ON THE NOTION OF GEOMETRY OVER $\mathbb{F}_1$

ALAIN CONNES AND CATERINA CONSANI

Abstract. We refine the notion of variety over the "field with one element" developed by C. Soulé by introducing a grading in the associated functor to the category of sets, and show that this notion becomes compatible with the geometric viewpoint developed by J. Tits. We then solve an open question of C. Soulé by proving, using results of J. Tits and C. Chevalley, that Chevalley schemes are examples of varieties over a quadratic extension of the above "field".

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1. Introduction

In his theory of buildings J. Tits obtained a broad generalization of the celebrated von Staudt reconstruction theorem in projective geometry, involving as groups of symmetries not only GL_n but the full collection of Chevalley algebraic groups. Among the axioms which characterize these constructions, a relevant one is played by the condition of "thickness" which states, in its simplest form, that a projective line contains at least three points. By replacing this requirement with its strong negation, *i.e.* by imposing that a line contains exactly two points, one still obtains a coherent "geometry" which is a degenerate form of classical projective geometry. In the case of buildings, this degenerate case is described by the theory of "thin" complexes and in particular by the structure of the apartments, which are the basic constituents of the theory of buildings. The degeneracy of the von Staudt field inspired to Tits the conviction that these degenerate forms of geometries are a manifestation of the existence of a hypothetical algebraic object that he named "the field of characteristic one". The richness and beauty of this geometric picture gives convincing evidence for the pertinence of a separate study of the degenerate case.

For completely independent reasons the need for a "field of characteristic one" (also called the field with one element) has also emerged in Arakelov geometry and *e.g.* in [17], in the context of a geometric interpretation of the zeros of zeta and L-functions. These speculative constructions aim for the description of a geometric framework analogous to the one used by Weil in the proof of the Riemann Hypothesis for function fields. More precisely, one seeks for a replacement of the surface $C \times_{\mathbb{F}_q} C$, where C is a (projective, smooth algebraic) curve over a finite field $\mathbb{F}_q$ and whose field of functions is the given global field. The main idea is to postulate the existence of the "absolute point" $\operatorname{Spec}\mathbb{F}_1$ over which any algebraic scheme would sit. In the particular case of $\operatorname{Spec}\mathbb{Z}$, one would then be able to use the spectrum of the tensor product $\mathbb{Z} \times_{\mathbb{F}_1} \mathbb{Z}$ as a substitute for the surface $C \times_{\mathbb{F}_q} C$. This viewpoint has given rise, in the recent past, to a number of interesting constructions (*cf.* [15], [17], [19], [20], [12], [13], [23]).

Our interest in the quest for $\mathbb{F}_1$ arose from the following equation

$$\mathbb{F}_{1^n} \otimes_{\mathbb{F}_1} \mathbb{Z} := \mathbb{Z}[T]/(T^n - 1), \qquad n \in \mathbb{N} \tag{1}$$

which was introduced in [20] and supplies a definition of the finite extension $\mathbb{F}_{1^n}$ of $\mathbb{F}_1$, after base change to $\mathbb{Z}$. The main point promoted in [8] is that the above equation (1) yields (without knowing the algebraic nature of $\mathbb{F}_1$, and after base change to $\mathbb{Z}$), an algebraic object which reflects the structure of the inductive limit $\mathbb{F}_{1^\infty} = \varinjlim \mathbb{F}_{1^n}$, by supplying also an analogue of the geometric Frobenius correspondence. This object is the integral model of a rational Hecke algebra which defines the quantum statistical mechanical system of [1]. It is known that our construction determines, after passing to the dual system, a spectral realization of the zeros of the Riemann zeta function, as well as a trace formula interpretation of the Riemann-Weil explicit formulas (see [1], [5], [6], [7], [18]).

In [8], we made use of the general definition of an algebraic variety over $\mathbb{F}_1$ as introduced by C. Soulé in [20]. Our goal in this paper is to give an answer to the following two natural questions

- Are Chevalley schemes examples of varieties that can be defined over $\mathbb{F}_1$?

• Is the notion of variety over $\mathbb{F}_1$ as in *op.cit.* compatible with the geometric viewpoint developed by Tits?

The first question was formulated in [20]. In this article we show that Chevalley schemes can be defined over the quadratic extension $\mathbb{F}_{1^2}$ (*cf.* Theorem 4.10). The second question originates naturally by working with the simplest example of a projective variety, namely the projective spaces $\mathbb{P}^d$.

At first sight, a very serious problem emerges in [20], since the definition of the set $\mathbb{P}^d(\mathbb{F}_1)$ does *not* appear to be naturally linked with the notion of a geometry over $\mathbb{F}_1$ as in [21]. In fact, in § 6 of [20] the cardinality of the set $\mathbb{P}^d(\mathbb{F}_{1^n})$ is shown to be $N(2n+1)$, where $N(x)$ is a polynomial function whose values at prime powers $q = p^r$ are given by the classical formula $N(q) = q^d + \ldots + q + 1$ giving the cardinality of $\mathbb{P}^d(\mathbb{F}_q)$. When $n = 1$, one obtains the integer $|\mathbb{P}^d(\mathbb{F}_1)| = N(3) = \frac{3^{d+1}-1}{2}$ which is incompatible with (and much larger than) the number $d+1$ of points of the set $\mathcal{P}_d$ on which Tits defines his notion of projective geometry of dimension d over $\mathbb{F}_1$ (*cf.* [21], § 13 p. 285).

After clarifying a few statements taken from [20] on the notion of variety over $\mathbb{F}_1$ and on the meaning of a natural transformation of functors (*cf.* § 2), we show how to resolve the aforementioned problem by introducing a suitable refinement of the notion of algebraic variety over $\mathbb{F}_1$. The main idea is to replace the category of sets (in which the covariant functor $\underline{X}$ of *op.cit.* takes values) by the category of $\mathbb{Z}_{\geq 0}$-*graded sets*. In § 2.2 (*cf.* Definition 2.6), we explain how to refine the covariant functor $\underline{X}$ into a graded functor $\underline{X} = \coprod_k \underline{X}^{(k)}$ defined by a disjoint union of homogeneous components which correspond, at the intuitive level to the terms of the Taylor expansion, at $q = 1$, of the counting function $N(q)$. The condition that $N(q)$ is a polynomial is very restrictive (it fails *e.g.* for elliptic curves) and was required in [20] (§6, Condition Z) to define the zeta function. In § 3 we check that in the case of a projective space, the set $\underline{\mathbb{P}}^d(\mathbb{F}_{1^n})$ coincides *in degree zero* with the $d+1$ points of the set $\mathcal{P}_d$, and this result shows the sought for agreement with the theory of Tits.

Our new definition of variety over $\mathbb{F}_1$ is described by the following data

(a) A covariant functor from the category of finite abelian groups to the category of graded sets

$$\underline{X} = \coprod_{k\geq 0} \underline{X}^{(k)} : \mathcal{F}_{ab} \to \mathcal{S}ets$$

(b) A variety $X_{\mathbb{C}}$ over $\mathbb{C}$

(c) A natural transformation e_X connecting $\underline{X}$ to the functor

$$\mathcal{F}_{ab} \to \mathcal{S}ets, \quad D \mapsto \mathrm{Hom}(\mathrm{Spec}\,\mathbb{C}[D], X_{\mathbb{C}}).$$

These data need to fulfill also a strong condition (*cf.* Definition 2.8) which determines uniquely a variety over $\mathbb{Z}$. To a point of $\mathrm{Spec}\,\mathbb{C}[D]$ is associated a character $\chi : D \to \mathbb{C}^*$ which assigns to a group element $g \in D$ a root of unity $\chi(g)$ in $\mathbb{C}$. For each such character, the map e_X provides a concrete interpretation of the elements of $\underline{X}(D)$ as points of $X_{\mathbb{C}}$.

In § 4.5 we test these ideas with Chevalley groups G and show that they can be defined over $\mathbb{F}_{1^2}$. Let $\mathfrak{G}$ be the algebraic group scheme over $\mathbb{Z}$ associated by Chevalley ([3], [11]) to a root system $\{L, \Phi, n_r\}$ of G (*cf.* § 4.1 and [22], § 4.1). In § 4.6 we prove that G can be defined over $\mathbb{F}_{1^2}$ in the above sense. For the proof, one needs to verify that the following conditions are satisfied

– The functor $\underline{G}$ (to graded sets) contains enough points so that, together with $G_{\mathbb{C}}$, it characterizes $\mathfrak{G}$.

– The cardinality of $\underline{G}(\mathbb{F}_{1^n})$ is given by a polynomial $P(n)$ whose value, for q a prime power and $n = q - 1$, coincides with the cardinality of $\mathfrak{G}(\mathbb{F}_q)$.

– The terms of lowest degree in $\underline{G}$ have degree equal to the rank of G and determine the group extension of the Weyl group of G by $\mathrm{Hom}(L, D)$, as constructed by Tits in [22].

The first condition ensures the compatibility with Soulé's original notion of variety over $\mathbb{F}_1$. The second statement sets a connection with the theory of zeta functions as in [17]. Finally, the third condition guarantees a link with the constructions of Tits. In fact, in [21] it was originally promoted the idea that the Weyl group of a Chevalley group G should be interpreted as the points of G which are rational over $\mathbb{F}_1$. For $G = \mathrm{GL}_d$, it was then shown in [15] that the points of G over $\mathbb{F}_{1^n}$ are described by the wreath product of the group of permutations of d letters with μ_n^d. It is important to notice that in our theory these groups are recast as the terms of lowest degree of $\underline{G}$. The terms of higher degree are more subtle to describe; to construct them we make use of the detailed theory of Chevalley as in [2], [3].

If G is a Chevalley group defined over the algebraic closure of a finite field, the cardinality of the set $\mathfrak{G}(\mathbb{F}_q)$ (*i.e.* the number of points of $\mathfrak{G}(\bar{\mathbb{F}}_q)$ which are rational[1] over $\mathbb{F}_q$), is given by the formula

$$|\mathfrak{G}(\mathbb{F}_q)| = (q-1)^\ell q^N \sum_{w \in W} q^{N(w)}, \tag{2}$$

where ℓ denotes the rank of G, N is the number of positive roots, W is the Weyl group and $N(w)$ is the number of positive roots r, such that $w(r) < 0$. The above formula (2) corresponds to a decomposition of $\mathfrak{G}(\mathbb{F}_q)$ as a disjoint union (over the Weyl group W) of products of the form

$$\mathfrak{G}(\mathbb{F}_q) = \coprod_{w \in W} \mathbb{G}_m(\mathbb{F}_q)^\ell \, \mathbb{A}^N(\mathbb{F}_q) \, \mathbb{A}^{N(w)}(\mathbb{F}_q). \tag{3}$$

This equality suggests the definition of the functor $\underline{G}$ by means of a sum of products of powers of the graded functors $\underline{\mathbb{G}}_m$ and $\underline{\mathbb{A}}$ (*cf.* § 3).

The most technical part of our construction is the definition of the natural transformation e_G as in (c), which involves the introduction of a lifting procedure from the Weyl group W to the complex group $G_{\mathbb{C}} = \mathfrak{G}(\mathbb{C})$. The solution to this problem is in fact already contained in [22]. Indeed, Tits introduced in that paper a functor $\mathcal{N}_{D,\epsilon}(L, \Phi)$ from pairs (D, ϵ) of an abelian group and an element of square one in D, to group extensions of the form

$$1 \to \mathrm{Hom}(L, D) \to \mathcal{N}_{D,\epsilon}(L, \Phi) \to W \to 1.$$

The definition of the graded functor $\underline{G}$ and the natural transformation e_G then follow by applying the original construction of Tits together with the Bruhat decomposition of G and working with a specific parametrization of its cells.

Incidentally, we find rather remarkable that the image of this lift of the Weyl group in $G_{\mathbb{C}}$ consists only of finite products of elements in the maximal torus with elements of the form $x_r(\mu)$, where μ is a root of unity in $\mathbb{C}$ and where the $x_r(t)$ generate (over

[1] There are in general more points in $\mathfrak{G}(\mathbb{F}_q)$ than in the Chevalley group $G_{\mathbb{F}_q}$ which is the commutator subgroup of $\mathfrak{G}(\mathbb{F}_q)$.

any field k) unipotent one-parameter subgroups associated to the roots r. The fact that $\underline{G}$ contains enough points so that, together with $G_{\mathbb{C}}$, it characterizes $\mathfrak{G}$, follows from an important result of Chevalley [3], by working only with the points in the big cell of G.

Finally in §5, we show that in all these examples of varieties X over $\mathbb{F}_1$ (more precisely $\mathbb{F}_{1^2}$) a much stronger property holds:

- The construction of the functor $\underline{X}$ extends to the category of pairs (D, ϵ) of an arbitrary abelian group and an element of square one.
- The construction of the natural transformation e_X extends to arbitrary commutative rings A and gives a map $e_{X,A} : \underline{X}(A^*) \to X(A)$.
- This natural transformation $e_{X,A}$ is a bijection when $A = K$ is a field.

These strong properties give, in particular, a conceptual reason for the equality of the number of points of $X(\mathbb{F}_q)$ and the cardinality of the set $\underline{X}(D)$ for $D = \mathbb{F}_q^*$.

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2. On the notion of variety over $\mathbb{F}_1$

In this section we review the notion of variety over $\mathbb{F}_1$ as in [20] and develop a refinement of this concept that will be applied to the case of Chevalley group schemes in §4.5, to show that these varieties can be defined over (an extension of) $\mathbb{F}_1$, thus establishing a link with Tits' geometries.

2.1. **Extension of scalars.** In this paragraph we shall use the same notation of [20]. Let k be a field and $k \subset \Omega$ be a field extension. One considers the functor of extension of scalars:

$$\cdot \otimes_k \Omega : \mathcal{A}_k \longrightarrow \mathcal{A}_\Omega, \qquad R \mapsto R_\Omega = R \otimes_k \Omega \tag{4}$$

from the category $\mathcal{A}_k$ of unital commutative k-algebras to the corresponding category $\mathcal{A}_\Omega$. This functor extends to the category of schemes over k and we use the same notation to denote it. If X is a scheme (of finite type) over k, one lets $X_\Omega = X \otimes_k \Omega$ the corresponding scheme over Ω. If $X = \mathrm{Spec}(R)$ is affine and corresponds to the k-algebra R, then X_Ω is also affine and corresponds to the Ω-algebra R_Ω. The natural homomorphism of algebras $R \to R_\Omega$ corresponds at the level of schemes to a surjective morphism

$$X_\Omega = \mathrm{Spec}(R_\Omega) \to X = \mathrm{Spec}(R)\,. \tag{5}$$

Let $\mathcal{S}et$ be the category of sets. Then we view a scheme X over k as a covariant functor

$$\underline{X} \,:\, \mathcal{A}_k \longrightarrow \mathcal{S}ets, \quad R \mapsto \underline{X}(R). \tag{6}$$

For affine schemes $X = \mathrm{Spec}(A)$, one has $\underline{X}(R) = \mathrm{Hom}(A, R)$. Note that the functor $X \to \underline{X}$ on schemes is covariant.

In [20] (*cf.* Proposition 1), one makes use of the following statement

Proposition 2.1. *(i) There exists a natural transformation of functors*

$$i \,:\, \underline{X} \to \underline{X}_\Omega\,, \quad X(R) \subset X_\Omega(R_\Omega) \tag{7}$$

(ii) For any scheme S over Ω and any natural transformation

$$\varphi \,:\, \underline{X} \to \underline{S}\,, \tag{8}$$

there exists a unique morphism φ_Ω *(over* Ω*) from* X_Ω *to* S *such that* $\varphi = \underline{\varphi_\Omega} \circ i$.

Notice that (7) seems to imply that by means of the covariance property of the functor $X \mapsto \underline{X}$, one should obtain a natural morphism of schemes $X \to X_\Omega$, and this is in evident contradiction with (5). This apparent inconsistency is due to an abuse of notation and it is easily fixed as follows. The functors $\underline{X_\Omega}$ and $\underline{S}$ which in *op.cit.* are defined as functors from $\mathcal{A}_k$ to $\mathcal{S}ets$ (*cf.* equation above Proposition 1), should instead be properly introduced as functors from the category $\mathcal{A}_\Omega$ to $\mathcal{S}ets$. The "hidden" operation in [20] is the composition with the functor of extension of scalars

$$\beta : \mathcal{A}_k \longrightarrow \mathcal{A}_\Omega, \quad \beta(R) = R_\Omega \tag{9}$$

Up-to replacing in the above proposition $\underline{X_\Omega}$ by $\underline{X_\Omega} \circ \beta$ and $\underline{S}$ by $\underline{S} \circ \beta$, one then obtains the following correct statement

Proposition 2.2. *(i) There exists a natural transformation of functors*

$$i \ : \ \underline{X} \to \underline{X_\Omega} \circ \beta\,, \quad X(R) \subset X_\Omega(R_\Omega) \tag{10}$$

(ii) For any scheme S *over* Ω *and any natural transformation*

$$\varphi \ : \ \underline{X} \to \underline{S} \circ \beta\,, \tag{11}$$

there exists a unique morphism φ_Ω *(over* Ω*) from* X_Ω *to* S *such that* $\varphi = \underline{\varphi_\Omega} \circ i$.

The proof is a simple translation of the one given in [20], with only a better use of notation.

Proof. We only consider the case of an affine scheme $X = \mathrm{Spec}(A)$. First, notice that the inclusion $X(R) \subset X_\Omega(R_\Omega)$ is simply described by the inclusion

$$i \ : \ \mathrm{Hom}_k(A, R) \subset \mathrm{Hom}_\Omega(A \otimes_k \Omega, R \otimes_k \Omega)\,, \quad f \mapsto i(f) = f \otimes_k id_\Omega\,.$$

Next, for any given natural transformation φ as in (11) and for any object R of $\mathcal{A}_k$, one gets a map $\varphi(R)$ from $\underline{X}(R)$ to $\underline{S}(R_\Omega)$. Thus, for $R = A$, one gets $id_A \in \underline{X}(A) = \mathrm{Hom}(A, A)$, with image

$$h = \varphi(A)(id_A) \in \underline{S}(A_\Omega) \tag{12}$$

(in the displayed formula just after Proposition 1 in [20] there is a typo: the term $X_\Omega(A_\Omega)$ should be replaced by $\underline{S}(A_\Omega)$). Thus one gets a morphism (over Ω) from X_Ω to S. If S is affine and associated to a Ω-algebra B, then $h = \varphi(A)(id_A) \in \mathrm{Hom}_\Omega(B, A_\Omega)$.
One has $\underline{X}(f)(id_A) = f \in \mathrm{Hom}(A, R) = \underline{X}(R)$ and the following formula results from the fact that φ is a natural transformation

$$\varphi(R)(f) = \varphi(R) \circ \underline{X}(f)(id_A) = (\underline{S} \circ \beta)(f) \circ \varphi(A)(id_A).$$

With $S = \mathrm{Spec}(B)$ and $h \in \mathrm{Hom}_\Omega(B, A_\Omega)$ given by (12), we thus obtain

$$\varphi(R)(f) = (f \otimes_k id_\Omega) \circ h \in \mathrm{Hom}_\Omega(B, R_\Omega)\,.$$

Moreover, one has $i(f) = f \otimes_k id_\Omega$ and $\underline{\varphi_\Omega} : \underline{X_\Omega} \to \underline{S}$ is given by

$$\underline{\varphi_\Omega}(t) = t \circ h \in \mathrm{Hom}_\Omega(B, Z)\,, \ \forall t \in \mathrm{Hom}_\Omega(A_\Omega, Z)\,, \ \forall Z \in \mathcal{A}_\Omega\,.$$

This shows that $\varphi = \underline{\varphi_\Omega} \circ i$.
To prove the uniqueness, one applies $\varphi = \underline{\varphi_\Omega} \circ i$ to the object A of $\mathcal{A}_k$. By construction $i(A)$ is given by the map

$$\mathrm{Hom}_k(A, A) \to \mathrm{Hom}_\Omega(A \otimes_k \Omega, A \otimes_k \Omega)\,, \quad f \mapsto i(f) = f \otimes_k id_\Omega\,.$$

Thus $i(id_A) = id_{A_\Omega}$. It follows that the morphism h that defines φ_Ω is determined as $h = \varphi(A)(id_A) \in \underline{S}(A_\Omega)$.
The extension to schemes which are no longer affine follows as in [20]. $\Box$

2.2. **Gadgets.** We keep the same notation as in the previous paragraph. Let $\mathcal{R}$ be the category of commutative rings which are finite and flat over $\mathbb{Z}$. Let us first recall the definition of "truc" given in [20].

Definition 2.3. *A truc over $\mathbb{F}_1$ consists of the following data*

- a pair $X = (\underline{X}, \mathcal{A}_X)$ of a covariant functor $\underline{X} : \mathcal{R} \longrightarrow \mathcal{S}ets$ and a $\mathbb{C}$-algebra $\mathcal{A}_X$
- for each object R of $\mathcal{R}$ and each homomorphism $\sigma : R \to \mathbb{C}$, an evaluation morphism ($\mathbb{C}$-algebra homomorphism)

$$e_{x,\sigma} : \mathcal{A}_X \to \mathbb{C}\,,\ \forall x \in \underline{X}(R),$$

which satisfies the functorial compatibility $e_{f(y),\sigma} = e_{y,\sigma\circ f}$, $\forall\ f : R' \to R$ morphism in $\mathcal{R}$ and $\forall\ y \in \underline{X}(R')$.

We shall reformulate slightly the above definition with the goal to:

- treat the archimedean place simultaneously with $\operatorname{Spec}\mathbb{Z}$
- replace the category $\mathcal{R}$ by the category $\mathcal{F}_{ab}$ of finite abelian groups
- put in evidence the role of the functor β.

Thus, we replace $\mathcal{R}$ by the category $\mathcal{F}_{ab}$ of finite abelian groups. There is a natural functor of extension of scalars from $\mathbb{F}_1$ to $\mathbb{Z}$ which is given by

$$\beta\ :\ \mathcal{F}_{ab} \longrightarrow \mathcal{R}, \quad \beta(D) = D \otimes_{\mathbb{F}_1} \mathbb{Z} = \mathbb{Z}[D] \tag{13}$$

and associates to an abelian group its convolution algebra over $\mathbb{Z}$. Let us understand the evaluation morphism as a natural transformation. We introduce the functor

$$\operatorname{Spec}_\infty(\mathcal{A}_X)\ : \mathcal{R} \longrightarrow \mathcal{S}ets, \qquad R \mapsto \operatorname{Hom}(\mathcal{A}_X, R \otimes_{\mathbb{Z}} \mathbb{C}) \tag{14}$$

and compose it with the functor $\beta : \mathcal{F}_{ab} \to \mathcal{R}$.

Lemma 2.4. *The evaluation morphism e as in Definition 2.3, determines a natural transformation of (covariant) functors*

$$e : \underline{X} \to \operatorname{Spec}_\infty(\mathcal{A}_X) \circ \beta.$$

Proof. For each object D of $\mathcal{F}_{ab}$ the evaluation map $e_{x,\sigma}$ can be viewed as a map of sets $\underline{X}(D) \to \operatorname{Hom}(\mathcal{A}_X, R \otimes_{\mathbb{Z}} \mathbb{C})$, where $R = \beta(D)$. Indeed, for $x \in \underline{X}(D)$, we get a map e_x from characters σ of R to characters of $\mathcal{A}_X$ which determines a morphism from $\mathcal{A}_X$ to $R \otimes_{\mathbb{Z}} \mathbb{C}$. $\Box$

We now reformulate Definition 2.3 as follows

Definition 2.5. *A gadget over $\mathbb{F}_1$ is a triple $X = (\underline{X}, X_{\mathbb{C}}, e_X)$ consisting of*

(a) a covariant functor $\underline{X} : \mathcal{F}_{ab} \to \mathcal{S}ets$ to the category of sets
(b) a variety $X_{\mathbb{C}}$ over $\mathbb{C}$
(c) a natural transformation $e_X : \underline{X} \to Hom(\operatorname{Spec}\mathbb{C}[-], X_{\mathbb{C}})$ from the functor $\underline{X}$ to the functor

$$Hom(\operatorname{Spec}\mathbb{C}[-], X_{\mathbb{C}}), \quad D \mapsto Hom(\operatorname{Spec}\mathbb{C}[D], X_{\mathbb{C}})\,. \tag{15}$$

2.2.1. *Example.* An affine variety V over $\mathbb{Z}$ defines a gadget $X = \mathcal{G}(V)$ over $\mathbb{F}_1$ by letting $X_{\mathbb{C}} = V_{\mathbb{C}} = V \otimes_{\mathbb{Z}} \mathbb{C}$. $\underline{X}(D)$ is the set of points of V in the convolution algebra $\mathbb{Z}[D]$ with the obvious natural transformation to $\mathrm{Hom}(\mathrm{Spec}\,\mathbb{C}[D], V_{\mathbb{C}})$.

Definition 2.6. *A gadget X over $\mathbb{F}_1$ is said to be graded when*

$$\underline{X} = \coprod_{k \geq 0} \underline{X}^{(k)} : \mathcal{F}_{ab} \to \mathcal{S}ets$$

takes values in the category of $\mathbb{Z}_{\geq 0}$-graded sets. It is finite when the set $\underline{X}(D)$ is finite $\forall\ D \in \mathcal{F}_{ab}$.

2.3. **Varieties over $\mathbb{F}_1$.** The notion of morphism of gadgets $\phi : X \to Y$ is essentially that of a natural transformation. More precisely, ϕ is determined by a pair $\phi = (\underline{\phi}, \phi_{\mathbb{C}})$, with

$$\underline{\phi} : \underline{X} \to \underline{Y}\,, \qquad \phi_{\mathbb{C}} : X_{\mathbb{C}} \to Y_{\mathbb{C}}. \tag{16}$$

$\underline{\phi}$ is a natural transformation of functors and $\phi_{\mathbb{C}}$ a morphism of varieties over $\mathbb{C}$. A required compatibility with the evaluation maps determines a commutative diagram

$$\begin{array}{ccc} \underline{X}(D) & \xrightarrow{\underline{\phi}(D)} & \underline{Y}(D) \\ \downarrow{\scriptstyle e_X(D)} & & \downarrow{\scriptstyle e_Y(D)} \\ \mathrm{Hom}(\mathrm{Spec}\,\mathbb{C}[D], X_{\mathbb{C}}) & \xrightarrow{\phi_{\mathbb{C}}} & \mathrm{Hom}(\mathrm{Spec}\,\mathbb{C}[D], Y_{\mathbb{C}}). \end{array} \tag{17}$$

As in [20], we introduce the following notion of *immersion* of gadgets

Definition 2.7. *A morphism of gadgets $\phi : X \to Y$ is said to be an immersion if $\phi_{\mathbb{C}}$ is an immersion and for any object D of $\mathcal{F}_{ab}$, the map $\underline{\phi} : \underline{X}(D) \to \underline{Y}(D)$ is injective.*

We can now re-state the key definition of an *affine* variety X over $\mathbb{F}_1$. In the formulation given in [20], it postulates the existence of a variety (of finite type) over $\mathbb{Z}$ which plays the role of the scheme $X \otimes_{\mathbb{F}_1} \mathbb{Z}$ and fulfills the universal property of Proposition 2.2.

Definition 2.8. *An* affine *variety X over $\mathbb{F}_1$ is a finite, graded gadget X such that there exists an affine variety $X_{\mathbb{Z}}$ over $\mathbb{Z}$ and an immersion $i : X \to \mathcal{G}(X_{\mathbb{Z}})$ of gadgets satisfying the following universal property: for any affine variety V over $\mathbb{Z}$ and any morphism of gadgets $\varphi : X \to \mathcal{G}(V)$, there exists a unique algebraic morphism*

$$\varphi_{\mathbb{Z}} : X_{\mathbb{Z}} \to V$$

of affine varieties such that $\varphi = \mathcal{G}(\varphi_{\mathbb{Z}}) \circ i$.

2.4. **Varieties over $\mathbb{F}_{1^n}$.** This small variant is obtained (following [20] §3.8.2) by replacing the category $\mathcal{F}_{ab}$ of finite abelian groups by the finer one $\mathcal{F}_{ab}^{(n)}$ whose objects are pairs (D, ϵ), where D is an abelian group and $\epsilon \in D$ fulfills $\epsilon^n = 1$. A morphism in $\mathcal{F}_{ab}^{(n)}$ is a homomorphism of abelian groups which sends $\epsilon \in D$ to $\epsilon' \in D'$. In Definition 2.5 one replaces $\mathcal{F}_{ab}$ by $\mathcal{F}_{ab}^{(n)}$ and in equation (15) one substitutes the group ring $\mathbb{C}[D]$ by the reduced group ring which is the tensor product of rings

$$\mathbb{C}[D, \epsilon] = \mathbb{Z}[D] \otimes_{\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]} \mathbb{C}. \tag{18}$$

The morphism $\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}] \to \mathbb{Z}[D]$ comes from the group morphism $\mathbb{Z}/n\mathbb{Z} \to D$ associated to ϵ and similarly the morphism $\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}] \to \mathbb{C}$ comes from the character given by the primitive root of unity $\xi = e^{2\pi i/n}$. Thus the characters χ of the algebra $\mathbb{C}[D, \epsilon]$ are the characters of $\mathbb{C}[D]$ such that

$$\chi(\epsilon) = \xi. \tag{19}$$

3. Elementary examples

In this section we apply Definition 2.8 by working out the explicit description of the graded functor $\underline{X}$ in several elementary examples of algebraic varieties over $\mathbb{F}_1$. The main new feature, with respect to [20], is the introduction of a grading. At the intuitive level, the underlying principle in the definition of the graded functor $\underline{X} = \coprod_{k\geq 0} \underline{X}^{(k)}$ is that of considering the Taylor expansion, at $q = 1$, of the function $N(q)$ counting the number of points of the scheme X over the finite field $\mathbb{F}_q$. The term of degree k (*i.e.* $a_k(q-1)^k$) in the expansion should agree with the cardinality of the set $\underline{X}^{(k)}(D)$, for $q - 1 = |D|$, $D \in \mathrm{obj}(\mathcal{F}_{ab})$.
The requirement that the function $N(q)$ counting the number of points of the scheme X over the finite field $\mathbb{F}_q$ is a polynomial in q is imposed in [20] in order to deal with the zeta function and is very restrictive. It fails for instance in general for elliptic curves but it holds for instance for Chevalley schemes. We shall first deal with a few concrete examples of simple geometric spaces for which it is easily checked. These are

(1) $\mathbb{G}_m$, $N(q) = q - 1$.
(2) The affine line $\mathbb{A}^1$, $N(q) = q$.
(3) The projective space $\mathbb{P}^d$, $N(q) = 1 + q + \ldots + q^d$.

In the following we shall consider each of these cases in details.

3.1. **The multiplicative group $\mathbb{G}_m$.** For the multiplicative group $\mathbb{G}_m$, the counting function is $N(q) = q - 1$: its Taylor expansion at $q = 1$ has just one term in degree 1. We define the functor $\underline{\mathbb{G}_m}$ from abelian groups to $\mathbb{Z}_{\geq 0}$-graded sets accordingly *i.e.*

$$\underline{\mathbb{G}_m} : \mathcal{F}_{ab} \longrightarrow \mathcal{S}ets, \quad \underline{\mathbb{G}_m}(D)^{(k)} = \begin{cases} \emptyset & \text{if } k \in \mathbb{Z}_{\geq 0} \setminus \{1\} \\ D & \text{if } k = 1. \end{cases} \tag{20}$$

In particular one sets

$$\underline{\mathbb{G}_m}(\mathbb{F}_{1^n})^{(k)} = \begin{cases} \emptyset & \text{if } k \in \mathbb{Z}_{\geq 0} \setminus \{1\} \\ \mathbb{Z}/n\mathbb{Z} & \text{if } k = 1. \end{cases} \tag{21}$$

Except for the introduction of the grading and for the replacement of the category of (commutative) rings finite and flat over $\mathbb{Z}$ (as in [20]) by that of finite abelian groups, the definition (20) is the same as the corresponding functor in *op.cit.*
We denote by $e_m : \underline{\mathbb{G}}_m \to \mathrm{Hom}(\mathrm{Spec}\,\mathbb{C}[-], \mathbb{G}_m(\mathbb{C}))$ the natural transformation from the functor $\underline{\mathbb{G}}_m$ to the functor

$$\mathrm{Hom}(\mathrm{Spec}\,\mathbb{C}[-], \mathbb{G}_m(\mathbb{C})), \quad D \mapsto \mathrm{Hom}(\mathrm{Spec}\,\mathbb{C}[D], \mathbb{G}_m(\mathbb{C})),$$

which assigns to a character χ associated to a point of $\mathrm{Spec}\,\mathbb{C}[D]$ the group homomorphism

$$D \to \mathbb{C}^*\,, \;\; e_m(D)(g) = \chi(g)\,. \tag{22}$$

It is now possible to adapt the proof of [20] (as in 5.2.2) and show that this gadget defines a variety over $\mathbb{F}_1$.

Proposition 3.1. *The gadget $\mathbb{G}_m = (\underline{\mathbb{G}}_m, \mathbb{G}_m(\mathbb{C}), e_m)$ defines a variety over $\mathbb{F}_1$.*

Proof. By construction $\mathbb{G}_m$ is a finite and graded gadget. It is easy to guess that $\mathbb{G}_{m,\mathbb{Z}} = \mathrm{Spec}(\mathbb{Z}[T, T^{-1}])$. Let us see that the condition of Definition 2.8 is also fulfilled. Let $V = \mathrm{Spec}(\mathcal{O})$ be an affine variety over $\mathbb{Z}$. Let $\phi : \mathbb{G}_m \to \mathcal{G}(V)$, be a morphism of (affine) gadgets. This means that we are given a pair $(\underline{\phi}, \phi_{\mathbb{C}})$, where $\phi_{\mathbb{C}}$ can be equivalently interpreted by means of the corresponding homomorphism of $\mathbb{C}$-algebras ($\mathcal{O}_{\mathbb{C}} = \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{C}$)

$$\phi^*_{\mathbb{C}} \;:\; \mathcal{O}_{\mathbb{C}} \to \mathbb{C}[T, T^{-1}].$$

Furthermore, $\underline{\phi}$ is a morphism of functors (natural transformation)

$$\underline{\phi}(D) \;:\; \underline{\mathbb{G}}_m(D) \to \mathrm{Hom}(\mathcal{O}, \beta(D))$$

which fulfills the following compatibility condition (*cf.* (17)). For any finite abelian group D the following diagram is commutative

$$\begin{array}{ccc} \underline{\mathbb{G}}_m(D) & \xrightarrow{\underline{\phi}(D)} & \mathrm{Hom}(\mathcal{O}, \beta(D)) \\ \downarrow{\scriptstyle e_m(D)} & & \downarrow{\scriptstyle e_{\mathcal{G}(V)}(D)} \\ \mathrm{Hom}(\mathbb{C}[T, T^{-1}], \beta(D)_{\mathbb{C}}) & \xrightarrow{\phi^*} & \mathrm{Hom}(\mathcal{O}_{\mathbb{C}}, \beta(D)_{\mathbb{C}}). \end{array} \tag{23}$$

It remains to show that $\phi^*(\mathcal{O}) \subset \mathbb{Z}[T, T^{-1}]$. Let $h \in \mathcal{O}$ and $f = \phi^*(h)$. Then by construction $f \in \mathbb{C}[T, T^{-1}]$. Let $D = \mathbb{Z}/n\mathbb{Z}$ be the cyclic group of order n with generator $\xi \in \mathbb{Z}/n\mathbb{Z}$, one has $\xi \in \underline{\mathbb{G}}_m(D)$ and $\underline{\phi}(D)(\xi) \in \mathrm{Hom}(\mathcal{O}, \beta(D))$. By evaluating on $h \in \mathcal{O}$ one gets

$$\underline{\phi}(D)(\xi)(h) \in \mathbb{Z}[D] \subset \mathbb{C}[D]\,.$$

It follows from the commutativity of the diagram (23) that this is the same as evaluating on $f \in \mathbb{C}[T, T^{-1}]$ the homomorphism $e_m(D)(\xi)$ which coincides with the quotient map

$$\theta_n :\; \mathbb{C}[T, T^{-1}] \to \mathbb{C}[\mathbb{Z}/n\mathbb{Z}]\,, \quad T \mapsto \xi \in \mathbb{Z}/n\mathbb{Z}\,.$$

This means that $\theta_n(f) \in \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$, for all n. For $f \in \mathbb{C}[T, T^{-1}]$ one can compute the coefficient of T^k as the limit

$$b_k = \lim_{n\to\infty} \frac{1}{n} \sum_{a=1}^{n} f(e^{2\pi i \frac{a}{n}}) e^{-2\pi i k \frac{a}{n}}.$$

When $f(x)=x^m$, the sum $\sum_{a=1}^n f(e^{2\pi i\frac{a}{n}})e^{-2\pi ik\frac{a}{n}}$ is either zero or n and the latter case only happens if $m-k$ is a multiple of n. Thus, $\frac{1}{n}\sum_{a=1}^n f(e^{2\pi i\frac{a}{n}})e^{-2\pi ik\frac{a}{n}}$, which only depends on $\theta_n(f)$ is a relative integer if $\theta_n(f)\in\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$. It follows that all the b_k are in $\mathbb{Z}$ and hence that $f\in\mathbb{Z}[T,T^{-1}]$. $\square$

3.2. **The affine space $\mathbb{A}^F$.** For the affine line $\mathbb{A}^1$, the number of points of $\mathbb{A}^1(\mathbb{F}_q)$ is $N(q)=q$. Thus, the Taylor expansion of $N(q)$ at $q=1$ is $q=1+(q-1)$ and has two terms in degree 0 and 1. This suggests to refine the definition of the corresponding functor of [20] as follows. We define $\underline{\mathbb{A}}^1$ as the graded functor

$$\underline{\mathbb{A}}^1:\mathcal{F}_{ab}\longrightarrow\mathcal{S}ets,\quad \underline{\mathbb{A}}^1(D)^{(k)}=\begin{cases}\{0\} & \text{if } k=0\\ D & \text{if } k=1\\ \emptyset & \text{if } k\geq 2.\end{cases}\tag{24}$$

More generally, one may introduce for any finite set F the graded functor

$$\underline{\mathbb{A}}^F(D)^{(k)}=\coprod_{\substack{Y\subset F,\\ |Y|=k}} D^Y\,.\tag{25}$$

which is just the graded product $(\{0\}\cup D)^F$.

Proposition 3.2. *Let $e_F:\underline{\mathbb{A}}^F\to Hom(\operatorname{Spec}\mathbb{C}[-],\mathbb{C}^F)$ be the natural transformation from the functor $\underline{\mathbb{A}}^F$ to the functor $D\mapsto Hom(\operatorname{Spec}\mathbb{C}[D],\mathbb{C}^F)$ which assigns to a point in $\operatorname{Spec}\mathbb{C}[D]$, i.e. to a character $\chi:\mathbb{C}[D]\to\mathbb{C}^*$, the following map*

$$\coprod_{Y\subset F} D^Y\to\mathbb{C}^F\,,\qquad e_F(D)((g_j)_{j\in Y})=(\xi_j)_{j\in F},\quad \xi_j=\begin{cases}\chi(g_j) & \text{if } j\in Y;\\ 0 & \text{if } j\notin Y.\end{cases}\tag{26}$$

Then, the gadget $\mathbb{A}^F=(\underline{\mathbb{A}}^F,\mathbb{C}^F,e_F)$ defines a variety over $\mathbb{F}_1$.

The proof is identical to that of [20].

3.3. **Projective space $\mathbb{P}^d$.** For the case of the projective space $\mathbb{P}^d$, we follow the construction of [20] by implementing the grading. More precisely, we define $\underline{\mathbb{P}}^d$ as the following graded functor

$$\underline{\mathbb{P}}^d:\mathcal{F}_{ab}\longrightarrow\mathcal{S}ets,\quad \underline{\mathbb{P}}^d(D)^{(k)}=\coprod_{\substack{Y\subset\{1,2,\ldots,d+1\}\\ |Y|=k+1}} D^Y/D,\quad k\geq 0\tag{27}$$

where the right action of D is the diagonal action. It follows that the points of lowest degree in $\mathbb{P}^d(\mathbb{F}_{1^n})$ are simply labeled by $\{1,2,\ldots,d+1\}$. Their number is evidently

$$\#\,\mathbb{P}^d(\mathbb{F}_{1^n})=d+1.\tag{28}$$

In particular, this shows that $\underline{\mathbb{P}}^d(\mathbb{F}_{1^n})$ coincides *in degree zero* with the $d+1$ points of the set $\mathcal{P}_d$ on which Tits defines his notion of projective geometry of dimension d over $\mathbb{F}_1$. It is striking that the right hand side of the formula (28) is independent of n. This result is also in agreement with the evaluation at $q=1^n$ of the counting function of the set $\mathbb{P}^d(\mathbb{F}_q)$, namely (with the evaluation at $q=1^n$) of the function $N(q)=\sum_{j=0}^d q^j$.

4. Chevalley group schemes

The main result of this section (Theorem 4.10) is the proof that Chevalley groups give rise naturally to affine varieties over (an extension of) $\mathbb{F}_1$. To achieve this result we shall need to apply the full theory of Chevalley groups both in the classical (*i.e.* Lie-theoretical) and algebraic group theoretical development.

If $\mathbb{K}$ is an algebraically closed field, a Chevalley group G over $\mathbb{K}$ is a connected, semi-simple, linear algebraic group over $\mathbb{K}$. By definition of a linear algebraic group over $\mathbb{K}$, G is isomorphic to a closed subgroup of some $GL_n(\mathbb{K})$. The coordinate ring of G, as affine linear algebraic variety over $\mathbb{K}$, is then $\mathbb{K}[G] = \mathbb{K}[x_{ij}, d^{-1}]/I$ *i.e.* a quotient ring of polynomials in n^2 variables with determinant d inverted by a prime ideal I.
As algebraic group over $\mathbb{K}$, G is also endowed with a group structure respecting the algebraic structure, *i.e.* G is endowed with the following two morphisms of varieties over $\mathbb{K}$

$$\mu : G \times G \to G,\ \mu(x, y) = xy; \quad \iota : G \to G,\ \iota(x) = x^{-1} \tag{29}$$

Notice that by construction $\mathbb{K}[G]$ is a Hopf algebra whose coproduct encodes the group structure.

Let k be the prime field of $\mathbb{K}$ and K an intermediate field: $k \subset K \subset \mathbb{K}$. Then, the group G is said to be defined over K (and in particular over k) if the affine variety G and the group structure are defined over K and, by extension of scalars, also over any field above it (*cf.* [16], Chapter 2 (2.1.1), p. 63). In terms of the Hopf algebra structure, one is given a Hopf algebra A over K such that, as Hopf algebras

$$\mathbb{K}[G] = A \otimes_K \mathbb{K}\,.$$

The property for the group G to be split over K means that some maximal torus $T \subset G$ is K-isomorphic to $\mathbb{G}_m \times \cdots \times \mathbb{G}_m$ (d copies, $d = \dim T$) (*cf.* [14], Chapter XII, §§ 34.3, 34.4 p. 219-20).
In this section, any connected, reductive, linear algebraic group over $\mathbb{K}$ is assumed to be defined and split over the prime subfield $k \subset \mathbb{K}$.
If G is a linear algebraic group over K, the set

$$G(K) = \mathrm{Hom}_K(A, K)$$

is a group called the group of *K-rational points* of G. One has an identification $G(K) = G \cap \mathbb{A}_K^{n^2}$ using K-polynomials to generate the ideal I (*cf.* *op.cit.* Chapter XII, § 34.1, p. 218).

To a reductive, connected algebraic group G defined over K and a K-split, maximal torus $T \subset G$, one associates the group $\mathrm{Hom}(T, \mathbb{G}_m)$ ($\mathbb{G}_m = GL_1(K)$): this is a free abelian group of rank equal to the dimension of T. The group $\mathbb{R} \otimes_{\mathbb{Z}} \mathrm{Hom}(T, \mathbb{G}_m)$ plays in this context, the role of the dual $\mathfrak{h}^*$ of a Cartan Lie-algebra.
One shows that there exist sub-tori $S \subset T$, $\dim S = \dim T - 1$, whose centralizers $Z(S)$ are of dimension $\dim T + 2$ and such that $Z(S)/S$ is isomorphic either to $SL_2(K)$ or $PGL_2(K)$. The study of these groups allows one to introduce pairs of elements $\pm\alpha \in \mathrm{Hom}(T, \mathbb{G}_m)$ and by varying S one defines a full set of roots Φ (*cf.* *op.cit.* Chapter XII, § 34.5 and [4], § 25.7).
If N denotes the normalizer of the torus $T \subset G$, then the (finite) group $W = N/T$ acts on $\mathbb{R} \otimes_{\mathbb{Z}} \mathrm{Hom}(T, \mathbb{G}_m)$ and is called the (K-)*Weyl group*.

The theory of Chevalley groups over a field K has been further extended in [3] and [11]. To every semi-simple, complex Lie group G, and more generally to an abstract root system, one associates canonically a group scheme $\mathfrak{G}$ over $\mathbb{Z}$, such that G gets identified with the group $\mathfrak{G}(\mathbb{C})$ of complex points of $\mathfrak{G}$. We shall return on this construction in § 4.3.

In the next paragraphs we shall first review and then apply a construction due to Tits (*cf.* [22]) which associates to an algebraic reductive group G defined over K and a K-split maximal torus $T \subset G$, a *canonical* extension of the Weyl group W, obtained by considering the groups of K-rational points of T and of its normalizer. This construction makes explicit use of a suitable extension of the Weyl group (so called the extended Weyl group) whose definition is *independent* of the field K and is given only in terms of the root system of G.
The notion of extended Weyl group is related to that of extended Coxeter group V associated to a Coxeter matrix M and a given abstract root system $\{L, \Phi, n_r\}$. The group V is a certain extension of the Coxeter group of M by a free abelian group whose rank equals the cardinality of the set of the reflections associated to the simple roots.

4.1. **Root systems and Coxeter groups.** In this paragraph we follow § 4.1 and § 2.2 of [22] and we shall review several fundamental notions associated to the notion of a root system.
A root system $\{L, \Phi, n_r\}$ is the data given by:

- a lattice L, *i.e.* a free abelian group of finite rank (the group of weights);
- a finite subset $\Phi \subset L$ (the set of roots);
- for each $r \in \Phi$, a $\mathbb{Z}$-valued linear form $n_r : L \to \mathbb{Z}$ (the co-root associated to r)

which satisfy the following axioms:

(1) $L \otimes \mathbb{Q}$ is generated, as $\mathbb{Q}$-vector space, by Φ and by the intersection of the kernels of the n_r;
(2) $n_r(r) = 2$, $\forall\ r \in \Phi$;
(3) the relations $r \in \Phi$, $ar \in \Phi$, $a \in \mathbb{Q}$ imply $a = \pm 1$;
(4) if $r, s \in \Phi$, then $r - n_s(r)s \in \Phi$.

For each $r \in \Phi$, the *reflection associated to* r is the map $s_r : L \to L$ defined by $s_r(x) = x - n_r(x) \cdot r$. The following equality holds $s_r = s_{-r}$.

One can always choose within Φ a system made by *simple roots*, *i.e.* a collection $\Phi^o = \{\rho_i, i \in \Pi\} \subset \Phi$ (Π = finite set) of linearly independent roots such that every root can be written as an integral linear combination of the ρ_i, with integer coefficients either all positive or all negative. Then, the square matrix $M = (m_{ij})$, $i, j \in \Pi$, with $2m_{ij}$ equal to the number of roots which are a linear combination of the ρ_i and ρ_j is a *Coxeter matrix i.e.* a symmetric square matrix with diagonal elements equal to 1 and off diagonal ones positive integers ≥ 2.
The *Coxeter group* $W = W(M)$ associated to M is defined by a system of generators $\{r_i, i \in \Pi\}$ (the *fundamental reflections*) and relations

$$(r_i r_j)^{m_{ij}} = 1\,,\ \forall i, j \in \Pi. \tag{30}$$

The group generated by the reflections associated to the roots is canonically isomorphic to the Coxeter group $W(M)$: the isomorphism is defined by sending the reflection associated to a simple root to the corresponding generator, *i.e.* $s_{\rho_i} \mapsto r_i$.

The root system $\{L, \Phi, n_r\}$ is said to be *simply connected* if the co-roots n_r generate the dual lattice $L' = \mathrm{Hom}(L, \mathbb{Z})$ of L. In this case, the co-roots n_{ρ_i} determine a basis of L'.
Given a root system $\{L, \Phi, n_r\}$, there exists a simply connected root system $\{\tilde{L}, \tilde{\Phi}, n_{\tilde{r}}\}$ and a homomorphism $\varphi : L \to \tilde{L}$ uniquely determined, up-to isomorphism, by the following two conditions:

$$\varphi(\Phi) = \tilde{\Phi}\,, \quad n_r = n_{\varphi(r)} \circ \varphi\,, \ \forall r \in \Phi\,. \tag{31}$$

The restriction of φ to Φ determines a bijection of Φ with $\tilde{\Phi}$.

We recall (*cf.* [11], Exposé XXI) that the lattice L can be endowed with a total order which divides the roots into two disjoint sets: positive and negative roots. The set of positive roots Φ^+ is a subset of Φ satisfying the conditions:

- if $r_1, r_2 \in \Phi^+$ then $r_1 + r_2 \in \Phi^+$
- for each $r \in \Phi$ exactly one of the conditions $r \in \Phi^+$, $-r \in \Phi^+$ holds.

One then lets $\Phi^o \subset \Phi^+$ be the set of indecomposable elements of Φ^+.

The *Braid group* $B = B(M)$ associated to a Coxeter matrix M is defined by generators $\{q_i, i \in \Pi\}$ and the relations

$$\mathrm{prod}(m_{ij}; q_i, q_j) := \underbrace{\cdots q_i q_j q_i q_j}_{m_{ij}} = \mathrm{prod}(m_{ij}; q_j, q_i), \quad \forall\ i, j \in \Pi\,. \tag{32}$$

The *extended Coxeter group* $V = V(M)$ associated to a Coxeter matrix M is the quotient of $B(M)$ by the commutator subgroup of the kernel $X(M)$ of the canonical surjective homomorphism $B(M) \to W(M)$. It is defined by generators and relations as follows. One lets S be the set of elements of W which are conjugate to one of the r_i, *i.e.* the set of reflections (*cf.* [22],§1.2). One considers two sets of generators:

$$\{q_i, i \in \Pi\}, \quad \{g(s), s \in S\}$$

The relations are given by (32) and the following:

(1) $q_i^2 = g(r_i), \quad \forall\ i \in \Pi$
(2) $q_i \cdot g(s) \cdot q_i^{-1} = g(r_i(s)), \quad \forall\ s \in S,\ i \in \Pi.$
(3) $[g(s), g(s')] = 1, \quad \forall s, s' \in S.$

Tits shows in Théorème 2.5 of [22] that the subgroup $U = U(M) \subset V$ generated by the $g(s)$ $(s \in S)$ is a free, abelian, normal subgroup of V that coincides with the kernel of the natural surjective map $f : V \to W$, $f(q_i) = r_i$ (and $f(g(s)) = 1$). The group $U = U(M)$ is the quotient of $X(M)$ by its commutator subgroup.

In the following paragraph we shall recall the construction of [22] of the extended Weyl group using the extended Coxeter group V.

4.2. **The group $\mathcal{N}_{D,\epsilon}(L, \Phi)$.** We keep the same notation as in the previous paragraph. In § 4.3 of [22] Tits introduces, for a given root system $\{L, \Phi, n_r\}$, a functor

$$(D, \epsilon) \to \{N, p, N_s; s \in S\} = \mathcal{N}$$

which associates to pair (D, ϵ) made by an abelian group and an element $\epsilon \in D$ with $\epsilon^2 = 1$, the data (*i.e.* an object $\mathcal{N}$ of a suitable category) given by a group $N = \mathcal{N}_{D,\epsilon}(L, \Phi)$, a surjective homomorphism of groups $p : N \to W = W(M)$ and for each reflection $s \in S$, a subgroup $N_s \subset N$ satisfying the following conditions:

(n1) $\mathrm{Ker}(p)$ is an abelian group;

(n2) For $s \in S$, $n \in N$ and $w = p(n)$, $nN_sn^{-1} = N_{w(s)}$;
(n3) $p(N_s) = \{1, s\}$, $\forall\ s \in S$.

A morphism connecting two objects $\mathcal{N}$ and $\mathcal{N}'$ is a homomorphism $a : N \to N'$ such that $p' \circ a = p$ and $a(N_s) \subset N'_s$ for all s (*cf.* § 3 of [22]).
Notice, in particular, that the data $\{V, f, V_s; s \in S\}$, characterizing the extended Coxeter group, where $V_s \subset V$ is the subgroup generated by $Q_s = \{v \in V, v^2 = g(s)\}$, satisfy (n1)-(n3).

The object $\mathcal{N}$ is obtained by the following canonical construction. One considers the abelian group $T = \mathrm{Hom}(L, D)$ endowed with the natural (left) action of W (denoted by $t \mapsto w(t)$, for $w \in W$, $t \in T$) induced by the corresponding action on L (generated by the reflections associated to the roots). For each $r \in \Phi$, let $s = s_r \in W$ be the reflection associated to the root r. One lets T_s be the subgroup of T made by homomorphisms of the form

$$L \ni x \mapsto a^{\nu(x)} \tag{33}$$

for some $a \in D$ and where $\nu : L \to \mathbb{Z}$ is a linear form proportional to n_r. Also, one defines (for $s = s_r$)

$$h_s(x) = \epsilon^{n_r(x)}, \quad x \in L. \tag{34}$$

(note that replacing $r \to -r$ does not alter the result since $\epsilon^2 = 1$). (34) determines a map $h : S \to T$. Then, the data $\{T, T_s, h_s; s \in S\}$ fulfill the following conditions $\forall\ w \in W,\ s \in S$ and $\ t \in T$:

(1) $w(T_s) = T_{w(s)}$;
(2) $w(h_s) = h_{w(s)}$;
(3) $h_s \in T_s$;
(4) $s(t) \cdot t^{-1} \in T_s$;
(5) $s(t) = t^{-1}, \quad \forall\ t \in T_s$.

One then obtains the object $\mathcal{N}$ as follows (*cf.* Proposition 3.4 of [22]). One defines the group $N = \mathcal{N}_{D,\epsilon}(L, \Phi)$ as the quotient of the product group $V \times T$ (V = extended Coxeter group) by the graph of the homomorphism $U \to T$ ($U = U(M)$) which extends the map $g(s) \mapsto h_s^{-1}$. One identifies T with its canonical image in N and for each $s \in S$, one lets N_s be the subgroup of N generated by the canonical image of $Q_s \times T_s$, where T_s is as above. The surjective group homomorphism $p : N \to W = W(M)$ is induced by $f \times 1$ where f is the canonical group homomorphism $f : V \to W = W(M)$. By construction one has a morphism connecting the data $\{V, f, V_s; s \in S\}$ to $\{N, p, N_s; s \in S\} = \mathcal{N}$ *i.e.* a homomorphism $a : V \to N$ such that

$$a(g(s)) = h_s\ , \ \forall s \in S\ . \tag{35}$$

More precisely, one has the following result (*cf.* § 3.4 and § 4.3 of [22])

Proposition 4.1. *The data $\mathcal{N} = \{N, p, N_s; s \in S\}$ satisfy the conditions (n1)-(n3). Moreover, every map*

$$\alpha : \{q_i, i \in \Pi\} \to N$$

such that $\alpha(q_i) \in N_{r_i} \setminus T_{r_i} = N_{r_i} \cap p^{-1}(r_i)$ extends to a homomorphism of groups $V \to N$.

We collect together, for an easy reference, the main properties of the construction of [22] reviewed in this paragraph.

Proposition 4.2. *Let $\{L, \Phi, n_r\}$ be a root system. To a pair (D, ϵ) of an abelian group and an element $\epsilon \in D$, $\epsilon^2 = 1$, corresponds a canonical extension $\mathcal{N}_{D,\epsilon}(L, \Phi)$ of the Coxeter group W by $T = Hom(L, D)$*

$$1 \to T \to N \xrightarrow{p} W \to 1 \tag{36}$$

and for each reflection $s \in S$ a subgroup $N_s \subset N$. These data satisfy the following properties

- *$nN_sn^{-1} = N_{w(s)}$, for $s \in S$, $n \in N$ and $w = p(n)$.*
- *$p(N_s) = \{1, s\}$, $\quad \forall\ s \in S$.*
- *$N_s \cap T = T_s$, $\quad \forall\ s \in S$.*
- *$a^2 = h_s \in T_s$, $\quad \forall\ a \in N_s \setminus T_s = p^{-1}(s)$.*
- *For each pair $i \neq j$ in Π, let $m = m_{ij}$ be the order of $r_ir_j \in W$, then*

$$\mathrm{prod}(m; a_i, a_j) = \mathrm{prod}(m; a_j, a_i)\,,\ \forall a_k \in N_{r_k}\,,\ p(a_k) = r_k \neq 1\,. \tag{37}$$

The canonical extension $\mathcal{N}_{D,\epsilon}(L, \Phi)$ of W by T is functorial in the pair (D, ϵ), with respect to morphisms $t : D \to D'$ such that $t(\epsilon) = \epsilon'$.

The meaning of equation (37) is the following one. Once a choice of a section $W \supset \Phi^o \ni s \mapsto \alpha(s) \in N_s$ of the map p has been made on the set of simple roots $\Phi^o \subset W$, this section admits a natural extension to all of W as follows. One writes $w \in W$ as a word of minimal length $w = \rho_1 \cdots \rho_k$, in the generators $\rho_j \in \Phi^o$. Then (37) ensures that the corresponding product $\alpha(w) = \alpha(\rho_1) \cdots \alpha(\rho_k) \in N$ is independent of the choice of the word of minimal length representing w (*cf.* [22] Proposition 2.1).

4.3. **Chevalley Schemes.** We keep the notation of § 4.1 and § 4.2. To a root system $\{L, \Phi, n_r\}$ one associates, following [3] and [11], a reductive group scheme $\mathfrak{G} = \mathfrak{G}\{L, \Phi, n_r\}$ over $\mathbb{Z}$: the *Chevalley scheme*. We denote by $\mathcal{T}$ a maximal torus that is part of a split structure of $\mathfrak{G}$ and by $\mathcal{N}$ its normalizer.
To a reflection $s \in S$ correspond naturally the following data: a one dimensional sub-torus $\mathcal{T}_s \subset \mathcal{T}$, a rank one semi-simple subgroup $\mathfrak{G}_s \subset \mathfrak{G}$ containing $\mathcal{T}_s$ as a maximal torus and a point $h_s \in \mathcal{T}_s$ (belonging to the center of $\mathfrak{G}_s$). One denotes by $\mathcal{N}_s$ the normalizer of $\mathcal{T}_s$ in $\mathfrak{G}_s$.
Let A be a commutative ring with unit and let A^* be its multiplicative group. We denote by $\mathfrak{G}(A)$, $\mathcal{T}(A)$, *resp.* $\mathcal{N}(A)$ the groups of points of $\mathfrak{G}$, $\mathcal{T}$, *resp.* $\mathcal{N}$ which are rational over A. The quotient $\mathcal{N}(A)/\mathcal{T}(A)$ is canonically isomorphic to $W = W(M)$. More precisely, there exists a unique surjective homomorphism $p_A : \mathcal{N}(A) \to W(M)$, whose kernel is $\mathcal{T}(A)$ so that the data $\{\mathcal{N}(A), p_A, \mathcal{N}_s(A); s \in S\}$ satisfy the conditions (n1)-(n3) of § 4.2.

The goal of this paragraph is to review a fundamental result of [22] which describes the data above only in terms of A and the root system $\{L, \Phi, n_r\}$. To achieve this result one makes use of the following facts:

- The group $\mathcal{T}(A)$ is canonically isomorphic to $\mathrm{Hom}_{\mathbb{Z}}(L, A^*)$ and the left action of $W(M) \simeq \mathcal{N}(A)/\mathcal{T}(A)$ on $\mathcal{T}(A)$ is induced from the natural action of W on L.
- If $s = s_r$ is the reflection associated to a root $r \in \Phi$, then

$$\mathcal{N}_{s_r}(A) \cap \mathcal{T}(A) = \mathcal{T}_{s_r}(A) = \{\rho \in \mathrm{Hom}(L, A^*)\,|\,\exists\, a \in A^*\,,\ \rho(x) = a^{\nu(x)}\,,\ \forall x \in L\}$$

where $\nu : L \to \mathbb{Z}$ is a linear form proportional to n_r (*cf.* § 4.2). Taking $a = -1$ and $\nu = n_r$, one gets the element $h_s \in \mathcal{T}_s(A)$.

- The normalizer $\mathcal{N}_s$ of $\mathcal{T}_s$ in $\mathfrak{G}_s$ is such that all elements of $\mathcal{N}_s(A)$ which are not in $\mathcal{T}_s(A)$ have a square equal to $h_s(A) \in \mathcal{T}_s(A)$.

We are now ready to state Theorem 4.4 of [22] which plays a key role in our construction.

Theorem 4.3. *The group extension*

$$1 \to \mathcal{T}(A) \to \mathcal{N}(A) \xrightarrow{p} W \to 1$$

is canonically isomorphic to the group extension

$$1 \to Hom(L, A^*) \to \mathcal{N}_{A^*,-1}(L, \Phi) \xrightarrow{p} W \to 1.$$

Here $\mathcal{N}_{A^*,-1}(L, \Phi)$ refers to the functorial construction of Proposition 4.2 for the group $D = A^*$ and $\epsilon = -1 \in D$. Note that the case $A = \mathbb{Z}$ corresponds to $D = \{\pm 1\}$, $\epsilon = -1$, and gives the extension $\mathcal{N}(\mathbb{Z})$ of W by $\mathrm{Hom}(L, \{\pm 1\}) \sim (\mathbb{Z}/2\mathbb{Z})^\ell$. This particular case contains the essence of the general construction since for any pair (D, ϵ) the group $\mathcal{N}_{D,\epsilon}(L, \Phi)$ is the amalgamated product

$$\mathrm{Hom}(L, D) \times_{\mathrm{Hom}(L,\{\pm 1\})} \mathcal{N}(\mathbb{Z})\,.$$

4.4. **Bruhat decomposition.** We keep the notation as in the earlier paragraphs of this section. To each root $r \in \Phi$ corresponds a root subgroup $\mathfrak{X}_r \subset \mathfrak{G}$ defined as the range of an isomorphism x_r from the additive group $\mathbb{G}_{a,\mathbb{Z}}$ to its image in $\mathfrak{G}$ and fulfilling the equation

$$t^{-1} x_r(\xi) t = x_r(r(t)\xi)\,, \ \forall\ t \in \mathcal{T}. \tag{38}$$

We recall the following standard notation and well-known relations:

$$n_r(t) = x_r(t) x_{-r}(-t^{-1}) x_r(t)\,, \ \forall t\ \in A^*\,, \quad n_r = n_r(1) \tag{39}$$

$$h_r(t) = n_r(t) n_r(-1)\,, \ \forall t\ \in A^*\,, h_r = h_r(1)\,, \quad h_r(t_1) h_r(t_2) = h_r(t_1 t_2). \tag{40}$$

Let $r, s \in \Phi$ be linearly independent roots and $t, u \in A$. The *commutator* of $x_s(u)$ and $x_r(t)$ is defined as

$$[x_s(u), x_r(t)] = x_s(u)^{-1} x_r(t)^{-1} x_s(u) x_r(t). \tag{41}$$

The following formula, due to Chevalley, expresses the above commutator as a product of generators corresponding to roots of the form $ir + js \in \Phi$, for $i, j > 0$

Lemma 4.4. *Let $r, s \in \Phi$ be linearly independent roots. Then there exist integers $C_{ijrs} \in \mathbb{Z}$ such that*

$$x_s(u)^{-1} x_r(t) x_s(u) x_r(t)^{-1} = \prod_{i,j} x_{ir+js}(t^i u^j)^{C_{ijrs}} \tag{42}$$

where the product is applied to pairs (i, j) of strictly positive integers such that $ir + js \in \Phi$, and the terms are arranged in order of increasing $i + j$.

The above formula holds in particular in the case that r, s belong to the subset $\Phi^+ \subset \Phi$ of the positive roots (*cf.* 4.1 and [11], Exposé XXI), in which case the product is taken over all positive roots of the form $ir + js$, $i > 0$, $j > 0$ in increasing order, for the chosen ordering of the lattice L (*cf.* [11], Exposé XXII, Lemme 5.5.6 p. 208).

Let $\mathcal{U}(A)$ be the subgroup of $\mathfrak{G}(A)$ generated by the elements $x_r(t)$ for $r \in \Phi^+$, $t \in A$. By construction, the subgroup $\mathcal{U} \subset \mathfrak{G}$ is generated by the root subgroups $\mathfrak{X}_r$

corresponding to the positive roots. For any $w \in W$, we let $\Phi_w = \{r \in \Phi^+ | w(r) < 0\}$ and we denote by $\mathcal{U}_w$ the subgroup generated by the root subgroups $\mathfrak{X}_r$ for $r \in \Phi_w$.
Chevalley proved in [2], Théorème 2 the existence of a canonical form for the elements of the group $\mathfrak{G}(K)$, when K is a field. We recall this result

Theorem 4.5. *Let K be a field. The group $\mathfrak{G}(K)$ is the disjoint union of the subsets (cells)*

$$C_w = \mathcal{U}(K)\, \mathcal{T}(K)\, n_w\, \mathcal{U}_w(K)$$

where for each $w \in W$, $n_w \in \mathcal{N}(K)$ is a chosen coset representative for w. The natural map

$$\varphi_w : \mathcal{U}(K) \times \mathcal{T}(K) \times \mathcal{U}_w(K) \to C_w\,, \quad \varphi_w(x, h, x') = xhn_w x' \tag{43}$$

is a bijection for any $w \in W$.

We refer to [11], Exposé XXI, Théorème 5.7.4 and Remarque 5.7.5.

4.5. **Chevalley group schemes as gadgets.** For the definition of the gadget over $\mathbb{F}_1$ associated to a Chevalley group G and in particular for the construction of the natural transformation e_G (*cf.* § 2.2), one needs to choose a Chevalley basis of the Lie algebra of G and a total ordering of the lattice L. We keep the notation as in § 4.4 and in § 4.1 through § 4.3.
Chevalley proved (*cf.* [2], Lemma 6) that, over any commutative ring A, each element of $\mathcal{U}(A)$ is *uniquely* expressible in the form

$$\prod_{r\in\Phi^+} x_r(t_r)\,, \quad t_r \in A\,,\ \forall r\ \in \Phi^+$$

where the product is taken over all positive roots in increasing order. More precisely one has the following

Lemma 4.6. *The map*

$$t = (t_r)_{r\in\Phi^+} \mapsto \psi(t) = \prod_{r\in\Phi^+} x_r(t_r) \tag{44}$$

establishes a bijection of the free A-module with basis the positive roots in Φ^+ with $\mathcal{U}(A)$.

The proof of this lemma applies without change to give the following variant, where we let $\Phi_w = \{r \in \Phi^+ | w(r) < 0, w \in W\}$ and we denote by $\mathcal{U}_w = \prod_{r\in\Phi_w} \mathfrak{X}_r$.

Lemma 4.7. *The map*

$$t = (t_r)_{r\in\Phi_w} \mapsto \psi_w(t) = \prod_{r\in\Phi_w} x_r(t_r) \tag{45}$$

establishes a bijection of the free A-module with basis Φ_w with $\mathcal{U}_w(A)$.

The key identity in the proof of Lemmas 4.6 and 4.7 is the commutator relation of Lemma 4.4.

We are now ready to apply the theory reviewed in the previous paragraphs to construct the functor

$$\underline{G} : \mathcal{F}_{ab}^{(2)} \to \mathcal{S}ets$$

from the category $\mathcal{F}_{ab}^{(2)}$ of pairs (D,ϵ) of a finite abelian group and an element of square one, to the category of graded sets.

Definition 4.8. *The functor $\underline{G}:\mathcal{F}_{ab}^{(2)}\to \mathcal{S}ets$ is defined as the graded product*

$$\underline{G}(D,\epsilon)=\underline{\mathbb{A}}^{\Phi^+}(D)\times\coprod_{w\in W}(p^{-1}(w)\times\underline{\mathbb{A}}^{\Phi_w}(D)) \tag{46}$$

where p is the projection $\mathcal{N}_{D,\epsilon}(L,\Phi)\xrightarrow{p} W$ as in §§4.2 and 4.3. All elements of $p^{-1}(w)$ have degree equal to the rank of $\mathfrak{G}$.

It follows immediately that there are no elements of degree less than the rank ℓ of $\mathfrak{G}$ and that the set of elements of degree ℓ is canonically identified with $\mathcal{N}_{D,\epsilon}(L,\Phi)$.
We now move to the definition of the natural transformation

$$e_G:\underline{G}\to \mathrm{Hom}(\mathrm{Spec}\,\mathbb{C}[-],G_{\mathbb{C}}),\quad (D,\epsilon)\mapsto \mathrm{Hom}(\mathrm{Spec}\,\mathbb{C}[D,\epsilon],G_{\mathbb{C}}).$$

For this part, we make use of the natural transformations e_F of (26) for $F=\Phi^+$ and $F=\Phi_w$ and of Theorem 4.3 to obtain, for a given character χ associated to a point in $\mathrm{Spec}\,\mathbb{C}[D,\epsilon]$, maps

$$e_{\Phi^+}\ :\ \underline{\mathbb{A}}^{\Phi^+}(D)\to\mathbb{C}^{\Phi^+} \tag{47}$$

$$e_{\Phi_w}\ :\ \underline{\mathbb{A}}^{\Phi_w}(D)\to\mathbb{C}^{\Phi_w} \tag{48}$$

$$e_{\mathcal{N}}\ :\ \mathcal{N}_{D,\epsilon}(L,\Phi)\to\mathcal{N}(\mathbb{C}). \tag{49}$$

The last map is compatible with the projection p, thus restricts to $p^{-1}(w)$ for any $w\in W$. We now make use of Lemmas 4.6 and 4.7 to obtain the natural transformation e_G defined as follows

$$e_G(a,n,b)=\psi(e_{\Phi^+}(a))\,e_{\mathcal{N}}(n)\,\psi_w(e_{\Phi_w}(b))\in\mathfrak{G}(\mathbb{C})=G_{\mathbb{C}} \tag{50}$$

where ψ and ψ_w are defined as in Lemmas 4.6 and 4.7, for $A=\mathbb{C}$.

4.6. **Proof that G determines a variety over $\mathbb{F}_{1^2}$.** In this paragraph we shall prove that the gadget $G=(\underline{G},G_{\mathbb{C}},e_G)$ associated to a Chevalley group G (or equivalently to its root system $\{L,\Phi,n_r\}$, *cf.* § 4.1) defines a variety (of finite type) over $\mathbb{F}_{1^2}$. We keep the earlier notation. We first recall the following important result of Chevalley (*cf.* [3], Proposition 1).

Proposition 4.9. *Let $w_0\in W$ be the unique element of the Weyl group such that $w_0(\Phi^+)=-\Phi^+$ and let w_0' be a lift of w_0 in $G_{\mathbb{Z}}$. Consider the following morphism, associated to the product in the group,*

$$\theta:\mathcal{U}\times p^{-1}(w_0)\times\mathcal{U}\to\mathfrak{G},\quad \theta(u,n,v)=unv \tag{51}$$

Then θ defines an isomorphism of $\mathcal{U}\times p^{-1}(w_0)\times\mathcal{U}$ with an open affine subscheme Ω of $\mathfrak{G}$, whose global algebra of coordinates is of the form

$$\mathcal{O}_\Omega=\mathcal{O}_{\mathfrak{G}}[d^{-1}] \tag{52}$$

where $d\in\mathcal{O}_{\mathfrak{G}}$ takes the value 1 on w_0'.

We refer also to proposition 4.1.2 page 172 in [11] combined with the next proposition 4.1.5.
The next theorem shows that the gadget $G=(\underline{G},G_{\mathbb{C}},e_G)$ over $\mathbb{F}_{1^2}$ fulfills the condition of Definition 2.8.

Theorem 4.10. *The gadget $G=(\underline{G},G_{\mathbb{C}},e_G)$ defines a variety over $\mathbb{F}_{1^2}$.*

Proof. By construction G is a finite and graded gadget. It is easy to guess that the sought for scheme $G_{\mathbb{Z}}$ over $\mathbb{Z}$ is the Chevalley scheme $\mathfrak{G}$ associated to the root system $\{L, \Phi, n_r\}$ (*cf.* eg [11], Corollary 1.2, Exposé XXV). One has by construction an immersion of gadgets $G \hookrightarrow \mathcal{G}(\mathfrak{G})$. It remains to be checked that it fulfills the universal property of Definition 2.8. Let $V = \mathrm{Spec}(\mathcal{O}(V))$ be an affine variety of finite type over $\mathbb{Z}$ and $\phi : G \to \mathcal{G}(V)$, be a morphism of gadgets. This means that we are given a pair $(\underline{\phi}, \phi_{\mathbb{C}})$ where

$$\phi_{\mathbb{C}} \ : \ \mathcal{O}_{\mathbb{C}}(V) \to \mathcal{O}_{\mathbb{C}}(\mathfrak{G})$$

is a homomorphism of $\mathbb{C}$-algebras, and $\underline{\phi}$ is a natural transformation of functors from abelian groups to sets

$$\underline{\phi}(D) \ : \ \underline{G}(D) \to \mathrm{Hom}(\mathcal{O}(V), \beta(D))$$

which satisfies the following compatibility condition (*cf.* (17)): for any finite abelian group D the following diagram commutes

$$\begin{array}{ccc} \underline{G}(D,\epsilon) & \xrightarrow{\ \underline{\phi}(D)\ } & \mathrm{Hom}(\mathcal{O}(V), \beta(D)) \\ \Big\downarrow{\scriptstyle e_G(D,\epsilon)} & & \Big\downarrow{\scriptstyle e_{\mathcal{G}(V)}(D)=\subset} \\ \mathrm{Hom}(\mathcal{O}_{\mathbb{C}}(\mathfrak{G}), \mathbb{C}[D,\epsilon]) & \xrightarrow{\ \phi_{\mathbb{C}}\ } & \mathrm{Hom}(\mathcal{O}_{\mathbb{C}}(V), \mathbb{C}[D,\epsilon]). \end{array} \qquad (53)$$

One needs to show that $\phi_{\mathbb{C}}(\mathcal{O}(V)) \subset \mathcal{O}(\mathfrak{G})$. Let $h \in \mathcal{O}(V)$ and $f = \phi_{\mathbb{C}}(h)$. Then by construction $f \in \mathcal{O}_{\mathbb{C}}(\mathfrak{G})$. By Proposition 4.9, the intersection $\mathcal{O}_{\mathbb{C}}(\mathfrak{G}) \cap \mathcal{O}_{\Omega}$ coincides with $\mathcal{O}(\mathfrak{G})$ since $\mathcal{O}_{\Omega} = \mathcal{O}_{\mathfrak{G}}[d^{-1}]$ while elements of $\mathcal{O}_{\mathbb{C}}(\mathfrak{G})$ have a trivial pole part in d^{-1}. Thus it is enough to show that the restriction of f to the open affine subscheme $\Omega \subset \mathfrak{G}$ belongs to $\mathcal{O}_{\Omega}$, to conclude that $f \in \mathcal{O}(\mathfrak{G})$. Let us choose a lift w_0' of w_0 in $\mathcal{N}_{\mathbb{Z}}$. In fact we can more precisely choose a lift w_0' of w_0 in $\mathcal{N}_{\mathbb{Z}/2\mathbb{Z},\epsilon}(L, \Phi)$, where $\mathbb{Z}/2\mathbb{Z}$ is the group of order two generated by ϵ and then take the image of w_0' under the map which sends ϵ to -1. We have $p^{-1}(w_0) = w_0'\mathcal{T}$. As in [3] (*cf.* § 4, Proposition 1), the algebra $\mathcal{O}_{\Omega}$ is the tensor product of the following three algebras:

- $\mathcal{O}(\mathcal{U})$ which is the algebra of polynomials over $\mathbb{Z}$ generated by the coordinates t_r of Lemma 4.6.
- $\mathcal{O}(\mathcal{T}) = \mathbb{Z}[L]$ the group ring of the abelian group L.
- Another copy of $\mathcal{O}(\mathcal{U})$.

We consider elements of $\underline{G}(D, \epsilon)$ of the form

$$g \in C = \underline{\mathbb{A}}^{\Phi^+}(D) \times p^{-1}(w_0) \times \underline{\mathbb{A}}^{\Phi^+}(D) \qquad (54)$$

and use the choice of $w_0' \in p^{-1}(w_0) \subset \mathcal{N}_{D,\epsilon}(L, \Phi)$ to identify the cosets $p^{-1}(w_0) = \mathrm{Hom}(L, D)w_0'$. Then, we choose generators v_j, $1 \leq j \leq \ell$, of the free abelian group L and use them to identify $\mathrm{Hom}(L, D)$ with the set of $(y_j)_{j \in \{1,\ldots,\ell\}}$, $y_j \in D$. Each map y from $Y = \Phi \cup \{1, \ldots, \ell\}$ to D defines uniquely an element $g(y) \in C$ by[2]

$$g(y) = (y_r)_{r \in \Phi^+} \times (y_j)_{j \in \{1,\ldots,\ell\}} \times (y_{-r})_{r \in \Phi^+} \qquad (55)$$

[2]not all elements of C are of this form

Then $g(y) \in \underline{G}(D)$ and $\underline{\phi}(D)(g(y)) \in \mathrm{Hom}(\mathcal{O}(V), \beta(D))$ so that by evaluating on $h \in \mathcal{O}(V)$ one gets

$$\underline{\phi}(D)(g(y))(h) \in \mathbb{Z}[D] \subset \mathbb{C}[D, \epsilon]\,. \tag{56}$$

By the commutativity of the diagram (53), this is the same as evaluating on $f \in \mathcal{O}_{\mathbb{C}}(\mathfrak{G})$ the homomorphism $e_G(D)(g(y))$. We denote by $k = f|_\Omega$ the restriction of f to Ω, it is given by a polynomial with complex coefficients

$$k = P(t_r, u_j, u_j^{-1}) \in \mathbb{C}[t_r, u_j, u_j^{-1}]\,,\ r \in \Phi\,,\ 1 \le j \le \ell\,. \tag{57}$$

Let n be an integer and $D = (\mathbb{Z}/n\mathbb{Z})^Y \times \mathbb{Z}/2\mathbb{Z}$ be the group of maps from $Y = \Phi \cup \{1, \ldots, \ell\}$ to the cyclic group of order n, times the cyclic group of order two with generator ϵ ($\epsilon^2 = 1$). We denote by ξ the generator of $\mathbb{Z}/n\mathbb{Z}$ and for $s \in Y$ we let $\xi_s \in D$ have all its components equal to $0 \in \mathbb{Z}/n\mathbb{Z}$ except for the component at s which is ξ. One has a homomorphism of algebras

$$\theta_n : \mathbb{C}[t_r, u_j, u_j^{-1}] \to \mathbb{C}[D] \sim \mathbb{C}[D, \epsilon]\,,\quad t_r \mapsto \xi_r\,,\quad u_j \mapsto \xi_j$$

Using (56), we know that for each n, $\theta_n(k) \in \mathbb{Z}[D]$. Now for $k \in \mathbb{C}[t_r, u_j, u_j^{-1}]$ one can compute the coefficients b_I of the polynomial as the Fourier coefficients

$$b_I = (2\pi)^{-d} \int_{(S^1)^d} k(e^{i\alpha_1}, \ldots, e^{i\alpha_d})\, e^{-iI.\alpha} \prod d\alpha_j$$

and hence as the limit

$$b_I = \lim_{n\to\infty} n^{-d} \sum_{1 \le a_j \le n} k(e^{2\pi i \frac{a_1}{n}}, \ldots, e^{2\pi i \frac{a_d}{n}})\, e^{-iI.\alpha}\,,\quad \alpha_j = 2\pi \frac{a_j}{n}$$

When $k(x) = \prod t_r^{m_r} \prod u_j^{m_j}$ is a monomial, the sum

$$\sum_{1 \le a_j \le n} k(e^{2\pi i \frac{a_1}{n}}, \ldots, e^{2\pi i \frac{a_d}{n}})\, e^{-iI.\alpha}$$

is either zero or n^d and the latter case only happens if all the components of the multi index $m - I$ are divisible by n. Thus

$$n^{-d} \sum_{1 \le a_j \le n} k(e^{2\pi i \frac{a_1}{n}}, \ldots, e^{2\pi i \frac{a_d}{n}})\, e^{-iI.\alpha}$$

only depends on $\theta_n(k)$ and is a relative integer if $\theta_n(k) \in \mathbb{Z}[D]$. It follows that all the b_I are in $\mathbb{Z}$ and hence $k \in \mathcal{O}(\Omega)$. □

4.7. **The distinction between G_k and $\mathfrak{G}(k)$.** Let $\mathfrak{G}$ be the Chevalley group scheme associated to a root system as in §4.3 and let k be a field.

The subgroups $\mathfrak{X}_r$ generate in the group $\mathfrak{G}(k)$ of points that are rational over k a subgroup G_k which is the *commutator subgroup* of the group $\mathfrak{G}(k)$. The subgroup $G_k \subset \mathfrak{G}(k)$ is often called a Chevalley group over k and is not in general an algebraic group. If $\mathfrak{G}$ is the universal Chevalley group, then one knows that $G_k = \mathfrak{G}(k)$, so that the distinction between the commutator subgroup G_k and the group $\mathfrak{G}(k)$ is irrelevant.

In the construction pursued in this paper of the gadget associated to a Chevalley group, one can take into account this subtlety between G_k and $\mathfrak{G}(k)$ by constructing the following sub-gadget. Let $(\tilde{L}, \tilde{\Phi}, n_{\tilde{r}})$ be the simply connected root system associated to (L, Φ, n_r) (*cf.* § 4.1) and let $\varphi : L \to \tilde{L}$ be the morphism connecting the two roots systems as follows

$$\varphi(\Phi) = \tilde{\Phi}\,, \quad n_r = n_{\varphi(r)} \circ \varphi\,,\ \forall r \in \Phi\,. \tag{58}$$

One simply replaces the term $\mathrm{Hom}(L, D)$ in the construction of the functor $\underline{G}(D)$ (*cf.* Definition 4.8) by the following subgroup

$$\{\chi \in \mathrm{Hom}(L, D) \mid \exists \chi' \in \mathrm{Hom}(\tilde{L}, D)\,,\ \chi = \chi' \circ \varphi\} \tag{59}$$

which is the range of the restriction map from $\mathrm{Hom}(\tilde{L}, D)$ to $\mathrm{Hom}(L, D)$.
Unlike for the group $\mathfrak{G}(k)$ the function $N(q)$ that counts the number of points of G_k for $k = \mathbb{F}_q$ is not in general a polynomial function of q.

5. Final remarks

The category $\mathcal{F}_{ab}$ of finite abelian groups that has been used in this paper in the definition of the graded functor $\underline{X}$ can be replaced by the larger category $\mathcal{G}_{ab}$ of abelian groups (not necessarily finite).
Also the construction of the natural transformation e_G can be extended to yield, for any commutative ring A and for $D = A^*$, an embedding of the points of $\underline{G}(D)$ in the group $G(A)$. If A is a field, the resulting map is a bijection.

Theorem 5.1. *Let $\mathfrak{G}$ be the Chevalley scheme over $\mathbb{Z}$ associated to a Chevalley group G.*

- *The construction* (46) *of the functor $\underline{G}$ extends to the category of pairs (D, ϵ) of an abelian group and an element of square one.*
- *The construction* (50) *of the natural transformation e_G extends to arbitrary commutative rings A to yield a map*

$$e_{G,A} :\ \underline{G}(A^*, -1) \to \mathfrak{G}(A) \tag{60}$$

- *When A is a field the map $e_{G,A}$ is a bijection.*

Proof. The first statement follows from the results of [22], § 4.3, in which the construction of $\mathcal{N}_{D,\epsilon}(L, \Phi)$ is done for all abelian groups D.
The second statement follows from Theorem 4.3, and the fact that the map e_G of Theorem 4.10 is defined over $\mathbb{Z}$.
The last statement follows from Bruhat's Theorem in the form of Theorem 4.5. □

Thus Theorem 5.1 suggests to strengthen the conditions imposed on a variety[3] over $\mathbb{F}_1$ by requiring that the natural transformation e_X in fact extends from the special case $A = \mathbb{C}$ to arbitrary abelian rings A and yields a map

$$e_{X,A} \ :\ \underline{X}(A^*) \to X(A) \tag{61}$$

which is bijective when A is a field. A nice feature of this requirement is that it ensures that the counting of points gives the correct answer. Indeed, the above condition ensures that the number of points over $\mathbb{F}_{1^n}$ which is given by the cardinality of $\underline{X}(D)$ for $D = \mathbb{Z}/n\mathbb{Z}$, agrees with the cardinality of $X(\mathbb{F}_q)$ when $n = q - 1$ and q is a prime power.

What we have shown in this paper is that Chevalley group schemes yield varieties over $\mathbb{F}_{1^2}$, but *not* that the group operation μ can be defined over $\mathbb{F}_{1^2}$. In fact, it is only the terms of lowest degree (equal to the rank of G) that yield a group, namely

[3]The variant of $\mathbb{F}_{1^n}$ is similar

the group $\mathcal{N}_{D,\epsilon}(L,\Phi)$ of J. Tits. The structure of the terms of higher order is more mysterious.

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A. Connes: Collège de France, 3, rue d'Ulm, Paris, F-75005 France, I.H.E.S. and Vanderbilt University

E-mail address: alain@connes.org

C. Consani: Mathematics Department, Johns Hopkins University, Baltimore, MD 21218 USA

E-mail address: kc@math.jhu.edu