Quick recap

Soulé's gadget: $X = (\underline{X}, \mathcal{A}_X)$

where $\underline{X} : \mathcal{R} \longrightarrow \mathcal{S}ets$ covariant functor

$\mathcal{A}_X$ complex alg.

$\forall R \in \mathcal{R}$, $\forall x \in \underline{X}(R)$, $\forall \sigma : R \to \mathbb{C}$ $ev_{\sigma,x} : \mathcal{A}_X \longrightarrow \mathbb{C}$ alg. map

s.t. $f : R' \longrightarrow R \Rightarrow ev_{\sigma, f(y)} = ev_{\sigma \circ f, y}$

A gadget is a (affine) variety (over $\mathbb{F}_1$) if

- $\forall R \in \mathcal{R}$ the set $\underline{X}(R)$ is finite
- $\exists$ $X_{\mathbb{Z}}$ alg. variety (of finite type) / $\mathbb{Z}$ with an immersion $i : X \hookrightarrow X_{\mathbb{Z}}$ s.t. $\forall$ V variety, $\forall \varphi : X \to V$ $\exists !$ $\tilde{\varphi} : X_{\mathbb{Z}} \longrightarrow V$ regular map with $\tilde{\varphi} \circ i = \varphi$

Some problems with this definition:

- The category $\mathcal{R}$ is arbitrary (no special reason for that)
- It does not address some motivating examples (Chevalley group-schemes)

In this lecture we'll talk about a refinement of Soulé's notion of variety introduced by Connes and Consani in September 2008.

Key point keeps being extension of scalars:

If $K \subseteq \Omega$ field extension, X scheme over K, then we have a natural transformation of functors $i : \underline{X} \longrightarrow \underline{X}_{\Omega} \circ \beta$ (w/ $\underline{X}(R) \subseteq \underline{X}_{\Omega}(R_{\Omega})$)

Moreover, $\forall$ S scheme over Ω and any natural transf. $\varphi : \underline{X} \longrightarrow \underline{S} \circ \beta$ there exist a unique morphism φ_{Ω} (over Ω) from X_{Ω} to S s.t. $\varphi = \varphi_{\Omega} \circ i$

(All above, $\beta : \mathcal{A}_K \longrightarrow \mathcal{A}_{\Omega}$ $\beta(R) = R \otimes_K \Omega$ is the functor of extension of scalars)

Replacing $\mathcal{R}$ by $\mathcal{F}_{ab}$

we want to use the category of finite abelian groups instead of the cat. $\mathcal{R}$

For this, we have a natural functor of "extension of scalars" from $\mathbb{F}_1$ to $\mathbb{Z}$,

given by
$$\beta : \mathcal{F}_{ab} \longrightarrow \mathcal{R}$$
$$D \longmapsto \beta(D) = D \otimes_{\mathbb{F}_1} \mathbb{Z} := \mathbb{Z}[D]$$

i.e., β takes to any group its integral groupring.

If $X = (\underline{X}, \mathcal{A}_X)$ is a Soulé gadget, the evaluation morphism ev determine a natural transformation of (covariant) functors
$$e : \underline{X} \longrightarrow \mathrm{Spec}_\infty(\mathcal{A}_X) \circ \beta$$

where $\mathrm{Spec}_\infty(\mathcal{A}_X) : \mathcal{R} \longrightarrow \mathrm{Sets}$ is the functor given by alg maps from $\mathcal{A}_X$ to the extension of scalars to $\mathbb{C}$ of R
$$R \longmapsto \mathrm{Hom}_{\mathbb{C}\text{-alg.}}(\mathcal{A}_X, R \otimes_{\mathbb{Z}} \mathbb{C})$$

so that
$$\mathrm{Spec}_\infty(\mathcal{A}_X) \circ \beta : \mathcal{F}_{ab} \longrightarrow \mathrm{Sets}$$
$$D \longmapsto \mathrm{Hom}_{\mathbb{C}}(\mathcal{A}_X, \mathbb{C}[D])$$

This natural transformation works as follows:

~~Consider~~ ~~$\mathrm{Hom}_{\mathbb{Z}}$~~ For $D \in \mathcal{F}_{ab}$, $x \in \underline{X}(D)$, $ev_{\cdot,x}$ defines a map
$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[D], \mathbb{C}) \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathcal{A}_X, \mathbb{C})$$

Consider also the set $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[D], \mathbb{C})$ $(= \mathrm{Spec}\,\mathbb{C}[D])$, every character $\sigma : \mathbb{C}[D] \longrightarrow \mathbb{C}$ determines a character $\sigma' : \mathbb{Z}[D] \longrightarrow D$ $(\sigma' = \sigma \circ i)$

so we get a map
$$\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[D], \mathbb{C}) \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathcal{A}_X, \mathbb{C})$$
$$\sigma \longmapsto ev_{\sigma', x}$$

But this is a map $\mathrm{Spec}(\mathbb{Z}[D] \otimes_{\mathbb{Z}} \mathbb{C}) \longrightarrow \mathrm{Spec}\,\mathcal{A}_X$, so it corresponds to a map $\mathcal{A}_X \longrightarrow \mathbb{Z}[D] \otimes_{\mathbb{Z}} \mathbb{C}$, as we wanted to show.

so, Connes-Consani reformulate the notion of <u>gadget</u> as a triple $X = (\underline{X}, X_{\mathbb{C}}, e_X)$ consisting of.

1. $\underline{X} : \mathcal{F}_{ab} \longrightarrow$ Sets covariant functor
2. $X_{\mathbb{C}}$ complex variety
3. A natural transformation $e_X : \underline{X} \longrightarrow \mathrm{Hom}(\mathrm{Spec}\, \mathbb{C}[-], X_{\mathbb{C}})$

<u>Ex</u>: V affine variety (over $\mathbb{Z}$) defines a gadget $X = \mathcal{G}(V)$

by $X_{\mathbb{C}} := V_{\mathbb{C}} = V \otimes_{\mathbb{Z}} \mathbb{C}$

$\underline{X}(D) = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Spec}\, \mathbb{Z}[D], V)$

w/ the obvious natural transformation to $\mathrm{Hom}(\mathrm{Spec}\, \mathbb{C}[D], V_{\mathbb{C}})$

A gadget X / $\mathbb{F}_1$ ~~is~~ is <u>graded</u> when it takes values on the category of $\mathbb{Z}_{\geq 0}$ graded sets, i.e, $\underline{X} = \coprod_{k \geq 0} \underline{X}^{(k)} : \mathcal{F}_{ab} \longrightarrow$ Sets

X is said to be <u>finite</u> if $\forall D \in \mathcal{F}_{ab}$ the set $\underline{X}(D)$ is finite

A <u>morphism</u> of gadgets $\phi : X \longrightarrow Y$ is given by $\phi = (\underline{\phi}, \phi_{\mathbb{C}})$, where

$\underline{\phi} : \underline{X} \longrightarrow \underline{Y}$ natural transform.

$\phi_{\mathbb{C}} : X_{\mathbb{C}} \longrightarrow Y_{\mathbb{C}}$ morphism of complex varieties

such that the following diagram is commutative $\forall D \in \mathcal{F}_{ab}$

$$\begin{tikzcd}
\underline{X}(D) \arrow[r, "\underline{\phi}(D)"] \arrow[d, "e_X(D)"'] & \underline{Y}(D) \arrow[d, "e_Y(D)"] \\
\mathrm{Hom}(\mathrm{Spec}\, \mathbb{C}[D], X_{\mathbb{C}}) \arrow[r, "\phi_{\mathbb{C}}^*"] & \mathrm{Hom}(\mathrm{Spec}\, \mathbb{C}[D], Y_{\mathbb{C}})
\end{tikzcd}$$

$\phi : X \longrightarrow Y$ is said to be an immersion if

- $\phi_{\mathbb{C}}$ is an immersion of complex varieties
- $\forall D \in \mathcal{F}_{ab}$ the map $\phi : \underline{X}(D) \longrightarrow \underline{Y}(D)$ is injective

Putting all this together, an <u>affine variety</u> over $\mathbb{F}_1$ is a finite graded gadget X such that exists an affine alg. variety $X_{\mathbb{Z}}$ and an immersion of gadgets

$$i : X \longrightarrow \mathcal{G}(X_{\mathbb{Z}})$$

with the following universal property:

For any affine variety V $(/\mathbb{Z})$ and any morphism of gadgets

$$\begin{tikzcd}
X \arrow[r, "\varphi"] \arrow[dr, "i"'] & \mathcal{G}(V) \arrow[r, squiggly, "\mathcal{G}"] & V \\
& \mathcal{G}(X_{\mathbb{Z}}) \arrow[u, dashed, "\mathcal{G}(\varphi_{\mathbb{Z}})"] \arrow[r, squiggly, "\mathcal{G}"'] & X_{\mathbb{Z}} \arrow[u, dashed, "\varphi_{\mathbb{Z}}"']
\end{tikzcd}$$

there exists a <u>unique</u> morphism of alg. varieties $\varphi_{\mathbb{Z}} : X_{\mathbb{Z}} \longrightarrow V$ such that $\boxed{\mathcal{G}(\varphi_{\mathbb{Z}}) \circ i = \varphi}$

<u>Varieties over $\mathbb{F}_{1^n}$</u>

In order to define varieties over the field extensions $\mathbb{F}_{1^n}$, we replace the category $\mathcal{F}_{ab}$ of finite abelian groups by the category $\mathcal{F}_{ab}^{(n)}$ with objects

$$\mathcal{F}_{ab}^{(n)} = \{ (D, \varepsilon) \mid D \in \mathcal{F}_{ab},\ \varepsilon \in D \text{ s.t. } \varepsilon^n = 1 \}$$

and having as morphisms ~~the maps~~ $(D, \varepsilon) \longrightarrow (D', \varepsilon')$ morphisms $\varphi : D \longrightarrow D'$ of groups such that $\varphi(\varepsilon) = \varepsilon'$

In this situation, we require the natural transformation e_X to be defined as $e_X : \underline{X} \longrightarrow \mathrm{Hom}(\mathrm{Spec}(\mathbb{Z}[D] \otimes_{\mathbb{Z}/n\mathbb{Z}} \mathbb{C}), X_{\mathbb{C}})$

where $\mathbb{Z}/n\mathbb{Z}$ acts on $\mathbb{Z}[D]$ by multiplication by powers of ε and in $\mathbb{C}$ by powers of the primitive n-th root of unity $\xi = e^{2\pi i/n}$.

Counting functions and zeta functions

For an algebraic variety defined over a finite field $\mathbb{F}_q$, the corresponding zeta function is obtained by using the number of points of the variety on the field extensions $\mathbb{F}_{q^n}$. In order to define zeta functions for varieties $/\mathbb{F}_1$, Soulé proposes using the number of points $\#\underline{X}(R_n)$. In order to develop his theory, he makes an (strong) extra assumption

(Z) There is a polynomial $N(x) \in \mathbb{Z}[X]$ such that, for every $n \geq 1$ we have $\#\underline{X}(R_n) = N(2n+1)$ // $N(q) = \#\underline{X}_{\mathbb{Z}}(\mathbb{F}_q)$ for many examples

This counting function is used to define a zeta function $\zeta_X(s)$. In terms of the polynomial $N(x) = a_0 + a_1 x + \cdots + a_d x^d$, Soulé shows that

$$\zeta_X(s) = s^{a_0}(s-1)^{a_1} \cdots (s-d)^{a_d}$$

Though right now we won't go into the theory of zeta functions, ~~they are~~ the underlying principle in the definition of the graded functor $\underline{X} = \coprod \underline{X}^{(k)}$ is the following:

"On the Taylor expansion at $q=1$ of the counting function $N(q) = \underline{X}(\mathbb{F}_q)$, the term of degree k (i.e. $a_k(q-1)^k$) ~~is the~~ should coincide with the cardinality of the set $\underline{X}^{(k)}(D)$ for $D \in \mathcal{F}_{ab}$, $|D| = q-1$"

As a remark, the requirement of $N(q)$ being polynomial fails for ~~many~~ many interesting situations, like for elliptic curves, so this is something that should be improved.

Some examples (redone)

- The multiplicative group $\mathbb{G}_m$

$$\mathbb{G}_m(\mathbb{F}_q) = \mathbb{F}_q^\times \Rightarrow N(q) = q-1 = \underbrace{1}_{\text{is}} \cdot \underbrace{(q-1)}_{\text{degree } 1}$$

So we define the functor $\underline{\mathbb{G}_m} : \mathcal{F}_{ab} \longrightarrow \text{Gr Sets}$

by $\underline{\mathbb{G}_m}(D)^{(k)} = \begin{cases} \emptyset & \text{if } k \neq 1 \\ D & \text{if } k = 1 \end{cases}$

In particular, one gets

$$\underline{\mathbb{G}_m}(\mathbb{F}_{1^n})^{(k)} = \begin{cases} \emptyset & \text{if } k \neq 1 \\ \mathbb{Z}/n\mathbb{Z} & \text{if } k = 1 \end{cases}$$

The "evaluation" is the natural transformation

$$e_m : \underline{\mathbb{G}_m} \longrightarrow \text{Hom}(\text{Spec}\, \mathbb{C}[-], \mathbb{G}_m(\mathbb{C}))$$

which associates to $\chi \in \text{Spec}\, \mathbb{C}[D]$ (i.e, $\chi : \mathbb{C}[D] \longrightarrow \mathbb{C}$)
$D \longrightarrow \mathbb{C}^*$

the group homomorphism $e_m(D)(g) = \chi(g)$

The gadget $(\underline{\mathbb{G}_m}, \mathbb{G}_m(\mathbb{C}), e_m)$ defines a variety over $\mathbb{F}_1$

Affine space $\mathbb{A}^n$

line: $N(q) = \# \mathbb{A}^1(\mathbb{F}_q) = q = \underbrace{1}_{\deg 0} + \underbrace{1 \cdot (q-1)}_{\deg 1}$

So we define $\underline{\mathbb{A}}^1 : \mathcal{F}_{ab} \longrightarrow \text{Gr Sets}$

$$\underline{\mathbb{A}}^1(D)^{(k)} = \begin{cases} \{0\} & \text{if } k = 0 \\ D & \text{if } k = 1 \\ \emptyset . & \text{if } k \geq 2 \end{cases}$$

For the general ~~proper~~ affine space $\mathbb{A}^n$ we have

$$\# \underline{\mathbb{A}}^n(\mathbb{F}_q) = q^n = (1+(q-1))^n = \sum_{k=0}^{n} \binom{n}{k} (q-1)^k = N(q)$$

So $\# \underline{\mathbb{A}}^n(D)^{(k)} = \binom{n}{k} |D|^k$, we define $\underline{\mathbb{A}}^n(D)^{(k)} = \coprod_{i=1,\dots,\binom{n}{k}} D^i$

Or, w/ a different notation, if F finite set ~~of~~ with $|F| = n$,

then $\underline{\mathbb{A}}^F(D)^{(k)} := \coprod_{\substack{Y \subseteq F \\ |Y| = k}} D^Y$ (Y is not just for book-keeping!) $D^Y = D \times D \times \overset{|Y|}{\cdots} \times D$

The natural transformation $e_F : \underline{\mathbb{A}}^F \longrightarrow \mathrm{Hom}(\mathrm{Spec}\, \mathbb{C}[-], \mathbb{C}^F)$

is ~~given by~~ ~~$D \mapsto \mathrm{Hom}(\mathrm{Spec}\, \mathbb{C}[D], \mathbb{C}^F)$~~

~~$q \mapsto e_F^q : \mathrm{Spec}\, \mathbb{C}[D] \longrightarrow \mathbb{C}^F$~~

~~$\chi \longmapsto e_F^q(\chi) :=$~~

given by assigning to a character $\chi : \mathbb{C}[D] \longrightarrow \mathbb{C}$ $(\chi : D \to \mathbb{C}^*)$

the map $\coprod_{Y \subset F} D^Y \longrightarrow \mathbb{C}^F$

$$e_F(D)((q_i)_{j \in Y}) := (\xi_j)_{j \in F}, \text{ where } \xi_j = \begin{cases} \chi(q_j) & \text{if } j \in Y \\ 0 & \text{if } j \notin Y \end{cases}$$

The gadget $\mathbb{A}^F = (\underline{\mathbb{A}}^F, \mathbb{C}^F, e_F)$ defines a variety over $\mathbb{F}_1$

Projective space $\mathbb{P}^d$

~~Note~~ $\# \mathbb{P}^d(\mathbb{F}_q) = \frac{q^{d+1} - 1}{q - 1}$; $q^d = \sum_{k=0}^{d+1} \binom{d+1}{k} (q-1)^d = 1 + \sum_{k=1}^{d} \binom{d+1}{k} (q-1)^d$

$$\Rightarrow q^{d+1} - 1 = \sum_{k=1}^{d+1} \binom{d+1}{k} (q-1)^k \Rightarrow \frac{q^{d+1} - 1}{q-1} = \sum_{k=1}^{d+1} \binom{d+1}{k} (q-1)^{k-1} =$$

$$= \sum_{j=0}^{d} \binom{d+1}{j+1} (q-1)^j = N(q)$$

We define the functor $\underline{\mathbb{P}}^d : \mathcal{F}_{ab} \longrightarrow \mathrm{Gr\,Sets}$ by

$$\underline{\mathbb{P}}^d(D)^{(k)} = \coprod_{\substack{Y \subseteq \{1, \dots, d+1\} \\ |Y| = k+1}} D^Y / D \quad \forall\, k \geq 0$$

where D acts on D^Y by diagonal action

In particular

$$\mathbb{P}^d(\mathbb{F}_1)^{(0)} = \{1, 2, \dots, d+1\}$$