

The rational BC-alg $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \rtimes \mathbb{N}$

given by $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ and generators

$$\mu_n, \mu_n^* \text{ with relations } \left. \begin{aligned} \mu_n^* \mu_n &= 1 \\ \mu_{nm} &= \mu_n \mu_m \\ \mu_{nm}^* &= \mu_n^* \mu_m^* \\ \mu_n \mu_m^* &= \mu_m^* \mu_n \end{aligned} \right\}$$

$$\mu_n e(\gamma) \mu_n^* = \frac{1}{n} \sum_{n\delta=\gamma} e(\delta)$$

Consider the maps $f_n: \mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \longrightarrow \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$
 $e(\gamma) \longmapsto \frac{1}{n} \sum_{n\delta=\gamma} e(\delta)$

and the idempotent $\pi_n = \frac{1}{n} \sum_{s \in \Gamma_n} e(s)$ $\Gamma_n \subseteq \mathbb{Q}/\mathbb{Z}$ n -torsion
 (with roots of 1)

$\pi_n^2 = \pi_n$, and $f_n: \mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \xrightarrow{\sim} \pi_n \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ isomorphism

Define $\sigma_n(e(\gamma)) := e(n\gamma)$

$\sigma_n f_n = \text{Id}$

$f_n \sigma_n = \pi_n$ (actually, "multiplication by π_n ")

σ_n 's restrict to maps $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \longrightarrow \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$

Let $\hat{f}_n := n f_n$, then if we replace μ_n by $\tilde{\mu}_n = n \mu_n$,

we get a new set of relations:

$$\left. \begin{aligned} \tilde{\mu}_n \times \mu_n^* &= \hat{f}_n(x) \\ \mu_n^* \times &= \sigma_n(x) \mu_n^* \\ x \tilde{\mu}_n &= \tilde{\mu}_n \sigma_n(x) \end{aligned} \right| \begin{aligned} \tilde{\mu}_{nm} &= \tilde{\mu}_n \tilde{\mu}_m \\ \mu_{nm}^* &= \mu_n^* \mu_m^* \\ \mu_n^* \tilde{\mu}_m &= n \\ \tilde{\mu}_n \mu_m^* &= \mu_m^* \tilde{\mu}_n \end{aligned} \quad \begin{aligned} & \text{(Realize that one lost} \\ & \text{unitarity!)} \\ & x \in \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \end{aligned}$$

Define $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \times \mathbb{N}$ by the generators and relations as above.

This is related to the tower of field extensions

$$\mathbb{F}_n \subseteq \mathbb{F}_m \subseteq \dots \subseteq \mathbb{F}_\infty \quad \text{nil}, \text{ with endomorphisms}$$

coming from Frobenius-type maps.

Let \mathcal{R} the full-subcat of the cat. of f.s. flat rings / \mathbb{Z} generated by the rings $A_n = \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$ and their tensor products (à la Soulé)

Recall: A gadget (\underline{X}, A, e) is given by . . .

1) $\underline{X} = \mathcal{R} \rightarrow \text{Sets}$

2) A complex alg.

3) $e : \underline{X} \Rightarrow (\mathcal{R} \rightarrow \text{Hom}(A, R \otimes_{\mathbb{Z}} C))$ natural transf.

The varieties $\mu^{(k)}$

$$\mu^{(k)}(\mathcal{R}) := \{x \in \mathcal{R} \mid x^k = 1\} \quad (k\text{-th roots of } 1 \text{ in } \mathcal{R})$$

$$\mu^{(k)} = F(\text{Spec}(A_k)) \quad (F: \{\text{Varieties}/\mathbb{Z}\} \rightarrow \{\text{Schemes}/\mathbb{F}_k\})$$

this gives us an inductive system of varieties / \mathbb{F}_k

$$\text{Define } \mu^{(\infty)}(\mathcal{R}) := \varprojlim \text{Hom}(A_k, \mathcal{R}) = \text{Hom}(\mathbb{Z}[\mathbb{Q}/\mathbb{Z}], \mathcal{R})$$

The maps
$$\begin{array}{ccc} \sigma_n : A_k & \longrightarrow & A_k \\ T & \longmapsto & T^n \end{array}$$

are morphisms of the inductive system in the limit

$$\sigma_n : A = \varinjlim A_k \longrightarrow A \quad \text{and here they are surjective.}$$

Prop: The σ_n are morphisms of \mathbb{F}_p -varieties

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If K perfect field, $p = \text{char } K > 0$, then the map

$$\begin{array}{ccc} \sigma_{p^e} \otimes (\sigma_{\mathbb{F}_p})^e : \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \otimes_{\mathbb{Z}} K & \longrightarrow & \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \otimes_{\mathbb{Z}} K \\ \text{acts as} & & \{ \longmapsto \} p^e \end{array}$$

~~acts by~~