$\mathbb{F}_1$-schemes and toric varieties

Anton Deitmar

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Author's address: Math. Inst., Auf der Morgenstelle 10, 72076 Tübingen, Germany

Abstract: In this paper it is shown that integral $\mathbb{F}_1$-schemes of finite type are essentially the same as toric varieties. A description of the $\mathbb{F}_1$-zeta function in terms of toric geometry is given. Etale morphisms and universal coverings are introduced.

Introduction

There are by now several attempts to make the theory of the field of one element $\mathbb{F}_1$ rigorous. In [10] the authors formalize the transition from rings to schemes on a categorial level and apply this machinery to the category of sets to obtain the category of $\mathbb{F}_1$-schemes as in [1]. In [3] and [5] the authors extend the definition of rings in order to capture a structure that deserves to be called $\mathbb{F}_1$. In [1] the author tried instead to fix the minimum properties any of these theories must share. The current paper extends this line of thought. We use terminology of [1] and [2].

In this paper, a ring will always be commutative with unit and a monoid will always be commutative. An *ideal* $\mathfrak{a}$ of a monoid A is a subset with $A\mathfrak{a} \subset \mathfrak{a}$. A *prime ideal* is an ideal $\mathfrak{p}$ such that $S_\mathfrak{p} = A \smallsetminus \mathfrak{p}$ is a submonoid of A. For a prime ideal $\mathfrak{p}$ let $A_\mathfrak{p} = S_\mathfrak{p}^{-1}A$ be the *localization* at $\mathfrak{p}$. The *spectrum* of a monoid A is the set of all prime ideals with the obvious Zariski-topology (see [1]). Similar to the theory of rings, one defines a structure sheaf $\mathcal{O}_X$ on $X = \mathrm{spec}\,(A)$, and one defines a *scheme over* $\mathbb{F}_1$ to be a topological space together with a sheaf of monoids, locally isomorphic to spectra of monoids.

A $\mathbb{F}_1$-scheme X is *of finite type*, if it has a finite covering by affine schemes $U_i = \mathrm{spec}\,(A_i)$ such that each A_i is a finitely generated monoid. For a ring R, we write X_R for the R-base-change of X, so $X_R = X_{\mathbb{Z}} \otimes_{\mathbb{Z}} R$.

For a monoid A we let $A \otimes \mathbb{Z}$ be the monoidal ring $\mathbb{Z}[A]$. This defines a functor from monoids to rings which is left adjoint to the forgetful functor that sends a ring R to the multiplicative monoid $(R, \times)$. This construction is compatible with gluing, so one gets a functor $X \mapsto X_{\mathbb{Z}}$ from $\mathbb{F}_1$-schemes to $\mathbb{Z}$-schemes. In [2] we have shown that X is of finite type if and only if $X_{\mathbb{Z}}$ is a $\mathbb{Z}$-scheme of finite type.

We say that the monoid A is *integral*, if it has the cancellation property, i.e., if $ab = ac$ implies $b = c$ in A. This is equivalent to saying that A injects into its quotient group or A is a submonoid of a group.

By a *module* of a monoid A we mean a set M together with a map $A \times M \to M$; $(a, m) \mapsto am$ with $1m = m$ and $(ab)m = a(bm)$. A *stationary point* of a module is a point $m \in M$ with $am = m$ for every $a \in A$. A *pointed module* is a pair (M, m_0) consisting of an A-module M and a stationary point $m_0 \in M$.

1 Flatness

Recall the tensor product of two modules M, N of A:

$$M \otimes N = M \otimes_A N = M \times N / \sim,$$

where $\sim$ is the equivalence relation generated by $(am, n) \sim (m, an)$ for every $a \in A, m \in M, n \in N$. The class of (m, n) is written as $m \otimes n$. The tensor product $M \otimes N$ becomes a module via $a(m \otimes n) = (am) \otimes n$. For example, the module $A \otimes M$ is isomorphic to M.

Let now (M, m_0) and (N, n_0) be two pointed modules of A, then $(M \otimes N, m_0 \otimes n_0)$ is a pointed module, called the pointed tensor product.

The category $\mathrm{Mod}_0(A)$ of pointed modules and pointed morphisms has a terminal and initial object 0, so it makes sense to speak of kernels and cokernels. It is easy to see that every morphism f in $\mathrm{Mod}_0(A)$ possesses both. One defines the *image* of f as $\mathrm{im}(f) = \ker(\mathrm{coker}(f))$ and the *coimage* as $\mathrm{coim}(f) = \mathrm{coker}(\ker(f))$.

A morphism is called *strong*, if the natural map from $\mathrm{coim}(f)$ to $\mathrm{im}(f)$ is an isomorphism. Kernels and cokernels are strong. If $A \xrightarrow{f} B \xrightarrow{g} C$ is given with g being strong and $gf = 0$, then the induced map $\mathrm{coker}(f) \to C$ is strong. Likewise, if f is strong and $gf = 0$, then the induced map $A \to \ker g$ is strong. A map is strong if and only if it can be written as a cokernel followed by a kernel.

The usual notion of exact sequences applies, and we say that a sequence of morphisms is *strong exact* if it is exact and all morphisms in the sequence are strong.

A module $F \in \mathrm{Mod}_0(A)$ is called *flat*, if the functor $X \mapsto F \otimes X$ is strong-exact, i.e., if for every strong exact sequence

$$0 \to M \to N \to P \to 0$$

the induced sequence

$$0 \to F \otimes M \to F \otimes N \to F \otimes P \to 0$$

is strong exact as well.

It is easy to see that a pointed module F is flat if and only if for every injection $M \hookrightarrow N$ of pointed modules the map $F \otimes M \to F \otimes N$ is an injection.

Examples. If A is a group, then every module is flat. Let S be a submonoid of A. Then the localization $S^{-1}A$ is a flat A-module. The direct sum $G \oplus F$ of two flat modules is flat. Finally, consider the free monoid in one generator $C_+ = \{1, \tau, \tau^2, \dots\}$, then an A-module M is flat if and only if $\tau m = \tau m'$ implies $m = m'$ for all $m, m' \in M$. This is equivalent to saying that M is a C_+-submodule of a module of the quotient group $C_\infty = \tau^{\mathbb{Z}}$ of C_+. The same characterization holds for every integral monoid.

A morphism $\varphi : A \to B$ of monoids is called flat if B is flat as an A-module. A morphism of $\mathbb{F}_1$-schemes $f : X \to Y$ is called flat if for every $x \in X$ the morphism of monoids $f^\# : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat.

The following is straightforward.

- A morphism of monoids $\varphi : A \to B$ is flat if and only if the induced morphism of $\mathbb{F}_1$-schemes $\operatorname{spec} B \to \operatorname{spec} A$ is flat.
- The composition of flat morphisms is flat.
- The base change of a flat morphism by an arbitrary morphism is flat.

Remark. It is easy to see that if $\mathbb{Z}[F]$ is flat as $\mathbb{Z}[A]$-module, then F is flat as A-module. The converse is already false if A is a group. As an example let k be a field and let A be the group of all matrices of the form $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ where $x \in k$. Let A act on k^2 in the usual way and trivially on k. Consider the exact sequence of $\mathbb{Z}[A]$-modules,

$$0 \longrightarrow k \xrightarrow{\alpha} k^2 \xrightarrow{\beta} k \longrightarrow 0,$$

where $\alpha(x) = \begin{pmatrix} x \\ 0 \end{pmatrix}$, $\beta\begin{pmatrix} x \\ y \end{pmatrix} = y$. Let $F = \{1\}$ the trivial A-module, then for every $\mathbb{Z}[A]$-module M one has $M \otimes_{\mathbb{Z}[A]} \mathbb{Z}[F] = H_0(A, M)$. Note that $H_0(A, k) = k$ and that $H_0(\alpha) = 0$, so it is not injective, hence $\mathbb{Z}[F]$ is not flat.

2 Algebraic extensions

Let A be a submonoid of B. An element $b \in B$ is called *algebraic over* A, if there exists $n \in \mathbb{N}$ with $b^n \in A$. The extension B/A is called *algebraic*, if every $b \in B$ is algebraic over A. An algebraic extension B/A is called *strictly algebraic*, if for every $a \in a$ the equation $x^n = a$ has at most n solutions in B.

If B/A is algebraic, then $\mathbb{Z}[B]/\mathbb{Z}[A]$ is an algebraic ring extension, but the converse is wrong in general, as the following example shows: Let $A = \mathbb{F}_1$ and B be the set of two elements, 1 and b with $b^2 = b$.

A monoid A is called *algebraically closed*, if every equation of the form $x^n = a$ with $a \in A$ has a solution in A. Every monoid A can be embedded into an algebraically closed one, and if A is a group, then there exists a smallest such embedding, called the *algebraic closure* of A. For example, the algebraic closure $\bar{\mathbb{F}}_1$ of $\mathbb{F}_1$ is the group μ_∞ of all roots of unity, which is isomorphic to $\mathbb{Q}/\mathbb{Z}$.

3 Etale morphisms

Recall that a homomorphism $\varphi\colon A \to B$ of monoids is called a *local* homomorphism, if $\varphi^{-1}(B^\times) = A^\times$ (every φ satisfies "$\supset$"). For a monoid A let $m_A = A \smallsetminus A^\times$ be its maximal ideal. It is easy to see that a homomorphism $\varphi\colon A \to B$ is local if and only if $\varphi(m_A) \subset m_B$.

A local homomorphism $\varphi\colon A \to B$ is called *unramified* if

- $\varphi(m_A)B = m_B$ and
- φ injects $A^\times$ into $B^\times$ and $B/\varphi(A)$ is a finite strictly algebraic extension.

Note that if φ is unramified, then so are all localizations $\varphi_\mathfrak{p} : A_{\varphi^{-1}(\mathfrak{p})} \to B_\mathfrak{p}$ for $\mathfrak{p} \in \operatorname{spec} B$.

A morphism $f : X \to Y$ of $\mathbb{F}_1$-schemes is called unramified, if for every $x \in X$ the local morphism $f^\# : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is unramified.

A morphism $f : X \to Y$ of $\mathbb{F}_1$-schemes is called *locally of finite type*, if every point in Y has an open affine neighborhood $V = \operatorname{spec} A$ such that $f^{-1}(V)$ is a union of open affines $\operatorname{spec} B_i$ with B_i finitely generated as a monoid over A. The morphism is *of finite type* if for every point in Y the number of B_i can be chosen finite. The morphism is called *finite*, if every $y \in Y$ has an open affine neighborhood $V = \operatorname{spec} A$ such that $f^{-1}(V)$ is affine, equal to $\operatorname{spec} B$, where B is finitely generated as A-module.

A morphism $f : X \to Y$ of finite type is called *étale*, if f is flat and unramified. It is called an *étale covering*, if it is also finite.

Proposition 3.1 *The étale coverings of* $\operatorname{spec} \mathbb{F}_1$ *are the morphisms of the form* $\operatorname{spec} A \to \operatorname{spec} \mathbb{F}_1$*, where A is a finite cyclic group. The scheme* $\operatorname{spec} \bar{\mathbb{F}}_1$ *has no non-trivial étale coverings.*

Proof: Clear. □

A connected scheme over $\mathbb{F}_1$, which has only the trivial étale covering, is called *simply connected*.

Proposition 3.2 *The schemes* $\operatorname{spec} \bar{\mathbb{F}}_1$, $\operatorname{spec} C_+ \times_{\mathbb{F}_1} \bar{\mathbb{F}}_1$ *and* $\mathbb{P}^1_{\bar{\mathbb{F}}_1}$ *are simply connected.*

Proof: The first has been dealt with. For the second, let $A = \mu_\infty \times C_+$. Then $\operatorname{spec} A = \operatorname{spec} C_+ \times_{\mathbb{F}_1} \bar{\mathbb{F}}_1$. Let $f : X \to \operatorname{spec} A$ be an étale covering. As f is finite, X is affine, say $X = \operatorname{spec} B$. Let $\varphi : A \to B$ denote the corresponding morphism of monoids. The space $\operatorname{spec} A$ consists of two points, the generic point η_A and the closed point c_A. Likewise, let η_B, c_B denote the generic and closed points of $\operatorname{spec} B$. One has $f(\eta_B) = \eta_A$. We will show that $f(c_B) = c_A$. Assume the contrary. Then $\varphi^{-1}(m_B)$ is empty, hence φ maps A to the unit group $B^\times$. The localization at the closed point c_B then maps $\mu_\infty \times C_\infty$ to $B^\times$ and is unramified, hence injective. But as $C_+ \to C_\infty$ is not finite, neither can φ be finite, a contradiction. So we conclude $f(c_B) = c_A$, and so the corresponding localization, which is φ itself, is unramified. Let $s = \varphi(\tau)$, where τ is the generator of C_+. Then $\varphi(m_A)B = m_B$ implies $m_B = sB$, and so $B = B^\times \dot\cup sB$ (disjoint union). Also, $B^\times$ is an algebraic extension of $A^\times \cong \mu_\infty$, hence equals $\varphi(A^\times)$. As B is finitely generated and flat as A-module, there are $b_1, \dots, b_r \in B$ with

$$sB = B^\times s^{\mathbb{N}} \dot\cup B^\times s^{\mathbb{N}} b_1 \dot\cup \dots \dot\cup B^\times s^{\mathbb{N}} b_r.$$

If we assume $r > 0$, then b_1 is algebraic over $\varphi(A) = B^\times \dot\cup B^\times s^{\mathbb{N}}$, so let N be the smallest number in $\mathbb{N}$ such that $b_1^N \in \varphi(A)$. Then $b_1^N \notin B^\times \cong \mu_\infty$, because, as the extension is strictly algebraic, then b_1 would be in $B^\times$ already. So $b_1^N \in B^\times s^{\mathbb{N}}$. As the group $B^\times$ is divisible, we can replace b_1 with a $B^\times$ multiple to get $b_1^N = s^M$ for some $M \in \mathbb{N}$. Then $b_1 \notin B^\times s^{\mathbb{N}} b_1$, as $b_1 = b^* s^k b_1$ leads to $s^M = b_1^N = (b^*)^N s^{kN+M}$ which contradicts the injectivity of φ. But then b_1 must be in one of the other $B^\times s^{\mathbb{N}}$-orbits,which contradicts the disjointness of these orbits. We conclude $r = 0$, i.e. $B = B^\times \dot\cup B^\times s^{\mathbb{N}} \cong A$ as claimed. The assertion for $\mathbb{P}^1_{\bar{\mathbb{F}}_1}$ is an easy consequence. □

4 Toric varieties

Recall a *toric variety* is an irreducible variety V over $\mathbb{C}$ together with an algebraic action of the r-dimensional torus GL_1^r, such that V contains an open orbit.

As toric varieties can be constructed via lattices it follows that every toric variety is the lift $X_{\mathbb{C}}$ of an $\mathbb{F}_1$-scheme X. For integral schemes of finite type there is a converse direction given in the following theorem, which shows that integral $\mathbb{F}_1$-schemes of finite type are essentially the same as toric varieties.

Theorem 4.1 *Let X be a connected integral $\mathbb{F}_1$-scheme of finite type. Then every irreducible component of $X_{\mathbb{C}}$ is a toric variety. The components of $X_{\mathbb{C}}$ are mutually isomorphic as toric varieties.*

Proof: Let $U = \operatorname{spec} A$ be an open affine subset of X. Let η be the generic point of X, then the localization $G = A_\eta$ is the quotient group of A. At the same time, G is the stalk $\mathcal{O}_{X,\eta}$, so G does not depend on the choice of U up to canonical isomorphism. Let $\varphi : A \to G$ be the quotient map, which is injective as X is integral. The $\mathbb{C}$-algebra homomorphism,

$$\begin{array}{rcl} \mathbb{C}[A] & \to & \mathbb{C}[G] \otimes \mathbb{C}[A] \\ a & \mapsto & \varphi(a) \otimes a \end{array}$$

defines an action of the algebraic group $\mathcal{G} = \operatorname{spec} \mathbb{C}[G]$ on $\operatorname{spec} \mathbb{C}[A]$. Since this is compatible with the restriction maps of the structure sheaf, we get an algebraic action of the group scheme $\mathcal{G}$ on $X_{\mathbb{C}}$. As X is integral, $\mathcal{G} = \operatorname{spec} \mathbb{C}[G] = \operatorname{spec} \mathbb{C}[A_\eta]$ also is an open subset $V_{\mathbb{C}}$ of $X_{\mathbb{C}}$, and for $U_{\mathbb{C}} = \operatorname{spec} \mathbb{C}[A]$ the map

$$\mathcal{O}(U_{\mathbb{C}}) = \mathbb{C}[A] \overset{\varphi}{\longrightarrow} \mathbb{C}[G] = \mathcal{O}(V_{\mathbb{C}})$$

is the restriction map of the structure sheaf $\mathcal{O}$ of $X_{\mathbb{C}}$. The map $\mathbb{C}[A] \to \mathbb{C}[G]$ is injective and $\mathbb{C}[G]$ has zero Jacobson radical, so it follows that $V_{\mathbb{C}}$ is dense in $X_{\mathbb{C}}$, so in particular it meets every irreducible component. The group G is a finitely generated abelian group, so $G \equiv \mathbb{Z}^r \times F$ for a finite abelian group

F. Hence $\mathcal{G} \equiv \mathrm{GL}_1^r \times F$ as a group-scheme. As $\mathcal{G}$ meets every component of $X_{\mathbb{C}}$, the latter are permuted by F. Whence the claim. □

To formulate the next result, we will briefly recall the standard construction of toric varieties, see [4]. Let N be a *lattice*, i.e., a group isomorphic to $\mathbb{Z}^n$ for some n. A *fan* Δ in N is a finite collection of *proper convex rational polyhedral cones* σ in the real vector space $N_{\mathbb{R}} = N \otimes \mathbb{R}$, such that every face of a cone in Δ is in Δ and the intersection of two cones in Δ is a face of each. (Here zero is considered a face of every cone.) We explain the notation further: A *convex cone* is a convex subset σ of $N_{\mathbb{R}}$ with $\mathbb{R}_{\geq 0}\sigma = \sigma$, it is *polyhedral*, if it is finitely generated and *rational*, if the generators lie in the lattice N. Finally, a cone is called *proper* if it does not contain a non-zero sub vector space of $N_{\mathbb{R}}$.

Let a fan Δ be given. Let $M = \mathrm{Hom}(N, \mathbb{Z})$ be the dual lattice. for a cone $\sigma \in \Delta$ the *dual cone* $\breve{\sigma}$ is the cone in the dual space $M_{\mathbb{R}}$ consisting of all $\alpha \in M_{\mathbb{R}}$ such that $\alpha(\sigma) \geq 0$. This defines a monoid $A_\sigma = \breve{\sigma} \cap M$. Set $U_\sigma = \mathrm{spec}\,(\mathbb{C}[A_\sigma])$. If τ is a face of σ, then $A_\tau \supset A_\sigma$, and this inclusion gives rise to an open embedding $U_\tau \hookrightarrow U_\sigma$. Along these embeddings we glue the affine varieties U_σ to obtain a variety X_Δ over $\mathbb{C}$, which has a given $\mathbb{F}_1$-structure. Then X_Δ is a toric variety, the torus being $U_0 \cong \mathrm{GL}_1^n$. Every toric variety is given in this way.

Lemma 4.2 *Let B be a submonoid of the monoid A of finite index. Then the map $\psi : \mathrm{spec}\, A \to \mathrm{spec}\, B$ defined by $\psi(\mathfrak{p}) = \mathfrak{p} \cap B$ is a bijection.*

Proof: Let $N \in \mathbb{N}$ be such that $a^N \in B$ for every $a \in A$. To see injectivity, let $\psi(\mathfrak{p}) = \psi(\mathfrak{q})$ and let $a \in \mathfrak{p}$. Then $a^N \in \mathfrak{q}$ and so $a \in \mathfrak{q}$ as $\mathfrak{q}$ is a prime ideal. This shows $\mathfrak{p} \subset \mathfrak{q}$ and by symmetry we get equality. For surjectivity, let $\mathfrak{p}_B \in \mathrm{spec}\, B$ and let $\mathfrak{p} = \{a \in A : a^N \in \mathfrak{p}_B\}$. Then $\psi(\mathfrak{p}) = \mathfrak{p}_B$. □

Proposition 4.3 *Suppose that Δ is a fan in a lattice of dimension n. For $j = 0, \dots, n$ let f_j be the number of cones in Δ of dimension j. Set*

$$c_j = \sum_{k=j}^{n} f_{n-k}(-1)^{k+j} \begin{pmatrix} k \\ j \end{pmatrix}.$$

Let X be the corresponding toric variety, then the $\mathbb{F}_1$-zeta function of X equals

$$\zeta_X(s) = s^{c_0}(s-1)^{c_1}\cdots(s-n)^{c_n}.$$

Proof: Let $\sigma \in \Delta$ be a cone of dimension k. Let F be a face of $\breve{\sigma}$. Let $\mathfrak{p}_F = A_\sigma \smallsetminus F$. Then $\mathfrak{p}_F$ is a non-empty prime ideal in A_σ. The map $F \mapsto \mathfrak{p}_F$ is a bijection between the set of all faces of $\breve{\sigma}$ and the set of non-empty prime ideals of A_σ. The set $S_\mathfrak{p} = A \smallsetminus \mathfrak{p}$ equals $M \cap F$. The quotient group $\mathrm{Quot}(S_\mathfrak{p})$ is isomorphic to $\mathbb{Z}^f$, where f is the dimension of F. There is a bijection between the set of faces of σ and the set of faces of $\breve{\sigma}$ mapping a face τ to the face F of all $\alpha \in \breve{\sigma}$ with $\alpha(\tau) = 0$. The dimension of F then equals $n - \dim(\tau)$. So let f_j^σ denote the number of faces of σ of dimension j. Then the zeta polynomial of X_σ equals

$$N_\sigma(x) = \sum_{k=0}^{n} f_k^\sigma (x-1)^{n-k}.$$

Let N_Δ be the zeta polynomial of X_Δ. We get

$$\begin{aligned} N_\Delta(x) &= \sum_{k=0}^{n} f_k (x-1)^{n-k} \\ &= \sum_{k=0}^{n} f_k \sum_{j=0}^{n-k} \binom{n-k}{j} x^j (-1)^{n-k-j} \\ &= \sum_{k=0}^{n} f_{n-k} \sum_{j=0}^{k} \binom{k}{j} x^j (-1)^{k-j} \\ &= \sum_{j=0}^{n} x^j \sum_{k=j}^{n} f_{n-k} \binom{k}{j} (-1)^{k-j}. \end{aligned}$$

This implies the claim. □

5 Valuations

On the infinite cyclic monoid $C_+ = \{1, \tau, \tau^2, \dots\}$ we have a natural linear order given by $\tau^k \le \tau^l \Leftrightarrow k \le l$. Let φ, ψ be two monoid morphisms from a monoid A to C_+. Then define $\varphi \le \psi \Leftrightarrow \varphi(a) \le \psi(a)\ \forall a \in A$. A *valuation* on A is a non-trivial homomorphism $v : A \to C_+$ which is minimal with respect to the order $\le$ among all non-trivial homomorphisms from A to C_+. Let $V(A)$ denote the set of valuations on A.

Lemma 5.1 *Let*

$$1 \longrightarrow A \longrightarrow B \stackrel{\varphi}{\longrightarrow} F \longrightarrow 1$$

be an exact sequence of monoids, where F is a finite abelian group. Then for every valuation $v \in V(A)$ there exists a unique valuation w on B and $k \in \mathbb{N}$ such that

$$w|_A = v^k.$$

Mapping v to w sets up a bijection from $V(A)$ to $V(B)$.

Proof: Let F' be a subgroup of F and let B' be the preimage of F' under φ. We get two exact sequences

$$1 \longrightarrow A \longrightarrow B' \longrightarrow F' \longrightarrow 1,$$

and

$$1 \longrightarrow B' \longrightarrow B \longrightarrow F/F' \longrightarrow 1.$$

Assume we have proven the lemma for each of these two sequences, then it follows for the original one. In this way we reduce the proof to the case when F is a finite cyclic group. We first show existence of w for given v. For this let f_0 be a generator of F and let l be its order. Choose a b_0 in the preimage $\varphi^{-1}(f_0)$. Then $b_0^l \in A$, and $v(b_0^l) = \tau^n$ for some $n \ge 0$. If $n = 0$, then set $k = 1$ and define $w : B \to C_+$ by $w(b_0^j a) = v(a)$ for $a \in A$ and $j \ge 0$. If $n > 0$, then set $k = l/\mathrm{gcd(l,n)}$ and let $w : B \to C_+$ be defined by $w(b_0^j a) = \tau^j v(a)^k$. This shows existence of the extension w. $\square$

6 Cohomology

Cohomology is not defined over $\mathbb{F}_1$. I am grateful to Ofer Gabber for bringing the following example to my attention. Let X be the topological space consisting of three points η, X_+, x_-. The open sets besides the trivial ones are

$U = \{\eta\}, U_+ = \{\eta, x_+\}, U_- = \{\eta, x_-\}$. Let A be a subgroup of the abelian group B and let $C = B/A$. Let $\mathcal{F}$ be the sheaf of abelian groups on X with $\mathcal{F}(U_\pm) = A$ and $\mathcal{F}(U) = B$ and the restriction being the inclusion. Let $\mathcal{G}$ be the constant sheaf B and let $\mathcal{H}$ be the quotient sheaf $\mathcal{G}/\mathcal{F}$. As $\mathcal{G}$ is flabby, the long cohomology sequence terminates and looks like this:

$$0 \to H^0(\mathcal{F}) \to H^0(\mathcal{G}) \to H^0(\mathcal{H}) \to H^1(\mathcal{F}) \to 0$$

In concrete terms this is

$$0 \to A \to B \to C \times C \to (C \times C)/\Delta \to 0,$$

where Δ means the diagonal in $C \times C$. Let $f : X \to X$ be the homeomorphism with $f(x_+) = x_-$, $f(x_-) = x_+$, and $f(\eta) = \eta$. There is a natural isomorphism $f_*\mathcal{F} \cong \mathcal{F}$ and for the other sheaves as well. On the global sections of $\mathcal{F}$ and $\mathcal{G}$ this induces the trivial map, whereas on $H^0(\mathcal{H})$ it induces the flip $(a, b) \mapsto (b, a)$, which on $H^1(\mathcal{F})$ amounts to the same as the inversion $a \mapsto -a$. The naturality of these isomorphisms means that if the sheaves and the cohomology groups are defined over $\mathbb{F}_1$, then so must be the flip. This, however, is not the case, as for a set S the inversion on the abelian group $\mathbb{Z}[S]$ is not induced by a self-map of S.

Even more convincing is the fact that in this example there are different injective resolutions which produce different cohomology groups.

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Mathematisches Institut
Auf der Morgenstelle 10
72076 Tübingen
Germany
deitmar@uni-tuebingen.de