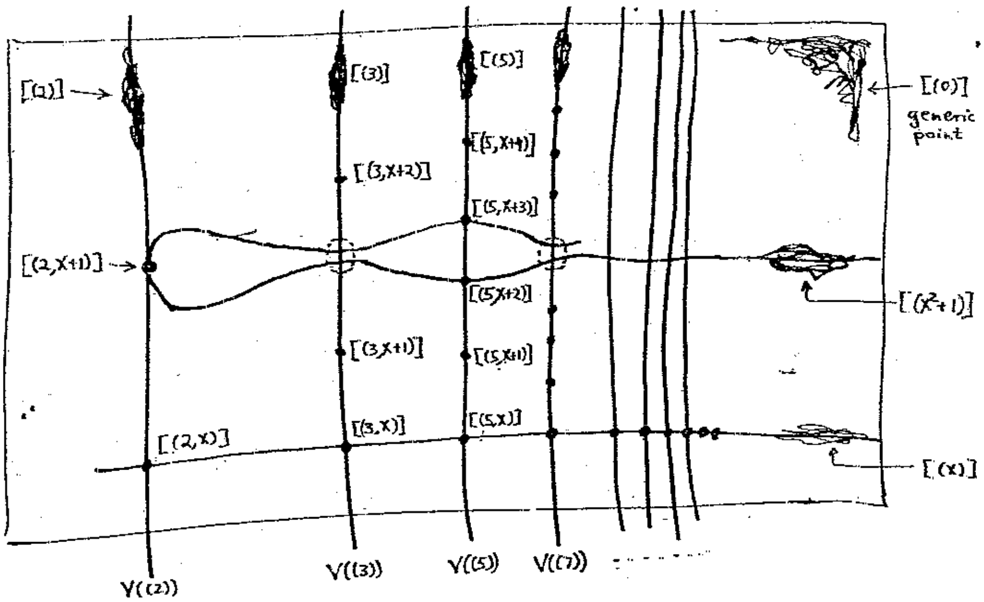
UN DESSIN~~S~~ D'ENFAN~~T~~CE



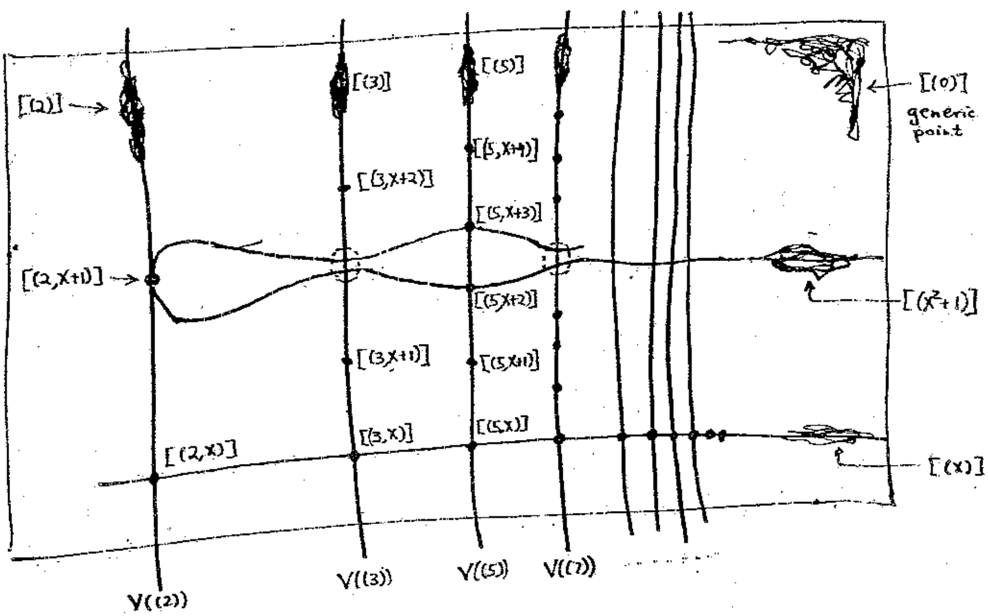
<u>Example H</u>: Spec($\mathbb{Z}[X]$). This is a so-called "arithmetic surface" and is the first example which has a real mixing of arithmetic and geometric properties. The prime ideals in $\mathbb{Z}[X]$ are:

i) (0),

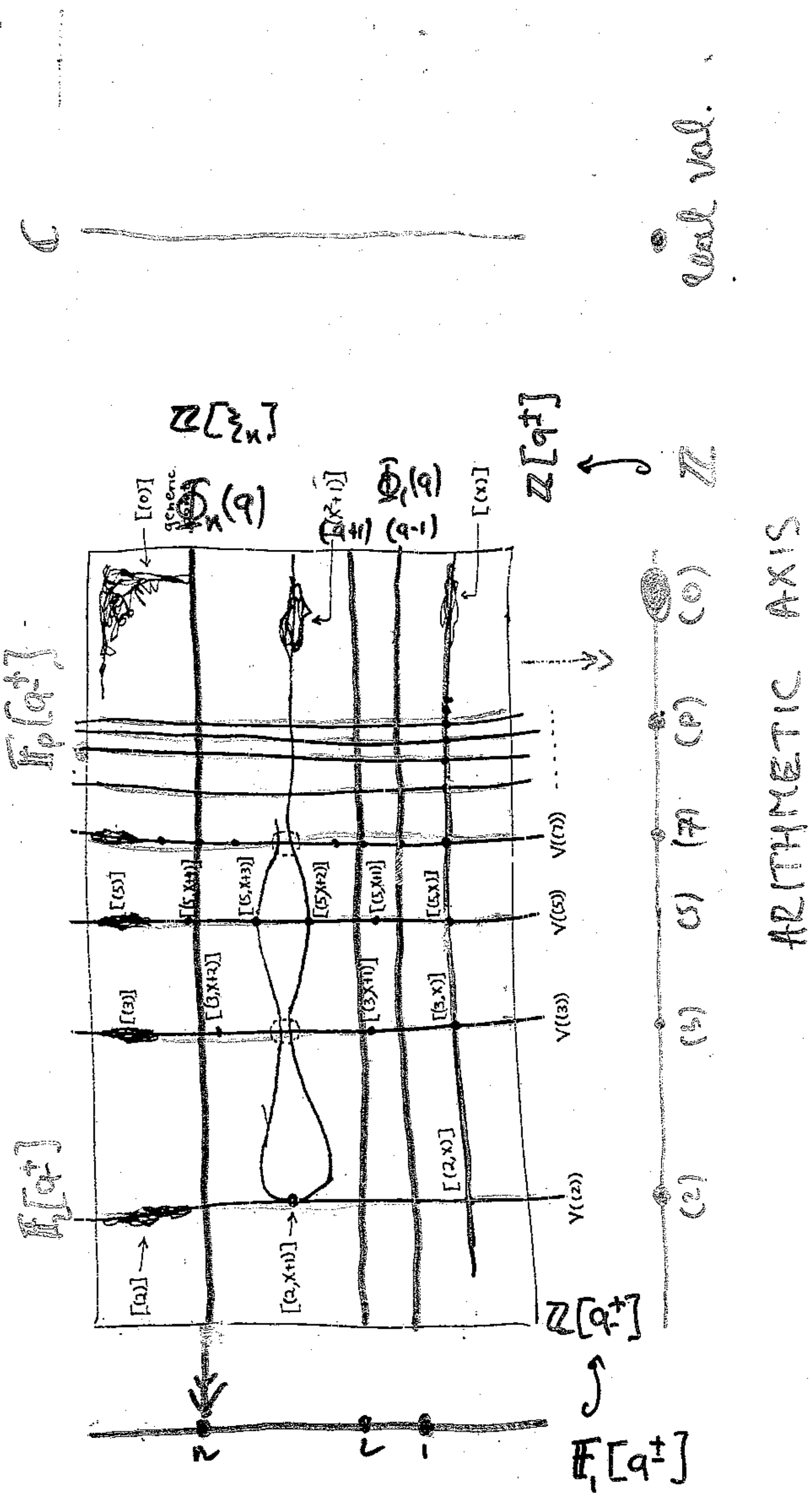
ii) principal prime ideals (f), where f is either a prime p, or a $\mathbb{Q}$-irreducible polynomial written so that its coefficients have g.c.d. 1,

iii) maximal ideals (p,f), p a prime and f a monic integral polynomial irreducible modulo p.

The whole should be pictured as follows:



cyclotomic integers
cyclotomic polys



GEOMETRIC AXIS

Manin 0809.1564 "cyclotomy & analytic geometry over $\mathbb{F}_1$"

BASIC ANALOGIES

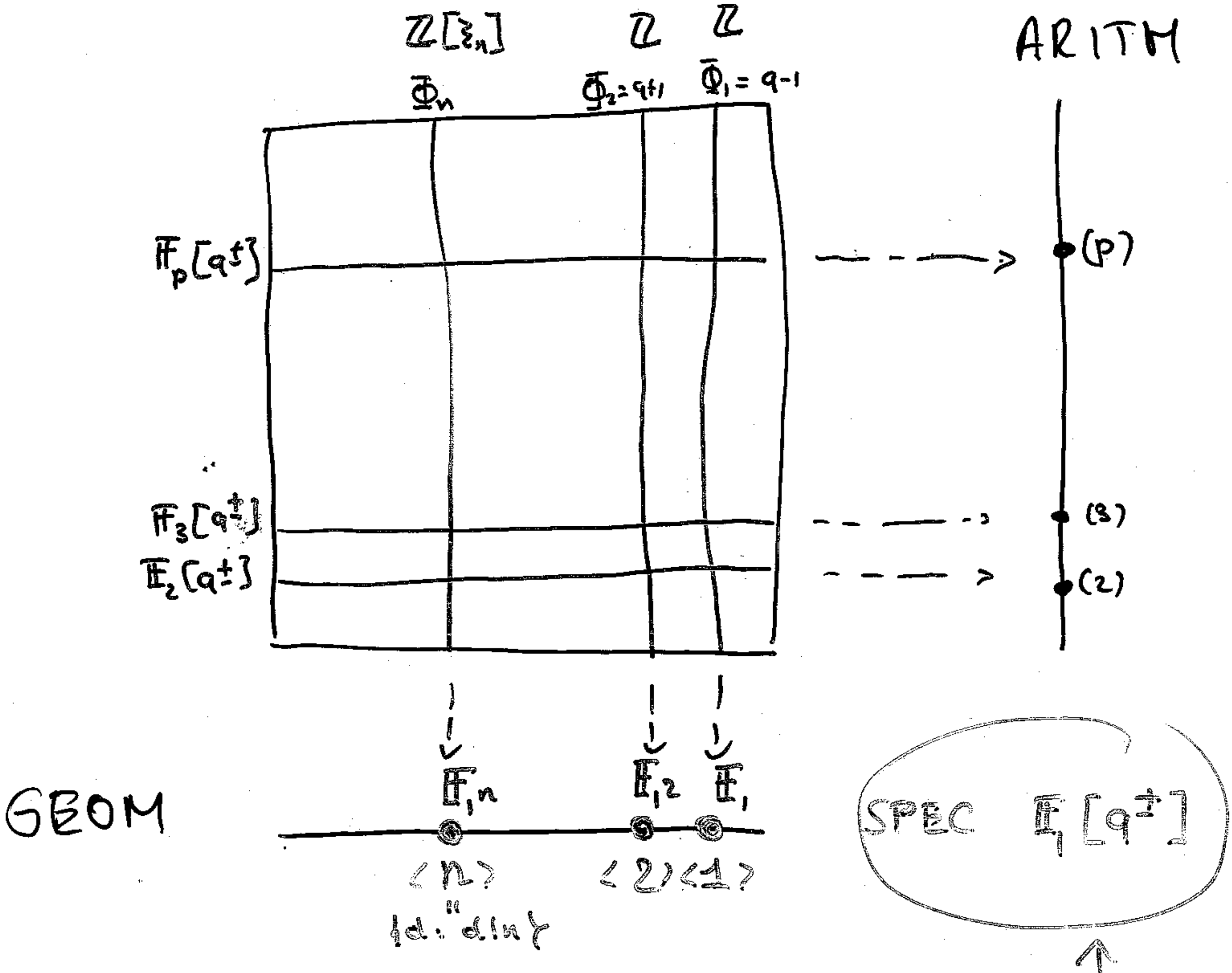
	Field k	Field $\mathbb{F}_1$
1	vector space	set
2	dimension	cardinality
3	$GL_n(k)$	S_n
4	G-representation $G \longrightarrow GL_n(k)$	$G \longrightarrow S_n$ permutation repr.
5	irreducible G-representation	transitive permutation repr.
6	k-field	cyclic group
7	k-algebra	abelian group
	(noncommutative)	(group)

MANTRA

$\mathbb{F}_1$-stuff acquires flesh after extending to $\mathbb{Z}$ (or $\mathbb{C}$)

SOULÉ - PROPOSAL

$$\boxed{\mathbb{F}_{1^n} \otimes_{\mathbb{F}_1} \mathbb{Z} \simeq \mathbb{Z} C_n}$$



WHAT?

$$\mathbb{F}_{1n} = \mathbb{F}_1[q^{\pm}]/(q^n-1)$$

$$\text{fiber} = (q^n-1) = \prod_{d|n} \Phi_d(q)$$

CONNES – CONSANI (arXiv: 0809.2926)

"On the notion of geometry over $\mathbb{F}_1$"

① F : (abelian) $\longrightarrow$ (sets) "functor of points"

② $\exists$ $\mathbb{C}$-algebra A s.t. $\forall G$ "realization w.r.t archimedean valuation"

$$F(G) \xrightarrow{ev} \mathrm{Alg}_{\mathbb{C}}(A, \mathbb{C}G)$$

③ $\exists!$ $\mathbb{Z}$-algebra B s.t. $\forall G$ "$\mathbb{Z}$-model"

"best" $$F(G) \overset{i}{\hookrightarrow} \mathrm{Alg}_{\mathbb{Z}}(B, \mathbb{Z}G)$$

$$\begin{tikzcd}
F(G) \arrow[rr, "ev"] \arrow[dr] \arrow[dd, "i"'] & & \mathrm{Alg}_{\mathbb{C}}(A, \mathbb{C}G) \arrow[dr, "-\circ\psi"] & \\
& \mathrm{Alg}_{\mathbb{Z}}(C, \mathbb{Z}G) \arrow[rr, "extns"'] & & \mathrm{Alg}_{\mathbb{C}}(C\otimes\mathbb{C}, \mathbb{C}G) \\
\mathrm{Alg}_{\mathbb{Z}}(B, \mathbb{Z}G) \arrow[ur, dashed, "-\circ\varphi", "\exists ?"'] \arrow[rr, "extns"'] & & \mathrm{Alg}_{\mathbb{C}}(B\otimes\mathbb{C}, \mathbb{C}G) \arrow[ur, dashed] &
\end{tikzcd}$$

$$\begin{tikzcd}
C \arrow[r, hook] & C\otimes\mathbb{C} \arrow[r, "\psi"] \arrow[d, Rightarrow, dashed] & A \\
C \arrow[r, "\varphi"'] & B &
\end{tikzcd}$$

[noncommutative $\mathbb{F}_1$-geometry : groups]

$$\mathbb{F}_1[q^{\pm}] : \begin{cases} F = \text{forgetfull} & F(G) = G \\ A = \mathbb{C}[q^{\pm}] & G \xrightarrow{ev} (\mathbb{C}G)^*, \quad g \mapsto e_g \\ B = \mathbb{Z}[q^{\pm}] & G \xhookrightarrow{i} (\mathbb{Z}G)^*, \quad g \mapsto e_g \end{cases}$$

$$\begin{tikzcd}
G \arrow[r, "ev"] \arrow[dr] \arrow[dd] & (\mathbb{C}G)^* \arrow[d, phantom, "\boxed{\bullet}"] \arrow[dr, bend left] & \\
 & Alg_{\mathbb{Z}}(C, \mathbb{Z}G) \arrow[r, "ext"] & Alg_{\mathbb{C}}(C\otimes\mathbb{C}, \mathbb{C}G) \\
(\mathbb{Z}G)^* \arrow[r, hook, "ext"'] & (\mathbb{C}G)^* &
\end{tikzcd}$$

$- \circ \psi : C \hookrightarrow C\otimes\mathbb{C} \xrightarrow{\psi} \mathbb{C}[q^{\pm}]$

$C \underset{?}{\to} \mathbb{Z}[q^{\pm}]$

<u>TRICK</u> : take $C_N = \langle c \rangle$ $N \gg$
and trace c around.

$$\begin{tikzcd}
C \arrow[r, hook] \arrow[d] & C\otimes\mathbb{C} \arrow[r, "\psi"] & \mathbb{C}[q^{\pm}] \arrow[d, "\pi"] \\
\mathbb{Z}C_N \arrow[rr, leftrightarrow] & & \mathbb{C}C_N = \dfrac{\mathbb{C}[q^{\pm}]}{(q^N - 1)}
\end{tikzcd}$$

$c \dashrightarrow \pi$

$\mathbb{F}_1[q^{\pm}]$ exists! (toric varieties, chevalley grps, proj space...)

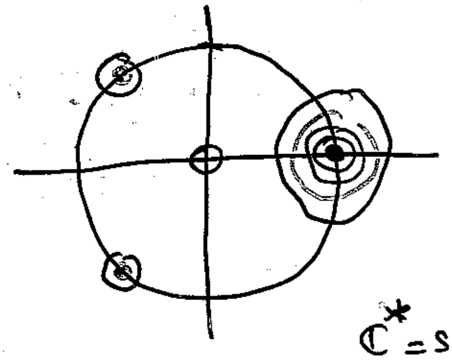
[dessins: $PSL_2(\mathbb{Z}) = C_2 * C_3 = \Gamma$; $\mathbb{C}\Gamma = A$, $B = \mathbb{Z}\Gamma$
$F : (\Gamma\text{-sum}) \to (\text{sets})$ $\quad G \to G_2 \times G_3$]

COMPLEX

C_3	e	c	c^2
x_1	1	1	1
x_2	1	ρ	ρ^2
x_3	1	ρ^2	ρ

spec $\mathbb{C}C_3$

$$\xrightarrow[ev]{}$$



$\mathbb{C}^* = \operatorname{spec}\mathbb{C}[q^{\pm}]$

$$\begin{pmatrix}\text{fin. dim}\\ \mathbb{C}\text{-alg}\end{pmatrix} \to (\text{sets})$$

$$\operatorname{spec}_{\text{Thin}}(A) = A^{o} = \left\{ \begin{array}{ccc} A & \xrightarrow[\text{linear}]{\lambda} & \mathbb{C} \\ \downarrow & \nearrow_{\lambda} & \\ \text{f.d.} \to A/I & & \end{array} \right\}$$

Thin spectrum

$$\left(\frac{\mathbb{C}[q^{\pm}]}{((q-\alpha_1)^{n_1}\cdots(q-\alpha_k)^{n_k})}\right)^* \cong \left(\frac{\mathbb{C}[q^{\pm}]}{(q-\alpha_1)^{n_1}}\right)^* \oplus \cdots \oplus \left(\frac{\mathbb{C}[q^{\pm}]}{(q-\alpha_k)^{n_k}}\right)^*$$

tg w/o

$$\mathbb{C}[q^{\pm}]^{o} = \boxed{\bigoplus_{\alpha\in\mathbb{C}^*}} U(T_{\mathbb{C}^*,\alpha})$$

Space = coroad

$$\mathbb{C}[q^{\pm}] \xrightarrow[\text{TAYLOR}]{} \mathbb{C}[q^{\pm}]^{o*} = \prod_{\alpha\in\mathbb{C}^*} \mathbb{C}[[q-\alpha]]$$

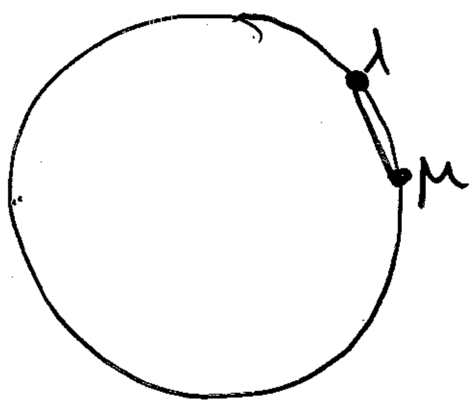
$\mathbb{F}_1$-geometry $\Rightarrow$ complex picture

① COORDINATE RING

$$\mathbb{F}_1[q^{\pm}]^{an} \underset{\mathbb{F}_1}{\otimes} \mathbb{C} = \bigcap_{\alpha \in \mu_\infty} \mathbb{C}[[q-\alpha]]$$

"holomorphic f'tions defined in all roots of 1"

② NEW TOPOLOGY ON μ_∞



λ and μ "nearby"

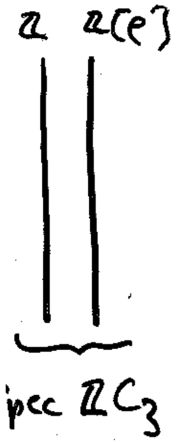
$\lambda \leftrightarrow \mu$

iff

$\frac{\lambda}{\mu}$ of order p^k, $k \in \mathbb{Z}$

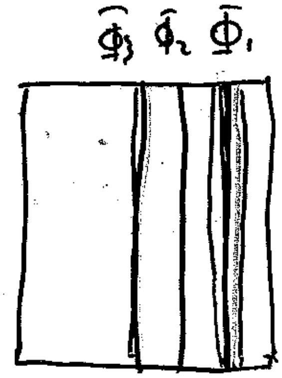
"nearness preserved under $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$"

(usual topology: not!)



INTEGRAL

$C_3 = \{e, c, c^2\}$



(f.g. proj $\mathbb{Z}$-rings) $\longrightarrow$ (sets)

$$\parallel$$

$$\mathrm{Spec}_{\mathrm{THIN}}(B) = B^c = \left\{ \begin{tikzcd} B \arrow[r, "\lambda"] \arrow[d] & \mathbb{Z} \\ B/I \arrow[ur, "\lambda"'] & \end{tikzcd} \right\}$$

$B/I \leftarrow$ f.g. proj $\mathbb{Z}$-ring

$$\left(\frac{\mathbb{Z}[q^{\pm}]}{(p_1(q)^{n_1} \cdots p_k(q)^{n_k})} \right)^* \leftrightarrows \left(\frac{\mathbb{Z}[q^{\pm}]}{(p_1(q))^{n_1}} \right)^* \oplus \cdots \oplus \left(\frac{\mathbb{Z}[q^{\pm}]}{(p_k(q))^{n_k}} \right)^*$$

$\leftarrow$ not necessarily co-maximal!

Π_1 - picture involves cyclotomic irred. $\Phi_n(q)$

$$\frac{m}{n} \neq p^k, k \in \mathbb{Z} \Rightarrow (\Phi_n(q), \Phi_m(q)) = 1$$

$$\frac{m}{n} = p^k, k \in \mathbb{N} \Rightarrow \Phi_m(q) \equiv \Phi_n(q)^d \mod (p)$$

(exist <u>extensions</u> $\mathbb{Z}[\xi_m] \supseteq \mathbb{Z}[\xi_n]$)

"clique"- relation on $\mathbb{N} = \{1, 2, 3, \ldots\}$ induced

$$n \longleftrightarrow m \quad \text{iff} \quad \frac{m}{n} = p^k, k \in \mathbb{Z}$$

$\mathbb{F}_1$-part of $\mathbb{Z}[q^\pm]^{\circ *}$ involves completions

$$\begin{cases} \widehat{\mathbb{Z}[q^\pm]}^S = \varprojlim_{p \in \Phi_S^*} \mathbb{Z}[q^\pm]/(p) \\ S \subset \mathbb{N} \qquad \Phi_{S^*} = \{\text{monics gener. by } \Phi_n(q),\ n \in S\} \end{cases}$$

K. HABIRO 0209324 "cyclotomic completions of polynomial rings"

① $S' \subset S$ s.t. every component of $\widehat{\text{clique}(S)}$ has el $\in S'$

$$\Rightarrow \quad \rho^S_{S'} : \widehat{\mathbb{Z}[q^\pm]}^S \hookrightarrow \widehat{\mathbb{Z}[q^\pm]}^{S'}$$

② if for every $n \in S$: $\langle n \rangle = \{m | n\} \subset S$, then

$$\widehat{\mathbb{Z}[q^\pm]}^S = \bigcap_{n \in S} \widehat{\mathbb{Z}[q^\pm]}^{\langle n \rangle} = \bigcap_{n \in S} \widehat{\mathbb{Z}[q^\pm]}_{(q^n-1)}$$

$\mathbb{F}_1$-geometry $\Rightarrow$ integral picture

$$\mathbb{F}_1[q^\pm]^{an} \otimes_{\mathbb{F}_1} \mathbb{Z} = \bigcap_{n \in \mathbb{N}} \widehat{\mathbb{Z}[q^\pm]}_{(q^n-1)}$$

$$= \widehat{\mathbb{Z}[q^\pm]}^{\mathbb{N}}$$

$\mathbb{C}: \bigcap_{\alpha \in \mu_\infty} \mathbb{C}[[q-\alpha]]$ holom. f'tions def. on μ_∞

$\mathbb{Z}: \widehat{\mathbb{Z}[q^\pm]}^{\mathbb{N}} = \varprojlim_N \mathbb{Z}[q^\pm] / ((q^N-1)(q^{N-1}-1)\cdots(q-1))$

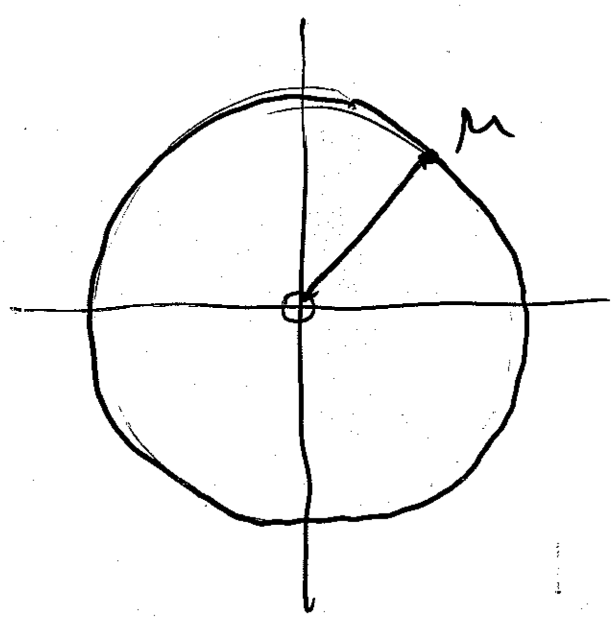
$= \left\{ \sum_{N=0}^{\infty} a_N(q)(q^N-1)(q^{N-1}-1)\cdots(q-1) \;;\; \deg a_N < N \right\} \subset \mathbb{Z}[[q^\pm]]$

WELL DEFINED ON μ_∞

GAL($\bar{\mathbb{Q}}/\mathbb{Q}$)

<u>Example</u>: Kontsevich - Zagier

$$\sum_{n=0}^{\infty} (1-q)(1-q^2)\cdots(1-q^n) = -\tfrac{1}{2} \sum_{n=0}^{\infty} n\,\chi(n)\, q^{\frac{n^2-1}{24}}$$

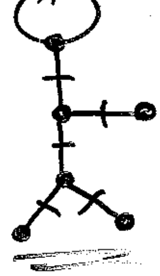


"DESSINS D'ENFANT"

$\mathbb{Z} = \langle c \rangle = C_\infty$ $\Gamma = PSL_2(\mathbb{Z}) = C_2 * C_3$

C_n dessins

$X \overset{12}{\downarrow} \mathbb{P}^1$ M_{12}



$C_3 = \{e, c, c^2\}$

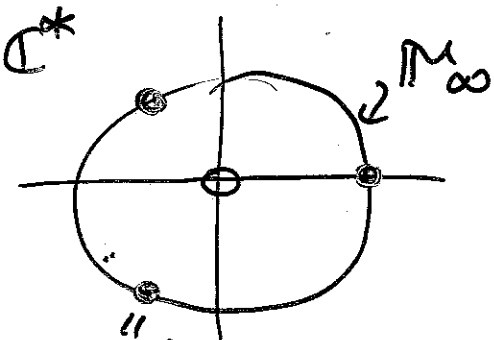
$\mathbb{F}_1$ $(\mathrm{ab}) \xrightarrow[\text{forget}]{F} (\mathrm{sets})$ $(\mathrm{groups}) \to (\mathrm{sets})$

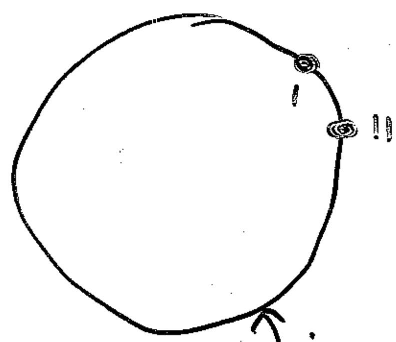
$G \to G_2 \times G_3$

$\mathbb{C}[q^\pm]$ $\mathbb{C}\Gamma$

$\mathbb{Z}[q^\pm]$ $\mathbb{Z}\Gamma$

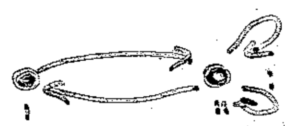
$\mathbb{C}$ $\mathbb{C}^*$





$\mathbb{C}[[q-1]] \oplus \mathbb{C}[[q-\rho]] \oplus \mathbb{C}[[q-\rho^2]]$

$\widehat{\mathbb{C}Q}$:



$\mathbb{C}[q^\pm]^\circ$ path coalg of ext-quiver

$\mathbb{F}_1[q^\pm]^{an} \otimes \mathbb{C}$ subcoalg on $\mathrm{irr}_{\mathbb{C}}\, \hat{\Gamma}$

$\mathbb{Z}$ $\widehat{\mathbb{Z}[q^\pm]}^{\mathbb{N}}$???