KAC-MOODY GROUPS, FINITE FIELDS AND TITS GEOMETRIES

Gallery

Curvature Flow

Starfish
Melting Matisse
(original cutout)

Dynamics on blowups of P^2

9 points : pencil
of cubics
10: dynamics /
cusp
11: dynamics /
cusp
11: dynamics /
conic+line
-- inverse
dynamics /
11 points: Siegel
disk
12: dynamics /
three lines
Invariant smooth
cubic
Stable/unstable
manifolds
Conway's map
(y,y-1/x)

Hyperbolic plane

Triangles and
curvature
Octagons
Pentagons
Decagons
SL_2 Z
Q/Z and unit disk
{{2,1},{1,1}}
geodesic
Gamma(2)
Ideal triangles
Farey packing
Geodesics
Horoballs
Torus lamination
Open Riemann

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In this talk we present the following viewpoint:

There is a notion of a 'field of characteristic 1' denoted $\mathbb{F}_1$.

Roughly speaking, if G is a group that is defined over a finite field $\mathbb{F}_q$ and G has a Dynkin diagram $\mathcal{D}$, then Tits provides us with a geometry X such that G is 'essentially' the group of automorphisms of X.

It makes sense to talk about G and X in the limit $\mathbb{F}_q \longrightarrow \mathbb{F}_1$ in such a way that G inherits a discrete structure and X inherits the geometry of this discrete structure.

These ideas are deeply embedded in the work of Tits concerning finite geometries, buildings and Kac-Moody group functors.

We will consider the following cases:

Finite groups: *G is a Chevalley group scheme over $\mathbb{F}_q$, X is the Bruhat-Tits spherical building of G.*

Finite dimensional groups: *G is a simple algebraic group over a nonarchimedean local field, X is the Bruhat-Tits affine building of G.*

'Infinite dimensional' groups: *- G is a Kac-Moody group functor of Tits over $\mathbb{F}_q$, X is the Tits building of G.*

Using the geometry of the Tits building over $\mathbb{F}_1$, I was able to prove:

○ Let G be a Kac-Moody group over a finite field that has 'type ∞', that is, the Weyl group W is a free product of $\mathbb{Z}/2\mathbb{Z}$'s. Then G acts 'nicely' on a simplicial tree. This gives a proof of the Kac-Peterson conjecture on the internal structure of G.

○ Let G be a Kac-Moody group over a finite field that has rank 2 or rank 3 noncompact hyperbolic type. Then G satisfies the Baum-Connes conjecture with coefficients in any C^*-algebra.

Chevalley groups over finite fields (1955)

Chevalley constructed a 'Chevalley basis', or '$\mathbb{Z}$-form' for the universal enveloping algebras of complex simple Lie algebras. This can be used to define the corresponding algebraic groups over $\mathbb{Z}$. It also leads to being able to take points of these algebraic groups with values in a finite field.

This gave the first unified construction of classical groups over fields other than $\mathbb{R}$ and $\mathbb{C}$, and also gave groups associated to E_6, E_7, E_8, F_4, and G_2 over finite fields.

This was an essential stage in the evolving classification of finite simple groups. After Chevalley's work, the distinction between classical groups occurring in the classification of Dynkin diagrams, and sporadic groups which did not, became sharp enough to be useful.

Tits geometries (1956)

Let K be a field, and G a finite dimensional simple Lie group over K.

Tits defined a 'geometry' $\Gamma_K(G)$ such that G is the group of 'transformations' of this geometry (preserving underlying axioms or properties of $\Gamma_K(G)$).

Sur les analogues algébriques des groupes semi-simples complexes, Colloque d'Algébre supérieure [1956, Bruxelles]

Tits' motivation was to find a 'geometric' interpretation of G, in contrast to the 'algebraic' analog of G proposed by Chevalley.

Tits' geometry associated to a classical group G was a precursor to the notion of a building by Bruhat and Tits (1972).

Fundamental example - projective geometry

Let $K = \mathbb{C}$ and let $G = PGL_{n+1}(\mathbb{C})$. Then Tits' geometry $\Gamma_K(G)$ is n-dimensional projective geometry $\mathcal{P}_n$ over $\mathbb{C}$.

This consists of subspaces $\mathcal{P}_i \subseteq \mathcal{P}_n$, $i = 0, \ldots, n$, such that

$\mathcal{P}_0 = $ 'points' $= 1$-dim subspaces of $\mathbb{C}^{n+1}$,
$\mathcal{P}_1 = $ 'lines' $= 2$-dim subspaces of $\mathbb{C}^{n+1}$, $\ldots$,
$\mathcal{P}_{n-1} = $ 'hyperplanes' $= (n-1)$-dim subspaces of $\mathbb{C}^{n+1}$,

with incidence given by inclusion as subspaces of $\mathbb{C}^{n+1}$. We have subgroups G_i of G

$G_0 = $ stabilizer of a point,
$G_1 = $ stabilizer of a line, $\ldots$
$G_{n-1} = $ stabilizer of a hyperplane,

and families $\mathcal{F}_i$

$\mathcal{F}_0 = G/G_0 \leftrightarrow$ points,
$\mathcal{F}_1 = G/G_1 \leftrightarrow$ lines, $\ldots$
$\mathcal{F}_{n-1} = G/G_{n-1} \leftrightarrow$ hyperplanes

which inherit the incidence relation.

The group G is then the group of automorphisms of $\Gamma_K(G)$ preserving families $\mathcal{F}_i$ and incidence.

Tits geometries over finite fields

Let G be a Chevalley group over a finite field $\mathbb{F}_q$. Let W be the Weyl group of G, defined as $W = N(T)/Z(T)$ where T is a maximal torus, $N(T)$ and $Z(T)$ are the normalizer and the centralizer of T in G.

If Φ is the root system of G, then W is a subgroup of the isometry group of Φ. Specifically, it is the subgroup which is generated by reflections in the hyperplanes orthogonal to the roots.

Tits suggests that there is a 'field of characteristic 1' denoted $\mathbb{F}_1$ such that in the limit $\mathbb{F}_q \longrightarrow \mathbb{F}_1$, G takes on the discrete structure of the Weyl group W

$$G(\mathbb{F}_q) \xrightarrow{q \mapsto 1} W,$$

and the finite geometry associated to G approaches the finite geometry of W

$$\Gamma(G) \xrightarrow{q \mapsto 1} \Gamma(W).$$

Example: $PGL_3(\mathbb{F}_2) \longrightarrow PGL_3(\mathbb{F}_1)$

Let $G = PGL_3(\mathbb{F}_2)$. Then G is a simple Lie group of type A_2 and order

$$2^3(2^3 - 1)(2^2 - 1) = 168$$

and W is the dihedral group of order 6.

The Tits geometry $\Gamma_{\mathbb{F}_2}(G)$ is the flag complex of a projective plane over $\mathbb{F}_2$. A projective plane is a 2 dimensional incidence geometry of points $\mathcal{P}_0$ and lines $\mathcal{P}_1$ satisfying the usual axioms:

$\mathcal{P}_0 = $ 'points' $= $ 1-dim subspaces of $\mathbb{F}_2^3$,
$\mathcal{P}_1 = $ 'lines' $= $ 2-dim subspaces of $\mathbb{F}_2^3$,

with incidence given by inclusion - a point $p \in \mathcal{P}_0$ is incident on a line $L \in \mathcal{P}_1$ if $p \subset L$ as subspaces of $\mathbb{F}_2^3$.

The flag complex is a graph where adjacent vertices correspond to pairwise incident elements.

$\Gamma_{\mathbb{F}_2}(G) = $ **flag complex of a projective plane over** $\mathbb{F}_2$
$$G = PGL_3(\mathbb{F}_2)$$

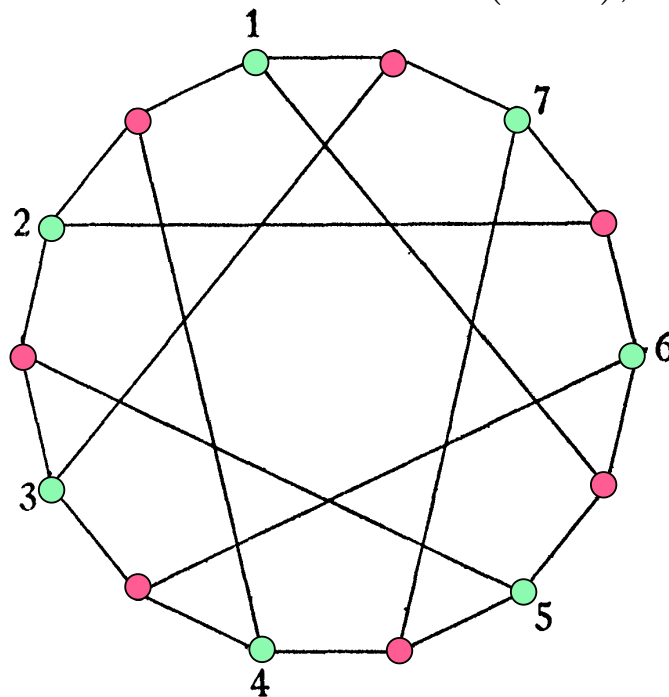- points $\leftrightarrow G/G_0$

- lines $\leftrightarrow G/G_1$

$G_0 = $ stabilizer of a point,
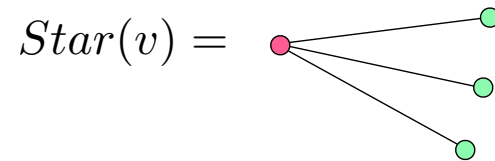$G_1 = $ stabilizer of a line

There are 7 points, 7 lines, 14 vertices and 21 edges.

The underlying diagram is taken from Coxeter's paper *Self-dual configurations and regular graphs* Bull. Amer. Math. Soc. 56 (1950), 413-455, where he names it the *'6-cage'*.

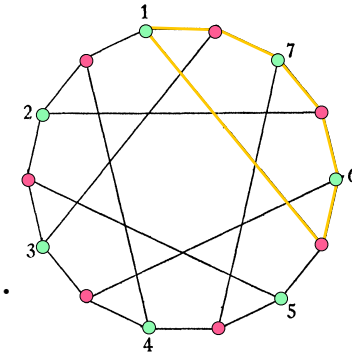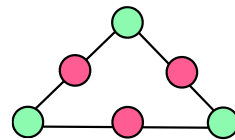# The limit $\mathbb{F}_2 \longrightarrow \mathbb{F}_1$

To understand the limit $\mathbb{F}_2 \longrightarrow \mathbb{F}_1$ we must understand the local picture in $X = \Gamma_{\mathbb{F}_2}(G)$. If the vertex v represents a point, then

$$Star(v) = $$

$| Star(v) | =$ no. of 1 dim subspaces of the 2 dim space $\mathbb{F}_2^2 = | \mathbb{P}^1(\mathbb{F}_2) | = 3$
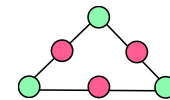
As $\mathbb{F}_2 \longrightarrow \mathbb{F}_1$, $| Star(v) | \longrightarrow | \mathbb{P}^1(\mathbb{F}_1) | = 2$

Geometry of $PGL_3(\mathbb{F}_2) \longrightarrow$

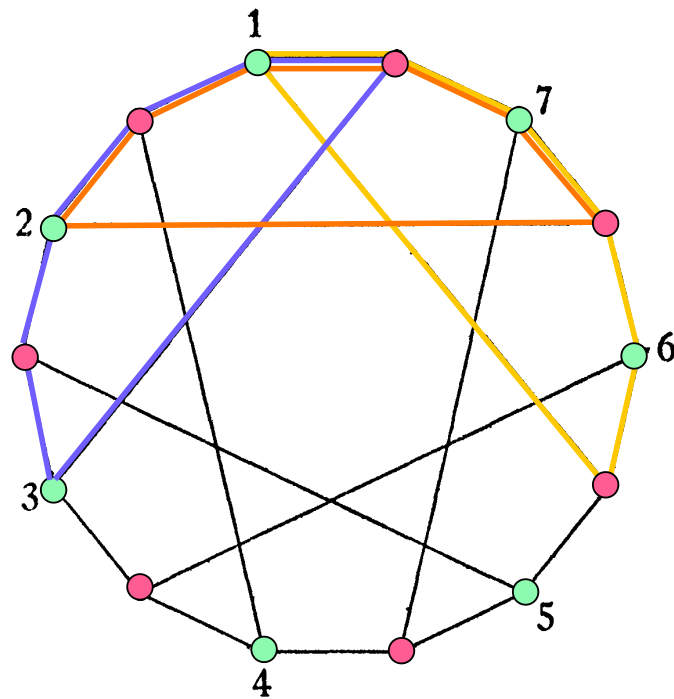This is the Geometry of W, the dihedral group of order 6.

And $PGL_3(\mathbb{F}_2) \longrightarrow$ group of type preserving automorphisms of

Thus $PGL_3(\mathbb{F}_2) \longrightarrow W$.

The limit $q \longrightarrow 1$ induces a *retraction* of X onto a single hexagon. This is known as *retracting a building onto an apartment*.

Conversely we can think of $X = \Gamma_{\mathbb{F}_2}(G)$ as gluing together copies of the geometry over $\mathbb{F}_1$: 3 hexagons along each edge

Tits geometries for infinite groups

Let G be

$\circ$ A simple algebraic group over a non-archimedean local field K ($K = \mathbb{Q}_p$, a finite extension of $\mathbb{Q}_p$ or $K = \mathbb{F}_q((t^{-1}))$),

$\circ$ A Kac-Moody group functor over a finite field $\mathbb{F}_q$.

Tits generalized his notion of finite geometries to such groups by associating a 'building' X to a collection of group theoretic data called a 'BN-pair' or 'Tits system'. In the finite dimensional case this was joint work with Bruhat (1972).

A BN-pair is a collection B and N of subgroups of G, such that $B \cup N$ generates G, $B \cap N$ is normal in N, the Weyl group W equals $N/B \cap N$ and this data satisfies other axioms.

A building is a simplicial complex X that can be expressed as a union of subcomplexes called apartments which are isomorphic Coxeter complexes satisfying certain axioms.

As in the case of finite Chevalley groups, one can see that $G(\mathbb{F}_q) \longrightarrow W$ as $q \longrightarrow 1$ and that the building X approaches the geometry of the Weyl group W of G, which is an infinite group here.

Whenever 'possible', the G is the automorphism group of the building X.

Example - a rank 2 affine group $G = SL_2(\mathbb{F}_q((t^{-1})))$

Let $G = SL_2(\mathbb{F}_q((t^{-1})))$. Then G has a BN-pair of Tits, where

$$B = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{F}_q[[t^{-1}]]) \mid c \equiv 0 \ mod(t^{-1})\},$$

$$N = G \cap \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \quad \cup \quad G \cap \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}.$$

We recall that $B \cup N$ generates G, $B \cap N$ is normal in N
and this data satisfies other axioms

Weyl group

The Weyl group W is the infinite dihedral group

$$W = N/(B \cap N) = <w_1> * <w_2> \cong \mathbb{Z} \rtimes \{\pm I\}$$

$$\text{where } w_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ w_2 = \begin{pmatrix} 0 & -t \\ 1/t & 0 \end{pmatrix}.$$

Parabolic subgroups

The subgroup B is called a *minimal parabolic subgroup*.

We define $P_1 := B \sqcup Bw_1B, \quad P_2 := B \sqcup Bw_2B.$

Then $P_1 = SL_2(\mathbb{F}_q[[t^{-1}]]), P_2 \cong SL_2(\mathbb{F}_q[[t^{-1}]])\}$. These are compact subgroups of G

and $P_1 \cap P_2 = B.$

Ihara-Bass-Serre-Tits amalgam decomposition

$$G \cong P_1 *_B P_2$$

Bruhat-Tits building of $G = SL_2(\mathbb{F}_q((t^{-1})))$

We define the Bruhat-Tits building of G as follows:

$$VX = G/P_1 \sqcup G/P_2,$$
$$EX = G/B \sqcup \overline{G/B},$$

G acts on X by left multiplication on cosets.

For $Q_1, Q_2 \in VX$, Q_1 and Q_2 are adjacent if and only if
$Q_1 \cap Q_2$ contains a conjugate of B.

Edges emanating from P_1 and P_2 may be indexed over $\mathbb{P}^1(\mathbb{F}_q)$:

$$Star(P_1) = \{B, \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} w_1 B/B \mid s \in \mathbb{F}_q\} \leftrightarrow \mathbb{P}^1(\mathbb{F}_q)$$

$$StarX(P_2) = \{B, \begin{pmatrix} 1 & 0 \\ s/t & 1 \end{pmatrix} w_2 B/B \mid s \in \mathbb{F}_q\} \leftrightarrow \mathbb{P}^1(\mathbb{F}_q)$$

X is a homogeneous bipartite tree of degree $[P_1 : B] = [P_2 : B] = q + 1$.

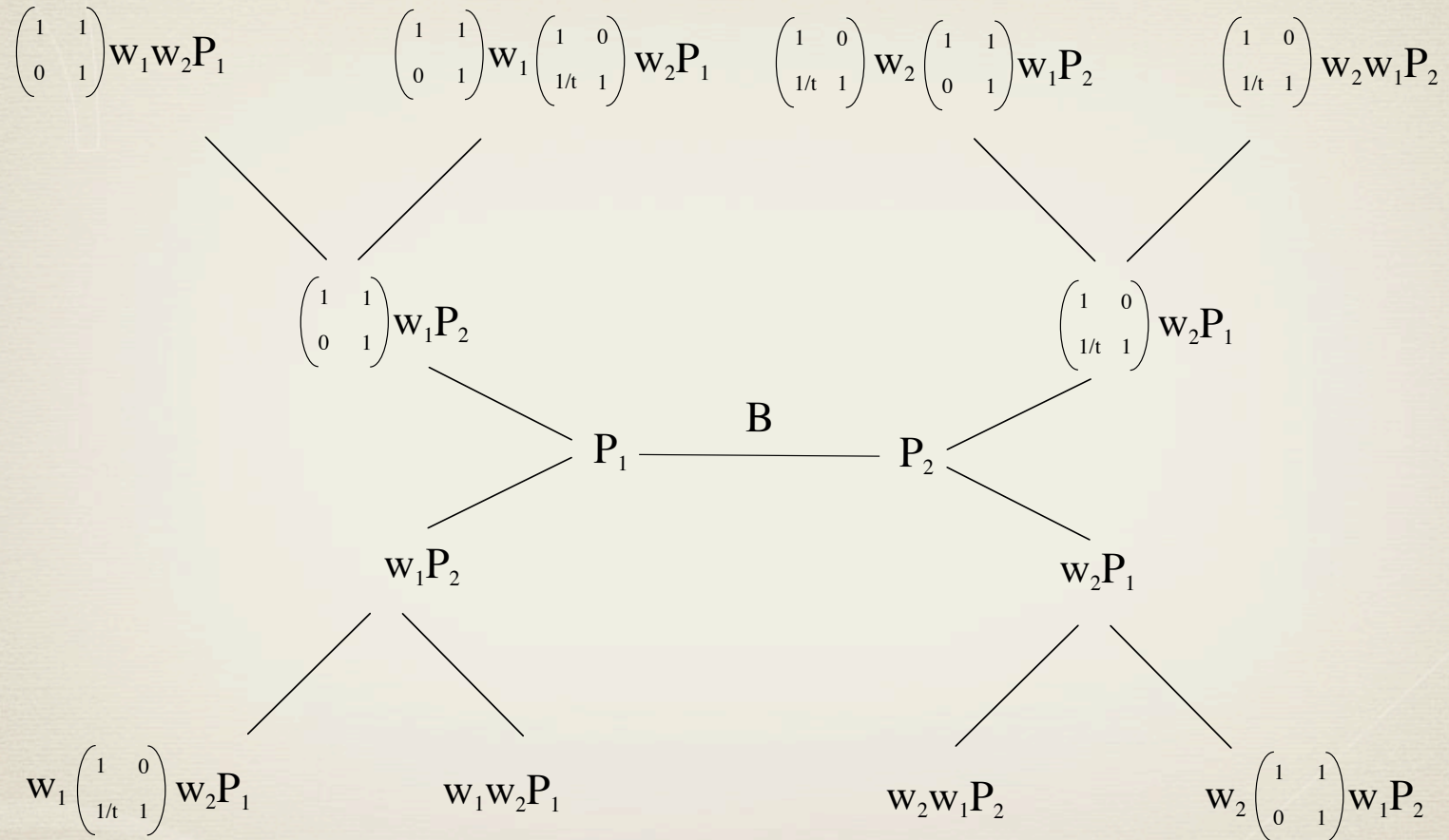The stabilizers in G for the vertices of X are conjugates of P_1 and P_2.
The stabilizers in G of edges are the conjugates of B.

G acts transitively on edges and has 2 orbits for vertices.

Here, G is no longer the full group of automorphisms of X.

For $G = SL_2(\mathbb{F}_q((t^{-1})))$, **the tree over the field of 2 elements**

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} w_1 w_2 P_1$

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} w_1 \begin{pmatrix} 1 & 0 \\ 1/t & 1 \end{pmatrix} w_2 P_1$

$\begin{pmatrix} 1 & 0 \\ 1/t & 1 \end{pmatrix} w_2 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} w_1 P_2$

$\begin{pmatrix} 1 & 0 \\ 1/t & 1 \end{pmatrix} w_2 w_1 P_2$

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} w_1 P_2$

$\begin{pmatrix} 1 & 0 \\ 1/t & 1 \end{pmatrix} w_2 P_1$

$P_1 \quad \overset{B}{\rule{3cm}{0.4pt}} \quad P_2$

$w_1 P_2$

$w_2 P_1$

$w_1 \begin{pmatrix} 1 & 0 \\ 1/t & 1 \end{pmatrix} w_2 P_1$

$w_1 w_2 P_1$

$w_2 w_1 P_2$

$w_2 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} w_1 P_2$

As $\mathbb{F}_q \longrightarrow \mathbb{F}_1$, $\mid Star(v) \mid \longrightarrow \mid \mathbb{P}^1(\mathbb{F}_1) \mid = 2$ and the geometry of X approaches the geometry of the infinite dihedral group W which is a bi-infinite line. This is the Coxeter complex of W, also called an apartment.
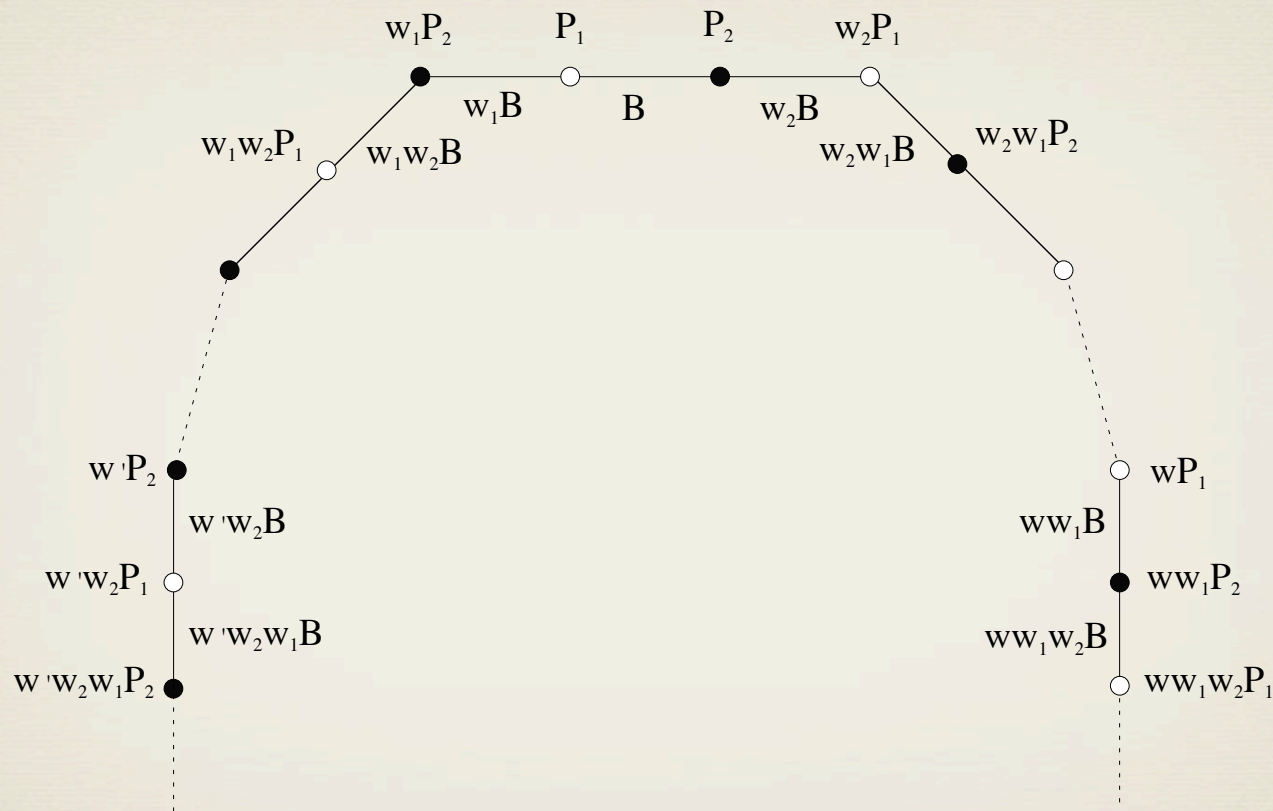
The BN-pair structure gives a Bruhat decomposition $G = \bigsqcup_{w \in W} BwB$.

As $\mathbb{F}_q \longrightarrow \mathbb{F}_1$, $B \longrightarrow \{1\}$ so by the Bruhat decomposition, $G \longrightarrow W$

KAC-MOODY GROUPS, INTRODUCTORY REMARKS

$\mathfrak{g}$ Kac-Moody Lie algebra over K, a field
$\mathfrak{g}$ has **finite, affine, or hyperbolic** type

If $\mathfrak{g}$ is affine or hyperbolic, $\mathfrak{g}$ is infinite dimensional
$\mathfrak{g}$ is the most natural generalization to infinite dimensions
of a finite dimensional simple Lie algebra.

G Kac-Moody group associated to $\mathfrak{g}$,
an abstract group, also constructed over K

No unique method for constructing G
Several constructions using a combination of techniques
and additional external data.

Most constructions use the Tits functor G

If $K = \mathbb{F}_q$ then G is locally compact and totally disconnected
G has an action on a locally finite Tits building X

Data for constructing Kac-Moody groups

The data used to construct a Kac-Moody group contains a generalized Cartan matrix $A = (A_{ij})_{i,j \in I}$. The entries satisfy the conditions $A_{ij} \in \mathbb{Z}$, $i, j \in I$, $A_{ii} = 2$, $i \in I$ and $A_{ij} \leq 0$ if $i \neq j$. We assume further that A is *symmetrizable:* there exist positive rational numbers $q_1, \ldots, q_l$, such that the matrix DA is symmetric, where $D = diag(q_1, \ldots, q_l)$.

Three possible types - Bourbaki definition

Finite type A is positive-definite. In this case A is the Cartan matrix of a finite dimensional semisimple Lie algebra.

Affine type A is positive-semidefinite, but not positive-definite.

Hyperbolic type if A is neither of finite nor affine type, but every proper, indecomposable submatrix is either of finite or of affine type.

Hyperbolic types

If A is of hyperbolic type, we say that A is of *compact hyperbolic type* if every proper, indecomposable submatrix is of finite type.

If A contains an affine submatrix, then we say that A is of *noncompact hyperbolic type*. This is equivalent to the condition that the fundamental chamber of the hyperbolic Weyl group $W(A)$ is not compact (but has finite volume).

The Tits geometry for a Kac-Moody group - a building

A Kac-Moody group G over a finite field is locally compact and totally disconnected and G admits an action on a locally finite building X.

G finite dimensional, *X finite (spherical) building*

G affine type, *X affine building*

G hyperbolic type, *X hyperbolic building*

G has rank 2 (affine or hyperbolic), *X is a tree*

Vertices correspond to cosets G/P_i, where P_i are the maximal parabolic subgroups of G. If the Weyl group W is infinite, by the Solomon-Tits theorem X is contractible. The group G acts on X by left translation of cosets.

If G is of hyperbolic type, apartments in X are hyperbolic spaces tessellated by the action of the hyperbolic Weyl group W.

We may use data from the action of G on X to obtain a decomposition theorem for G as a free product of standard parabolic subgroups amalgamated over their intersections. Obtain also structure theorems, generators and relations for subgroups of G.

The Tits building as a symmetric space

As well as giving an infinite analog of the notion of a flag complex or projective plane, the Tits building plays the role of a symmetric space 'G/K' for a Kac-Moody group G and maximal compact subgroup K.

Let G be a locally compact Kac-Moody group, K a maximal compact subgroup. Since G is totally disconnected and K is open, G/K is discrete.

We 'repair' G/K to play the role of a symmetric space by making the discrete set G/K the vertices of a simplicial complex, the Tits building.

If Γ is a nonuniform lattice subgroup, we refer to $\Gamma \backslash G/K$ as an 'arithmetic quotient'.

KAC-MOODY COSET SPACES G/K

Motivation:

○ *The action of G and its subgroups on G/K* should reveal much about the structure of these groups.

○ *'Arithmetic quotients'* $\Gamma\backslash G/K$ for G locally compact and Γ a nonuniform lattice subgroup are the source of automorphic forms and other arithmetic information.

○ *Scalars fields in supergravity theories* always parametrize a coset space G/H. After dimensional reduction to dimensions $n \geq 3$, G is a noncompact real form of the exceptional Lie group E_{11-n} and H is its maximal compact subgroup. In dimensions ≤ 2, G is a hyperbolic Kac-Moody group and H is generated by Kac-Moody generators invariant under a certain involution. Very mysterious. (Cremmer and Julia, 1978).

○ *Geometrical objects of 11 dimensional supergravity* correspond to coordinates in the coset space $E_{10}/K(E_{10})$, where $K(E_{10})$ is the maximal compact subgroup of the canonical real form of the hyperbolic Kac-Moody group E_{10} (Damour, Henneaux and Nicolai also Brown, Ganor and Helfgott).

Tits geometries for Kac-Moody groups

We consider here the following class of examples. Let G be a Kac-Moody group over a finite field that has has 'type ∞', that is, the Weyl group W is a free product of $\mathbb{Z}/2\mathbb{Z}$'s.

This coincides with the class of all Kac-Moody groups corresponding to generalized Cartan matrices $A = (A_{ij})_{i,j \in I\}}$ with all $A_{ij}A_{ji} \geq 4$, $i \neq j$. In particular, this includes all rank 2 Kac-Moody groups, whose generalized Cartan matrices form the infinite family

$$A = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}, \ ab \in \mathbb{Z}_{\geq 4},$$

which is affine if $ab = 4$ and hyperbolic if $ab > 4$.

If $rank(G) \geq 3$, then the fundamental chamber for W lies in hyperbolic space and is not compact but has finite volume. In fact, the the fundamental chamber for W is an ideal simplex. This class of Kac-Moody groups should fall under the category 'noncompact hyperbolic type'.

Example   Let $A = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}$

A has Dynkin diagram
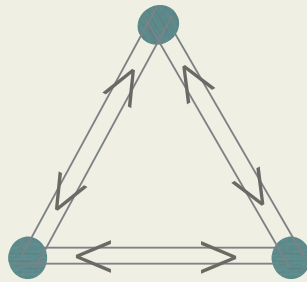
The Weyl group of A is the (∞, ∞, ∞)-triangle group:

$W = <w_1, w_2, w_3 \mid w_1^2 = w_2^2 = w_3^2 = 1> \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z},$

which contains $PGL_2(\mathbb{Z})$ as a subgroup of index 6.

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Tessellation of Poincare disk by (∞, ∞, ∞)-triangle group, a rank 3 Kac-Moody Weyl group of noncompact hyperbolic type

Example. $A = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}$, $\quad W(A) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$
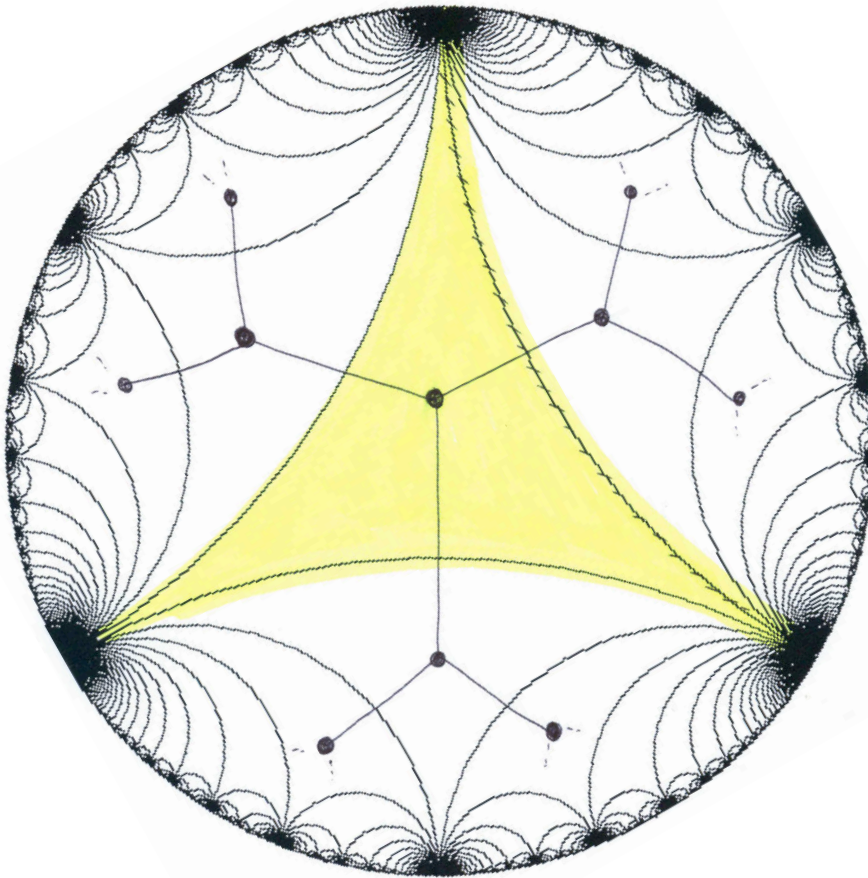

Diagram shows the geometry of the Tits building X over $\mathbb{F}_1$. This is an apartment (Poincaré disk) with $W(A)$ - tessellation by ideal triangles and a naturally inscribed tree

The Tits building can also be viewed as the universal covering of a complex of groups over the ideal simplex. However the theory of Tits already carries the data from the action of the Kac-Moody group on this complex of groups without referring to it explicitly.

Full Tits building over $\mathbb{F}_q$

Constructed by 'gluing' $(q+1)$ apartments along each chamber
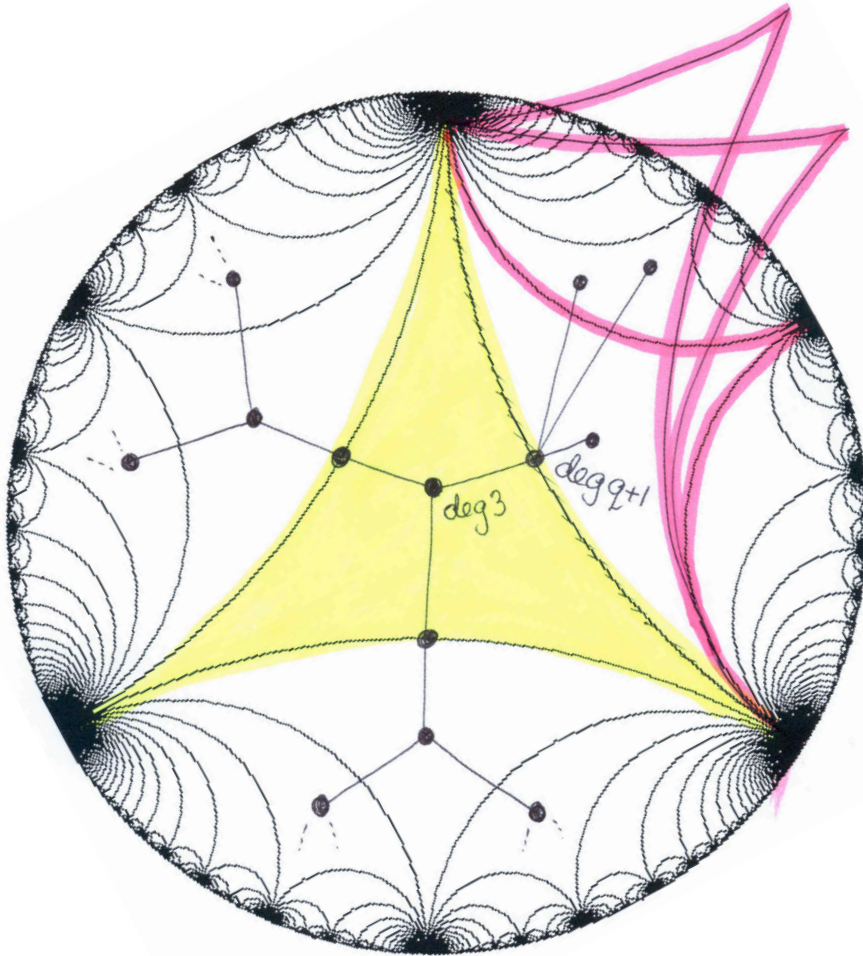
The inscribed tree $\mathcal{X}$ in the full Tits building X is a bi-homogeneous tree

The Kac-Moody group G acts on the Tits building X and on the inscribed tree $\mathcal{X}$

THE HAAGERUP PROPERTY AND PROPERTY (T)

Let G be a locally compact group. We say that there is a continuous, isometric action of G on some affine Hilbert space H if there is a a continuous map $G \longrightarrow Isom(H)$.

We say that the action of G on H is metrically proper if for any bounded subset B in H the set $K(G, B) := \{g \in G \text{ s.t. } gB \cap B \neq \emptyset\}$ has compact closure in G.

The locally compact group G satisfies the *Haagerup property*, (or is *a-T-menable*) if it admits a continuous, isometric, proper action on an affine Hilbert space.

A locally compact group G has Property (T) if and only if every continuous action of G by isometries on a Hilbert space has a fixed point. Other equivalent definitions in representation theory and ergodic theory all indicate that the Haagerup property is a strong negation of Kazhdan's Property (T).

Cherix, Martin, and Valette in 'Spaces with measured walls, the Haagerup property and Property (T)' (2004) showed that a group acting properly on a tree has the Haagerup property. Here 'properly' means that the action has finite vertex stabilizers.

The Haagerup property, the Baum-Connes conjecture and Property (T) for hyperbolic Kac-Moody groups

For all locally compact Kac-Moody groups of rank 2 or rank 3 noncompact hyperbolic type, we can construct a proper action on a tree, that is, with finite vertex stabilizers. This implies:

Theorem [C] *Let G be a Kac-Moody group over a finite field that has rank 2 or rank 3 noncompact hyperbolic type. Then G has the Haagerup property.*

Work of Higson and Kasparov shows that this is sufficient for the Baum-Connes conjecture. Thus we have revealed a new class of locally compact groups satisfying the strongest form of the Baum-Connes conjecture, that is with coefficients in any C^*-algebra.

Using the work of Dymara and Januszkiewicz we have also deduced that if G has compact hyperbolic type or if $rank(G) > 3$ and G has noncompact hyperbolic type then G has Kazhdan's Property (T).

CONCLUSION

The ingenuity of Tits shows itself in this subject in many different ways.

His invention of a 'geometry' on which a group acts gives the analog of a symmetric space for a wide class of groups.

This has had a profound impact on the development of both geometric and algebraic approaches to group theory.

The concept of a 'field of characteristic 1' is deeply and naturally embedded in Tits' work and is very useful in applications.