http://matrix.cmi.ua.ac.be/fun/

A bit of history

Jacques Tits, 1957, studying Chevalley groups G / finite fields $\mathbb{F}_q$ through certain geometries realized that the geom. objects still made sense when taking limit $q \to 1$.
The resulting geometry was naturally asociated to the corresponding ~~Weyl~~ Weyl group W. Tits proposed then thinking of W as the group of points on G belonging to a "field of characteristic one".

A pretty example (nice degeneration of geom) of this situation goes as follows:

Let $\mathbb{P}^n(\mathbb{F}_q)$ be the n-dim ~~n-dim~~ projective space ~~plane~~ / $\mathbb{F}_q$ $(q = p^m)$

~~Regardless of the field~~, $\mathbb{P}^n(\mathbb{F}_q)$ contains ~~n+1 n-dim affine spaces.~~ $(q^{n+1}-1)/(q-1)$ points
~~Each of these affine spaces contains q^n points~~ $= 1+q+\cdots+q^n = [n]_q$

Taking $q \to 1$ each ~~affine subspace of $\mathbb{P}^n(\mathbb{F}_1)$ should contain exactly 1 point !~~ we obtain $\# \mathbb{P}^n(\mathbb{F}_1) = n+1$

~~Thus, $\mathbb{P}^n(\mathbb{F}_1)$ will be a set with n+1 elements~~

~~Or, more reasonably~~ Thus, sets w/ ~~n~~ $n+1$ points might be interpreted as projective spaces in a "limit" sense.

Yuri Manin, 1995, describing some analogies between arithmetics over $\mathbb{F}_q$ and over $\mathbb{Z}$, and studying certain families of ζ-functions, introduces an "absolute Tate motive" this motive suggests an algebraic geometry over a one-element field.

Recently
(~~Ked~~ Kapranov - Smirnov, Deitmar, Toen - Vaquie, Soulé)
different notions for such a geometry appeared.

Even more recently, Connes, Consani and Marcolli stablished some relations between those geometries and the BC-system used to study the Riemann ζ-function.

Our goal in this seminar will initially be an approach to this reinterpretation of the BC-system.

Some folklore about $\mathbb{F}_1$

- $\mathbb{F}_1$ does not really exist as a well defined object, so we are not allowed to use it directly.

Some general principles that apply to $\mathbb{F}_1$ are:

- Doing things over $\mathbb{F}_1$ somehow feel like "forgetting addition"
- Vector spaces / $\mathbb{F}_1$ are just sets. Sometimes (to define quotients, exact sequences and so on) it is convenient to add a "formal" 0 to the set. The dimension of the vector space S is just $\#(S)$
- ~~The general linear group $Gl_n(\mathbb{F}_1)$ is identified w/ the symmetric group in n letters~~
- Linear maps are just set maps. In particular, linear isomorphisms between S and itself are permutations on the set. Thus $GL_n(\mathbb{F}_1) = S_n$ permutation grp. in n letters.

A fancy way of saying this is

"Linear algebra over $\mathbb{F}_1$ is the same thing as combinatorics on finite sets"

If at any moment we need to consider a v.s. S over $\mathbb{F}_1$ w/ the additional formal 0, we'll write $S^\bullet := S \cup \{0\}$

Since every finite field admits a unique finite extension of any given degree, it makes sense to consider extensions $\mathbb{F}_{1^n}$ of $\mathbb{F}_1$ for any n.

As a mental picture, we can think of $\mathbb{F}_{1^n}$ as the set of n-th roots of 1 (plus the formal 0), or in our former notation, $\boxed{\mathbb{F}_{1^n} := \mu_n^\bullet}$

RK: With this interpretation, $\mathbb{F}_1$ itself would have not one, but two elements (0 and 1) which is kind of perverse. However, we are not allowed to take addition or to use the 0 barely for anything, so $\mathbb{F}_1$ has one "real" element (the 1) and a "phantom" one (the 0)

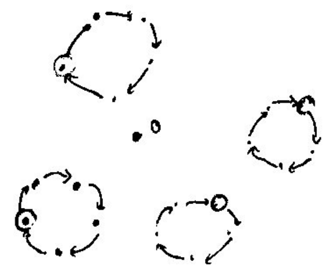
NO [We will say that a scheme X is defined over $\mathbb{F}_{1^n}$ if the ring of regular functions on X contains n-th roots of 1.

A vector space V over $\mathbb{F}_{1^n}$ will be a set V endowed w/ a free action $\mu_n \times V \longrightarrow V$

(if we add the 0, we'll assume that it stays fixed by this action)

If V is finite, since the action is free it must have $n \cdot d$ elements (all the orbits must be μ_n - torsors) In this case, we'll say that V has dimension d, and a basis of V will consist on choosing one representant for every orbit.

Mental pic of a v.s over $\mathbb{F}_{1^n}$:



Each "cycle" represents one of the μ_s orbits (only the action of the generator is drawn)
Elements in green are a possible choice of a basis

The "phantom" 0 stays fixed by the action (and separated from the orbits)

The group of automorphisms ~~$GL_d(\mathbb{F}_{1^n})$~~ of a d-dim $\mathbb{F}_{1^n}$-v.s is $GL_d(\mathbb{F}_{1^n}) \cong S_d \wr (\mu_n)^d$

~~through identification given by~~

that also can be viewed as the group of $d \times d$ matrices, containing exactly one non-zero element at every row and column, and that element is a root of unity in μ_n.

Though this is not really our goal, it can be shown that "linear algebra" can be fully developed in this context by completely forgetting about addition and only using scalar multiplication. Moreover, all traditional results of linear algebra over $\mathbb{F}_q$ are recovered from the new definitions using the vector-space identification

$$\mathbb{F}_q = (\mathbb{F}_q^{\times})^{\bullet} = \mathbb{F}_{1^n} \quad , \quad \text{where} \quad n = q-1$$

Soulé's geometry over $\mathbb{F}_1$

Soulé's idea of a geometry over $\mathbb{F}_1$ is deeply enrooted on the notion of extension of scalars.

Initial idea is

" A variety (of finite type) 'X' over $\mathbb{F}_1$ should have an extension of scalars to $\mathbb{Z}$, $X_{\mathbb{Z}}$, that will be a scheme of finite type / $\mathbb{Z}$"

~~For~~ In order to define schemes over $\mathbb{F}_1$, one should have a nice collection of rings (/ $\mathbb{Z}$) that can be considered extensions of scalars of algebras defined over $\mathbb{F}_1$. First obvious canditates for rings defined over $\mathbb{F}_1$ are of course the field extensions $\mathbb{F}_{1^n}$, obtained by ~~adding~~ "adjoining roots of 1". This induces to ~~think~~ propose the following axiomatic equation:

$$\boxed{\mathbb{F}_{1^n} \otimes_{\mathbb{F}_1} \mathbb{Z} := \mathbb{Z}[t]/(t^n - 1)} = R_n$$

This rings R_n give us a first test for a definition of a variety over $\mathbb{F}_1$, if X is such a variety (seen as a functor $\underline{X}$: Rings $\to$ Sets) then ~~we must have~~ (because of the well known behavior of schemes under extension of scalars : $X(R) \subseteq X_\Omega(R \otimes_k \Omega)$) we must have

$$\boxed{X(\mathbb{F}_{1^n}) \subseteq \underbrace{(X \otimes_{\mathbb{F}_1} \mathbb{Z})}_{X_{\mathbb{Z}}} (R_n)}$$

Actually, we'll assume that the ~~category~~ extension of scalars from $\mathbb{F}_1$ to $\mathbb{Z}$ induces an equivalence of categories between the category of $\mathbb{F}_1$-algs. and the category $\mathcal{R}$ of finite flat rings over $\mathbb{Z}$ (i.e, rings whose underlying group is a finite rank lattice)

The definitions

A gadget (from french "truc") over $\mathbb{F}_1$ is given by a couple $X = (\underline{X}, A_X)$ where $\underline{X} : \mathcal{R} \longrightarrow \mathrm{Sets}$ is a covariant functor and A_X is a (complex) algebra, such that, for every (unital) ring morphism

$$\sigma : R \longrightarrow \mathbb{C} \qquad (w/\ R \in \mathcal{R})$$

and every $x \in \underline{X}(R)$ we have an "evaluation morphism" given by an algebra map

$$ev_{\sigma, x} : A_X \longrightarrow \mathbb{C}$$

satisfying, for any $f : R' \longrightarrow R$ morphism in $\mathcal{R}$ and any $y \in \underline{X}(R')$, the equality

$$ev_{(\sigma, f(y))} = ev_{\sigma \circ f, y} \qquad \forall\ \sigma : R \longrightarrow \mathbb{C}$$

Rk : The category $\mathcal{R}$ is quite arbitrary here. Possible refinements on the definition are

- The full subcat. of $\mathcal{R}$ generated by R_n and their tensor products
- The category $\mathcal{F}_{ab}$ of finite abelian groups (Connes - Consani)
- The cat. Grp. of all groups (being developed now by LeBruyn)

Rk 2: The alg. A_X represent a "given topology at infinity"
So far we'll make no assumptions on A_X (its concrete properties will be determined by the needs of our geometry)

A gadget $X = (\underline{X}, \mathcal{A}_X)$ is <u>finite</u> if all sets $\underline{X}(R)$, $R \in \mathcal{R}$ are finite.

A <u>morphism of gadgets</u> $\varphi : X \longrightarrow Y$ is a couple $\varphi = (\underline{\varphi}, \varphi^*)$

where $\underline{\varphi} : \underline{X} \longrightarrow \underline{Y}$ is a natural transformation

and $\varphi^* : \mathcal{A}_Y \longrightarrow \mathcal{A}_X$ is an algebra map

such that $\forall R \in \mathcal{R}$, $\sigma : R \longrightarrow \mathbb{C}$ alg. map, $x \in \underline{X}(R)$

we have
$$ev_{\sigma, \underline{\varphi}(x)}(\alpha) = ev_{\sigma, x}(\varphi^*(\alpha)) \qquad \forall \alpha \in \mathcal{A}_Y$$

(in other words, $ev_{\sigma, \underline{\varphi}(x)} = ev_{\sigma, x} \circ \varphi^*$ as maps $\mathcal{A}_Y \longrightarrow \mathbb{C}$)

Composition of morphisms
$$\varphi : X \longrightarrow Y \qquad \varphi = (\underline{\varphi}, \varphi^*)$$
$$\psi : Y \longrightarrow Z \qquad \psi = (\underline{\psi}, \psi^*)$$

is given by $\psi \circ \varphi = (\underline{\psi} \circ \underline{\varphi}, \varphi^* \circ \psi^*)$

This gives us the <u>category of gadgets</u> over $\mathbb{F}_1$

<u>Ex</u>: Let V an alg. variety (of finite type) over $\mathbb{Z}$. To V we can associate a gadget $V = (\underline{V}, \mathcal{O}(V_\mathbb{C}))$ as follows:

- $\forall R \in \mathcal{R}$, $\underline{V}(R) := \mathrm{Hom}_\mathbb{Z}(\mathrm{Spec}(R), V)$ (regular maps)

$$f : R' \longrightarrow R \quad \Rightarrow \quad \underline{V}(f) : \underline{V}(R') \longrightarrow \underline{V}(R)$$
$$\varphi \longmapsto \underline{V}(f)(\varphi) := \varphi \circ f$$

- $\mathcal{O}(V_\mathbb{C})$ is the alg. of global functions on $V_\mathbb{C} = V \otimes_\mathbb{Z} \mathbb{C}$
- For $\sigma : R \longrightarrow \mathbb{C}$ alg. map, $x \in \underline{V}(R)$ $\Rightarrow \sigma(x) := x(\ker \sigma) \in V_\mathbb{C}$

and we define $ev_{\sigma, x} : \mathcal{O}(V_\mathbb{C}) \longrightarrow \mathbb{C}$
$$f \longmapsto ev_{\sigma, x}(f) := f(\sigma(x))$$

An *affine variety* over $\mathbb{F}_1$ is a finite gadget X over $\mathbb{F}_1$ such that there exists an affine alg. variety (of finite type) $X_{\mathbb{Z}}$ over $\mathbb{Z}$, and an immersion

$$i : X \longrightarrow X_{\mathbb{Z}}$$

of gadgets ($\varphi = (\underline{\varphi}, \varphi^*)$ is an *immersion* if φ^* injective and $\forall R \in \mathcal{R}$, $\varphi : \underline{X}(R) \to \underline{Y}(R)$ is injective)

satisfying the following condition:

$\forall$ V affine alg. variety (over $\mathbb{Z}$), $\forall$ $\varphi : X \longrightarrow V$ map of gadgets, there exists a unique regular function

$$\varphi_{\mathbb{Z}} : X_{\mathbb{Z}} \longrightarrow V$$

such that $\varphi = \varphi_{\mathbb{Z}} \circ i$

~~In this~~ Under the above conditions, the alg. variety $X_{\mathbb{Z}}$ is uniquely determined by X, and we'll denote it by $X_{\mathbb{Z}} = X \otimes_{\mathbb{F}_1} \mathbb{Z}$

Moreover, given X, Y affine varieties over $\mathbb{F}_1$, $f : X \longrightarrow Y$ morphism of gadgets, there is a unique morphism $f_{\mathbb{Z}}$ of alg. varieties such that the diagram

$$\begin{tikzcd} X_{\mathbb{Z}} \arrow[r, "f_{\mathbb{Z}}"] & Y_{\mathbb{Z}} \\ X \arrow[u] \arrow[r, "f"] & Y \arrow[u] \end{tikzcd}$$

is commutative.

The class of all affine varieties over $\mathbb{F}_1$ is a full subcategory of the category of gadgets over $\mathbb{F}_1$. We'll ~~denothe~~ denote this category by

$\mathcal{A} = \mathcal{A}_{\mathbb{F}_1}$

An object over $\mathbb{F}_1$ is a couple $X = (\underline{\underline{X}}, \mathcal{A}_X)$ given by a contravariant functor

$$\underline{\underline{X}} : \mathcal{A} \longrightarrow \mathcal{S}ets$$

and a complex algebra $\mathcal{A}_X$, together with an alg. map

$$ev_x : \mathcal{A}_X \longrightarrow \mathcal{A}_A \qquad \forall x \in \underline{\underline{X}}(A)$$
$$\forall A \in \mathcal{A}$$

such that for every morphism $f : A \longrightarrow B$ in $\mathcal{A}$ and $x \in \underline{\underline{X}}(B)$ we have

$$ev_{f^*(x)} = f^* \circ ev_x$$

(remember $f : A \longrightarrow B$ morphism of gadgets $\Rightarrow f = (\underline{f}, f^*)$)

where $f^*(x) := \underline{\underline{X}}(f)(x) \in \underline{\underline{X}}(A)$

An object over $\mathbb{F}_1$ is finite if $\forall R \in \mathcal{R}$, $\underline{\underline{X}}(\mathrm{Spec}R)$ is finite

A morphism $\varphi : X \longrightarrow Y$ between objects over $\mathbb{F}_1$ is given by a natural transformation $\underline{\underline{\varphi}} : \underline{\underline{X}} \longrightarrow \underline{\underline{Y}}$
plus an alg. map $\varphi^* : \mathcal{A}_Y \longrightarrow \mathcal{A}_X$ such that
$\forall A \in \mathcal{A}$, $\forall x \in \underline{\underline{X}}(A)$ we have $ev_{\underline{\underline{\varphi}}(x)} = ev_x \circ \varphi^*$

If both $\underline{\underline{\varphi}}$ and φ^* are injective we'll say that φ is an inmersion.

In this way we obtain a category $\mathcal{O}$ of objects over $\mathbb{F}_1$

Ex V alg. variety $(/\mathbb{Z})$ $\rightsquigarrow$ $V = (\underline{V}, \mathcal{O}(V_{\mathbb{C}}))$ object $/\mathbb{F}_1$

via

$$\underline{V}(A) := \mathrm{Hom}_{\mathbb{Z}}(A_{\mathbb{Z}}, V) \qquad \forall A \in \mathcal{A}$$

(we already saw $V \in \mathcal{A}$)

$$\underline{V}(f)(\varphi) := \varphi \circ f_{\mathbb{Z}}$$

and evaluation morphism $e_x := i^* \circ x^*$

where $\mathcal{O}(V_{\mathbb{C}}) \xrightarrow{x^*} \mathcal{O}(A_{\mathbb{C}}) \xrightarrow{i^*} \mathcal{A}_A \qquad \forall x \in \underline{V}(A)$

Associating to each $f \in \mathrm{Hom}_{\mathbb{Z}}(V, W)$ morphism of composition w/ f

and the inverse image $f^* : \mathcal{O}(W_{\mathbb{C}}) \longrightarrow \mathcal{O}(V_{\mathbb{C}})$

we obtain a functor

$$Var_{\mathbb{Z}} \longrightarrow \mathcal{O}$$

Def A <u>variety</u> over $\mathbb{F}_1$ is given by an object $X \in \mathcal{O}$ over $\mathbb{F}_1$

such that $\exists$ $X_{\mathbb{Z}}$ alg. variety and $i : X \longrightarrow X_{\mathbb{Z}}$ immersion in $\mathcal{O}$

such that $\forall$ $V \in Var_{\mathbb{Z}}$, $\forall \varphi : X \longrightarrow V$ in $\mathcal{O}$

$\exists !$ $\varphi_{\mathbb{Z}} : X_{\mathbb{Z}} \longrightarrow V$ regular map

such that $\varphi = \varphi_{\mathbb{Z}} \circ i$

Henceforth, the variety $X_{\mathbb{Z}}$ is determined (up to unique isomorphism) by the variety $/\mathbb{F}_1$, X. So we'll write $X_{\mathbb{Z}} = X \otimes_{\mathbb{F}_1} \mathbb{Z}$ and will call it the extension of scalars of X to $\mathbb{Z}$. Moreover, if X, Y varieties $/\mathbb{F}_1$, $f : X \longrightarrow Y$ in $\mathcal{O}$, then $\exists !$ $f_{\mathbb{Z}} \in \mathrm{Hom}_{\mathbb{Z}}(X_{\mathbb{Z}}, Y_{\mathbb{Z}})$ that induces f over X. In other words, we have a faithful functor

$$\{\text{Varieties } /\mathbb{F}_1\} \longrightarrow \{\text{Varieties } /\mathbb{Z}\}$$

Remarks

·) Possible restrictions to the algebra A_X could be, for instance, impossing that it must be a Banach algebra (thus ~~impossing~~ forcing it to have some topology)

·) By replacing the categories $\mathcal{R}$ and $Var_{\mathbb{Z}}$ by the sub-categories of flat R_n-algebras of finite type and the one of schemes over R_n, the former construction gives us a definition of varieties over the extension $\mathbb{F}_{1^n}$

Properties of varieties / $\mathbb{F}_1$

$\boxed{\cdot)\quad \mathcal{R}^{op} \subseteq \mathcal{A} \quad (\text{affine } \mathbb{F}_1\text{-varieties})}$

More concretely:

a) $\forall\ R \in \mathcal{R}$, X $\underbrace{\text{gadget}}_{\mathcal{G}}$ $/\mathbb{F}_1$, $\underline{X}(R) \simeq \mathrm{Hom}_{\mathcal{G}}(\mathrm{Spec}\,R, X)$ (as sets)

b) $\forall\ R \in \mathcal{R}$, the gadget associated to $\mathrm{Spec}(R)$ is an affine variety $/\mathbb{F}_1$ moreover, the extension of this variety to $\mathbb{Z}$ coincides w/ $\mathrm{Spec}(R)$

Hence, we have a fully faithful contravariant functor $\mathcal{R} \longrightarrow \mathcal{A}$

$\boxed{\cdot)\ \text{Every gadget } /\mathbb{F}_1 \text{ defines an object } /\mathbb{F}_1}$

More precisely, we have a fully faithful functor $\varepsilon : \mathcal{G} \longrightarrow \mathcal{O}$ associating a gadget $X = (\underline{X}, A_X)$ the couple $(\underline{\underline{X}}, A_X)$, where $\underline{\underline{X}}$ is the functor on $\mathcal{A}$ represented by X, i.e.,

$$\underline{\underline{X}}(A) = \mathrm{Hom}_{\mathcal{G}}(A, X)$$

And evaluation in $u \in \underline{\underline{X}}(A)$ given by inverse image

$$ev_u = u^* : A_X \longrightarrow A_A$$

Moreover, the essential image of $\mathcal{A}$ by $\mathcal{E}$ is the category of varieties / $\mathbb{F}_1$ such that their extension to $\mathbb{Z}$ is affine.

Remark

·) A variety over $\mathbb{Z}$ may well be the extension of scalar of several different varieties over $\mathbb{F}_1$!

·) If $V = \bigcup_{i \in I} U_i$ open covering of V, s.t. each U_i comes from a suitable variety / $\mathbb{F}_1$, then also V does.

Examples of $\mathbb{F}_1$ varieties

1. The affine line $\mathbb{A}^1$ over $\mathbb{F}_1$ is the (affine) variety / $\mathbb{F}_1$ defined by $\boxed{\underline{\mathbb{A}}^1(R) = \mu(R) \cup \{0\}} =: \mu(R)^\bullet \quad \forall R \in \mathcal{R}$

where $\mu(R)$ = roots of unit in R

and $\mathcal{A}_{\mathbb{A}^1}$ is the alg of continuous functions on the unit disk

$D = \{z \in \mathbb{C} \mid |z| \le 1\}$ which are holomorphic in the interior

and evaluation $\boxed{ev_{\sigma, x}(f) := f(\sigma(x))}$

For the affine line we have $\boxed{\mathbb{A}^1 \otimes_{\mathbb{F}_1} \mathbb{Z} = \mathbb{A}^1_{\mathbb{Z}}}$

2. The multiplicative group $\mathbb{G}_m$ over $\mathbb{F}_1$ is the affine variety given by

$$\underline{\mathbb{G}}_m(R) = \mu(R) \quad \forall R \in \mathcal{R}$$
$$\mathcal{A}_{\mathbb{G}_m} = C(S^1), \quad ev_{\sigma,x}(f) := f(\sigma(x))$$

Of course, the scalar extension $\mathbb{G}_m \otimes_{\mathbb{F}_1} \mathbb{Z} = \mathbb{G}_{m,\mathbb{Z}}$ is the multiplicative group scheme over $\mathrm{Spec}(\mathbb{Z})$

3. The projective space $\mathbb{P}^d$ over $\mathbb{F}_1$

~~$\mathbb{P}^d(R)$ is~~

$\underline{\mathbb{P}}^d(\mathrm{Spec}(R)) = \mathbb{P}^d_{\mathbb{Z}}(R)$ where we can choose a system of homogeneous coordinates in $(\mu(R) \cup \{0\})^{d+1}$

$\mathcal{A}_{\mathbb{P}^d} = \mathbb{C}$ (constant functions)

4. Λ lattice of finite rank ($\Lambda \cong \mathbb{Z}^k$ for some k)

$\|\cdot\|$ a hermitian norm on $\Lambda \otimes_{\mathbb{Z}} \mathbb{C}$

Let $B = \{x \in \Lambda \otimes_{\mathbb{Z}} \mathbb{C} \,/\, \|x\| \leq 1\}$ the unit ball of $\Lambda \otimes_{\mathbb{Z}} \mathbb{C}$

$\Phi \subseteq (B \cap \Lambda) \setminus \{0\}$ s.t. for a given $v \in (B \cap \Lambda) \setminus \{0\}$
either $v \in \Phi$ or $-v \in \Phi$ (and only one of the two)

For $R \in \mathcal{R}$, we define

$$\underline{X}(R) := \{ x = \sum_{v \in \Phi} v \otimes \zeta_v \in \Lambda \otimes \mathbb{C} \mid \zeta_v \in \mu(R) \cup \{0\} \}$$

∙) Realize that $\underline{X}(R)$ does not depend on Φ !

this gives us a functor $\underline{X} : \mathcal{R} \longrightarrow \mathcal{S}ets$

~~$\mathcal{A} = \mathcal{A}lg$ of functions continuous on $\mathcal{C} = \dagger$~~

Λ_0 the sublattice of Λ generated by Φ

$$\mathcal{C} := \{ \sum_{v \in \Phi} v \otimes \zeta_v \in \Lambda_0 \otimes \mathbb{C} \mid \| \zeta - \| \leq \#(\Phi) \}$$

Now, we define

$$\mathcal{A} = \{ f : \mathcal{C} \longrightarrow \mathbb{C} \mid f \text{ continuous}, \ f_{|\mathcal{C}^\circ} \text{ holomorphic} \}$$

Since, for every $x \in \underline{X}(R)$, $\sigma : R \longrightarrow \mathbb{C}$

$$x = \sum_{v \in \Phi} v \otimes \underset{\substack{\cap \\ \mu(R) \cup \{0\}}}{\zeta_v} \Rightarrow \Big\| \sum_{v \in \Phi} v \otimes \underbrace{\sigma(\zeta_v)}_{\substack{\cap \\ \mathbb{C}}} \Big\| \leq t$$

so we can define $\boxed{ev_{\sigma, x}(f) := f(\sigma(x))}$

the pair $(\underline{X}, \mathcal{A})$, with the preceding evaluation, is an affine variety over $\mathbb{F}_1$. The extension by scalars to $\mathbb{Z}$ of this variety is the spectrum $X_{\mathbb{Z}}$ of the symmetric algebra of Λ_σ^*