Fun seminar

The BC system and Riemann zeta function (Talk by BRAM MESLAND)

A C^*-dynamical system is a pair (A, σ) where A is a C^*-alg.
and $\sigma : \mathbb{R} \longrightarrow Aut(A) := \{*\text{-automorphisms of } A\}$ (continuous) group homomorphism
(σ is called the "time evolution") i.e, $\{\sigma_t\}_{t\in\mathbb{R}}$ is a 1-parameter group of automorphisms of A

Ex: D self adjoint (unbounded) op. on a Hilbert space $\mathcal{H}$, $A = \mathcal{B}(\mathcal{H})$,
assume $e^{-tD} \in \mathcal{L}^1(\mathcal{H})$ $\forall\, t>0$. Then $\sigma_t(a) := e^{itD} a e^{-itD}$ gives a
C^*-dynamical system.

A state on a C^*-alg A is a linear map $\varphi : A \longrightarrow \mathbb{C}$ such that $\|\varphi\| = 1$
$\varphi(aa^*) \geq 0$ $\forall a \in A$

Let (A, σ) be a C^*-dyn. sys. A state $\varphi : A \to \mathbb{C}$ is ~~pure~~ a KMS_β-state
(for $\beta > 0$, $\beta \in \mathbb{R}$) if there is a function $F : \{0 \leq \operatorname{Im} z \leq \beta\} \longrightarrow \mathbb{C}$
holomorphic in the interior (and continuous in the boundary) such that

$$\forall\, t \in \mathbb{R},\ a, b \in A \qquad F(t) = \varphi(a\,\sigma_t(b))$$
$$F(t + i\beta) = \varphi(\sigma_t(b)\,a)$$

Rk In quantum systems, KMS_β states describe "equilibrium" states at temperature $1/\beta$

In our former example, $\varphi_\beta(a) = \dfrac{Tr(a\,e^{-\beta D})}{T(e^{-\beta D})}$ gives a KMS_β-state

For each $\beta > 0$, the KMS_β-states form a simplex, and any KMS_β-state is
a unique convex combination of extremal KMS_β-states

We are interested in one particular example of such a system, that is related with the Riemann zeta function (and eventually with the field with one element)

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Let $\Gamma_0 \subseteq \Gamma$ be an inclusion of groups such that Γ_0 is <u>almost normal</u> in Γ (i.e, the orbits of the right action of Γ on Γ/Γ_0 is finite)

For such a pair (Γ_0, Γ) we can define a <u>Hecke algebra</u>

$$\mathcal{H}(\Gamma, \Gamma_0) = \mathcal{H} := \{ f : \Gamma_0 \backslash \Gamma / \Gamma_0 \longrightarrow \mathbb{C} \mid \mathrm{supp}(f) \text{ is finite} \}$$

with the convolution product

$$(f * g)(\gamma) = \sum_{\delta \in \Gamma_0 \backslash \Gamma} f(\gamma \delta^{-1}) \, g(\delta) \qquad [*]$$

and involution $\quad f^*(\gamma) := \overline{f(\gamma^{-1})}$

The formula $[*]$ defines a $*$rep. of $\mathcal{H}(\Gamma, \Gamma_0)$ on $\ell^2(\Gamma_0 \backslash \Gamma)$

Define $\quad A = \overline{\mathcal{H}(\Gamma, \Gamma_0)}$ (closure wrt this representation)

Set $L(\gamma)$ = cardinality of the Γ_0-orbit of $[\gamma] \in \Gamma/\Gamma_0$

$R(\gamma) := L(\gamma^{-1})$ (= cardinality of Γ_0-orbit of $[\gamma] \in \Gamma_0 \backslash \Gamma$)

<u>Prop</u>: The following formula

$$\sigma_t(f)(\gamma) := \left(\frac{L(\gamma)}{R(\gamma)} \right)^{-it} f(\gamma)$$

defines a 1-parameter group of automorphisms of A.

The Bost-Connes system is obtained by applying the above construction to the pair $(P^+_{\mathbb{Q}}, P^+_{\mathbb{Z}})$, where

$$P^+_{\mathbb{Q}} = \left\{ \begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix} \;\middle|\; b \in \mathbb{Q},\ a \in \mathbb{Q}_{>0} \right\}$$

$$P^+_{\mathbb{Z}} = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \;\middle|\; n \in \mathbb{Z} \right\}$$

The map

$$\mathbb{Q}/\mathbb{Z} \longrightarrow P^+_{\mathbb{Z}} \backslash P^+_{\mathbb{Q}} / P^+_{\mathbb{Z}}$$
$$[q] \longmapsto \begin{bmatrix} 1 & q \\ 0 & 1 \end{bmatrix}$$

is well defined, and extends to an injection

$$\begin{array}{ccc} \mathbb{C}[\mathbb{Q}/\mathbb{Z}] & \longrightarrow & \mathcal{H}(P^+_{\mathbb{Q}}, P^+_{\mathbb{Z}}) \\ \cap & & \cap \\ C^*(\mathbb{Q}/\mathbb{Z}) & \longrightarrow & A \end{array}$$

If $q = \frac{a}{b} \in \mathbb{Q}/\mathbb{Z}$, $a, b \in \mathbb{Z}$, $\gcd(a,b) = 1$, $b > 0$

$b = \prod_{p \in \mathcal{P}} p^{k_p}$ prime decomposition, then for $\beta > 0$, define

$$\Psi_\beta(q) := \prod_{p \in \mathcal{P}} p^{-\beta k_p} (1 - p^{\beta - 1})(1 - p^{-1})^{-1}$$

Theorem (BC '95)

Classification of KMS_β-states for (A, σ)

1. For $0 < \beta \leq 1$, the functions Ψ_β extend uniquely to KMS_β-states for (A, σ) and every KMS_β-state is obtained in this way.
2. For $\beta \in (1, \infty)$, the extreme points of the simplex Δ_β of KMS_β-states are parametrized by embeddings $\chi : \mathbb{Q}[\mathbb{Q}/\mathbb{Z}] = \mathbb{Q}^{ab} \longrightarrow \mathbb{C}$, and their restrictions to $\mathbb{C}[\mathbb{Q}/\mathbb{Z}] \subset A$ are given by

$$\boxed{\varphi_{\beta,\chi}(\gamma) = \zeta(b)^{-1} \sum_{n \geq 1} n^{-\beta} \chi(\gamma)^n}$$

(ζ = Riemann zeta function)

Define the <u>adeles</u> $\mathbb{A}_{\mathbb{Q}} := \{(\alpha_\infty, \alpha_2, \alpha_3, \dots)\} \subseteq \mathbb{R} \times \prod_{p \in P} \mathbb{Q}_p$

where all but finitely many of the α_p are in $\mathbb{Z}_p$ (p-adic integers)

$\mathbb{Q}^\times$ acts diagonally on $\mathbb{A}_{\mathbb{Q}}$: $q \cdot (\alpha_0, \alpha_2, \alpha_3, \dots) = (q\alpha_0, q\alpha_2, q\alpha_3, \dots)$

$$\hat{\mathbb{Z}} = \prod_{p \in P} \mathbb{Z}_p \subseteq \mathbb{A}_{\mathbb{Q}}$$

We can define the action groupoid

$$\mathbb{A}_{\mathbb{Q}} \rtimes \mathbb{Q}^\times = \mathbb{A}_{\mathbb{Q}} \times \mathbb{Q}^\times \quad \text{with product} \quad (\alpha, q)(\alpha q, q') := (\alpha, qq')$$

The set $J = \{(\alpha, q) \mid \alpha \in \hat{\mathbb{Z}}, \alpha q \in \hat{\mathbb{Z}}\} \subseteq \mathbb{A} \rtimes \mathbb{Q}^\times$ is a subgroupoid of $\mathbb{A} \rtimes \mathbb{Q}^\times$

the convolution algebra $C_c(J)$ carries the product

$$(f * g)(\alpha, q) := \sum_{\substack{(\alpha', q') \\ \alpha' q' = \alpha}} f(\alpha', q')\, g(\alpha, q'^{-1} q)$$

and involution $f^*(\alpha, q) := \overline{f(\alpha q, q^{-1})}$

<u>Prop</u> $\boxed{C^*(J) \simeq A(P_{\mathbb{Q}}^+, P_{\mathbb{Z}}^+)}$

<u>the BC-alg. over $\mathbb{Q}$</u>

The group alg. $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ is generated by symbols

$e(\gamma)$ / $\gamma \in \mathbb{Q}/\mathbb{Z}$ (characteristic function of γ)

Now consider the extra generators μ_n, $n \in \mathbb{N}^+$, with relations

$\mu_n^* \mu_n = 1$

$\mu_{nm} = \mu_n \mu_m$ $\qquad \mu_n e(\gamma) \mu_n^* = \frac{1}{n} \sum_{n\delta = \gamma} e(\delta)$

$\mu_{nm}^* = \mu_n^* \mu_m^*$

$\mu_n \mu_m^* = \mu_m^* \mu_n$

Define $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \rtimes \mathbb{N}^+$ to be the alg. generated by $e(\gamma)$, μ_n, μ_n^* ~~and~~ with the above relations.