Homological algebra over belian categories and cohomology of $\mathbb{F}_1$-schemes

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Introduction

Following ideas of Manin [10] and Soulé [14], and inspired by Kurokawa-Ochiai-Wakayama [8], the author introduced the notion of $\mathbb{F}_1$-schemes in [2]. This notion turned out to be a generalisation of a "fan" introduced by Kato [6] in the context of logarithmic schemes.

For (hopefully) possible number theoretical applications it is necessary to develop homological algebra over $\mathbb{F}_1$-schemes. As the descent from $\mathbb{Z}$-schemes to $\mathbb{F}_1$-schemes comes about by "forgetting additivity", the categories in question are no longer additive categories, therefore fail the usual approach to homological algebra. The author first tried to establish a *homotopical algebra* á la Quillen [11] in this case, but failed to verify the axioms. The more traditional approach through resolutions, however, finally worked out. This forms the first part of the paper.

Homological algebra has been extended to more general settings by many authors, for instance, see [1, 3, 5, 7, 13, 12], but sooner or later each of the papers known to the author introduces assumptions (pre-additivity, existence of biproducts) which are not satisfied in the case of interest here. Thus it was necessary to develop homological algebra on belian categories from scratch. We proceed rather slowly here and people used to this kind of arguments might urge for more speed, but readers, like myself, who are very much used to additive categories and want to make sure that no additivity is used indeed, might feel relieved that all details are worked out.

To counterbalance failing additivity we have to impose heavy conditions on the category otherwise, such as having enough injectives and projectives. These conditions have to be verified in applications.

In the second part of the paper we verify the conditions in the context of sheaves over $\mathbb{F}_1$-schemes. Where the first part is more algebraic in nature, this part is more geometric. We prove some of the results one might expect, like vanishing of cohomology in degrees above the dimension or that cohomology can be computed using flabby resolutions. Finally, the quite useful compatibility with base change is proved. This allows one to compute the $\mathbb{Z}$-lift of cohomology by means of ordinary Zariski-sheaf cohomology.

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1 Belian categories

A category is called *balanced* if every morphism which is a monomorphism as well as an epimorphism, already has an inverse, i.e., is an isomorphism. For example, the category of groups is balanced, but the category of fields is not.

Let $\mathcal{B}$ be a category. An object $I \in \mathcal{B}$ is called *injective* if for every monomorphism $M \hookrightarrow N$ the induced map $\mathrm{Mor}(N, I) \to \mathrm{Mor}(M, I)$ is surjective. Dually, an object $P \in \mathcal{B}$ is called *projective* if for every epimorphism $M \twoheadrightarrow N$ the induced map $\mathrm{Mor}(P, M) \to \mathrm{Mor}(P, N)$ is surjective. We say that $\mathcal{B}$ *has enough injectives* if for every $A \in \mathcal{B}$ there exists a monomorphism $A \hookrightarrow I$, where I is an injective object. Likewise, we say that $\mathcal{B}$ *has enough projectives* if for every $A \in \mathcal{B}$ there is an epimorphism $P \twoheadrightarrow A$ with P projective.

A category $\mathcal{C}$ is *pointed* if it has an object 0 such that for every object X the sets $\mathrm{Mor}(X, 0)$ and $\mathrm{Mor}(0, X)$ have exactly one element each. The zero object is uniquely determined up to unique isomorphism. In every set $\mathrm{Mor}(X, Y)$ there exists a unique morphism which factorises over the zero object, this is called the zero morphism. In a pointed category it makes sense to speak of kernels and cokernels. Kernels are always mono and cokernels are always epimorphisms.

Assume that kernels and cokernels always exist. Then every kernel is the kernel of its cokernel and every cokernel is the cokernel of its kernel. For a morphism f let $\mathrm{im}\,(f) = \ker(\mathrm{coker}\,(f))$ and $\mathrm{coim}(f) = \mathrm{coker}\,(\ker(f))$. If $\mathcal{C}$ has enough projectives, then the canonical map $\mathrm{im}\,(f) \to \mathrm{coim}(f)$ has zero kernel and if $\mathcal{C}$ has enough injectives, then this map has zero cokernel.

A *belian category* is a balanced pointed category $\mathcal{B}$ which

- contains fibre products and cofibre products, and
- has the property that every morphism with zero cokernel is an epimorphism.

Every abelian category is belian.

As a special case of fibre and cofibre products, a belian category contains finite products, finite sums (=co-products), kernels and cokernels.

The third axiom says that a morphism with zero cokernel is an epimorphism and consequently a monomorphism with zero cokernel is an isomorphism. However, not every morphism with zero kernel is a monomorphism. We will call a morphism with zero kernel a *weak monomorphism*. Likewise, a morphism with zero kernel and cokernel will be called a *weak isomorphism*.

The third axiom implies that for every morphism f the canonical map from $\mathrm{coim}(f)$ to $\mathrm{im}\,(f)$ is an epimorphism. If this map is indeed an isomorphism, we call f a *strong morphism*. Monomorphisms and cokernels are strong. If $A \xrightarrow{f} B \xrightarrow{g} C$ is given with g being strong and $gf = 0$, then the induced map $\mathrm{coker}\,(f) \to C$ is strong. Likewise, if f is strong and $gf = 0$, then the induced map $A \to \ker g$ is strong.

As many results of this section are formulated for strong morphisms, one might wonder why to bother with non-strong morphisms at all. The reason is that the class of strong morphisms is, in general, not closed under fibre products and the precise conditions which would then replace this axiom are quite messy to formulate and to verify in applications. Further, much of what we do depends on having enough projectives, another property that fails in applications if one restricts to strong morphisms.

Note that in a belian category, although one cannot add morphisms, one can "add" morphisms from direct sums thanks to the universal property of direct sums: Suppose given two morphisms $\varphi_i \colon M_i \to N$, $i = 1, 2$. Then there exists a unique morphism

$$\varphi_1 \oplus \varphi_2 \colon M_1 \oplus M_2 \ \to \ N$$

such that $M_i \longrightarrow M_1 \oplus M_2 \xrightarrow{\varphi_1 \oplus \varphi_2} N$ equals φ_i for $i = 1, 2$.

The simplest example of a belian category is the category Set_0 of pointed sets. Objects are pairs (X, x_0) where X is a set and $x_0 \in X$ is an element. A morphism $\varphi \in \mathrm{Mor}((X, x_0), (Y, y_0))$ is a map $\varphi \colon X \to Y$ with $\varphi(x_0) = y_0$. Any singleton $(\{x_0\}, x_0)$ is a zero object. The kernel of a morphism $\varphi \colon X \to Y$ is the inverse image $\varphi^{-1}(\{y_0\})$ of the special point and the cokernel is $Y/\varphi(X)$, where the image $\varphi(X)$ is collapsed to a point. The product is the Cartesian product and the coproduct is the disjoint union with the special points identified. A morphism $\varphi \in \mathrm{Mor}((X, x_0), (Y, y_0))$ is strong if and only if φ is injective outside $\varphi^{-1}(\{y_0\})$.

Other examples include the category of pointed simplicial sets, pointed CW-complexes, or the categories of sheaves of these.

If $\mathcal{B}$ is a belian category, then for $X, Y \in \mathcal{B}$ the set $\mathrm{Mor}_{\mathcal{B}}(X, Y)$ is a pointed set, the special point being the zero morphism.

1.1 Complexes

In a belian category a sequence of morphisms,

$$\cdots \longrightarrow M^i \xrightarrow{d^i} M^{i+1} \xrightarrow{d^{i+1}} \cdots$$

is called a *complex* if $d^{i+1} \circ d^i = 0$ for every i. In that case there is an induced morphism $\mathrm{im}\, d^i \to \ker d^{i+1}$ which is a monomorphism since the maps $\mathrm{im}\,(d^i) \to M^{i+1}$ and $\ker(d^{i+1}) \to M^{i+1}$ are monomorphisms. We call the complex *exact*, if this morphism is an isomorphism. For a given complex let

$$H^i(A^\bullet) \ \stackrel{\mathrm{def}}{=} \ \mathrm{coker}\ \left(\mathrm{im}\, d^i \to \ker d^{i+1}\right) \ \in \ \mathcal{B}$$

be the *cohomology* of the complex $M^\bullet$. Then the cohomology is zero if and only if the complex is exact.

A complex is called a *strong complex* if every differential d^i is strong.

Let $\mathcal{B}$ be a belian category and let $\mathcal{C}(\mathcal{B})$ be the category of complexes over $\mathcal{B}$. Morphisms in $\mathcal{C}(\mathcal{B})$ are morphisms $f\colon X \to Y$ of complexes, i.e., f is a sequence $f^i\colon X^i \to Y^i$ of morphisms is $\mathcal{B}$ such that every square

$$\begin{array}{ccc} X^i & \longrightarrow & X^{i+1} \\ \big\downarrow{\scriptstyle f^i} & & \big\downarrow{\scriptstyle f^i} \\ Y^i & \longrightarrow & Y^{i+1} \end{array}$$

is commutative.

Let $\mathcal{C}_+(\mathcal{B})$ be the full subcategory of complexes Y which are bounded below, i.e., $Y^i = 0$ for $i << 0$. Further $\mathcal{C}_-(\mathcal{B})$ denotes the subcategory of complexes which are bounded above and finally let $\mathcal{C}_b(\mathcal{B}) = \mathcal{C}_+(\mathcal{B}) \cap \mathcal{C}_-(\mathcal{B})$ be the category of bounded complexes.

1.2 Pull-backs and push-outs

Lemma 1.1 *Let $\mathcal{B}$ be a category and let*

$$\begin{array}{ccc} A & \xrightarrow{f'} & X \\ {\scriptstyle g'}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ B & \xrightarrow[f]{} & Y \end{array}$$

be a Cartesian square in $\mathcal{B}$.

- *If f is a monomorphism, then so is f'.*
- *If $\mathcal{B}$ contains enough projectives and f is an epimorphism, then f' is an epimorphism.*
- *If $\mathcal{B}$ is belian and contains enough projectives, and if g is a monomorphism and f is strong, then f' is strong.*

Likewise, let

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ {\scriptstyle j}\big\downarrow & & \big\downarrow{\scriptstyle j'} \\ C & \xrightarrow{h'} & P \end{array}$$

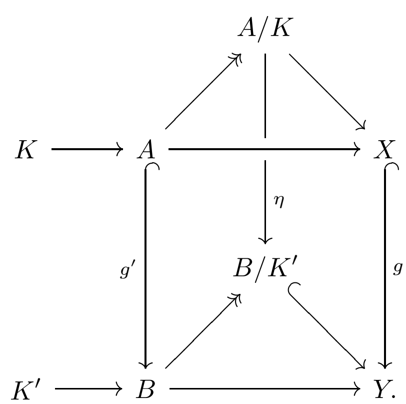
be co-Cartesian.

- *If h is an epimorphism, then so is h'.*
- *If $\mathcal{B}$ contains enough injectives and h is a monomorphism, then h' is a monomorphism.*
- *If $\mathcal{B}$ is belian and contains enough injectives, h is a monomorphism and j a strong morphism then j' is strong.*

Proof: Assume the first situation and let α, β be two morphisms $Z \to A$ with $f'\alpha = f'\beta$. We have to show $\alpha = \beta$. Since $fg'\alpha = fg'\beta$ and f is injective, we have $g'\alpha = g'\beta$. The square being Cartesian implies $\alpha = \beta$ as claimed. For the second assertion, let $\alpha\colon P \to X$ be an epimorphism with P projective. The resulting morphism $P \to Y$ can be lifted to B, giving a commutative square

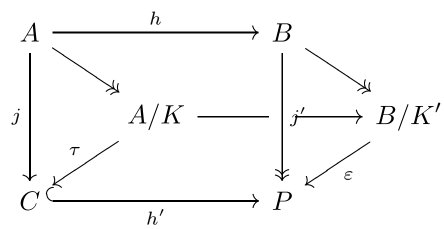
$$\begin{array}{ccc} P & \longrightarrow & X \\ \downarrow & & \downarrow \\ B & \longrightarrow & Y. \end{array}$$

Since the original square was Cartesian, the epimorphism $P \to X$ factorises as $P \to A \xrightarrow{f'} X$, hence f' is an epimorphism. For the third assertion, let K, K' be the kernels of f and f' respectively. Write A/K for $\operatorname{coker}(\ker f)$ and B/K' for $\operatorname{coker}(\ker f')$. We have the diagram



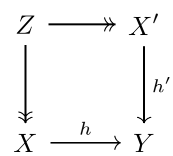
We claim that η is a monomorphism. For this let α, β be morphisms from some Z to A/K with $\eta\alpha = \eta\beta$. Replacing Z with a projective cover if necessary we may assume that Z is projective. Since g is injective, the induced morphisms from Z to X coincide. Since Z is projective, the morphism $Z \to B/K'$ can be lifted to $Z \to B$. Since the diagram is Cartesian, there is a unique morphism $Z \to A$ making the diagram commutative. This uniqueness implies that $\alpha = \beta$, so η is indeed a monomorphism and so is $A/K \to X$ which implies that f' is strong.

The first two assertions for co-Cartesian squares follow by reversing the arrows. For the third let $A/K = \operatorname{coker}(\ker(j))$ and $B/K' = \operatorname{coker}(\ker(j'))$. We have the commutative diagram

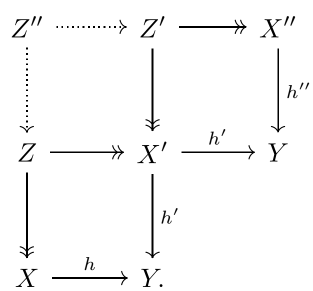


It is easy to see that P also is the cofibre product of B/K' and C over A/K. Since j is strong, τ is a monomorphism, hence by the above, ε is a monomorphism, so j' is strong. □

Let $\mathcal{B}$ be a category which contains fibre-products and has enough projectives. Let Y be an object in $\mathcal{B}$. On the class of morphisms $h\colon X \to Y$ we define an equivalence relation as follows. We say that $(h, X) \sim (h', X')$ if there exists a commutative diagram



where the arrows emanating at Z are epimorphisms. One has to check that this indeed is an equivalence. The only problem is transitivity. For this assume $(h, X) \sim (h', X')$ and $(h', X') \sim (h'', X'')$. This means that we have the solid arrows in the following diagram,



Let Z'' be the fibre-product so that the upper left square is cartesian. Then by the last lemma the dotted arrows are epimorphisms and so are the arrows $Z'' \to X$ and $Z'' \to X''$. This proves that $\sim$ is an equivalence relation.

1.3 Snake Lemma

A functor between belian categories is called *strong-exact* if it maps strong exact sequences to strong exact sequences.

Let $\mathcal{B}$ be a belian category. An *ascent functor* F is functor from $\mathcal{B}$ to an abelian category $\mathcal{C}$, which is faithful, strong-exact, and preserves fibre- and cofibre-products. We observe that a sequence in $\mathcal{B}$ is strong exact if and only if its image under F is exact. To see the "if"-part let

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

be a sequence in $\mathcal{B}$ which is exact after applying F. Then $0 = F(\beta)F(\alpha) = F(\beta\alpha)$ and the faithfullness implies $\beta\alpha = 0$. So we get a natural map $f : \operatorname{im}\alpha \to \ker\beta$. Then $F(\operatorname{im}\alpha) = \operatorname{im}F(\alpha)$ and $F(\ker\beta) = \ker F(\beta)$ and $F(f)$ is the natural map coming from the sequence in $\mathcal{C}$, hence it is an isomorphism. As F is faithful, f is epi and mono, hence also an isomorphism as $\mathcal{B}$ is balanced. So the sequence is strong-exact.

Lemma 1.2 *(Snake Lemma) Let $\mathcal{B}$ be a category which is belian, has enough injectives and projectives, and admits an ascent functor. Given a strong commutative diagram with exact rows*

$$\begin{array}{ccccccc} & & X_1 & \xrightarrow{g_1} & X_2 & \xrightarrow{g_2} & X_3 & \longrightarrow & 0 \\ & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \\ 0 & \longrightarrow & Y_1 & \xrightarrow{h_1} & Y_2 & \xrightarrow{h_2} & Y_3. & & \end{array}$$

Then the induced sequences

$$\ker(f_1) \to \ker(f_2) \to \ker(f_3)$$

and

$$\operatorname{coker}(f_1) \to \operatorname{coker}(f_2) \to \operatorname{coker}(f_3)$$

are strong and exact, and there is a natural strong morphism $\delta\colon \ker(f_3) \to \operatorname{coker}(f_1)$ *making the whole sequence exact.*

Proof: The first part of the Lemma is a standard verification. We will now construct the snake morphism δ. Note first that, applying an ascent functor $F : \mathcal{B} \to \mathcal{C}$, one gets a snake morphism in $\mathcal{C}$. So we only have to show that the standard construction of this snake morphism already works in $\mathcal{B}$, gives a strong morphism, and is compatible with F. Once this is achieved, the exactness of the sequence follows from the corresponding exactness in $\mathcal{C}$.

To construct δ, extend the diagram as follows:

$$
\begin{array}{ccccccccc}
 & & X_1 & \xrightarrow{t} & Z & \xrightarrow{s} & \ker(f_3) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle k} & & \\
 & & X_1 & \xrightarrow{g_1} & X_2 & \xrightarrow{g_2} & X_3 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\
0 & \longrightarrow & Y_1 & \xrightarrow{h_1} & Y_2 & \xrightarrow{h_2} & Y_3 & & \\
 & & \twoheaddownarrow{\scriptstyle k'} & & \downarrow{\scriptstyle l'} & & \downarrow{\scriptstyle \mathrm{Id}} & & \\
0 & \longrightarrow & \operatorname{coker}(f_1) & \xrightarrow{s'} & Z' & \xrightarrow{t'} & Y_3. & &
\end{array}
$$

Here Z is the fibre product of $\ker(f_3)$ and X_2 over X_3 and Z' is the cofibre product of $\operatorname{coker}(f_1)$ and Y_2 over Y_1. By Lemma 1.1, s is an epimorphism and s' is a monomorphism. The morphism t is the fibre product of g_1 and the zero map. We claim that the first row is exact. Since $st = 0$ it remains to show that t is surjective on $\ker(s)$. Now g_1 is surjective on $\ker(g_2)$. Replacing g_1 by $\ker(g_2)$ amounts to the same as assuming that g_1 is injective. It suffices to prove the claim under that assumption. Indeed, then t is the kernel of s. To see this, let $W \xrightarrow{w} Z$ be a morphism with $sw = 0$. We shall show that w factorises uniquely over t. The induced arrow $W \to X_3$ is zero, therefore there is a unique morphism $r\colon W \to X_1$ such that the solid arrow diagram

$$
\begin{array}{ccc}
W & \xrightarrow{w} & Z \\
\downarrow{\scriptstyle r} & \overset{t}{\nearrow} & \downarrow{\scriptstyle l} \\
X_1 & \longrightarrow & X_2
\end{array}
$$

is commutative. We have to show that it remains commutative when t is inserted. We have two morphisms $w, tr\colon W \to Z$ with $lw = ltr$ and $sw = str$, where the second equality stems from the fact that k is a monomorphism. By the universal property of the fibre product Z it follows that $w = tr$, hence the diagram commutes and so t is indeed the kernel of s, in particular, t is a strong morphism. Further, by construction the morphisms k and k' are strong, so by Lemma 1.1 the morphisms s and l' are strong.

In the last row the morphism t' is the cofibre product of h_2 and zero. The exactness of this row and the strongness of t' follows from the previous part by reversing all arrows. So the rows are exact and all morphisms are strong. Consider the morphism $\varepsilon = l' f_2 l \colon Z \to Z'$. It satisfies $\varepsilon t = s' k' f_1 = 0$ and $t' \varepsilon = f_3 k s = 0$. Since the top and the bottom row are exact, there exists a unique morphism $\delta \colon \ker(f_3) \to \operatorname{coker}(f_1)$ such that $\varepsilon = s' \delta s$. We claim that δ is strong. For this consider the strong diagram with exact rows

$$\begin{array}{ccccccccc}
 & & X & \xrightarrow{t} & Z & \xrightarrow{s} & \ker f_3 & \xrightarrow{0} & \\
 & & & \searrow^{0} & \downarrow{\scriptstyle \varepsilon} & \searrow^{0} & & & \\
0 & \longrightarrow & \operatorname{coker} f_1 & \xrightarrow{s'} & Z' & \xrightarrow{t'} & Y_3. & &
\end{array}$$

Firstly,the induced morphism $\delta_1 : \ker f_3 \to Z'$ such that $\varepsilon = \delta_1 s$ is strong, as there are natural isomorphisms $\operatorname{coim}\delta_1 \cong \operatorname{coim}\varepsilon$ and $\operatorname{im}\delta_1 \cong \operatorname{im}\varepsilon$ identifying the natural map $\operatorname{coim}\delta_1 \to \operatorname{im}\delta_1$ with $\operatorname{coim}\varepsilon \to \operatorname{im}\varepsilon$ which is an isomorphism. Similarly, the natural map δ such that $\delta_1 = s'\delta$ is strong.

So we have made clear that the construction of δ works in $\mathcal{B}$ and that δ is strong. As the ascent functor F is strong exact, it translates the snake diagram to a snake diagram. As it preserves fibre- and cofibre-products, it translates the extended diagram to the extended diagram in $\mathcal{C}$. This implies that $F(\delta)$ is the snake morphism in $\mathcal{C}$ and that the sequence

$$\ker(f_2) \xrightarrow{\tilde{g}_2} \ker(f_3) \xrightarrow{\delta} \operatorname{coker}(f_1) \xrightarrow{\overline{h_1}} \operatorname{coker}(f_2)$$

is exact after applying F. As F is an ascent functor, the above sequence is strong exact. □

As an application we will show the existence of a long exact cohomology sequence attached to a short exact sequence of complexes. We assume that $\mathcal{B}$ has enough injectives and projectives and admits ascent. Let

$$0 \to F \xrightarrow{e} F \xrightarrow{f} G \to 0$$

be a strong and exact sequence of complexes over the belian category $\mathcal{B}$. Assume further that e is a monomorphism. At each stage $i \in \mathbb{Z}$ one gets a strong commutative and exact diagram

$$\begin{array}{ccccccccc}
 & & \operatorname{coker} d_E^{i-1} & \longrightarrow & \operatorname{coker} d_F^{i-1} & \longrightarrow & \operatorname{coker} d_G^{i-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker d_E^{i+1} & \longrightarrow & \ker d_F^{i+1} & \longrightarrow & \ker d_G^{i+1} & &
\end{array}$$

and the snake lemma gives a long exact sequence

$$\cdots \to H^i(F) \to H^i(G) \xrightarrow{\delta} H^{i+1}(E) \to H^{i+1}(F) \to \cdots .$$

1.4 Delta functors

Let $\mathcal{B}, \mathcal{B}'$ be belian categories. A *delta functor* from $\mathcal{B}$ to $\mathcal{B}'$ is a sequence of functors $(F^n)_{n\geq 0}$ and to each strong exact sequence

$$0 \to A' \to A \to A'' \to 0$$

in $\mathcal{B}$, an associated family of strong morphisms

$$\delta^n \colon F^n(A'') \to F^{n+1}(A'), \qquad n \geq 0,$$

satisfying the following conditions.

D1. For each short exact sequence as above the induced sequence

$$0 \to F^0(A') \to F^0(A) \to F^0(A'') \xrightarrow{\delta} F^1(A') \to \dots$$

is exact.

D2. For each commutative strong diagram in $\mathcal{B}$ with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & A_1' & \longrightarrow & A_1 & \longrightarrow & A_1'' & \longrightarrow & 0 \end{array}$$

the δ's give commutative diagrams

$$\begin{array}{ccc} F^n(A'') & \xrightarrow{\delta} & F^{n+1}(A') \\ \big\downarrow & & \big\downarrow \\ F^n(A_1'') & \xrightarrow{\delta} & F^{n+1}(A_1'). \end{array}$$

A functor $F \colon \mathcal{B} \to \mathcal{B}'$ of belian categories is called *strong* if F maps strong morphisms to strong morphisms. A delta functor (F^n) is called strong if all the F^n are strong for $n = 0, 1, 2, \dots$. This definition will be used later.

A δ-functor (F^n) is called *universal* if given another δ-functor from $\mathcal{B}$ to $\mathcal{B}'$ and a morphism of functors $f_0 \colon F^0 \to G^0$ there exists a unique sequence of morphisms $f_n \colon F^n \to G^n$, $n \geq 0$ which commute with the δ^n for each short exact sequence. Given two universal δ-functors (F^n) and (G^n) with $F^0 \cong G^0$ it follows that $F^n \cong G^n$ for every n.

A functor F from $\mathcal{B}$ to $\mathcal{B}'$ is called *erasable*, if to each object B there exists a monomorphism $u \colon B \to I$ in $\mathcal{A}$ with $F(u) = 0$.

Theorem 1.3 *Suppose that the categories $\mathcal{B}, \mathcal{B}'$ are belian and $\mathcal{B}$ has enough injectives. Let (F^n) be a δ-functor. If F^n is erasable for $n \geq 1$, then (F^n) is universal.*

Proof: Let G be another δ-functor and given $f_0\colon F^0 \to G^0$. Given an object A, we erase it with and object I and we get a strong exact seqeunce

$$0 \to A \xrightarrow{u} I \xrightarrow{v} C \to 0$$

with $F^1(u) = 0$. This yields the following solid arrow commutative and exact diagram,

$$\begin{array}{ccccccc}
F^0(I) & \xrightarrow{F^0(v)} & F^0(C) & \xrightarrow{\delta_F} & F^1(A) & \longrightarrow & 0 \\
\downarrow{\scriptstyle f_0(I)} & & \downarrow{\scriptstyle f_0(C)} & & \vdots{\scriptstyle f_1(A)} & & \\
G^0(I) & \xrightarrow{G^0(v)} & G^0(C) & \xrightarrow{\delta_G} & G^1(A). & &
\end{array}$$

By exactness and strongness it follows that $\delta_F = \operatorname{coker}(F^0(v))$. Since the second row is exact, we get $\delta_G G^0(v) f_0(I) = 0$ and thus $\delta_G f_0(C) F^0(v) = 0$. Hence there exists a unique map $f_1(A)$ making the whole diagram commutative.

We show now that $f_1(A)$ is functorial in A. For this let $\varphi\colon A \to B$ be a morphism in $\mathcal{B}$. We consider the cofibre product P,

$$\begin{array}{ccc}
A & \xhookrightarrow{u} & I \\
\downarrow{\scriptstyle \varphi} & & \downarrow \\
B & \longrightarrow & P.
\end{array}$$

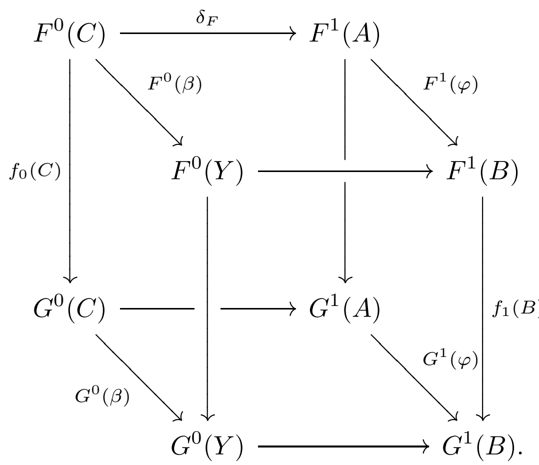
Since u is a monomorphism, Lemma 1.1 implies that the map $B \to P$ also is mono. Next let $P \to N$ be a monomorphism which erases P. This yields a commutative strong and exact diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & I & \longrightarrow & C & \longrightarrow & o \\
 & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \\
0 & \longrightarrow & B & \longrightarrow & N & \longrightarrow & Y & \longrightarrow & 0,
\end{array}$$

where $B \to N$ is the composite $B \to P \to N$ and Y is the cokernel. Functoriality in A means that the following diagram is commutative

$$\begin{array}{ccc} F^1(A) & \xrightarrow{F^1(\varphi)} & F^1(B) \\ \downarrow{\scriptstyle f_1(A)} & & \downarrow{\scriptstyle f_1(B)} \\ G^1(A) & \xrightarrow{G^1(\varphi)} & G^1(B). \end{array}$$

This is the right hand face of the following cube



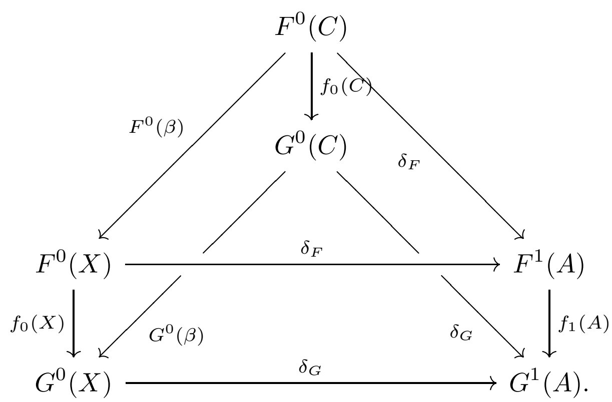
All faces of the cube are commutative except possibly the right hand one. But since δ_F is an epimorphism, also the last face must be commutative. This shows functoriality. Next we have to show that f_1 commutes with the connection morphism δ. Let

$$0 \to A \to B \to C \to 0$$

be a strong exact sequence in $\mathcal{B}$. The same cofibre construction as before yields an erasing monomorphism $A \to I$ and a commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \\ 0 & \longrightarrow & A & \longrightarrow & I & \longrightarrow & X & \longrightarrow & 0. \end{array}$$

Consider the diagram



Our aim is to prove that the right hand face is commutative. The triangles on top and bottom are commutative by the definition of a δ-functor. The left hand square is commutative since f_0 is a morphism of functors. The front square is commutative by the definition of f_1. This implies that the last face also is commutative. An iteration of the argument with index pair $(n, n+1)$ instead of $(0, 1)$ implies the Theorem. □

1.5 Derived functors

Let $\mathcal{B}$ be a belian category. An *injective class* in $\mathcal{B}$ is a class $\mathcal{I}$ of injective objects in $\mathcal{B}$ such that

- every object of $\mathcal{B}$ injects into an object in $\mathcal{I}$, and
- $\mathcal{I}$ is closed under finite products.

Note that every belian category $\mathcal{B}$ with enough injectives admits injective classes.

A *resolution* of an object X in $\mathcal{B}$ is a strong exact sequence

$$0 \to X \to I_X^0 \to I_X^1 \to \cdots .$$

Let $\mathcal{I}$ be an injective class, an $\mathcal{I}$-resolution is a resolution with all I_X^j in $\mathcal{I}$. We show that $\mathcal{I}$-resolutions always exist. For given $X \in \mathcal{B}$ choose a monomorphism $X \hookrightarrow I_X^0$ with I_X^0 in $\mathcal{I}$. This starts the inductive construction. Suppose $0 \to X \to I_X^0 \stackrel{d^0}{\to} \cdots \stackrel{d^{j-1}}{\to} I_X^j$ already constructed. Choose a monomorphism $\operatorname{coker}(d^{j-1}) \hookrightarrow I_X^{j+1}$ for some $I_X^{j+1} \in \mathcal{I}$. The induced morphism $d^j \colon I_X^j \to I_X^{j+1}$ gives a strong exact sequence.

Dually we define a *projective class* to be a class $\mathcal{P}$ of projective objects such that

- for every object X there exists an epimorphism $P \to X$ with $P \in \mathcal{P}$,
- $\mathcal{P}$ is closed under finite sums.

A functor $F\colon \mathcal{B} \to \mathcal{B}'$ between belian categories is called *left strong-exact* if for every strong exact sequence

$$0 \to X \overset{\varphi}{\hookrightarrow} Y \to Z$$

in $\mathcal{B}$, the sequence

$$0 \to F(X) \to F(Y) \to F(Z)$$

is strong exact in $\mathcal{B}'$.

Let $F\colon \mathcal{B} \to \mathcal{B}'$ be left strong-exact and assume that $\mathcal{B}$ has enough injectives. Fix an $\mathcal{I}$-resolution $X \to I_X$ for every $X \in \mathcal{B}$. For $j = 0, 1, \ldots$ define

$$R^j F(X) \ \stackrel{\text{def}}{=}\ H^j(F(I_X)).$$

One finds that $R^j F$ defines a functor $\mathcal{B} \to \mathcal{B}'$, called the *j-th derived functor of* F.

Lemma 1.4 *There is a natural isomorphism $R^0 F \cong F$.*

Proof: Let $0 \to X \to I_X^0 \to \cdots$ be the chosen resolution of $X \in \mathcal{B}$. Since F is left strong-exact, the sequence $0 \to F(X) \hookrightarrow F(I_X^0) \to F(I_X^1)$ is exact. Therefore there exists a natural functorial isomorphism,

$$R^0 F(X) = H^0 F(I_X) \ \cong\ F(X). \qquad \square$$

By construction, the derived functors $R^j F$ depend on the choice of the resolutions. We will now give a criterion which implies that the derived functors depend on this choice only up to canonical isomorphism.

The ascent functor F is said to be *$\mathcal{I}$-injective* if it maps objects in $\mathcal{I}$ to injective objects and to be *$\mathcal{P}$-projective* if it maps all objects in $\mathcal{P}$ to projectives. Here $\mathcal{I}$ and $\mathcal{P}$ are an injective and a projective class respectively. If we can choose $\mathcal{I}$ to be the class of all injective objects we simply say that F *preserves injectives* and likewise in the projective case.

Example. Let Set_0 be the category of pointed sets as before. For a ring R and a pointed set (M, m_0) let $R[M]$ be the free R-module generated by M and let $R[M]_0 = R[M]/Rm_0$. Then $F\colon M \mapsto R[M]_0$ from Set_0 to the category of R-modules is an ascent functor which is $\mathcal{P}$-projective for any projective class $\mathcal{P}$. If R is a field, it will also be $\mathcal{I}$-injective for every injective class $\mathcal{I}$. Note that this functor indeed is strong exact but not exact.

Let now $F\colon \mathcal{B} \to \mathcal{B}'$ be a left strong-exact functor and assume that $\mathcal{B}$ and $\mathcal{B}'$ are equipped with ascent functors

$$\begin{array}{rcl} \mathrm{Asc}_{\mathcal{B}}\colon \mathcal{B} & \to & \mathcal{C}_{\mathcal{B}} \\ \mathrm{Asc}_{\mathcal{B}'}\colon \mathcal{B}' & \to & \mathcal{C}_{\mathcal{B}'} \end{array}$$

for some abelian categories $\mathcal{C}_{\mathcal{B}}$ and $\mathcal{C}_{\mathcal{B}'}$. We say that F is *compatible with ascent* if F lifts to an additive left exact functor $F\colon \mathcal{C}_{\mathcal{B}} \to \mathcal{C}_{\mathcal{B}'}$ such that the diagram of functors

$$\begin{array}{ccc} \mathcal{C}_{\mathcal{B}} & \xrightarrow{F} & \mathcal{C}_{\mathcal{B}'} \\ \uparrow{\scriptstyle \mathrm{Asc}_{\mathcal{B}}} & & \uparrow{\scriptstyle \mathrm{Asc}_{\mathcal{B}'}} \\ \mathcal{B} & \xrightarrow{F} & \mathcal{B}' \end{array}$$

commutes.

An *ascent datum* is a pair $(\mathcal{I}, \mathrm{Asc})$ consisting of an injective class $\mathcal{I}$ and an ascent functor Asc which is $\mathcal{I}$-injective.

Lemma 1.5 *Let $(\mathcal{I}, \mathrm{Asc}_{\mathcal{B}})$ be an ascent datum on $\mathcal{B}$ and $\mathrm{Asc}_{\mathcal{B}'}$ be an ascent functor on $\mathcal{B}'$ Assume that the left strong-exact functor F is compatible with ascent. Then the derived functors R^jF depend on the choice of the ascent datum and the injective resolutions only up to canonical isomorphism.*

Proof: Let $(\mathcal{I}, \mathrm{Asc}_{\mathcal{B}})$ and $(\mathcal{I}', \mathrm{Asc}'_{\mathcal{B}})$ be two different ascent data and let $\mathrm{Asc}_{\mathcal{B}'}$ as well as $\mathrm{Asc}'_{\mathcal{B}'}$ be two ascent functors on $\mathcal{B}'$ such that F is compatible with both ascents. Let $I_X \in \mathcal{I}$ and $I'_X \in \mathcal{I}'$ be two different choices of injective resolutions and let R^jF and R'^jF be the corresponding derived functors. The injectivity implies the existence of morphisms φ^j making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & I_X^0 & \longrightarrow & I_X^1 & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \varphi^0} & & \downarrow{\scriptstyle \varphi^1} & & \\ 0 & \longrightarrow & X & \longrightarrow & {I'}_X^0 & \longrightarrow & {I'}_X^1 & \longrightarrow & \cdots \end{array}$$

commutative. We view this as a morphism φ of complexes from I_X to I'_X. This morphism φ is not uniquely determined, but for any other choice ψ it follows that $\mathrm{Asc}'_{\mathcal{B}}(\varphi)$ and $\mathrm{Asc}'_{\mathcal{B}}(\psi)$ are homotopic [9]. This implies that both induce the same map $R^jF(X) \to R'^jF(X)$ which then must be an isomorphism. □

Theorem 1.6 *Let $F\colon \mathcal{B} \to \mathcal{B}'$ be a left strong-exact functor on belian categories compatible with ascent. Assume that the $\mathcal{B}$-ascent is $\mathcal{I}$-injective for an injective class of $\mathcal{B}$, and that $\mathcal{B}'$ contains enough injectives and projectives. Then the sequence $(R^nF)_n$ is a strong universal δ-functor.*

Proof: We show that $R^\bullet F$ is a delta functor and that $R^n F$ is erasable by objects in $\mathcal{I}$ for $n \geq 1$. By Theorem 1.3 it will then follow that $R^\bullet F$ is universal.

We will now construct the δ-homomorphisms. Given a strong exact sequence $0 \to X \hookrightarrow Y \to Z \to 0$ in $\mathcal{B}$ let I_X and I_Y be given $\mathcal{I}$-resolutions of X and Z. Consider the diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X & \hookrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \beta} & & \downarrow & & \\ 0 & \longrightarrow & I_X^0 & \xrightarrow{\alpha} & I_X^0 \times I_Z^0 & \xrightarrow{\gamma} & I_Z^0 & \longrightarrow & 0, \end{array}$$

where α is the natural map given by the universal property of the product and the maps $I_X^0 \stackrel{\text{id}}{\to} I_X^0$ and $I_X^0 \stackrel{0}{\to} I_Z^0$. For the definition of β recall that since I_X^0 is injective, the map $X \to I_X^0$ extends to $Y \to I_X^0$ and β is given by this map and the composition $Y \to Z \to I_Z^0$. Finally, γ is the projection onto the second factor. The commutativity of the diagram is immediate. We claim that it is strong and exact everywhere. The commutativity of

$$\begin{array}{ccc} I_X^0 & \xrightarrow{\alpha} & I_X^0 \times I_Z^0 \\ & {\scriptstyle \text{id}} \searrow & \downarrow{\scriptstyle pr_1} \\ & & I_X^0 \end{array}$$

implies that α is a monomorphism. Since γ is a projection, it is an epimorphism. It is easy to see that β is a weak monomorphism, so indeed, the diagram is strong and exact. Since $\mathcal{I}$ is an injective class, $I_X^0 \times I_Z^0$ lies in $\mathcal{I}$.

We write $I_Y^0 = I_X^0 \times I_Z^0$ and extend the diagram by the corresponding cokernels X', Y', Z' to get a commutative strong exact diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \hookrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I_X^0 & \longrightarrow & I_Y^0 & \longrightarrow & I_Z^0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

One uses diagram chase to verify the exactness of this diagram. We repeat the procedure with the exact sequence $0 \to X' \to Y' \to Z' \to 0$. Iteration leads to a commutative and exact diagram of injective resolutions

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \hookrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I_X & \hookrightarrow & I_Y & \longrightarrow & I_Z & \longrightarrow & 0.
\end{array}$$

Applying F to this diagram yields a strong exact sequence of complexes,

$$0 \to F(I_X) \hookrightarrow F(I_Y) \to F(I_Z) \to 0.$$

To verify the exactness recall that by construction I_Y^j is the direct product of I_X^j and I_Z^j. For any two objects A, B in $\mathcal{B}$ the map $A \stackrel{\mathrm{id}\times 0}{\to} A \times B \to A$ is an automorphism of A. Hence the same is true for $F(A) \to F(A \times B) \to F(A)$, so the map $F(A \times B) \to F(A)$ is an epimorphism and $F(A) \to F(A \times B)$ is a monomorphism.

To this sequence of complexes we now apply the snake lemma to get a long exact sequence

$$\cdots \to R^i F(Y) \to R^i F(Z) \stackrel{\delta}{\to} R^{i+1} F(X) \to R^{i+1} F(Y) \to \cdots$$

which is the first ingredient of a delta functor. The functoriality comes from the naturality of the snake construction. So $(R^n F)$ indeed is a delta functor. It is erasable, as $R^i F(I) = 0$ for every object I in $\mathcal{I}$ and every $i > 0$ by construction. □

Fix the situation as in the theorem. An object X in $\mathcal{B}$ is called *F-acyclic* if $R^i F(X) = 0$ for every $i > 0$.

Theorem 1.7 *Let $0 \to X \to A^0 \to A^1 \to \cdots$ be a resolution by F-acyclics. Then $R^i F(X) \cong H^i(F(A^\bullet))$, so cohomology can be computed using resolutions by arbitrary acyclics..*

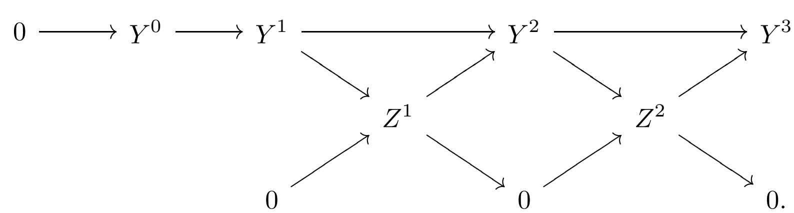
Proof: We need a lemma.

Lemma 1.8 *Let $0 \to Y^0 \to Y^1 \to \cdots$ be a strong exact sequence of F-acyclics. Then the sequence $0 \to F(Y^0) \to F(Y^1) \to \dots$ is exact.*

Proof: Since F is left strong-exact, the sequence

$$0 \to F(Y^0) \to F(Y^1) \to F(Y^2)$$

is exact. Let $Z^j = \operatorname{coker}(Y^{j-1} \to Y^j)$. Since the morphisms $Y^j \to Y^{j+1}$ are strong we get an exact, strong, and commutative diagram



Applying F we get an exact sequence

$$0 \to F(Y^0) \to F(Y^1) \to F(Z^1) \to R^1 F(Y^0) = 0,$$

and thus an epimorphism $\operatorname{coker}(F(Y^0) \to F(Y^1)) \twoheadrightarrow F(Z^1)$. Next the exact sequence $0 \to Z^1 \to Y^2 \to Y^3$ gives exactness of

$$0 \to F(Z^1) \to F(Y^2) \to F(Y^3).$$

Thus we get an exact sequence

$$\operatorname{coker}(F(Y^0) \to F(Y^1)) \to F(Y^2) \to F(Y^3),$$

which is the desired exactness at $F(Y^2)$. We conclude by induction. □

To finish the proof of the theorem we choose an $\mathcal{I}$-resolution

$$0 \to X \to I^0 \to I^1 \to \dots$$

such that we get a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \longrightarrow & A^0 & \longrightarrow & A^1 & \longrightarrow & \cdots \\
 & & \Big\downarrow{\scriptstyle =} & & \Big\downarrow & & \Big\downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & \cdots
\end{array}$$

where the vertical maps can be chosen injective by enlarging I^j is necessary. Let (Y^j) be the sequence of cokernels so that we get an exact, strong, commutative diagram,

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \Big\downarrow & & \Big\downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & A^0 & \longrightarrow & A^1 & \longrightarrow & \cdots \\
 & & \Big\downarrow{\scriptstyle =} & & \Big\downarrow & & \Big\downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & \cdots \\
 & & & & \Big\downarrow & & \Big\downarrow & & \\
 & & 0 & \longrightarrow & Y^0 & \longrightarrow & Y^1 & \longrightarrow & \cdots \\
 & & & & \Big\downarrow & & \Big\downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since A^i and I^i are acyclic, the exact sequence
$R^kF(I^i) \to R^kF(Y^i) \to R^{k+1}F(A^i)$ tells us that Y^i is acyclic. Applying F we obtain a short exact seqeunce of complexes

$$0 \to F(A) \hookrightarrow F(I) \to F(Y) \to 0.$$

The corresponding cohomology sequence reads

$$H^{i-1}F(Y) \to H^iF(A) \to H^iF(I) \to H^iF(Y).$$

Both ends are zero by Lemma 1.8, so we get an isomorphism in the middle, i.e.,

$$H^iF(A) \;\cong\; R^iF(X).$$

□

2 Pointed modules and sheaves

Let A be a commutative monoid. A *module* over A is a set M together with an action $A \times M \to M$ sending (a, m) to am and satisfying $(ab)m = a(bm)$ and $1m = m$ for all $a, b \in A$ and every $m \in M$. Let $N \subset M$ be a sub-module, then we define the quotient module M/N by collapsing N: as a set, M/N equals $M/\sim$, where $\sim$ is the equivalence relation with the equivalence classes $\{m\}$, $m \notin N$ and N. The module structure is defined by $a[m] = [am]$, where $[m]$ is the class of $m \in M$.

An element $m_0 \in M$ is called *stationary* if $am = m$ for every $a \in A$. A *pointed module* is a pair (M, m_0) consisting of an A-module M and a stationary point $m_0 \in M$. A *homomorphism of pointed modules* from (M, m_0) to (N, n_0) is an A-module homomorphism φ with $\varphi(m_0) = n_0$. Let $\mathrm{Mod}_0(A)$ denote the category of pointed modules and their homomorphisms. The special point m_0 of a pointed module M is also denoted by 0_M or 0 if no confusion is likely. It is called the *zero element* of M.

If M is a module over A, we define the pointed module M^+ to be $M \cup \{0\}$, where 0 is a new stationary point which we choose to be the special point of M^+.

The category $\mathrm{Mod}_0(A)$ contains a terminal and initial object, the zero module $\{0\}$, also written 0. A morphism $\varphi : M \to N$ is called zero if φ factors over zero. This is equivalent to $\varphi(M) = \{0_N\}$.

The category $\mathrm{Mod}_0(A)$ contains products and coproducts. Products are the usual Cartesian products and coproducts are given as follows: Let $(M_i)_{i\in I}$ be a family of objects in $\mathrm{Mod}_0(A)$, then the coproduct is

$$\coprod_{i\in I} M_i = \dot{\bigcup_{i\in I}} M_i \Big/ \sim$$

where the union means the disjoint union of the M_i and the equivalence relation just identifies all zeros 0_{M_i} to one. We also write coproducts as direct sums.

2.1 Limits

Proposition 2.1 *The category* $\mathrm{Mod}_0(A)$ *contains direct and inverse limits.*

Proof: Let I be a small category and $F\colon I \to \mathrm{Mod}_0(A)$ be a functor. Write M_i for $F(i)$, $i \in I$. Define

$$M \stackrel{\text{def}}{=} \coprod_{i\in I} M_i/\sim,$$

where $\sim$ is the equivalence relation given by $m \sim F(\varphi)(m)$ whenever $m \in M_i$ and $\varphi\colon i \to j$ is a morphism in I. A straightforward verification shows that M is a direct limit.

Likewise,

$$N \stackrel{\text{def}}{=} \left\{ x \in \prod_{i \in I} M_i \;\middle|\; x_j = F(\varphi)(x_i)\ \forall \varphi \in \mathrm{Mor}_I(i,j) \right\}$$

is an inverse limit. □

Lemma 2.2 *A morphism* $\varphi\colon X \to Y$ *in* $\mathrm{Mod}_0(A)$ *is an epimorphism if and only if* φ *is a surjective map.*

Proof: Suppose φ is an epimorphism, then $Y/\mathrm{im}\,\varphi$ is zero, so $\mathrm{im}\,\varphi = Y$, i.e., φ is surjective. The rest is clear. □

2.2 Injectives and projectives, ascent

Theorem 2.3 *The category* $\mathrm{Mod}_0(A)$ *is a belian category with enough injectives and enough projectives.*

Proof: We start with the existence of enough injectives. For any set X we have an A-module structure on the set $\mathrm{Map}(A, X)$ of all maps $\alpha\colon A \to X$ given by

$$a\alpha(b) = \alpha(ab).$$

Further, if X is a pointed set, then $\mathrm{Map}(A, X)$ is a pointed module, the special point being α_0 with $\alpha_0(a) = x_0$, where x_0 is the special point of X. For a given pointed module M we define I_M to be

$$I_M \stackrel{\text{def}}{=} \mathrm{Map}(A, M).$$

We have a natural embedding $M \hookrightarrow \mathrm{Map}(A, M)$ of A-modules given by $m \mapsto \alpha_m$ with $\alpha_m(a) = am$. The theorem will follow if we show that $\mathrm{Map}(A, M)$ is indeed injective. For this note that for any A-module P and any set X there is a functorial isomorphism of A-modules

$$\psi\colon \mathrm{Map}(P, X) \to \mathrm{Hom}_A(P, \mathrm{Map}(A, X))$$

given by

$$\psi(\alpha)(p)(a) = \alpha(ap).$$

The inverse is given by

$$\psi^{-1}(\beta)(p) = \beta(p)(1).$$

Now let $P \hookrightarrow N$ be an injective A-module homomorphism, then for any set X one has the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_A(N, \mathrm{Map}(A,X)) & \longrightarrow & \mathrm{Hom}_A(P, \mathrm{Map}(A,X)) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Map}(N,X) & \longrightarrow & \mathrm{Map}(P,X) \end{array}$$

The second horizontal map is surjective, therefore the first horizontal map is surjective as well. For $X = M$ this implies the first part of the theorem.

For the existence of enough projectives, consider A as a module over itself. Let $P_M = \bigoplus_{m \in M} A_m^+$ be a direct sum of copies of A^+. Then the pointed module P_M is projective as a straightforward verification shows. For a given module M define a map

$$\begin{array}{rcl} \varphi\colon P_M & \to & M \\ a \in A_m & \mapsto & am, \\ 0 & \mapsto & m_0. \end{array}$$

Then $\varphi\colon P_M \to M$ is the desired surjection. □

Theorem 2.4 *The category* $\mathrm{Mod}_0(A)$ *admits an ascent functor which preserves injectives and projectives.*

Proof: One can choose for example the category $\mathcal{C}$ of $\mathbb{Q}$-vector spaces and $\mathcal{F}\colon \mathrm{Mod}_0(A) \to \mathcal{C}$ mapping M to the vector space $\mathbb{Q}[M]/\mathbb{Q}m_0$, where m_0 is the special point of M. Since every object in $\mathcal{C}$ is injective as well as projective, the theorem follows. □

2.3 Pointed sheaves

Let X be a monoidal space, i.e., a topological space with a sheaf $\mathcal{O}_X$ of monoids. A given topological space can be made monoidal by defining $\mathcal{O}_X$ to be the constant sheaf $\mathcal{O}_X(U) = \{1\}$. A *pointed sheaf* is a sheaf of pointed $\mathcal{O}_X$-modules where the restrictions are assumed to preserve the special points. Let $\mathrm{Mod}_0(X)$ denote the category of pointed sheaves.

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism in $\mathrm{Mod}_0(X)$. Then $\ker\varphi : U \mapsto \ker\varphi_U$ is a sheaf, where φ_U is the induced morphism from $\mathcal{F}(U)$ to $\mathcal{G}(U)$. We call it the kernel sheaf $\ker\varphi$. Further, $U \mapsto \mathrm{coker}\,\varphi_U$ and $U \mapsto \mathrm{im}\,\varphi_U$ are pre-sheaves the sheafifications of which we call the cokernel and image sheaf.

Proposition 2.5 *The category* $\mathrm{Mod}_0(X)$ *is belian and contains enough injectives.*

Proof: The zero object is the zero sheaf. Consider a diagram of pointed sheaves over X,

$$\begin{array}{ccc} & & \mathcal{F} \\ & & \downarrow \\ \mathcal{G} & \longrightarrow & \mathcal{H}. \end{array}$$

Let $\mathcal{P}(U) = \mathcal{G}(U) \times_{\mathcal{H}(U)} \mathcal{F}(U)$. Then $\mathcal{P}$ forms a sheaf, which is the fibre product of $\mathcal{G}$ and $\mathcal{F}$ over $\mathcal{H}$. Similarly one shows that cofibre products exist. To verify the last axiom let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a morphism with zero cokernel and let $\mathcal{G} \rightrightarrows \mathcal{Z}$ be two morphisms such that the induced morphisms from $\mathcal{F}$ to $\mathcal{Z}$ agree. For any $x \in X$ one has the exact sequence of the stalks $\mathcal{F}_x \rightrightarrows \mathcal{G}_x \to 0$. Therefore φ_x is an epimorphism and thus the two maps $\mathcal{G}_x \rightrightarrows \mathcal{Z}_x$ agree. Since this holds for every $x \in X$, the two morphisms $\mathcal{G} \rightrightarrows \mathcal{Z}$ agree, so φ is an epimorphism. The existence of enough injectives is proved in the next subsection. □

Lemma 2.6 *The following holds in* $\mathrm{Mod}_0(X)$*.*

(a) *A morphism $f\colon \mathcal{F} \to \mathcal{G}$ is strong if an only if all fibres $f_x\colon \mathcal{F}_x \to \mathcal{G}_x$, $x \in X$, are strong.*

(b) *A sequence $\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H}$ is exact if and only if all the sequences at the fibres $\mathcal{F}_x \xrightarrow{f_x} \mathcal{G}_x \xrightarrow{g_x} \mathcal{H}_x$, $x \in X$, are exact.*

Proof:

(a) A morphism f in a belian category is strong if and only if the induced $\tilde{f}$: $\mathrm{coim} f \to \mathrm{im}\, f$ is an isomorphism. This map is always a weak isopmorphism. If f is a morphism in $\mathrm{Mod}_0(X)$, then for every $x \in X$ one has $(\tilde{f})_x = \tilde{f}_x$. Replacing f by $\tilde{f}$ it therefore suffices to show that f is a monomorphism if and only if all its fibres f_x are.

Let's assume that f is a monomorphism and let $x \in X$. We have to show that f_x is injective. For this assume $f_x(s_x) = f_x(t_x)$ for some $s_x, t_x \in \mathcal{F}_x$. Then there exists an open neighbourhood U of x and representatives $s_U, t_U \in \mathcal{F}_U$ with $f_U(s_U) = f_U(t_U)$ in $\mathcal{G}(U)$. We can consider $\mathcal{O}|_U$ as an $\mathcal{O}_U$-module, but not a pointed one in general. To make it pointed we add an extra stationary point ω_V to $\mathcal{O}_V$ for every open $V \subset U$. Thus we get a pointed $\mathcal{O}_U$-module $\mathcal{Z} = (\mathcal{O}|_U)_0$. We extend this module by zero outside the open set U to obtain a pointed $\mathcal{O}_X$-module which we likewise denote by $\mathcal{Z}$. We define a morphism $\alpha\colon \mathcal{Z} \to \mathcal{F}$ as follows. For $V \subset U$ open, $\alpha_V\colon \mathcal{Z}(V) \to \mathcal{F}(V)$ is defined as $\alpha_v(a) = a s_V$ for $a \in \mathcal{O}_V$ and $\alpha_V(\omega_V) = 0$. This defines a morphism α in $\mathrm{Mod}_0(X)$. Using t instead of s we define $\beta\colon \mathcal{Z} \to \mathcal{F}$ in the same manner. Then $f\alpha = f\beta$ and since f is a monomorphism, $\alpha = \beta$, hence $s_U = t_U$ and so $s_x = t_x$. The other direction is trivial.

(b) This assertion is shown in the same way as for sheaves of abelian groups. □

2.4 Injectives and ascent

Proposition 2.7 *The category* $\mathrm{Mod}_0(X)$ *has enough injectives. In particular, the class* $\mathcal{I}$ *of products of skyscraper sheaves with injective stalks is an injective class.*

Proof: To see that there are enough injectives, let $\mathcal{F}$ be a pointed $\mathcal{O}_X$-module. For each point $x \in X$ the stalk $\mathcal{F}_x$ is a pointed $\mathcal{O}_{X,x}$-module. Therefore there is an injection $\mathcal{F}_x \hookrightarrow I_x$ into an injective $\mathcal{O}_{X,x}$-module. Let i_x denote the injection of x in X and consider the sheaf $\mathcal{I} = \prod_{x\in X} i_{x,*}I_x$. For any $\mathcal{O}_X$-module $\mathcal{G}$ we have $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G},\mathcal{I}) \cong \prod_x \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, i_{x,*}I_x)$ and for every $x \in X$ also $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, i_{x,*}I_x) \cong \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_x, I_x)$. So there is a monomorphism $\mathcal{F} \hookrightarrow \mathcal{I}$ obtained from the maps $\mathcal{F}_x \hookrightarrow I_x$. Also it follows that $\mathcal{I}$ is injective and hence the claim. □

Proposition 2.8 *The category* $\mathrm{Mod}_0(X)$ *admits an ascent datum* $(\mathcal{I}, \mathrm{Asc})$.

Proof: For injective class we choose the class $\mathcal{I}$ of all products of skyscraper sheaves with injective stalks. This means that the stalk $\mathcal{F}_x$ has to be injective as $\mathcal{O}_{X,x}$-module.

Let $\mathcal{C}$ be the category of all sheaves of $\mathbb{Q}$-vector spaces on X. The ascent functor

$$\mathrm{Asc}\colon \mathrm{Mod}_0(X) \to \mathcal{C}$$

maps a sheaf $\mathcal{F}$ to

$$\mathrm{Asc}(\mathcal{F})(U) = \mathbb{Q}[\mathcal{F}(U)]/\mathbb{Q}x_0(U),$$

where $x_0(U)$ is the special point of $\mathcal{F}(U)$. Since Asc maps products of skyscraper sheaves to products of skyscraper sheaves the claim follows. □

2.5 Sheaf cohomology

Let X be a monoidal space and set $A = \mathcal{O}_X(X)$. We consider the global sections functor $\Gamma(X,\cdot)$ from $\mathrm{Mod}_0(X)$ to $\mathrm{Mod}_0(A)$.

Lemma 2.9 *The global sections functor* $\Gamma(X,\cdot)$ *is left strong-exact.*

Proof: Let $0 \to \mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H}$ be an exact sequence in $\mathrm{Mod}_0(X)$, where f is strong, i.e., f is a monomorphism. We have to show that the sequence

$0 \to \mathcal{F}(X) \stackrel{f_X}{\to} \mathcal{G}(X) \stackrel{g_X}{\to} \mathcal{H}(X)$ is exact in $\mathrm{Mod}_0(A)$. So let $\alpha \in \ker(f_X)$. Then $\mathcal{O}_X \cdot \alpha$ is a subsheaf of $\mathcal{F}$ which by f is mapped to zero. Hence $\mathcal{O}_X \cdot \alpha = 0$, so $\alpha = 0$. Next let $\beta \in \ker(g_X)$. Then $\mathcal{O}_X \cdot \beta$ is a subsheaf of $\mathcal{G}$ mapped to zero by g. Now f, being a monomorphism, gives an isomorphism $\mathcal{F} \to \ker(g)$. Hence the map $\mathcal{O}_X \cdot \beta \to \mathcal{G}$ factorizes over f as claimed. □

we define the cohomology of a sheaf $\mathcal{F} \in \mathrm{Mod}_0(X)$ by

$$H^i(X, \mathcal{F}) \stackrel{\text{def}}{=} R^i\Gamma(X, \mathcal{F}), \qquad i = 0, 1, \ldots$$

A sheaf $\mathcal{F}$ is called *flabby* if for any two open sets $U \subset V$ the restriction map $\mathcal{F}(V) \to \mathcal{F}(U)$ is surjective.

Lemma 2.10 *Every injective sheaf is flabby.*

Proof: For any open set $U \subset x$ let $\mathcal{O}_U$ denote the sheaf $j_!(\mathcal{O}_X|_U)$, which is the restriction of $\mathcal{O}_X$ to U, extended by zero outside U. Now let I be an injective $\mathcal{O}_X$-module and let $U \subset V$ be open sets. We have an inclusion $\mathcal{O}_U \hookrightarrow \mathcal{O}_V$ and since I is injective we get a surjection $\mathrm{Hom}(\mathcal{O}_V, I) \to \mathrm{Hom}(\mathcal{O}_U, I)$. But $\mathrm{Hom}(\mathcal{O}_V, I) \cong I(V$ and $\mathrm{Hom}(\mathcal{O}_U, I) \cong I(U)$, so I is flabby. □

Lemma 2.11 *Let* $0 \to \mathcal{F} \stackrel{f}{\to} \mathcal{H} \stackrel{h}{\to} \mathcal{G} \to 0$ *be a strong exact sequence in* $\mathrm{Mod}_0(X)$.

(a) If $\mathcal{F}$ is flabby, then for every open set $U \subset X$ the sequence

$$0 \to \mathcal{F}(U) \stackrel{f_U}{\to} \mathcal{H}(U) \stackrel{h_U}{\to} \mathcal{G}(U) \to 0$$

is exact.

(b) If $\mathcal{F}$ and $\mathcal{H}$ are flabby, then so is $\mathcal{G}$.

Proof: After applying the ascent functor, the claim follows from the corresponding result for sheaves of abelian groups [4]. □

Proposition 2.12 *If $\mathcal{F}$ is a flabby sheaf in* $\mathrm{Mod}_0(X)$*, then $H^i(X, \mathcal{F}) = 0$ for $i > 0$.*

Proof: Embed $\mathcal{F}$ in an object $I \in \mathcal{I}$ and let $\mathcal{G}$ be the quotient. Then the sequence

$$0 \to \mathcal{F} \to I \to \mathcal{G} \to 0$$

is strong and exact. Since $\mathcal{F}$ and I are flabby, so is $\mathcal{G}$. Since $\mathcal{F}$ is flabby, the sequence

$$0 \to H^0(X,\mathcal{F}) \to H^0(X,I) \to H^0(X,\mathcal{G}) \to 0$$

is exact. Since $I \in \mathcal{I}$, we have $H^i(X,I) = 0$ for $i > 0$ and so the long exact cohomology sequence shows that $H^1(X,\mathcal{F}) = 0$ and $H^i(X,\mathcal{F}) \cong H^{i-1}(X,\mathcal{G})$ for $i \geq 2$. But $\mathcal{G}$ is also flabby, so the claim follows by induction. □

Note that according to Theorem 1.7 we now can compute cohomology using flabby resolutions.

This has interesting consequences. For instance, it shows that vanishing of cohomology does not depend on the sheaf of monoids $\mathcal{O}_X$ as the following Lemma shows.

Lemma 2.13 *Let* For *be the forgetful functor from the category* $\mathrm{Mod}_0(A)$ *to* $\mathrm{Set}_0 \cong \mathrm{Mod}_0(1)$*. Then the isomorphy class of* $\mathrm{For}(H^i(X,\mathcal{F}))$ *in* Set_0 *does not depend on the choice of the sheaf* $\mathcal{O}_X$*.*

Proof: Let $\mathrm{Set}_0(X)$ denote the category of pointed sheaves over X for the trivial structure sheaf $\mathcal{O}_X = const$. To compute the cohomology, use flabby resolutions in $\mathrm{Mod}_0(X)$. They will remain flabby in $\mathrm{Set}_0(X)$. □

2.6 Noetherian Spaces

We say that a monoid A is *noetherian* if every chain of ideals $I_1 \subset I_2 \subset I_3 \subset \dots$ is eventually stationary, i.e., there exists an index j_0 such that $I_j = I_{j_0}$ for every $j \geq j_0$. A topological space X is called *noetherian* if every sequence of closed subsets $Y_1 \supset Y_2 \supset Y_3 \supset \dots$ is eventually stationary. The *dimension* of a topological space is the supremum of the lengths of strictly descending chains of closed subsets. A noetherian topological space is not necessarily of finite dimension.

If $X = \mathrm{Spec}\,\mathbb{F}_A$, then X is noetherian if and only if A is. An $\mathbb{F}_1$-scheme X is called *noetherian* if X can be covered by finitely many affine schemes $\mathrm{Spec}\,(A_i)$ where each monoid A_i is noetherian. A noetherian scheme is noetherian and finite dimensional as topological space.

Let $(\mathcal{F}_\alpha)$ be a direct system of pointed sheaves. By $\varinjlim \mathcal{F}_\alpha$ we denote the sheafification of the presheaf $U \mapsto \varinjlim \mathcal{F}_\alpha(U)$.

Let X be a monoidal space.

Lemma 2.14 *Let* $(\mathcal{F}_\alpha)_{\alpha\in I}$ *be a direct system of flabby sheaves and assume that* X *is noetherian. Then* $\varinjlim \mathcal{F}_\alpha$ *is flabby.*

Proof: As in the group valued case one proves that if X is noetherian, then the presheaf $U \mapsto \lim\limits_{\to} \mathcal{F}_\alpha(U)$ already is a sheaf. For every $\alpha \in I$ and every inclusion $V \subset U$ of open sets the restriction $\mathcal{F}_\alpha(U) \to \mathcal{F}_\alpha(V)$ is surjective. This implies that $\lim\limits_{\to} \mathcal{F}_\alpha(U) \to \lim\limits_{\to} \mathcal{F}_\alpha(V)$ also is surjective. Since X is noetherian we have $\lim\limits_{\to} \mathcal{F}_\alpha(U_= (\lim\limits_{\to} \mathcal{F}_\alpha)(U)$, so $\lim\limits_{\to} \mathcal{F}_\alpha$ is flabby. □

Proposition 2.15 *Let X be a noetherian monoidal space and $(\mathcal{F}_\alpha)_{\alpha \in I}$ a direct system of pointed sheaves on X. Then for every $i \geq 0$ there is a natural isomorphism*

$$\lim_{\to} H^i(X, \mathcal{F}_\alpha) \ \to \ H^i(X, \lim_{\to} \mathcal{F}_\alpha).$$

Proof: For every $\beta \in I$ we have a natural map $\mathcal{F}_\beta \to \lim\limits_{\to} \mathcal{F}_\alpha$. This induces a map on cohomology and we take the direct limit of these maps. For $i = 0$ the result is clear. For $i > 0$, consider the category $\mathrm{ind}_I(\mathrm{Mod}_0(X))$ of all direct systems in $\mathrm{Mod}_0(X)$ indexed by I. This category is belian. Furthermore, $\lim\limits_{\to}$ is an exact functor and so one has a natural transformation of δ-functors

$$\lim_{\to} H^i(X, \cdot) \ \to \ H^i(X, \lim_{\to} \cdot)$$

from $\mathrm{ind}_I(\mathrm{Mod}_0(X))$ to $\mathrm{Mod}_x(A)$, where $A = \Gamma(X, \mathcal{O}_X)$. This transformation is the identity for $i = 0$, so it suffices to show that both functors are erasable for $i > 0$. So let $(\mathcal{F}_\alpha)$ be in $\mathrm{ind}_I(\mathrm{Mod}_0(X))$. For each α let $\mathcal{G}_\alpha$ be the sheaf of discontinuous sections of $\mathcal{F}_\alpha$, i.e.,

$$\mathcal{G}_\alpha(U) = \{s \colon U \to \bigcup_{u \in U} (\mathcal{F}_\alpha)_u : s(u) \in (\mathcal{F}_\alpha)_u \ \forall u \in U\}.$$

Then $\mathcal{G}_\alpha$ is flabby and there is a natural inclusion $\mathcal{F}_\alpha \hookrightarrow \mathcal{G}_\alpha$. Furthermore, the $\mathcal{G}_\alpha$ form a direct system and we obtain a monomorphism $(\mathcal{F}_\alpha) \hookrightarrow (\mathcal{G}_\alpha)$ in the category $\mathrm{ind}_I(\mathrm{Mod}_0(X))$. All $\mathcal{G}_\alpha$ are flabby and so is their limit. This implies that both functors are indeed erasable. □

Let Y be a closed subset of X and $\mathcal{F}$ a pointed sheaf on Y. Let $j_*\mathcal{F}$ be the extension by zero outside Y. Then one has $H^i(Y, \mathcal{F}) = H^i(X, j_*\mathcal{F})$ as a flabby resolution $\mathcal{J}^\bullet$ of $\mathcal{F}$ gives a flabby resolution $j_*\mathcal{J}^\bullet$ of $j_*\mathcal{F}$.

Theorem 2.16 *Let X be noetherian of dimension n. Then for every $i > n$ and every pointed sheaf $\mathcal{F}$ on X we have $H^i(X, \mathcal{F}) = 0$.*

Proof: By Lemma 2.13 we may assume that $\mathcal{O}_X$ is the trivial sheaf of monoids. For a closed subsed Y of X and a pointed sheaf $\mathcal{F}$ on X we write $\mathcal{F}_Y$ for $j_*(\mathcal{F}|_Y)$. If $U \subset X$ is open, we write $\mathcal{F}_U = i_!(\mathcal{F}|_U)$. Then, if $U - X \setminus Y$, we have an exact sequence

$$0 \to \mathcal{F}_U \hookrightarrow \mathcal{F} \to \mathcal{F}_Y \to 0,$$

as one easily checks.

We next reduce the proof to the case when X is irreducible. For assume X is reducible, then $X = Y \cup Z$ with closed sets Y, Z both different from X. Let $U = X \smallsetminus Y$ and consider the exact sequence

$$0 \to \mathcal{F}_U \hookrightarrow \mathcal{F} \to \mathcal{F}_Y \to 0.$$

By the long exact sequence of cohomology it suffices to show $H^i(X, \mathcal{F}_U) = 0$ and $H^i(X, \mathcal{F}_Y) = 0$. Now $\mathcal{F}_U$ can be viewed as a sheaf on Z and so the proof if reduced to the components Y and Z. By induction on the number of components we can now assume that X is irreducible.

We prove the Theorem by induction on $n = \dim X$. If $n = 0$ then X has only two open sets, itself and the empty set. Then $\Gamma(X, \cdot)$ is exact, so the claim follows. Now for the induction step let X be irreducible of dimension $N > 0$ and let $\mathcal{F}$ be a pointed sheaf on X. Since every pointed sheaf is a direct limit of sheaves which are generated by finitely many sections, we are reduced by Proposition 2.15 to the case of $\mathcal{F}$ being finitely generated. By an induction argument it suffices to assume that $\mathcal{F}$ is generated by a single section in $\mathcal{F}(U)$, say, for an open set U. Let $\mathcal{Z}$ be the constant sheaf with fibre $\mathbb{Z}/2\mathbb{Z}$. Then $\mathcal{F}$, being generated by a single section, is a quotient of $\mathcal{Z}_U$. So we have an exact sequence,

$$0 \to \mathcal{R} \hookrightarrow \mathcal{Z}_U \to \mathcal{F} \to 0.$$

By the long exact cohomology sequence it suffices to show the vanishing of the cohomology of $\mathcal{R}$ and $\mathcal{Z}_U$. If $\mathcal{R} \neq 0$, then there exists an open set $V \subset U$ such that $\mathcal{R}_V \cong \mathcal{Z}_V$. So we have an exact sequence

$$0 \to \mathcal{Z}_V \hookrightarrow \mathcal{R} \to \mathcal{R}/\mathcal{Z}_V \to 0.$$

The sheaf $\mathcal{R}/\mathcal{Z}_V$ is supported in $\overline{U \smallsetminus V}$ which has dimension $< n$ since X is irreducible. So it follows that $H^i(X, \mathcal{R}/\mathcal{Z}_V) = 0$ for $i > n$ by induction hypothesis. It remains to show vanishing of cohomology for $\mathcal{Z}_V$. We show that for every open $U \subset X$ we have $H^i(X, \mathcal{Z}_U) = 0$ for $i > n$. Let $Y = X \smallsetminus U$. We have an exact sequence

$$0 \to \mathcal{Z}_U \hookrightarrow \mathcal{Z} \to \mathcal{Z}_Y \to 0.$$

Since X is irreducible, we have $\dim Y < n$. So by induction hypothesis we have $H^i(X, \mathcal{Z}_Y) = 0$ for $i \geq n$. On the other hand, $\mathcal{Z}$ is flabby as it is a constant sheaf on an irreducible space. Hence $H^i(X, \mathcal{Z}) = 0$ for $i > 0$. So the long exact cohomology sequence gives the claim. □

2.7 Base change

Now assume that X is an $\mathbb{F}_1$-scheme. Let $X_{\mathbb{Z}} = X \otimes \mathbb{Z}$ be the base change to $\mathbb{Z}$. Instead of $\mathbb{Z}$ one could take any other ring here. Let $\mathcal{F}$ be a pointed sheaf over X. For a pointed module (M, m_0) over a monoid A write $M_{\mathbb{Z}}$ for the $\mathbb{Z}[A]$-module $\mathbb{Z}[M]/\mathbb{Z}m_0$. Every open set U in X defines an open set $U_{\mathbb{Z}}$ in $X_{\mathbb{Z}}$ as follows. If $X = \operatorname{Spec}_{\mathbb{F}_1} A$ is affine, then U defines an ideal $\mathfrak{a}$ of A. Then $\mathbb{Z}[\mathfrak{a}]$ is an ideal of $\mathbb{Z}[A]$ which defines an open set $U_{\mathbb{Z}}$ of $X_{\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[A]$. For non-affine X define $U_{\mathbb{Z}}$ locally and take the union. We define the sheaf $\mathcal{F}_{\mathbb{Z}}$ to be the sheafification of the presheaf

$$U \mapsto \varinjlim_{V_{\mathbb{Z}} \supset U} \mathcal{F}(V)_{\mathbb{Z}}.$$

here the inductive limit is taken over all open sets in $X_{\mathbb{Z}}$ which contain U and are of the form $V_{\mathbb{Z}}$ for some V open in X.

If $\mathcal{F}$ is a skyscraper sheaf in $x \in X$, then the closed set $\bar{x} = \overline{\{x\}}$ is given by an ideal sheaf which base changes to an ideal sheaf of $X_{\mathbb{Z}}$ which defines a closed subset $\bar{x}_{\mathbb{Z}}$ of $X_{\mathbb{Z}}$. It turns out that $\mathcal{F}_{\mathbb{Z}}$ is a constant sheaf on $\bar{x}_{\mathbb{Z}}$ extended by zero outside $\bar{x}_{\mathbb{Z}}$. In particular, $\mathcal{F}_{\mathbb{Z}}$ is flabby.

The functor $\mathcal{F} \mapsto \mathcal{F}_{\mathbb{Z}}$ is an ascent functor from $\operatorname{Mod}_0(X)$ to $\operatorname{Mod}(X_{\mathbb{Z}})$ which maps sheaves in the injective class $\mathcal{I}$ to flabby sheaves, hence $\mathcal{I}$-resolutions are mapped to flabby resolutions.

Theorem 2.17 *As functors in $\mathcal{F}$,*

$$H^p(X, \mathcal{F})_{\mathbb{Z}} \;\cong\; H^p(X_{\mathbb{Z}}, \mathcal{F}_{\mathbb{Z}}).$$

Proof: For $p = 0$ the claim follows from the definitions. Both sides define δ-functors from $\operatorname{Mod}_0(X)$ to $\operatorname{Mod}(X_{\mathbb{Z}})$. The right hand side is clearly universal. The left hand side is also universal since if $\mathcal{F} \in \mathcal{I}$, then $\mathcal{F}_{\mathbb{Z}}$ is flabby, we conclude that $\mathcal{F} \mapsto H^p(X_{\mathbb{Z}}, \mathcal{F}_{\mathbb{Z}})$ is erasable for $p > 0$. □

Corollary 2.18 *If $X = \operatorname{Spec}_{\mathbb{F}_1}(A)$ is affine and M is a pointed A-module, then $H^p(X, \tilde{M}) = 0$ for $p > 0$.*

Proof: Since $(\tilde{M})_{\mathbb{Z}} \cong \widetilde{M_{\mathbb{Z}}}$, the claim follows from the corresponding claim for schemes. □

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