding lemma, they intersect in a plane Ω . So $M_1, \dots, M_{q+1} \subset \Omega$. So there is a plane Ω containing $q^2 + q$ contact points. Every plane of \mathcal{F} must share at least one point with Ω , since every plane of \mathcal{F} has one point of a line M_i . No plane of \mathcal{F} shares a line with Ω , else we obtain points of Ω on 2 planes of \mathcal{F} . This contradicts with the fact that they are contact points. This argument shows that Ω extends \mathcal{F} to a set of $q^2 + q + 1$ planes pairwise intersecting in a point, where still three planes have an empty intersection, so $\mathcal{F} \cup \{\Omega\}$ is the set of tangent planes to a Veronesean surface V_2^4 in $PG(5, q)$, q odd [7] and [4, Theorem 25.2.12.]. So we have proven the following extension result.

Theorem 15

A set of $q^2 + q$ planes \mathcal{F} in $PG(5, q)$, q odd, $q > 3$, such that

- (i) the planes generate $PG(5, q)$,
- (ii) every two planes intersect in a point,
- (iii) every three planes are skew,

can be extended to a set of $q^2 + q + 1$ planes in $PG(5, q)$ having the same properties.

Equivalently, such a set of $q^2 + q$ planes is a set of $q^2 + q$ tangent planes to a Veronesean variety V_2^4 of $PG(5, q)$, q odd, $q > 3$.

4 An algebraic characterisation of the regular generalised dual arc with parameters $(9, 5, 2, 0)$, q odd, $q > 3$

We are going to use the extension result of the previous section in order to study the regular generalised dual arc \mathcal{D} with parameters $(9, 5, 2, 0)$, q odd. So we have a set \mathcal{F} of $q^2 + q + 1$ distinct 5-spaces in $PG(9, q)$ that generate $PG(9, q)$. Furthermore, the 5-spaces intersect in planes and the planes coming from the intersections of a given 5-space Ω with the other 5-spaces of \mathcal{D} span Ω . We know from Theorem 15 that the $q^2 + q$ intersection planes in a 5-space Ω are tangent planes to a Veronesean variety V_2^4 in this 5-space Ω . This will play a crucial role in the characterisation result. Also the following observation is of great importance.

Remark. The Veronesean surface V_2^4 , q odd, can be determined uniquely in the following way, as indicated in the proof of Theorem 25.2.12 in [4].

Let Q_{00}, Q_{11} and Q_{22} be three distinct points of V_2^4 which are not contained in a plane of V_2^4 . This means that their corresponding tangent planes π_{00}, π_{11} and π_{22} are not lying in a common hyperplane with $q + 1$ tangent planes of V_2^4 . Let $Q_{ij} = Q_{ji} = \pi_{ii} \cap \pi_{jj}$, $i, j \in \{0, 1, 2\}$. Then the plane generated by Q_{ii}, Q_{jj} and Q_{ij} contains a conic $C_{ij} = C_{ji}$ of contact points.

Then for a point U of V_2^4 , $U \notin C_{01} \cup C_{02} \cup C_{12}$, select the coordinates such that $Q_{00} = e_0, Q_{11} = e_1, Q_{22} = e_2, Q_{01} = e_3, Q_{02} = e_4$ and $Q_{12} = e_5$, where e_i is the vector having coordinate 1 in position i and 0 in all other positions, and $U = (1, 1, \dots, 1)$. Then the unique Veronesean surface V_2^4 , passing through U , having Q_{00}, Q_{11} and Q_{22} as contact points and π_{00}, π_{11} and π_{22} as tangent planes in Q_{00}, Q_{11} and Q_{22} , is the Veronesean surface V_2^4 in standard form

$$(x_0^2, x_1^2, x_2^2, x_0x_1, x_0x_2, x_1x_2).$$

Lemma 16

For q odd and $q > 3$, let Ω and Ω' be two of the 5-dimensional spaces of \mathcal{D} . In each of the 5-dimensional spaces, we see a configuration of $q^2 + q$ planes, such

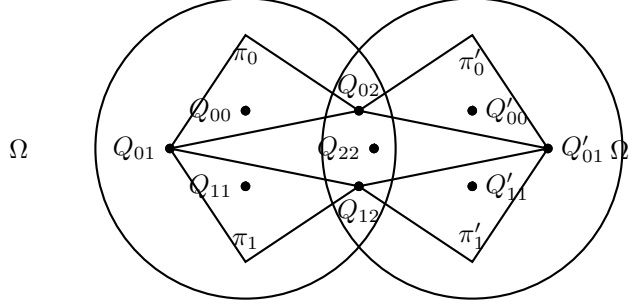


Figure 1: The 5-spaces Ω and Ω' with the extra planes π_0 and π'_0 .

that each two intersect in a point, and each three are skew. By Theorem 15, this configuration can be extended by a plane to a set of $q^2 + q + 1$ tangent planes to a Veronesean surface. Let π_0 and π'_0 denote these extension planes in Ω and Ω' , respectively, and let the respective sets of $q^2 + q + 1$ planes be the tangent planes to the Veronesean surfaces V_2^4 and $V_2^{4'}$ in Ω and Ω' respectively.

Then π_0 and π'_0 are skew.

Proof. Consider Ω and Ω' . The plane $\pi_2 = \Omega \cap \Omega'$ has 2 contact points. Assume that the extension planes in Ω and Ω' use the same contact point Q_{02} in $\Omega \cap \Omega'$.

Step 1: The coordinates in Ω .

Let Q_{22} be the other contact point in the plane $\pi_2 = \Omega \cap \Omega'$. In the extra plane π_0 , we have the contact point Q_{00} and $Q_{02} = \pi_0 \cap \pi_2$. In π_2 , select a point Q_{12} not on $Q_{02}Q_{22}$. This point lies in a plane π_1 of Ω ; this plane π_1 contains the contact point Q_{11} , and the point $Q_{01} = \pi_0 \cap \pi_1$. Take $U \in V_2^4$ in Ω , not in the planes $C_{01} = \langle Q_{01}, Q_{00}, Q_{11} \rangle$, $C_{02} = \langle Q_{02}, Q_{00}, Q_{22} \rangle$, $C_{12} = \langle Q_{12}, Q_{11}, Q_{22} \rangle$. Then choose the coordinates as indicated in the remark above, so that V_2^4 is equal to the Veronesean surface in standard form, i.e.:

$$(x_0^2, x_1^2, x_2^2, x_0x_1, x_0x_2, x_1x_2, 0, 0, 0, 0).$$

Step 2: The coordinates in Ω' .

Let the plane π'_1 be the second tangent plane of the Veronesean surface $V_2^{4'}$ in Ω' through Q_{12} . Then also Q'_{11} and Q'_{01} are uniquely determined as the contact point in π'_1 and the intersection point $\pi'_0 \cap \pi'_1$. Then since π'_0 is fixed, also the contact point Q'_{00} is fixed. So the six points $Q'_{00}, Q'_{11}, Q'_{22} = Q_{22}, Q'_{01}, Q'_{02} = Q_{02}, Q'_{12} = Q_{12}$ are fixed. It is possible to take in $V_2^{4'}$ in Ω' the point U' corresponding to the same point V in π_2 as U .

Indeed, if we have chosen U in Ω , then we find after projection a point V in π_2 . Let U vary over $V_2^4 \setminus (C_{01} \cup C_{02} \cup C_{12})$, so $q^2 + q + 1 - 3q = q^2 - 2q + 1$ choices for U . In π_2 , V cannot lie on the three lines defined by the points Q_{02}, Q_{12} and Q_{22} , so we also have $q^2 - 2q + 1$ choices for V . Furthermore, a direct calculation

shows that the point $U = (a^2, b^2, c^2, ab, ac, bc, 0, 0, 0, 0)$, with $a, b, c \neq 0$, projects on $V = (0, 0, c^2, 0, ac, bc, 0, 0, 0, 0) = (0, 0, c, 0, a, b, 0, 0, 0, 0)$. So different points U give different projections V .

We select U' to be the unit point in Ω' ; we have the Veronesean variety $V_2^{4'}$ in Ω' in standard form defined by $Q'_{02} = e_4, Q'_{12} = e_5, Q'_{22} = e_2, Q'_{00} = e_6, Q'_{11} = e_7, Q'_{01} = e_8$. Then $V_2^{4'}$ can be represented in coordinates in the following way

$$(0, 0, x_2'^2, 0, x_0'x_2', x_1'x_2', x_0'^2, x_1'^2, x_0'x_1', 0).$$

Step 3: Take a line l in π_0 through Q_{02} , but not through Q_{00} , and let P_1 be a point of l different from Q_{02} . There is a 5-space $\Omega_1 \in \mathcal{D}$, different from Ω , through P_1 since $P_1 \neq Q_{00}$. There cannot be two such 5-spaces since P_1 lies in the extra plane of Ω . Let $\Omega \cap \Omega_1 = \pi_{P_1}$. Then $\pi_{P_1} \cap \pi_2$ is a point $R_1 = \Omega \cap \Omega' \cap \Omega_1$. It must lie in a second plane π'_{P_1} of Ω' . Now the intersection of Ω_1 and Ω' is a plane of the induced Veronesean $\mathcal{V}_2^{4'}$ in Ω' through R_1 , so $\pi'_{P_1} = \Omega' \cap \Omega_1$. Set $P'_1 = \pi'_0 \cap \pi'_{P_1}$.

Step 4: The geometrical properties we know are: P_1 defines a second tangent plane π_{P_1} of V_2^4 in Ω . This second plane which lies in Ω_1 , intersects π_2 in a point R_1 . This point R_1 lies in a second tangent plane of $V_2^{4'}$ in Ω' , and this second plane is π'_{P_1} , and P'_1 is $\pi'_0 \cap \pi'_{P_1}$. This correspondence between the points P_1 and R_1 is bijective when P_1 varies over $\pi_0 \setminus \{Q_{00}, Q_{02}\}$. We have the same correspondence for the points P'_1 in π'_0 . It is the same function since V_2^4 and $V_2^{4'}$ are both in standard form. The line $P_1P'_1$ lies in Ω_1 . If P_1 has coordinates $(a, 0, 0, b, c, 0, 0, 0, 0, 0)$, then P'_1 has coordinates $(0, 0, 0, 0, c, 0, a, 0, b, 0)$. Hence, it is easy to see that all these lines have a point in common if we let P_1 vary over a fixed line through Q_{02} in π_0 . This yields a contradiction since every four 5-dimensional spaces of \mathcal{D} are skew. \square

Theorem 17

Every regular generalised dual arc \mathcal{D} with parameters $(9, 5, 2, 0)$ in $PG(9, q)$, q odd and $q > 3$, is isomorphic to the one given by Construction 1, discussed in detail in Example 3.

Proof.

Step 1: Selection of Ω_0, Ω_1 and Ω_2 .

Choose any two 5-spaces Ω_0 and Ω_1 of \mathcal{D} . They intersect in a plane π_{01} which contains two contact points.

Assume that the q^2 other 5-spaces Ω of \mathcal{D} that intersect π_{01} in a point not collinear with the two contact points of π_{01} are all contained in the 8-dimensional space spanned by Ω_0 and Ω_1 . Let Ω' be a 5-space of \mathcal{D} which generates together with Ω_0 and Ω_1 the whole space $PG(9, q)$. The other 5-spaces intersect Ω' in a plane and at least $q^2 + 2$ of the planes must lie in the 4-space $\Omega' \cap \langle \Omega_0, \Omega_1 \rangle$. But this contradicts Lemma 5.

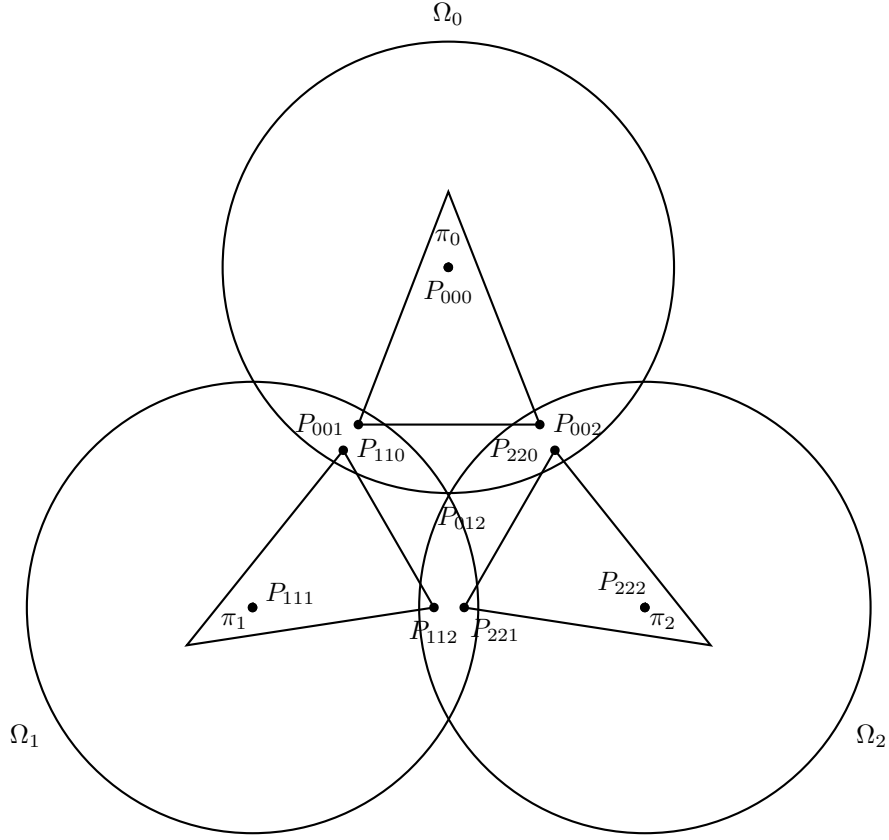


Figure 2: The three 5-spaces Ω_0 , Ω_1 , Ω_2 together with the extension planes π_0 , π_1 and π_2 .

Let π_i be the extra plane of Ω_i ($i = 0, 1, 2$) which exists by Theorem 15. Then π_i and the $q^2 + q$ intersection planes of Ω_i with the other elements in \mathcal{D} are the tangent planes to a Veronesean surface V_i in Ω_i . Denote by P_{iii} the contact point of π_i with respect to the Veronesean surface V_i in Ω_i . The plane π_i intersects the plane $\pi_{ij} = \Omega_i \cap \Omega_j$ in P_{ijj} . Note that by the proof of Lemma 16, P_{ijj} is the contact point of Π_{ij} to V_j in Ω_j ; alternatively, it is the intersection of Π_i and Π_{ij} ; so again by Lemma 16, π_i and π_j are skew, implying $P_{ijj} \neq P_{jji}$.

Thus there exists a 5-space Ω_2 which intersects π_{01} in a point not collinear with the contact points P_{001} and P_{110} , and such that $\Omega_0, \Omega_1, \Omega_2$ span $PG(9, q)$.

Let π_{02} be the intersection plane of Ω_0 and Ω_2 , and let π_{12} be the intersection plane of Ω_1 and Ω_2 . The intersection point of $\Omega_0, \Omega_1, \Omega_2$ is P_{012} . We determine the conic plane α in the 4-space spanned by π_{01} and π_{12} in Ω_1 . This is the plane that is generated by the two contact points P_{001} , P_{221} and the intersection point $P_{012} = \pi_{01} \cap \pi_{12}$. The arguments of Lemma 5 show that every tangent plane to the Veronesean surface V_1 in Ω_1 is either skew to α or intersects α in a line.

Since P_{012} , P_{001} and P_{110} are chosen to be non-collinear, π_1 cannot intersect the conic plane generated by the points P_{012} , P_{001} , and P_{221} in a line. Otherwise this line should intersect the line $P_{001}P_{012}$ in a point. The only possible intersection point is $\pi_1 \cap \pi_{01} = P_{110}$, but this is not collinear with P_{001} and P_{012} . Thus P_{012} , P_{221} and P_{112} are non-collinear. The same argument in Ω_2 shows that P_{012} , P_{002} and P_{220} are non-collinear.

Step 2: Construction of the coordinates.

Since we have chosen Ω_0 , Ω_1 and Ω_2 such that P_{012} , P_{001} and P_{110} are non-collinear, the planes π_{01} , π_{02} and π_0 must span Ω_0 by the structure of the Veronesean surface V_0 in Ω_0 . For, the only candidate for their conic plane when they would define a 4-space is the plane generated by the points P_{001} , P_{012} and P_{002} , which does not contain the contact point P_{110} of π_{01} with respect to V_0 . Furthermore, π_{01} is spanned by P_{012} , P_{001} , P_{110} , π_{02} is spanned by P_{012} , P_{002} , P_{220} , and π_0 is spanned by P_{001} , P_{002} , P_{000} , because the contact point P_{000} does not lie in the 4-space defined by Π_{01} and Π_{02} .

Thus Ω_0 is spanned by P_{000} , P_{001} , P_{002} , P_{110} , P_{220} and P_{012} ; the points with at least one index zero. Similarly, Ω_1 is spanned by P_{111} , P_{110} , P_{112} , P_{001} , P_{221} and P_{012} , and Ω_2 is spanned by P_{222} , P_{220} , P_{221} , P_{112} , P_{002} and P_{012} , which are the points with at least one index one or two, respectively. Thus the ten points P_{000} , P_{111} , P_{222} , P_{001} , P_{002} , P_{110} , P_{112} , P_{220} , P_{221} and P_{012} span $PG(9, q)$. Choose these points as the vectors e_0, \dots, e_9 in this order.

Choose a 5-space Ω of \mathcal{D} different from Ω_0 , Ω_1 and Ω_2 . We may choose Ω such that $\Omega \cap \Omega_0 \cap \Omega_1$ is a point that does not lie on the lines $P_{001}P_{012}$, $P_{110}P_{012}$ and $P_{001}P_{110}$.

Then Ω intersects Ω_0 , Ω_1 and Ω_2 in the planes $\bar{\pi}_0$, $\bar{\pi}_1$ and $\bar{\pi}_2$ respectively. Let $U_0 = (1, 0, 0, 1, 1, 1, 0, 1, 0, 1)$, $U_1 = (0, 1, 0, 1, 0, 1, 1, 0, 1, 1)$, $U_2 = (0, 0, 1, 0, 1, 0, 1, 1, 1, 1)$ be the contact points of the Veronesean surfaces V_0 , V_1 , V_2 in Ω_0 , Ω_1 , Ω_2 in the respective planes $\bar{\pi}_0$, $\bar{\pi}_1$, $\bar{\pi}_2$. This indeed is possible since $\bar{\pi}_i$ and $\bar{\pi}_j$ intersect π_{ij} in the same point, namely $\Omega_i \cap \Omega_j \cap \Omega$.

With these choices, the Veronesean surface in Ω_0 is in standard form and has the equation:

$$V_0 = (x_0^2, 0, 0, x_0x_1, x_0x_2, x_1^2, 0, x_2^2, 0, x_1x_2).$$

Similarly, the Veronesean surfaces in Ω_1 and Ω_2 have the equations

$$V_1 = (0, x_1^2, 0, x_0^2, 0, x_1x_0, x_1x_2, 0, x_2^2, x_0x_2)$$

$$V_2 = (0, 0, x_2^2, 0, x_0^2, 0, x_1^2, x_2x_0, x_2x_1, x_0x_1).$$

Step 3: Identification of the 5-spaces.

Now let Ω be a 5-space of \mathcal{D} different from Ω_0 , Ω_1 and Ω_2 . Then Ω intersects π_{01} in a point Q_{01} with coordinates $(0, 0, 0, a, 0, b, 0, 0, 0, c)$.

In Ω_0 , the point Q_{01} lies in the tangent plane $\tilde{\pi}_0$ of the Veronesean surface V_0 with equation

$$\tilde{\pi}_0 : (ax_0, 0, 0, ax_1 + bx_0, ax_2 + cx_0, bx_1, 0, cx_2, 0, bx_2 + cx_1).$$

By the same arguments, we find that the intersection plane $\tilde{\pi}_1$ of Ω with Ω_1 has the equation

$$\tilde{\pi}_1 : (0, bx_1, 0, ax_0, 0, ax_1 + bx_0, cx_1 + bx_2, 0, cx_2, cx_0 + ax_2).$$

Now $\tilde{\pi}_0$ is the intersection of Ω with Ω_0 , and Ω intersects π_{02} in the point $Q_{02} = \pi_{02} \cap \tilde{\pi}_0$ with coordinates $(0, 0, 0, 0, a, 0, 0, c, 0, b)$. Consequently, from the description of this point and V_2 , the intersection plane $\tilde{\pi}_2$ of Ω and Ω_2 has the equation

$$\tilde{\pi}_2 : (0, 0, cx_2, 0, ax_0, 0, bx_1, cx_0 + ax_2, cx_1 + bx_2, bx_0 + ax_1).$$

Then Ω intersects π_{12} in the point Q_{12} with coordinates $(0, 0, 0, 0, 0, 0, b, 0, c, a)$, and Ω also contains the points Q_0 , Q_1 and Q_2 with coordinates

$$\begin{aligned} Q_0 &: (a, 0, 0, b, c, 0, 0, 0, 0, 0) \in \tilde{\pi}_0, \\ Q_1 &: (0, b, 0, 0, 0, a, c, 0, 0, 0) \in \tilde{\pi}_1, \\ Q_2 &: (0, 0, c, 0, 0, 0, 0, a, b, 0) \in \tilde{\pi}_2. \end{aligned}$$

As we can see from the coordinates, the points Q_{01} , Q_{02} , Q_{12} , Q_0 , Q_1 , Q_2 are independent if at least two of the three values a , b and c are non-zero. But this is the case since Ω intersects π_{01} neither in P_{001} , P_{110} or P_{012} . Thus Ω is uniquely defined by the points Q_{01} , Q_{02} , Q_{12} , Q_0 , Q_1 , Q_2 .

Now we can check the definition of $D(P)$ in Equation (1) to see that the 5-space Ω is the space $D((a, b, c))$. Alternatively, it is possible to use the trilinear form θ from Equation (2) to check that Q_{01} , Q_{02} , Q_{12} , Q_0 , Q_1 , Q_2 are the points $\theta((a, b, c), e_i, e_j)$, (e_i, e_j are basis vectors).

This proves that Ω is of the form as defined in Construction 1, and discussed in Example 3. \square

We know that in every 5-space Π of the regular generalised dual arc \mathcal{D} in $PG(9, q)$, q odd, $q > 3$, with parameters $(9, 5, 2, 0)$, there is one plane extending the set of $q^2 + q$ intersection planes of Π with the other 5-spaces of the generalised dual arc to a set of tangent planes of a Veronesean variety V_2^4 in Π .

As indicated in Example 3, it might be possible that these $q^2 + q + 1$ extension planes in the $q^2 + q + 1$ distinct 5-spaces of the generalised dual arc define a Veronesean variety in a 5-space $\tilde{\Pi}$, extending the generalised dual arc of $q^2 + q + 1$ distinct 5-spaces to a generalised dual arc of $q^2 + q + 2$ distinct 5-spaces.

This however is impossible, as was shown in Example 3. So we have found the maximal size for a regular generalised dual arc in $PG(9, q)$, q odd, $q > 3$, with parameters $(9, 5, 2, 0)$.

Corollary 18

A regular generalised dual arc in $PG(9, q)$, q odd, $q > 3$, with parameters $(9, 5, 2, 0)$ contains at most $q^2 + q + 1$ elements.

Proof. Assume that the dual arc contains at least $q^2 + q + 1$ elements. By Theorem 17, these $q^2 + q + 1$ elements form a configuration isomorphic to the configuration of Example 3. But we have seen in Example 3 that this configuration cannot be extended. \square

5 Applications to cryptography

In this section, we describe an application of generalised dual arcs in cryptography.

Let us recall the definition of a message authentication code [5].

Definition 19

A message authentication code (MAC) is a 4-tuple $(\mathcal{S}, \mathcal{A}, \mathcal{K}, \mathcal{E})$ with

1. \mathcal{S} a finite set of source states (messages).
2. \mathcal{A} a finite set of authentication tags.
3. \mathcal{K} a finite set of keys.
4. For each $K \in \mathcal{K}$, we have an authentication rule $e_K \in \mathcal{E}$ with $e_K : \mathcal{S} \rightarrow \mathcal{A}$.

The security of a MAC is measured by the following probabilities.

Definition 20

Let p_i denote the probability of an attacker to construct a pair $(s, e_K(s))$ without knowledge of the key K , if he only knows i different pairs $(s_j, e_K(s_j))$. The smallest value r for which $p_{r+1} = 1$ is called the order of the scheme.

For $r = 1$, the probability p_0 is also known as the probability of an impersonation attack and the probability p_1 is called the probability of a substitution attack.

The next theorem bounds the number of keys by the attack probabilities. For $r = 1$ and $p_0 = p_1$, it is stated in [2], and for arbitrary r with $p_0 = p_1 = \dots = p_r$, it was proven in [1].

Theorem 21

If a MAC has attack probabilities $p_i = 1/n_i$ ($0 \leq i \leq r$), then $|\mathcal{K}| \geq n_0 \dots n_r$.

Proof. Suppose that we send the messages $(s_1, e_K(s_1)), \dots, (s_r, e_K(s_r))$. Let \mathcal{K}_i be the set of all keys which give the same authentication tag for the first i messages, i.e.

$$\mathcal{K}_i = \{\hat{K} \in \mathcal{K} \mid e_{\hat{K}}(s_j) = e_K(s_j) \text{ for } j \leq i\}.$$

By definition, we have $\mathcal{K}_0 = \mathcal{K}$. Formally, we define $\mathcal{K}_{r+1} = \{K\}$.

An attacker who knows the first i messages can create a false signature by guessing a key $\hat{K} \in \mathcal{K}_i$ and computing $e_{\hat{K}}(s_{i+1})$. The attack is successful if $\hat{K} \in \mathcal{K}_{i+1}$. Therefore

$$p_i \leq \frac{|\mathcal{K}_{i+1}|}{|\mathcal{K}_i|}.$$

Multiplying these inequalities proves the theorem. \square

A MAC that satisfies this theorem with equality is called *perfect*.

Theorem 22

Let $p_i = 1/n_i$, with $n_i \in \mathbb{N}$. If a MAC has $|\mathcal{K}| = n_0 \cdot \dots \cdot n_r$, then $|\mathcal{S}| \leq \frac{n_{r-1}n_r-1}{n_r-1} + r - 1$.

Proof. After $r - 1$ messages, the number of possible keys is reduced to $n_{r-1}n_r$. After $r - 1$ messages, we call the possible keys *points*. A set of points that produce the same authentication tag for an r -th message will be called a *block*.

Since the MAC is perfect, we know that two blocks have at most one common point, because otherwise the probability $p_r \geq 2/n_r$. The equation $p_r = 1/n_r$ says that each block contains at least n_r points, and $p_{r-1} = 1/n_{r-1}$ says that each block belongs to a parallel class of at least n_{r-1} blocks. It follows that every point lies on at most $(n_{r-1}n_r - 1)/(n_r - 1)$ blocks. This bounds the number of remaining messages, since every message defines a unique block. \square

Now we show how to use generalised dual arcs to construct perfect MACs.

Theorem 23

Let Π be a hyperplane of $PG(n + 1, q)$ and let \mathcal{D} be a generalised dual arc of order l in Π with parameters (n, d_1, \dots, d_{l+1}) .

The elements of \mathcal{D} are the messages and the points of $PG(n + 1, q)$ not in Π are the keys. The authentication tag that belongs to a message and a key is the generated $(d_1 + 1)$ -dimensional subspace.

This defines a perfect MAC of order $r = l + 1$ with attack probabilities

$$p_i = q^{d_{i+1} - d_i}.$$

Proof. After i message tag pairs $(m_1, t_1), \dots, (m_i, t_i)$ are sent, the attacker knows that the key must lie in the $(d_i + 1)$ -dimensional space $\pi = t_1 \cap \dots \cap t_i$. This space contains $q^{d_i + 1}$ different keys. A message m_{i+1} intersects $m_1 \cap \dots \cap m_i$ in a

d_{i+1} -dimensional space π' . Two keys K and \bar{K} generate the same authentication tag if and only if K and \bar{K} generate together with π' the same $(d_{i+1} + 1)$ -dimensional space. Thus the keys form groups of size $q^{d_{i+1}+1}$ and keys from the same group give the same authentication tag.

The attacker has to guess a group. The probability to guess the correct group is $p_i = q^{d_{i+1}+1}/q^{d_i+1}$. \square

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