New Approach to Arakelov Geometry

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Introduction

The principal aim of this work is to provide an alternative algebraic framework for Arakelov geometry, and to demonstrate its usefulness by presenting several simple applications. This framework, called *theory of generalized rings and schemes*, appears to be useful beyond the scope of Arakelov geometry, providing a uniform description of classical scheme-theoretical algebraic geometry ("schemes over $\operatorname{Spec}\mathbb{Z}$"), Arakelov geometry ("schemes over $\operatorname{Spec}\mathbb{Z}_\infty$ and $\widehat{\operatorname{Spec}\mathbb{Z}}$"), tropical geometry ("schemes over $\operatorname{Spec}\mathbb{T}$ and $\operatorname{Spec}\mathbb{N}$") and the geometry over the so-called field with one element ("schemes over $\operatorname{Spec}\mathbb{F}_1$"). Therefore, we develop this theory a bit further than it is strictly necessary for Arakelov geometry.

The approach to Arakelov geometry developed in this work is completely *algebraic*, in the sense that it doesn't require the combination of scheme-theoretical algebraic geometry and complex differential geometry, traditionally used in Arakelov geometry since the works of Arakelov himself.

However, we show that our models $\bar{\mathscr{X}}/\widehat{\operatorname{Spec}\mathbb{Z}}$ of algebraic varieties $X/\mathbb{Q}$ define both a model $\mathscr{X}/\operatorname{Spec}\mathbb{Z}$ in the usual sense and a (possibly singular) Banach (co)metric on (the smooth locus of) the complex analytic variety $X(\mathbb{C})$. This metric cannot be chosen arbitrarily; however, some classical metrics like the Fubini–Study metric on $\mathbb{P}^n$ do arise in this way. It is interesting to note that "good" models from the algebraic point of view (e.g. finitely presented) usually give rise to not very nice metrics on $X(\mathbb{C})$, and conversely, nice smooth metrics like Fubini–Study correspond to models with "bad" algebraic properties (e.g. not finitely presented).

Our algebraic approach has some obvious advantages over the classical one. For example, we never need to require X to be smooth or proper, and we can deal with singular metrics.

In order to achieve this goal we construct a theory of *generalized rings*, commutative or not, which include classical rings (always supposed to be associative with unity), then define *spectra* of such (commutative) generalized rings, and construct *generalized schemes* by patching together spectra of generalized rings. Of course, these generalized schemes are *generalized ringed spaces*, i.e. topological spaces, endowed with a sheaf of generalized rings. Then the "compactified" $\operatorname{Spec}\mathbb{Z}$, denoted by $\widehat{\operatorname{Spec}\mathbb{Z}}$, is constructed as a (pro-)generalized scheme, and our models $\bar{\mathscr{X}}/\widehat{\operatorname{Spec}\mathbb{Z}}$ are (pro-)generalized schemes as well.

All the "generalized" notions we discuss are indeed generalizations of corresponding "classical" notions. More precisely, "classical" objects (e.g. commutative rings) always constitute a full subcategory of the category of corresponding "generalized" objects (e.g. commutative generalized rings). In this way we can always treat for example a classical scheme as a generalized scheme, since no new morphisms between classical schemes arise in the larger category of generalized schemes.

In particular, the category of (commutative) generalized rings contains all classical (commutative) rings like $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}_p$, …, as well as some new objects, such as $\mathbb{Z}_\infty$ (the "archimedian valuation ring" of $\mathbb{R}$, similar to p-adic integers $\mathbb{Z}_p \subset \mathbb{Q}_p$), $\mathbb{Z}_{(\infty)}$ (the "non-completed localization at ∞", or the "archimedian valuation ring" of $\mathbb{Q}$), $\bar{\mathbb{Z}}_\infty$ ("the integral closure of $\mathbb{Z}_\infty$ in $\mathbb{C}$"). Furthermore, once these "archimedian valuation rings" are constructed, we can define some other generalized rings, such as $\mathbb{F}_{\pm 1} := \mathbb{Z}_\infty \cap \mathbb{Z}$, or the "field with one element" $\mathbb{F}_1$. Tropical numbers $\mathbb{T}$ and other semirings are also generalized rings, thus fitting nicely into this picture as well.

In this way we obtain not only a "compact model" $\widehat{\operatorname{Spec}\mathbb{Z}}$ of $\mathbb{Q}$ (called also "compactification of $\operatorname{Spec}\mathbb{Z}$"), and models $\mathscr{X} \to \widehat{\operatorname{Spec}\mathbb{Z}}$ of algebraic varieties $X/\mathbb{Q}$, but a geometry over "the field with one element" as well. For example, $\widehat{\operatorname{Spec}\mathbb{Z}}$ itself is a pro-generalized scheme over $\mathbb{F}_1$ and $\mathbb{F}_{\pm 1}$.

In other words, we obtain rigorous definitions both of the "archimedian local ring" $\mathbb{Z}_\infty$ and of the "field with one element" $\mathbb{F}_1$. They have been discussed in mathematical folklore for quite a long time, but usually only in a very informal fashion.

We would like to say a few words here about some applications of the theory of generalized rings and schemes presented in this work. Apart from defining generalized rings, their spectra, and generalized schemes, we discuss some basic properties of generalized schemes, essentially transferring some results of EGA I and II to our case. For example, we discuss projective (generalized) schemes and morphisms, study line and vector bundles, define Picard groups and so on.

Afterwards we do some homological (actually homotopic) algebra over generalized rings and schemes, define perfect simplicial objects and cofibrations (which replace perfect complexes in this theory), define K_0 of perfect simplicial objects and vector bundles, briefly discuss higher algebraic K-theory (Waldhausen's construction seems to be well-adapted to our situation), and construct Chow rings and Chern classes using the γ-filtration on K_0, in the way essentially known since Grothendieck's proof of Riemann–Roch theorem.

We apply the above notions to Arakelov geometry as well. For example, we compute Picard group, Chow ring and Chern classes of vector bundles over $\widehat{\operatorname{Spec}\mathbb{Z}}$, and construct the moduli space of such vector bundles. In particular, we obtain the notion of *(arithmetic) degree* $\deg \mathscr{E} \in \log \mathbb{Q}_+^\times$ of a vector bundle $\mathscr{E}$ over $\widehat{\operatorname{Spec}\mathbb{Z}}$; it induces an isomorphism $\deg : \operatorname{Pic}(\widehat{\operatorname{Spec}\mathbb{Z}}) \to \log \mathbb{Q}_+^\times$. We also prove that any affine or projective algebraic variety X over $\mathbb{Q}$ admits a finitely presented model $\mathscr{X}$ over $\widehat{\operatorname{Spec}\mathbb{Z}}$, and show (under some natural conditions) that rational points $P \in X(\mathbb{Q})$ extend to uniquely determined sections $\sigma_P : \widehat{\operatorname{Spec}\mathbb{Z}} \to \mathscr{X}$. We show that when X is a closed subvariety of the projective space $\mathbb{P}^n_{\mathbb{Q}}$, and its model $\mathscr{X}$ is chosen accordingly (e.g. $\mathscr{X}$ is the "scheme-theoretical closure" of X in $\mathbb{P}^n_{\widehat{\operatorname{Spec}\mathbb{Z}}}$), then the (arithmetic) degree of the pullback $\sigma_P^*(\mathscr{O}_{\mathscr{X}}(1))$ of the ample line bundle of $\mathscr{X}$ equals the logarithmic height of point $P \in X(\mathbb{Q}) \subset \mathbb{P}^n(\mathbb{Q})$.

There are several reasons to believe that our "algebraic" Arakelov geometry can be related to its more classical variants, based on Kähler metrics, differential forms and Green currents, as developed first by Arakelov himself, and then in the series of works of H. Gillet, C. Soulé and their colloborators. The simplest reason is that our "algebraic" models give rise to some (co)metrics, and classical metrics like the Fubini–Study do appear in this way. More sophisticated arguments involve comparison with the non-archimedian variant of classical Arakelov geometry, developed in [BGS] and [GS]. This non-archimedian Arakelov geometry is quite similar to (classical) archimedian Arakelov geometry, and at the same time admits a natural interpretation in terms of models of algebraic varieties over discrete valuation rings. Analytic torsion corresponds in this picture to torsion in the special fiber (i.e. lack of flatness).

Therefore, one might hope to transfer eventually the results of these two works to archimedian context, using our theory of generalized schemes, thus establishing a direct connection between our "algebraic" and classical "analytic" variant of Arakelov geometry.

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Contents

Overview

We would like to start with a brief overview of the rest of this work, discussing chapter after chapter. Purely technical definitions and statements will be omitted or just briefly mentioned, while those notions and results, which we consider crucial for the understanding of the remainder of this work, will be explained at some length.

0.1. (Motivation.) Chapter **1** is purely motivational. Here we discuss proper smooth models both of functional and number fields, and indicate why non-proper models (e.g. the affine line $\mathbb{A}^1_k$ as a model of $k(t)$, or $\operatorname{Spec}\mathbb{Z}$ as a model of $\mathbb{Q}$) are not sufficient for some interesting applications. We also introduce some notations. For example, $\widehat{\operatorname{Spec}\mathbb{Z}}$ denotes the "compactification" of $\operatorname{Spec}\mathbb{Z}$. Its closed points must correspond to all valuations of $\mathbb{Q}$, archimedian or not, i.e. we expect $\widehat{\operatorname{Spec}\mathbb{Z}} = \operatorname{Spec}\mathbb{Z} \cup \{\infty\}$ as a set, where ∞ denotes a new "archimedian point". We denote by $\mathbb{Z}_\infty \subset \mathbb{Q}_\infty := \mathbb{R}$ and $\mathbb{Z}_{(\infty)} \subset \mathbb{Q}$ the completed and non-completed local rings of $\widehat{\operatorname{Spec}\mathbb{Z}}$ at ∞, analogous to classical notations $\mathbb{Z}_p \subset \mathbb{Q}_p$ and $\mathbb{Z}_{(p)} \subset \mathbb{Q}$.

0.1.1. It is important to notice here that $\widehat{\operatorname{Spec}\mathbb{Z}}$, $\mathbb{Z}_\infty$ and $\mathbb{Z}_{(\infty)}$ are not defined in this chapter. Instead, they are used in an informal way to describe the properties we would expect these objects to have. In this way we are even able to explain the classical approach to Arakelov geometry, which insists on defining an Arakelov model $\bar{\mathscr{X}}$ of a smooth projective algebraic variety $X/\mathbb{Q}$ as a flat proper model $\mathscr{X} \to \operatorname{Spec}\mathbb{Z}$ together with a metric on $X(\mathbb{C})$ subject to certain restrictions (e.g. being a Kähler metric).

0.1.2. Another thing discussed in this chapter is that the problem of constructing models over $\widehat{\operatorname{Spec}\mathbb{Z}}$ can be essentially reduced to the problem of constructing $\mathbb{Z}_{(\infty)}$-models of algebraic varieties $X/\mathbb{Q}$, or $\mathbb{Z}_\infty$-models of algebraic varieties $X/\mathbb{R}$. In other words, we need a notion of a $\mathbb{Z}_\infty$-structure on an algebraic variety X over $\mathbb{R}$; if $X = \operatorname{Spec} A$ is affine, this is the same thing as a $\mathbb{Z}_\infty$-structure on an $\mathbb{R}$-algebra A. So we see that a proper understanding of "compactified" models of algebraic varieties over $\mathbb{Q}$ must include an understanding of $\mathbb{Z}_\infty$-structures on $\mathbb{R}$-algebras and vector spaces.

0.2. ($\mathbb{Z}_\infty$-structures.) Chapter **2** is dedicated to a detailed study of $\mathbb{Z}_\infty$-structures on real vector spaces and algebras. We start from the simplest cases and extend our definitions step by step, arriving at the end to the "correct" definition of $\mathbb{Z}_\infty$-*Mod*, the category of $\mathbb{Z}_\infty$-modules. In this way we learn what the $\mathbb{Z}_\infty$-modules are, without still having a definition of $\mathbb{Z}_\infty$ itself.

The main method employed here to obtain "correct" definitions is the comparison with the p-adic case.

0.2.1. ($\mathbb{Z}_\infty$-lattices: classical description.) The first step is to describe $\mathbb{Z}_\infty$-structures on a finite-dimensional real vector space E, i.e. $\mathbb{Z}_\infty$*-lattices* $A \subset E$.

The classical solution is this. In the p-adic case a $\mathbb{Z}_p$-lattice A in a finite-dimensional $\mathbb{Q}_p$-vector space E defines a maximal compact subgroup $K_A := \mathrm{Aut}_{\mathbb{Z}_p}(A) \cong GL(n, \mathbb{Z}_p)$ in locally compact group $G := \mathrm{Aut}_{\mathbb{Q}_p}(E) \cong GL(n, \mathbb{Q}_p)$, all maximal compact subgroups of G arise in this way, and $K_A = K_{A'}$ iff A' and A are similar, i.e. $A' = \lambda A$ for some $\lambda \in \mathbb{Q}_p^\times$.

Therefore, it is reasonable to expect similarity classes of $\mathbb{Z}_\infty$-lattices inside real vector space E to be in one-to-one correspondence with maximal compact subgroups K of locally compact group $G := \mathrm{Aut}_{\mathbb{R}}(E) \cong GL(n, \mathbb{R})$. Such maximal compact subgroups are exactly the orthogonal subgroups $K_Q \cong O(n, \mathbb{R})$, defined by positive definite quadratic forms Q on E, and $K_Q = K_{Q'}$ iff Q and Q' are proportional, i.e. $Q' = \lambda Q$ for some $\lambda > 0$.

In this way the classical answer is that a $\mathbb{Z}_\infty$-structure on a finite dimensional real space E is just a positive definite quadratic form on E, and similarly, a $\bar{\mathbb{Z}}_\infty$-structure on a finite dimensional complex vector space is a positive definite hermitian form. This point of view, if developed further, explains why classical Arakelov geometry insists on equipping (complex points of) all varieties and vector bundles involved with hermitian metrics.

0.2.2. ($\mathbb{Z}_\infty$-structures on finite $\mathbb{R}$-algebras.) Now suppose that E is a finite $\mathbb{R}$-algebra. We would like to describe $\mathbb{Z}_\infty$-structures on this algebra, i.e. $\mathbb{Z}_\infty$-lattices $A \subset E$, compatible with the multiplication and unit of E. In the p-adic case this would actually mean $1 \in A$ and $A \cdot A \subset A$, but if we want to obtain "correct" definitions in the archimedean case, we must re-write these conditions for A in terms of corresponding maximal compact subgroup $K_A \subset G$.

And here a certain problem arises. These compatibility conditions cannot be easily expressed in terms of maximal compact subgroups of automorphism groups even in the p-adic case. However, if we consider maximal compact submonoids of endomorphism monoids instead, this problem disappears.

0.2.3. (Maximal compact submonoids: p-adic case.) Thus we are induced to describe $\mathbb{Z}_\infty$-lattices A in a finite-dimensional real space E in terms of maximal compact submonoids M_A of locally compact monoid $M := \mathrm{End}_{\mathbb{R}}(E) \cong M(n, \mathbb{R})$. When we study the corresponding p-adic problem, we see that all maximal compact submonoids of $\mathrm{End}_{\mathbb{Q}_p}(E)$ are of form $M_A := \{\varphi : \varphi(A) \subset A\} \cong \mathrm{End}_{\mathbb{Z}_p}(A) \cong M(n, \mathbb{Z}_p)$ for a $\mathbb{Z}_p$-lattice $A \subset E$, and that $M_A = M_{A'}$ iff A and A' are similar, i.e. in the p-adic case maximal compact submonoids of

End(E) are classified by similarity classes of lattices $A \subset E$, exactly in the same way as maximal compact subgroups of Aut(E).

0.2.4. (Maximal compact submonoids: archimedian case.) However, in the archimedian case there are much more maximal compact submonoids M_A inside $M := \mathrm{End}_{\mathbb{R}}(E)$ than positive definite quadratic forms. Namely, we can take any symmetric compact convex body $A \subset E$ (a convex *body* is always required to be absorbent, i.e. $E = \mathbb{R} \cdot A$), and put $M_A := \{\varphi \in \mathrm{End}_{\mathbb{R}}(E) : \varphi(A) \subset A\}$. Any such M_A is a maximal compact submonoid in $M = \mathrm{End}_{\mathbb{R}}(E)$, all maximal compact submonoids in M arise in this way, and $M_A = M_{A'}$ iff $A' = \lambda A$ for some $\lambda \in \mathbb{R}^\times$.

0.2.5. ($\mathbb{Z}_\infty$-lattices.) This leads us to define a *$\mathbb{Z}_\infty$-lattice* A in a finite-dimensional real space E as a symmetric compact convex body A. It is well-known that any such A is the unit ball with respect to some Banach norm $\|\cdot\|$ on E; in other words, *a $\mathbb{Z}_\infty$-structure on E is essentially a Banach norm on E.*

Next, we can define the *category $\mathbb{Z}_\infty$-Lat of $\mathbb{Z}_\infty$-lattices* as follows. Objects of $\mathbb{Z}_\infty$-*Lat* are couples $A = (A_{\mathbb{Z}_\infty}, A_{\mathbb{R}})$, where $A_{\mathbb{R}}$ is a finite-dimensional real vector space, and $A_{\mathbb{Z}_\infty} \subset A_{\mathbb{R}}$ is a symmetric compact convex body in $A_{\mathbb{R}}$. Morphisms $f : A \to B$ are couples $(f_{\mathbb{Z}_\infty}, f_{\mathbb{R}})$, where $f_{\mathbb{R}} : A_{\mathbb{R}} \to B_{\mathbb{R}}$ is an $\mathbb{R}$-linear map, and $f_{\mathbb{Z}_\infty} : A_{\mathbb{Z}_\infty} \to B_{\mathbb{Z}_\infty}$ is its restriction.

Of course, this definition is again motivated by the p-adic case. Notice, however, that in the p-adic case we might define the category of $\mathbb{Z}_p$-lattices without any reference to ambient $\mathbb{Q}_p$-vector spaces, while we are still not able to describe $\mathbb{Z}_\infty$-lattices without reference to a real vector space.

Another interesting observation is that $\mathbb{Z}_\infty$-*Lat* is essentially the category of finite-dimensional Banach vector spaces, with $\mathbb{R}$-linear maps of norm ≤ 1 as morphisms. While this description establishes a connection to Banach norms, we don't insist on using it too much, since the p-adic case suggests that we should concentrate our attention on $A_{\mathbb{Z}_\infty}$, not on ambient space $A_{\mathbb{R}}$.

0.2.6. ($\mathbb{Z}_\infty$-structures on finite $\mathbb{R}$-algebras.) Now we are able to define a $\mathbb{Z}_\infty$-structure A on a finite $\mathbb{R}$-algebra E, i.e. a $\mathbb{Z}_\infty$-lattice $A \subset E$, compatible with the multiplication and unit of E. The key idea here is to express this compatibility in terms of corresponding maximal compact submonoid $M_A \subset \mathrm{End}(E)$ first in the p-adic case, and then to transfer the conditions on M_A thus obtained to the archimedian case.

The final result is that we must consider symmetric compact convex bodies $A \subset E$, such that $1 \in \partial A$ and $A \cdot A \subset A$. In the language of Banach norms this means $\|1\| = 1$ and $\|xy\| \leq \|x\| \cdot \|y\|$, i.e. we recover the notion of a finite-dimensional real Banach algebra.

0.2.7. (From lattices to torsion-free modules.) Next step is to embed $\mathbb{Z}_\infty$-*Lat* into a larger category of *torsion-free* $\mathbb{Z}_\infty$*-modules* $\mathbb{Z}_\infty$-*Fl.Mod*. Comparing to the p-adic case, we see that $\mathbb{Z}_\infty$-*Fl.Mod* might be constructed as the category $\operatorname{Ind}(\mathbb{Z}_\infty\text{-}Lat)$ of ind-objects over $\mathbb{Z}_\infty$-*Lat* (cf. SGA 4 I). This category consists of "formal" inductive limits "$\varinjlim$" M_α of $\mathbb{Z}_\infty$-lattices, taken along filtered ordered sets or small categories, with morphisms given by $\operatorname{Hom}(\text{“}\varinjlim\text{”}\, M_\alpha, \text{“}\varinjlim\text{”}\, N_\beta) = \varprojlim_\alpha \varinjlim_\beta \operatorname{Hom}(M_\alpha, N_\beta)$.

0.2.8. (Direct description of $\mathbb{Z}_\infty$-*Fl.Mod*.) However, the category $\mathbb{Z}_\infty$-*Fl.Mod* of torsion-free $\mathbb{Z}_\infty$ admits a more direct description, similar to **0.2.5**. Namely, one can define it as the category of couples $(A_{\mathbb{Z}_\infty}, A_\mathbb{R})$, where $A_\mathbb{R}$ is a real vector space (not required more to be finite-dimensional), and $A_{\mathbb{Z}_\infty} \subset A_\mathbb{R}$ is a symmetric convex body (not required to be compact, but still required to be absorbent: $A_\mathbb{R} = \mathbb{R} \cdot A_{\mathbb{Z}_\infty}$). Morphisms are defined in the same way as for $\mathbb{Z}_\infty$-lattices, and it is immediate from this construction that $\mathbb{Z}_\infty$-*Lat* is a full subcategory of $\mathbb{Z}_\infty$-*Fl.Mod*.

0.2.9. (Inductive and projective limits in $\mathbb{Z}_\infty$-*Fl.Mod*.) We show that arbitrary inductive and projective limits exist in $\mathbb{Z}_\infty$-*Fl.Mod*. For example, product $A \times B$ equals $(A_{\mathbb{Z}_\infty} \times B_{\mathbb{Z}_\infty}, A_\mathbb{R} \times B_\mathbb{R})$, and the direct sum (i.e. coproduct) $A \oplus B$ can be computed as $(\operatorname{conv}(A_{\mathbb{Z}_\infty} \cup B_{\mathbb{Z}_\infty}), A_\mathbb{R} \oplus B_\mathbb{R})$, where $\operatorname{conv}(S)$ denotes the convex hull of a subset S in a real vector space.

0.2.10. (Tensor structure on $\mathbb{Z}_\infty$-*Fl.Mod*.) We define an ACU $\otimes$-structure on $\mathbb{Z}_\infty$-*Fl.Mod* and $\mathbb{Z}_\infty$-*Lat*, having the following property. An algebra (A, μ, ε) in $\mathbb{Z}_\infty$-*Lat* is the same thing as a couple $(A_{\mathbb{Z}_\infty}, A_\mathbb{R})$, consisting of a finite dimensional $\mathbb{R}$-algebra $A_\mathbb{R}$, and a $\mathbb{Z}_\infty$-lattice $A_{\mathbb{Z}_\infty} \subset A_\mathbb{R}$, compatible with the algebra structure of $A_\mathbb{R}$ in the sense of **0.2.6**.

This tensor structure is extremely natural in several other respects. For example, when translated into the language of (semi)norms on real vector spaces, it corresponds to Grothendieck's projective tensor product of (semi)norms.

The unit object for this tensor structure is $\mathbb{Z}_\infty := ([-1,1], \mathbb{R})$. One should think of this $\mathbb{Z}_\infty$ as "$\mathbb{Z}_\infty$, considered as a left module over itself", not as "the ring $\mathbb{Z}_\infty$".

0.2.11. (Underlying set of a torsion-free $\mathbb{Z}_\infty$-module.) Once we have a unit object $\mathbb{Z}_\infty$, we can define the "forgetful functor" $\Gamma : \mathbb{Z}_\infty\text{-}Fl.Mod \to Sets$, $A \mapsto \operatorname{Hom}(\mathbb{Z}_\infty, A)$. It is natural to say that $\Gamma(A)$ is the *underlying set of* A. Direct computation shows that $\Gamma(A) = A_{\mathbb{Z}_\infty}$ for $A = (A_{\mathbb{Z}_\infty}, A_\mathbb{R})$, thus suggesting that we ought to concentrate our attention on $A_{\mathbb{Z}_\infty}$, not on auxiliary vector space $A_\mathbb{R}$.

0.2.12. (Free $\mathbb{Z}_\infty$-modules. Octahedral combinations.) Since arbitrary direct sums exist in $\mathbb{Z}_\infty$-*Fl.Mod*, we can construct *free* $\mathbb{Z}_\infty$*-modules* $\mathbb{Z}_\infty^{(S)}$, by taking the direct sum of S copies of $\mathbb{Z}_\infty$. It is immediate that $L_{\mathbb{Z}_\infty} : S \mapsto \mathbb{Z}_\infty^{(S)}$ is a left adjoint to $\Gamma = \Gamma_{\mathbb{Z}_\infty} : \mathbb{Z}_\infty\text{-}\mathit{Fl.Mod} \to \mathit{Sets}$. One can describe this $\mathbb{Z}_\infty^{(S)} = (\Sigma_\infty(S), \mathbb{R}^{(S)})$ explicitly. Its vector space component is simply $\mathbb{R}$-vector space $\mathbb{R}^{(S)}$ freely generated by S. Its standard basis elements will be denoted by $\{s\}$, $s \in S$. Then the symmetric convex subset $\Sigma_\infty(S) \subset \mathbb{R}^{(S)}$ consists of all *octahedral (linear) combinations* of these basis vectors:

$$\Sigma_\infty(S) = \operatorname{conv}\bigl(\pm\{s\} \,:\, s \in S\bigr) \tag{0.2.12.1}$$

$$= \Bigl\{\sum_s \lambda_s\{s\} \,:\, \sum_s |\lambda_s| \leq 1,\ \text{almost all } \lambda_s = 0\Bigr\} \tag{0.2.12.2}$$

In other words, $\Sigma_\infty(S)$ is the *standard octahedron* in $\mathbb{R}^{(S)}$.

0.2.13. (Notation: finite sets and basis elements.) We would like to mention here some notation, used throughout this work. If M is a free "object", generated by a set S (e.g. $M = R^{(S)}$ for some ring R), we denote by $\{s\}$ the "basis element" of M corresponding to $s \in S$. Thus $s \mapsto \{s\}$ is the natural embedding $S \to M$.

Another notation: we denote by $\mathbf{n}$ the *standard finite set* $\{1, 2, \ldots, n\}$, where $n \geq 0$ is any integer. For example, $\mathbf{0} = \varnothing$, and $\mathbf{2} = \{1, 2\}$. Furthermore, whenever we have a functor Σ defined on the category of sets, we write $\Sigma(n)$ instead of $\Sigma(\mathbf{n})$, for any $n \geq 0$. In this way $\mathbb{R}^{(n)} = \mathbb{R}^n$ is the standard n-dimensional real vector space, with standard basis $\{k\}$, $1 \leq k \leq n$, and $\Sigma_\infty(n) \subset \mathbb{R}^n$ is the standard n-dimensional octahedron.

0.2.14. (From $\mathbb{Z}_\infty$-*Fl.Mod* to $\mathbb{Z}_\infty$-*Mod*.) Our next step is to recover the category $\mathbb{Z}_\infty$-*Mod* of all $\mathbb{Z}_\infty$-modules, starting from the category $\mathbb{Z}_\infty$-*Fl.Mod* of torsion-free $\mathbb{Z}_\infty$-modules. We use adjoint functors $L_{\mathbb{Z}_\infty} : \mathit{Sets} \leftrightarrows \mathbb{Z}_\infty\text{-}\mathit{Fl.Mod} : \Gamma$ for this. Namely, we observe that this pair of adjoint functors defines a *monad* structure $(\Sigma_\infty, \mu, \varepsilon)$ on endofunctor $\Sigma_\infty := \Gamma L_{\mathbb{Z}_\infty} : \mathit{Sets} \to \mathit{Sets}$. However, functor Γ happens not to be monadic, i.e. the arising "comparison functor" $I : \mathbb{Z}_\infty\text{-}\mathit{Fl.Mod} \to \mathit{Sets}^{\Sigma_\infty}$ from $\mathbb{Z}_\infty$-*Fl.Mod* into the category of Σ_∞-algebras in *Sets* (which we prefer to call Σ_∞*-modules*) is not an equivalence of categories, but just a fully faithful functor.

The p-adic analogy suggests to define $\mathbb{Z}_\infty\text{-}\mathit{Mod} := \mathit{Sets}^{\Sigma_\infty}$. Then category $\mathbb{Z}_\infty$-*Fl.Mod* can be identified with a full subcategory of $\mathbb{Z}_\infty$-*Mod* with the aid of functor I, and the forgetful functor $\Gamma_{\mathbb{Z}_\infty} : \mathbb{Z}_\infty\text{-}\mathit{Mod} \to \mathit{Sets}$ is now monadic by construction, similarly to the forgetful functor on any category defined by an algebraic system (e.g. category R-*Mod* of modules over an associative ring R).

0.2.15. (Explicit description of $\mathbb{Z}_\infty$-modules.) We can obtain a more explicit description of $\mathbb{Z}_\infty$-modules, i.e. of objects $M = (M, \alpha) \in \operatorname{Ob} \mathbb{Z}_\infty\text{-}\mathit{Mod} = \operatorname{Ob} \mathit{Sets}^{\Sigma_\infty}$. Indeed, by definition $M = (M, \alpha)$ consists of a set M, together with a "$\mathbb{Z}_\infty$-structure", i.e. a map $\alpha : \Sigma_\infty(M) \to M$, subject to certain conditions. Since $\Sigma_\infty(M)$ consists of formal octahedral combinations $\sum_x \lambda_x\{x\}$, $\sum_x |\lambda_x| \leq 1$, of elements of M, such a map α should be thought of as a way of evaluating such formal octahedral combinations. Thus we write $\sum_x \lambda_x x$ instead of $\alpha(\sum_x \{x\})$. We use this notation for finite sums as well, e.g. $\lambda x + \mu y$ actually means $\alpha(\lambda\{x\} + \mu\{y\})$, for any x, $y \in M$, λ, $\mu \in \mathbb{R}$, $|\lambda| + |\mu| \leq 1$. Notice, however, that the $+$ sign in expression $\lambda x + \mu y$ is completely formal: in general we don't get any addition operation on M.

As to the conditions for α, they are exactly all the usual relations satisfied by octahedral combinations of elements of real vector spaces. For example, $\nu(\lambda x + \mu y) = \nu\lambda \cdot x + \nu\mu \cdot y$.

In this way we see that a $\mathbb{Z}_\infty$-module is nothing else than a set M, together with a way of evaluating octahedral combinations of its elements, in such a way that all usual relations hold in M.

0.2.16. (Torsion modules.) Category $\mathbb{Z}_\infty\text{-}\mathit{Mod}$ is strictly larger than the category $\mathbb{Z}_\infty\text{-}\mathit{Fl.Mod}$ of torsion-free $\mathbb{Z}_\infty$-modules. In fact, it contains objects like $\mathbb{F}_\infty := \operatorname{Coker}(\mathfrak{m}_\infty \rightrightarrows \mathbb{Z}_\infty)$, where $\mathfrak{m}_\infty = ((-1, 1), \mathbb{R})$ in $\mathbb{Z}_\infty\text{-}\mathit{Fl.Mod}$, and the two morphisms $\mathfrak{m}_\infty \to \mathbb{Z}_\infty$ are the natural inclusion and the zero morphism. Notice that this cokernel is zero in $\mathbb{Z}_\infty\text{-}\mathit{Fl.Mod}$, but not in $\mathbb{Z}_\infty\text{-}\mathit{Mod}$: in fact, this $\mathbb{F}_\infty$ is a three-element set with a certain $\mathbb{Z}_\infty$-module structure. This gives an example of a non-trivial torsion $\mathbb{Z}_\infty$-module.

0.2.17. (Arakelov affine line.) Among other things discussed in Chapter **2**, we would like to mention the study of "Arakelov affine line" $\operatorname{Spec} \mathbb{Z}_\infty[T]$, carried both from the point of view of (co)metrics and from that of (generalized) schemes.

When we compute the naturally arising cometric on $\mathbb{A}^1_\mathbb{R}$ or $\mathbb{A}^1_\mathbb{C}$, coming from this $\mathbb{Z}_\infty$-structure $\mathbb{Z}_\infty[T]$ on $\mathbb{R}[T]$, it turns out to be identically zero outside unit disk $\{\lambda : |\lambda| < 1\}$, i.e. all points outside this disk are at infinite distance from each other. Inside the disk we obtain a continuous piecewise smooth cometric, very similar to Poincaré model of hyperbolic plane in the unit disk.

On the other hand, we can use the $\otimes$-structure defined on $\mathbb{Z}_\infty\text{-}\mathit{Fl.Mod}$ (and actually on all of $\mathbb{Z}_\infty\text{-}\mathit{Mod}$) to define $\mathbb{Z}_\infty$-algebras A, modules over them, ideals and prime ideals inside them and so on, thus obtaining a definition of prime spectrum $\operatorname{Spec} A$. For example, topological space $\operatorname{Spec} \mathbb{Z}_\infty$ looks like the spectrum of a DVR. When we study the "Arakelov affine line" $\operatorname{Spec} \mathbb{Z}_\infty[T]$ from this point of view, we observe some unexpected phenomena, e.g. Krull dimension of this topological space turns out to be infinite.

0.3. (⊗-categories, algebras and monads.) Chapter **3** collects some general definitions and constructions, related to ⊗-categories, algebras and monads. It is quite technical, but nevertheless quite important for the next two chapters. Most results collected here are well-known and can be found in [MacLane]; however, we want to fix the terminology, and discuss the generalization to the topos case.

0.3.1. (AU ⊗-categories and ⊗-actions.) We discuss AU ⊗-categories, i.e. categories $\mathcal{A}$, equipped with a tensor product functor $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and a unit object $\mathbf{1} \in \operatorname{Ob}\mathcal{A}$, together with some *associativity constraint* $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$ and *unit constraints* $\mathbf{1} \otimes X \cong X \cong X \otimes \mathbf{1}$, subject to certain axioms (e.g. the pentagon axiom). Roughly speaking, these axioms ensure that multiple tensor products $X_1 \otimes X_2 \otimes \cdots \otimes X_n$ are well-defined and have all the usual properties of multiple tensor products, apart from commutativity.

After that we discuss *external (left) ⊗-actions* $\oslash : \mathcal{A} \times \mathcal{B} \to \mathcal{B}$ of an AU ⊗-category $\mathcal{A}$ on a category $\mathcal{B}$. Here we have to impose some external associativity and unit constraints $(X \otimes Y) \oslash M \cong X \oslash (Y \oslash M)$ and $\mathbf{1} \oslash M \cong M$, subject to similar relations.

Of course, any AU ⊗-category $\mathcal{A} = (\mathcal{A}, \otimes)$ admits a natural ⊗-action on itself, given by $\oslash := \otimes$.

0.3.2. (Algebras and modules.) Whenever we have an AU ⊗-category $\mathcal{A}$, we can consider *algebras* $A = (A, \mu, \varepsilon)$, $A \in \operatorname{Ob}\mathcal{A}$, $\mu : A \otimes A \to A$, $\varepsilon : \mathbf{1} \to A$, always supposed to be associative with unity (but not commutative – in fact, commutativity doesn't make sense without a commutativity constraint on $\mathcal{A}$). Thus we obtain a *category of algebras in* $\mathcal{A}$, denoted by $Alg(\mathcal{A})$.

Next, if $\oslash : \mathcal{A} \times \mathcal{B} \to \mathcal{B}$ is an external ⊗-action, and A is an algebra in $\mathcal{A}$, we denote by A-*Mod* or $\mathcal{B}^A$ the category of A-*modules in* $\mathcal{B}$, consisting of couples (M, α), $M \in \operatorname{Ob}\mathcal{B}$, $\alpha : A \oslash M \to M$, subject to classical relations $\alpha \circ (\varepsilon \oslash 1_M) = 1_M$ and $\alpha \circ (1_A \oslash \alpha) = \alpha \circ (\mu \oslash 1_M)$.

0.3.3. (Monads over a category $\mathcal{C}$.) Now if $\mathcal{C}$ is an arbitrary category, the category of endofunctors $\mathcal{A} := \mathit{Endof}(\mathcal{C}) = \mathit{Funct}(\mathcal{C}, \mathcal{C})$ admits a natural AU ⊗-structure, given by composition of functors: $F \otimes G := F \circ G$. Furthermore, there is a natural ⊗-action of $\mathcal{A}$ on $\mathcal{C}$, defined by $F \oslash X := F(X)$.

Then a *monad* $\Sigma = (\Sigma, \mu, \varepsilon)$ over a category $\mathcal{C}$ is simply an algebra in this category of endofunctors $\mathcal{A}$, i.e. $\Sigma : \mathcal{C} \to \mathcal{C}$ is an endofunctor and $\mu : \Sigma^2 \to \Sigma$, $\varepsilon : \mathrm{Id}_{\mathcal{C}} \to \Sigma$ are natural transformations, subject to associativity and unit relations: $\mu \circ (\Sigma \star \mu) = \mu \circ (\mu \star \Sigma)$ and $\mu \circ (\varepsilon \star \Sigma) = \mathrm{id}_\Sigma = \mu \circ (\Sigma \star \varepsilon)$, or equivalently, $\mu_X \circ \Sigma(\mu_X) = \mu_X \circ \mu_{\Sigma(X)} : \Sigma^3(X) \to \Sigma(X)$, and $\mu_X \circ \varepsilon_{\Sigma(X)} =$

$\mathrm{id}_{\Sigma(X)} = \mu_X \circ \Sigma(\varepsilon_X)$, for any $X \in \mathrm{Ob}\,\mathcal{C}$.

Similarly, the category $\mathcal{C}^\Sigma$ of Σ-modules (in $\mathcal{C}$) is defined as the category of Σ-modules in $\mathcal{C}$ with respect to the external $\otimes$-action of $\mathcal{A}$ on $\mathcal{C}$ just discussed. In this way $\mathcal{C}^\Sigma$ consists of couples $M = (M, \alpha)$, with $M \in \mathrm{Ob}\,\mathcal{C}$, $\alpha : \Sigma(M) \to M$, such that $\alpha\circ\Sigma(\alpha) = \alpha\circ\mu_M : \Sigma^2(M) \to M$, and $\alpha\circ\varepsilon_M = \mathrm{id}_M$.

0.3.4. (Monads and adjoint functors.) Whenever we have a monad Σ over a category $\mathcal{C}$, we get a forgetful functor $\Gamma_\Sigma : \mathcal{C}^\Sigma \to \mathcal{C}$, $(M, \alpha) \mapsto M$, which admits a left adjoint $L_\Sigma : \mathcal{C} \to \mathcal{C}^\Sigma$, $S \mapsto (\Sigma(S), \mu_S)$, such that $\Gamma_\Sigma \circ L_\Sigma = \Sigma$. Conversely, given any two adjoint functors $L : \mathcal{C} \leftrightarrows \mathcal{D} : \Gamma$, we obtain a canonical monad structure on $\Sigma := \Gamma \circ L : \mathcal{C} \to \mathcal{C}$, together with a "comparison functor" $I : \mathcal{D} \to \mathcal{C}^\Sigma$, such that $\Gamma = \Gamma_\Sigma \circ I$. Functor Γ is said to be *monadic* if I is an equivalence of categories.

0.3.5. (Examples: R-modules, $\mathbb{Z}_\infty$-modules…) For example, if $\mathcal{C} = \mathit{Sets}$, $\mathcal{D} = R\text{-}\mathit{Mod}$ is the category of left modules over a ring R (always supposed to be associative with unity), then the forgetful functor $\Gamma : R\text{-}\mathit{Mod} \to \mathit{Sets}$ turns out to be monadic. The corresponding monad $\Sigma_R : \mathit{Sets} \to \mathit{Sets}$ transforms a set S into the underlying set of free R-module $R^{(S)}$ generated by S, i.e. into the set of all formal R-linear combinations of basis elements $\{s\}$, $s \in S$. In this way an R-module M is just a set M together with a method $\alpha : \Sigma_R(M) \to M$ of evaluating formal R-linear combinations of its elements, subject to some natural conditions.

Similarly, $\mathbb{Z}_\infty\text{-}\mathit{Mod}$ was defined to be $\mathit{Sets}^{\Sigma_\infty}$ for a certain monad Σ_∞ over Sets, so the set $\Sigma_\infty(S)$ of formal octahedral combinations of elements of S should be thought of as the set of all formal $\mathbb{Z}_\infty$-linear combinations of elements of S, or as the underlying set of free $\mathbb{Z}_\infty$-module $\mathbb{Z}_\infty^{(S)}$. This was the way we've defined Σ_∞ in the first place.

This analogy between $R\text{-}\mathit{Mod} = \mathit{Sets}^{\Sigma_R}$ and $\mathbb{Z}_\infty\text{-}\mathit{Mod} = \mathit{Sets}^{\Sigma_\infty}$ suggests that our category of generalized rings, which is expected to contain all classical rings R as well as exotic objects like $\mathbb{Z}_\infty$, might be constructed as a certain full subcategory of the category of monads over Sets. Then we might put $\Sigma\text{-}\mathit{Mod} := \mathit{Sets}^\Sigma$ for any monad Σ from this full subcategory, thus obtaining a reasonable definition of Σ-modules without any special considerations.

0.3.6. (Topos case: inner endofunctors and inner monads.) Apart from the things just discussed, Chapter **3** contains some technical definitions and statements, related to *inner endofunctors* and *inner monads* over a topos $\mathcal{E}$. These notions are used later to transfer the definitions of generalized rings and modules over them from the category of sets into arbitrary topoi, e.g. categories of sheaves of sets over a topological space X. This is necessary to obtain a monadic interpretation of sheaves of generalized rings and of modules over them. Without such notions we wouldn't be able to discuss generalized ringed spaces and in particular generalized schemes. However, we suggest to the reader to skip these technical topos-related pages during the first reading.

0.4. (Algebraic monads as non-commutative generalized rings.) Chapter **4** is dedicated to the study of *algebraic* endofunctors and monads over *Sets* (and algebraic *inner* endofunctors and monads over a topos $\mathcal{E}$ as well). This notion of *algebraicity* is actually the first condition we need to impose on monads over *Sets* in order to define the category of generalized rings. In fact, *algebraic monads are non-commutative generalized rings.* For example, any monad Σ_R defined by a classical ring R (as usual, associative with unity) is algebraic, as well as monad Σ_∞, used to define $\mathbb{Z}_\infty$-*Mod*.

0.4.1. (Algebraic monads vs. algebraic systems.) Another important remark is that *an algebraic monad over Sets is essentially the same thing as an algebraic system.* We discuss this correspondence in more detail below. In some sense algebraic systems are something like presentations (by a system of generators and relations) of algebraic monads. Thus different (but equivalent) algebraic systems may correspond to isomorphic algebraic monads, and algebraic monads provide an invariant way of describing algebraic systems. In this way the study of algebraic monads might be thought of as the study of algebraic systems from a categorical point of view, i.e. a categorical approach to *universal algebra*.

0.4.2. (Algebraic endofunctors over *Sets*.) We say that an endofunctor $\Sigma : \mathit{Sets} \to \mathit{Sets}$ is *algebraic* if it commutes with filtered inductive limits: $\Sigma(\varinjlim_\alpha S_\alpha) \cong \varinjlim_\alpha \Sigma(S_\alpha)$. Since any set is a filtered inductive limit of its finite subsets, we obtain $\Sigma(S) = \varinjlim_{(\mathbf{n}\xrightarrow{\varphi}S)\in\underline{\mathbb{N}}_{/S}} \Sigma(n)$, where $\underline{\mathbb{N}} = \{\mathbf{0}, \mathbf{1}, \ldots, \mathbf{n}, \ldots\}_{n\geq 0}$ denotes the category of *standard finite sets*, considered as a full subcategory of *Sets*. Therefore, *any algebraic endofunctor* $\Sigma : \mathit{Sets} \to \mathit{Sets}$ *is completely determined by its restriction* $\Sigma|_{\underline{\mathbb{N}}} : \underline{\mathbb{N}} \to \mathit{Sets}$. In fact, this restriction functor $\Sigma \mapsto \Sigma|_{\underline{\mathbb{N}}}$ induces an equivalence between the category of algebraic endofunctors $\mathcal{A}_{alg} \subset \mathcal{A} = \mathit{Endof}(\mathit{Sets})$ and $\mathit{Funct}(\underline{\mathbb{N}}, \mathit{Sets}) = \mathit{Sets}^{\underline{\mathbb{N}}}$. This means that *an algebraic endofunctor* Σ *is essentially the same thing as a countable collection of sets* $\{\Sigma(n)\}_{n\geq 0}$*, together with maps* $\Sigma(\varphi) : \Sigma(n) \to \Sigma(m)$*, defined for any* $\varphi : \mathbf{n} \to \mathbf{m}$*, such that* $\Sigma(\psi \circ \varphi) = \Sigma(\psi) \circ \Sigma(\varphi)$ *and* $\Sigma(\mathrm{id}_\mathbf{n}) = \mathrm{id}_{\Sigma(n)}$.

0.4.3. (Algebraic monads.) On the other hand, if two endofunctors Σ and Σ' commute with filtered inductive limits, the same is true for their composition $\Sigma \otimes \Sigma' = \Sigma \circ \Sigma'$, i.e. $\mathcal{A}_{alg} \cong \mathit{Sets}^{\underline{\mathbb{N}}}$ *is a full* $\otimes$*-subcategory of* $\mathcal{A} = \mathit{Endof}(\mathit{Sets})$. Therefore, we can define an *algebraic monad* $\Sigma = (\Sigma, \mu, \varepsilon)$ as an algebra in $\mathcal{A}_{alg}$. Of course, an algebraic monad is just a monad, such that its underlying endofunctor commutes with filtered inductive limits.

0.4.4. (Why algebraic?) Notice that monads Σ_R defined by an associative ring R are algebraic, i.e. $\Sigma_R(S) = \bigcup_{\text{finite } I \subset S} \Sigma_R(I)$, just because any element of $\Sigma_R(S) = R^{(S)}$, i.e. any formal R-linear combination of elements of S, involves only finitely many elements of S, hence comes from $\Sigma_R(I)$ for some finite subset $I \subset S$. Similarly, Σ_∞ is algebraic, because any octahedral combination $\sum_{s\in S} \lambda_s\{s\} \in \Sigma_\infty(S)$ has only finitely many $\lambda_s \neq 0$.

We can express this by saying that we consider only "operations" depending on finitely many arguments. For example, we might remove the requirement "$\lambda_s \neq 0$ only for finitely many $s \in S$" and consider "infinite octahedral combinations" $\sum \lambda_s\{s\}$, with the only requirement $\sum_s |\lambda_s| \leq 1$. In this way we obtain a larger monad $\hat{\Sigma}_\infty \supset \Sigma_\infty$, which coincides with Σ_∞ on finite sets, but is different on larger sets. A $\hat{\Sigma}_\infty$-structure on a set M is essentially a way of computing such "infinite octahedral combinations" of elements of M. This is definitely not an algebraic operation, and $\hat{\Sigma}_\infty$ is not an algebraic monad.

Therefore, word *"algebraic"* means here something like "expressible in terms of operations involving only finitely many arguments".

0.4.5. (Algebraic monads and operations.) Now let Σ be an algebraic endofunctor (over *Sets*), M be a set, $\alpha : \Sigma(M) \to M$ be any map of sets. For example, we might take an algebraic monad Σ and a Σ-module M.

Let $t \in \Sigma(n)$ for some integer n. Take any n-tuple $x = (x_1, \ldots, x_n) \in M^n$ of elements of M. It can be considered as a map $\tilde{x} : \mathbf{n} \to M$, $k \mapsto x_k$, hence we get a map $\Sigma(\tilde{x}) : \Sigma(n) \to \Sigma(M)$. Now we can apply $\alpha \circ \Sigma(\tilde{x})$ to t, thus obtaining an element of M:

$$t(x_1, \ldots, x_n) = [t]_M(x_1, \ldots, x_n) := (\alpha \circ \Sigma(\tilde{x}))(t) \tag{0.4.5.1}$$

In this way any $t \in \Sigma(n)$ defines an n-ary operation $[t]_M : M^n \to M$ on M. That's why we say that $\Sigma(n)$ is *the set of n-ary operations of* Σ, and call its elements *n-ary operations of* Σ. Furthermore, we say that $[t]_M : M^n \to M$ is the *value* of operation t on M. Of course, when $n = 0$, we speak about *constants* and their values (any constant $c \in \Sigma(0)$ has a value $[c]_M \in M$), and when $n = 1, 2, 3, \ldots$ we obtain *unary, binary, ternary, … operations*.

One can show that giving a map $\alpha : \Sigma(M) \to M$ is actually *equivalent* to giving a family of "evaluation maps" $\alpha^{(n)} : \Sigma(n) \times M^n \to M$, $(t, x_1, \ldots, x_n) \mapsto [t]_M(x_1, \ldots, x_n)$, $n \geq 0$, satisfying natural compatibility relations, which can be written as

$$[\varphi_*(t)]_M(x_1, \ldots, x_n) = [t]_M(x_{\varphi(1)}, \ldots, x_{\varphi(m)}), \quad \forall t \in \Sigma(m), \varphi : \mathbf{m} \to \mathbf{n} \tag{0.4.5.2}$$

Here φ_* is a shorthand for $\Sigma(\varphi)$.

0.4.6. (Elementary description of algebraic monads.) The above description is applicable to maps $\mu_n : \Sigma(\Sigma(n)) \to \Sigma(n)$. We see that such a map is completely determined by a sequence of "evaluation" or "substitution" maps $\mu_n^{(k)} : \Sigma(k) \times \Sigma(n)^k \to \Sigma(n)$, $(t, t_1, \ldots, t_k) \mapsto t(t_1, \ldots, t_k) = [t]_{\Sigma(n)}(t_1, \ldots, t_k)$. In this way we obtain an "elementary" description of an algebraic monad Σ, consisting of a sequence of sets $\{\Sigma(n)\}_{n \geq 0}$, transition maps $\Sigma(\varphi) : \Sigma(m) \to \Sigma(n)$, defined for all $\varphi : \mathbf{m} \to \mathbf{n}$, evaluation maps $\mu_n^{(k)} : \Sigma(k) \times \Sigma(n)^k \to \Sigma(n)$, and a "unit element" $\boldsymbol{e} := \varepsilon_{\mathbf{1}}(1) \in \Sigma(1)$.

In fact, this collection completely determines the algebraic monad Σ, and conversely, if we start from such a collection, satisfying some natural compatibility conditions (certain "associativity" and "unit" axioms), we obtain a uniquely determined algebraic monad Σ.

Of course, Σ-modules $M = (M, \alpha)$ also admit such an "elementary" description, consisting of a set M, and a collection of "evaluation maps" $\alpha^{(n)} : \Sigma(n) \times M^n \to M$, subject to certain compatibility, associativity and unit conditions (e.g. $[\boldsymbol{e}]_M = \mathrm{id}_M$).

Therefore, we might have defined algebraic monads and modules over them in such an "elementary" fashion. We didn't do this just because arising definitions and especially relations seem to be quite complicated and not too enlightening, unless one knows that they come from the definition of algebraic monads, i.e. they are the algebra relations in $\mathcal{A}_{alg}$, written in explicit form.

0.4.7. (Unit and basis elements.) Notice that the unit element $\boldsymbol{e} = \varepsilon_1(1) \in \Sigma(1)$ actually completely determines the unit $\varepsilon : \mathrm{Id}_{Sets} \to \Sigma$, since $\varepsilon_X(x) = (\Sigma(i_x))(\boldsymbol{e})$ for any $x \in X$, where $i_x : \mathbf{1} \to X$ is the map $1 \mapsto x$.

Recall that we denote by $\{k\}_{\mathbf{n}}$ or simply by $\{k\}$ the "basis element" $\varepsilon_n(k) \in \Sigma(n)$, $1 \leq k \leq n$. For example, $\boldsymbol{e} = \{1\}_{\mathbf{1}}$. These basis elements have some natural properties, e.g. $\varphi_*\{k\}_{\mathbf{m}} = \{\varphi(k)\}_{\mathbf{n}}$, for any $\varphi : \mathbf{m} \to \mathbf{n}$, $1 \leq k \leq m$. More interesting properties come from the unit axioms for Σ and Σ-modules. For example, $\mu \circ (\Sigma \star \varepsilon) = \mathrm{id}$ implies (and actually is equivalent to)

$$t(\{1\}, \{2\}, \ldots, \{n\}) = [t]_{\Sigma(n)}(\{1\}_{\mathbf{n}}, \ldots, \{n\}_{\mathbf{n}}) = t, \quad \forall t \in \Sigma(n). \qquad (0.4.7.1)$$

The other unit axiom $\mu \circ (\varepsilon \star \Sigma) = \mathrm{id}$ is actually equivalent to $\boldsymbol{e}(t) = t$ for all $n \geq 0$, $t \in \Sigma(n)$.

Unit axiom for a Σ-module M is equivalent to $[\{k\}_{\mathbf{n}}]_M = \mathrm{pr}_k : M^n \to M$, or just to $[\boldsymbol{e}]_M = \mathrm{id}_M$.

0.4.8. (Special notation for unary operations.) If $u \in \Sigma(1)$ is a unary operation, and $t \in \Sigma(n)$ is an n-ary operation, we usually write ut or $u \cdot t$ instead of $[u]_{\Sigma(n)}(t) \in \Sigma(n)$, and similarly $ux := [u]_M(x)$ for any Σ-module M and any $x \in M$. This notation is unambiguous because of associativity relations $(ut)(x_1, \ldots, x_n) = u \cdot t(x_1, \ldots, x_n)$ for any $u \in \Sigma(1)$, $t \in \Sigma(n)$, $x_i \in M$ or $\Sigma(m)$.

In this way we obtain on set $|\Sigma| := \Sigma(1)$ a monoid structure with identity $\boldsymbol{e}$, and a monoid action of $|\Sigma|$ on the underlying set of any Σ-module M.

0.4.9. (Special notation for binary operations.) Of course, when we have a binary operation, denoted by a sign like $+$, $*$, $\ldots$, usually written in infix form, we write $x + y$, $x * y$ etc. instead of $[+](x, y)$, $[*](x, y)$ etc.

Since $t = t(\{1\}, \ldots, \{n\})$ for any $t \in \Sigma(n)$, we can write $[+] = \{1\} + \{2\} \in \Sigma(2)$ when we want to point out the corresponding element of $\Sigma(2)$.

0.4.10. (Free modules.) Notice that $L_\Sigma(n) = (\Sigma(n), \mu_n)$ is the free Σ-module of rank n, i.e. $\mathrm{Hom}_\Sigma(L_\Sigma(n), M) \cong M^n$ for any Σ-module M. Of course, the map $\mathrm{Hom}_\Sigma(L_\Sigma(n), M) \to M^n$ is given by evaluating $f : L_\Sigma(n) \to M$ on basis elements $\{k\}_{\mathbf{n}}$. We also denote $L_\Sigma(n)$ simply by $\Sigma(n)$, when no confusion can arise.

0.4.11. (Set of unary operations.) For any algebraic monad Σ the set $|\Sigma| := \Sigma(1)$ has two natural structures: that of a monoid, and that of a Σ-module (free of rank one). If $\Sigma = \Sigma_R$ for a classical associative ring R, then $|\Sigma|$ is the underlying set of R, its monoid structure is given by the multiplication of R, and its module structure is the natural left R-module structure on R.

In this way $|\Sigma|$ plays the role of the underlying set of algebraic monad Σ. Notice that it is always a monoid under multiplication, but in general it doesn't have any addition, i.e. *multiplication is in some sense more fundamental than addition.*

Another interesting observation is that, while the addition of a classical ring R indeed corresponds to a binary operation $[+] = (1, 1) \in \Sigma_R(2) = R^2$, the multiplication of R doesn't correspond to any element of $\Sigma_R(n)$. Instead, it is built in the "composition maps" $\mu_n^{(k)}$, i.e. it is part of the more fundamental structure of algebraic monad.

0.4.12. (Matrix description. Comparison to Shai Haran's approach.) Since $\mathrm{Hom}_\Sigma(\Sigma(n), \Sigma(m)) \cong \Sigma(m)^n$, we *define* the set of $m \times n$-matrices over Σ by $M(m, n; \Sigma) := \Sigma(m)^n$. Composition of morphisms defines maps $M(n, k; \Sigma) \times M(k, m; \Sigma) \to M(n, m; \Sigma)$, i.e. we obtain a well-defined *product of matrices.* Putting here $m = 1$, we get maps $\Sigma(n)^k \times \Sigma(k) \to \Sigma(n)$, which are nothing else than the "structural maps" $\mu_n^{(k)}$ of Σ, up to a permutation of the two arguments. This matrix language is often quite convenient. For example, the "associativity relations" for maps $\mu_n^{(k)}$ are essentially equivalent to associativity of matrix products, and the "unit relations" actually mean that the identity matrix $I_n := (\{1\}_{\mathbf{n}}, \ldots, \{n\}_{\mathbf{n}}) \in M(n,n;\Sigma)$ is indeed left and right identity with respect to matrix multiplication.

One might actually try to describe generalized rings in terms of collections of "matrix sets" $\{M(m,n;\Sigma)\}_{m,n\geq 0}$, together with composition (i.e. matrix multiplication) maps as above, and direct sum maps $M(m,n;\Sigma) \times M(m',n';\Sigma) \to M(m+m',n+n';\Sigma)$, $(u,v) \mapsto u \oplus v$. Pursuing this path one essentially recovers Shai Haran's notion of an $\mathbb{F}$-*algebra* (or rather its non-commutative counterpart, since we don't say anything about tensor products at this point), defined in [ShaiHaran]. Furthermore, this notion of $\mathbb{F}$-algebra is more general than our notion of generalized ring, since Shai Haran never requires $M(m,n;\Sigma) = M(1,n;\Sigma)^m$.

In fact, once we impose this condition (which is quite natural if we want to have $\Sigma(m) \oplus \Sigma(n) = \Sigma(m+n)$ for directs sums, i.e. coproducts of free modules) on (non-commutative) $\mathbb{F}$-algebras, we recover our notion of generalized ring. However, Shai Haran doesn't impose such restrictions. Actually, he doesn't define a module over an $\mathbb{F}$-algebra A as a set Q with some additional structure (in fact, any algebraic structure on a set is described by some algebraic monad, so our theory of generalized rings is the largest algebraic theory of ring-like objects, which admit a notion of module over them), but considers infinite collections $\{M(m,n;Q)\}_{m,n\geq 0}$ instead, thought of as "$m \times n$-matrices with entries in Q". This more complicated notion of module roughly corresponds to our "Σ-bimodules in $\mathcal{A}_{alg}$", i.e. to algebraic endofunctors, equipped with compatible left and right actions of Σ.

It is not clear whether one might transfer more sophisticated constructions of our work to Shai Haran's more general case, since these constructions involve forgetful functors $\Gamma : \Sigma\text{-}\mathit{Mod} \to \mathit{Sets}$, i.e. expect Σ-modules to be sets with additional structure, and heavily rely on formulas like $\Sigma(m) \oplus \Sigma(n) = \Sigma(m+n)$ for categorial coproducts.

Unfortunately, we cannot say much more about this right now, since we haven't found any publications of Shai Haran on this topic apart from his original preprint [ShaiHaran], which deals only with the basic definitions of his theory.

0.4.13. (Initial and final algebraic monads.) Notice that the category of algebraic monads has an initial object $\mathbb{F}_\varnothing$, given by the only monad structure on $\mathrm{Id}_{\mathit{Sets}}$, as well as a final object $\mathbf{1}$, given by the constant functor with value $\mathbf{1}$, equipped with its only monad structure.

This initial algebraic monad $\mathbb{F}_\varnothing$ has the property $\mathbb{F}_\varnothing(n) = \mathbf{n}$, i.e. a free $\mathbb{F}_\varnothing$-module of rank n consists only of basis elements $\{k\}_{\mathbf{n}}$, $1 \leq k \leq n$. Furthermore, any set admits a unique $\mathbb{F}_\varnothing$-module structure, i.e. $\mathbb{F}_\varnothing\text{-}\mathit{Mod} = \mathit{Sets}$.

As to $\mathbf{1}$-*Mod*, the only sets which admit a $\mathbf{1}$-module structure are the one-element sets, hence $\mathbf{1}$-*Mod* is equivalent to the "final category" $\mathbf{1}$.

0.4.14. (Projective limits of algebraic monads. Submonads.) Projective limits of algebraic monads can be computed componentwise: $(\varprojlim_\alpha \Sigma_\alpha)(n) = \varprojlim_\alpha \Sigma_\alpha(n)$, for any $n \geq 0$. For example, $(\Sigma \times \Sigma')(n) = \Sigma(n) \times \Sigma'(n)$. An immediate consequence is that an algebraic monad homomorphism $f : \Sigma' \to \Sigma$ is a *monomorphism* iff all maps $f_n : \Sigma'(n) \to \Sigma(n)$ are injective. (Notice that injectivity of $f_1 = |f| : |\Sigma'| \to |\Sigma|$ does not suffice.) Therefore, Σ' is an *(algebraic) submonad* of Σ iff $\Sigma'(n) \subset \Sigma(n)$ for all $n \geq 0$, $\boldsymbol{e}_{\Sigma'} = \boldsymbol{e}_\Sigma$, and the "composition maps" $\mu_n^{(k)}$ of Σ' are restrictions of those of Σ.

We can easily compute intersections of algebraic submonads inside an algebraic monad. For example, we can intersect $\mathbb{Z}_\infty \subset \mathbb{R}$ and $\mathbb{Q} \subset \mathbb{R}$ (here we denote by $\mathbb{Z}_\infty$ the algebraic monad previously denoted by Σ_∞, and identify classical associative rings R with corresponding algebraic monads Σ_R), thus obtaining the "non-completed local ring at ∞":

$$\mathbb{Z}_{(\infty)} := \mathbb{Z}_\infty \cap \mathbb{Q} \tag{0.4.14.1}$$

0.4.15. (Classical associative rings as algebraic monads.) In fact, the functor $R \mapsto \Sigma_R$, transforming a classical ring into corresponding algebraic monad, is fully faithful, and we have R-*Mod* $= \Sigma_R$-*Mod*. Therefore, we can safely identify R with Σ_R and treat classical associative rings as algebraic monads, i.e. (non-commutative) generalized rings.

0.4.16. (Underlying set of an algebraic monad.) We denote by $\|\Sigma\|$ the "total" or "graded" underlying set of an algebraic monad Σ, given by

$$\|\Sigma\| := \bigsqcup_{n \geq 0} \Sigma(n) \tag{0.4.16.1}$$

This is a $\mathbb{N}_0$-graded set, i.e. a set together with a fixed decomposition into a disjoint union indexed by $\mathbb{N}_0$, or equivalently, a set $\|\Sigma\|$ together with a "degree map" $r : \|\Sigma\| \to \mathbb{N}_0$. In our case we say that r is the *arity map* of Σ.

0.4.17. (Free algebraic monads.) The underlying set functor $\Sigma \mapsto \|\Sigma\|$ from the category of algebraic monads into the category $\mathit{Sets}_{/\mathbb{N}_0}$ of $\mathbb{N}_0$-graded sets admits a left adjoint, called the *free algebraic monad functor* and denoted by $S \mapsto \langle S \rangle$ or $S \mapsto \mathbb{F}_\varnothing\langle S \rangle$. If S is a finite set $\{f_1, \ldots, f_n\}$, consisting of elements f_i of "degrees" or "arities" $r_i = r(f_i)$, we also write $\mathbb{F}_\varnothing\langle f_1^{[r_1]}, \ldots, f_n^{[r_n]}\rangle$ or just $\langle f_1^{[r_1]}, \ldots, f_n^{[r_n]}\rangle$.

This $\Sigma = \mathbb{F}_\varnothing\langle S \rangle$ is something like an algebra of polynomials over $\mathbb{F}_\varnothing$ in non-commuting indeterminates from S. Notice, however, that the structure of Σ depends on the choice of arities of indeterminates from S: if all of them are unary, the result is indeed very much like an algebra of polynomials in non-commuting variables, while non-unary free algebraic monads often exhibit more complicated behaviour.

Existence of free algebraic monads $\Sigma = \mathbb{F}_\varnothing\langle S\rangle$ is shown as follows. We explicitly define $\Sigma(X)$ as the set of all terms, constructed from free variables $\{x\}$, $x \in X$, with the aid of "formal operations" $f \in S$. In other words, $\Sigma(X)$ is the set of all *terms* over X, defined by structural induction as follows:

- Any $\{x\}$, $x \in X$, is a term.
- If $f \in S$ is a formal generator of arity r, and $t_1, \ldots, t_r$ are terms, then $f\, t_1 \ldots t_r$ is also a term.

Of course, this is just the formal construction of free algebraic systems, borrowed from mathematical logic. When we don't want to be too formal, we write $f(t_1, \ldots, t_n)$ instead of $f\, t_1 \ldots t_n$, and $t_1 * t_2$ instead of $*\, t_1\, t_2$, if $*$ is a binary operation, traditionally written in infix form.

0.4.18. (Generators of an algebraic monad.) Given any (graded) subset $S \subset \|\Sigma\|$, we can always find the smallest algebraic submonad $\Sigma' \subset \Sigma$, containing S (i.e. such that $\|\Sigma'\| \supset S$), for example by taking the intersection of all such algebraic submonads. Another description: Σ' is the image of the natural homomorphism $\mathbb{F}_\varnothing\langle S\rangle \to \Sigma$ from the free algebraic monad generated by S into Σ, induced by the embedding $S \to \|\Sigma\|$. Therefore, $\Sigma'(n)$ consists of all operations which can be obtained by applying operations from S to the basis elements $\{k\}_{\mathbf{n}}$ finitely many times.

If $\Sigma' = \Sigma$, i.e. if $\mathbb{F}_\varnothing\langle S\rangle \to \Sigma$ is surjective, we say that S *generates* Σ.

0.4.19. (Relations and strict quotients of algebraic monads.) Notice that epimorphisms of algebraic monads needn't be surjective, as illustrated by epimorphism $\mathbb{Z} \to \mathbb{Q}$. However, an algebraic monad homomorphism $f : \Sigma \to \Sigma'$ is a *strict* epimorphism (i.e. coincides with the cokernel of its kernel pair) iff all components $f_n : \Sigma(n) \to \Sigma'(n)$ are surjective. Therefore, strict quotients $\Sigma \twoheadrightarrow \Sigma'$ of Σ are in one-to-one correspondence with compatible (algebraic) equivalence relations $R \subset \Sigma \times \Sigma$. Any such equivalence relation is completely determined by $\|R\| \subset \|\Sigma \times \Sigma\|$, i.e. by the collection of equivalence relations $R(n) \subset \Sigma(n) \times \Sigma(n)$. *Compatibility* of such a family of equivalence relations with the algebraic monad structure of Σ essentially means compatibility with all maps $\Sigma(\varphi) : \Sigma(n) \to \Sigma(m)$ and $\mu_n^{(k)} : \Sigma(k) \times \Sigma(n)^k \to \Sigma(n)$.

Given any "system of equations" $E \subset \|\Sigma \times \Sigma\|$, we can construct the smallest compatible algebraic equivalence relation $R = \langle E\rangle \subset \Sigma \times \Sigma$ containing E, e.g. by taking the intersection of all such equivalence relations. The corresponding strict quotient $\Sigma \twoheadrightarrow \Sigma/\langle E\rangle$ is universal among all homomorphisms $\Sigma \xrightarrow{f} \Sigma'$, such that all relations from $f(E)$ are satisfied in Σ'.

0.4.20. (Presentations of algebraic monads.) This is applicable in particular to subsets $E \subset \|\mathbb{F}_\varnothing\langle S\rangle \times \mathbb{F}_\varnothing\langle S\rangle\|$, where $S \xrightarrow{r} \mathbb{N}_0$ is any $\mathbb{N}_0$-graded set. Corresponding strict quotient $\mathbb{F}_\varnothing\langle S\rangle/\langle E\rangle$ will be denoted by $\mathbb{F}_\varnothing\langle S \,|\, E\rangle$ or $\langle S \,|\, E\rangle$, and called the *free algebraic monad generated by S modulo relations from E*. When S or E are finite sets, we can replace S or E in $\langle S \,|\, E\rangle$ by an explicit list of generators (with arities) or relations. Furthermore, when we write relations from E, we often replace standard "free variables" $\{1\}$, $\{2\}$, ..., by lowercase letters like x, y, ..., thus writing for example $\mathbb{F}_{\pm 1} = \langle 0^{[0]}, -^{[1]} \,|\, -0 = 0, -(-x) = x\rangle$.

Whenever $\Sigma \cong \langle S \,|\, E\rangle$, we say that (S, E) is a *presentation* of algebraic monad Σ. Clearly, any algebraic monad Σ admits a presentation: we just have to choose any system of generators $S \subset \|\Sigma\|$ (e.g. $\|\Sigma\|$ itself), and take any system of equations E generating the kernel $R \subset \mathbb{F}_\varnothing\langle S\rangle \times \mathbb{F}_\varnothing\langle S\rangle$ of the canonical surjection $\mathbb{F}_\varnothing\langle S\rangle \twoheadrightarrow \Sigma$ (e.g. $E = R$). If both S and E can be chosen to be finite, we say that Σ *is finitely presented* (absolutely, i.e. over $\mathbb{F}_\varnothing$).

0.4.21. (Inductive limits of algebraic monads.) Presentations of algebraic monads are very handy when we need to compute inductive limits of algebraic monads. For example, the coproduct $\Sigma \boxtimes \Sigma'$ of two algebraic monads $\Sigma = \langle S \,|\, E\rangle$ and $\Sigma' = \langle S' \,|\, E'\rangle$ can be computed as $\langle S, S \,|\, E, E'\rangle$. Since the cokernel of a couple of morphisms can be computed as a strict quotient modulo a suitable compatible algebraic equivalence relation, and filtered inductive limits of algebraic monads can be computed componentwise, similarly to the projective limits, we can conclude that *arbitrary inductive and projective limits exist in the category of algebraic monads.*

0.4.22. (Endomorphism monad of a set.) Let X be any set. We denote by $\mathrm{END}(X)$ its *endomorphism monad*, constructed as follows. We put $(\mathrm{END}(X))(n) := \mathrm{Hom}_{\mathcal{S}ets}(X^n, X)$, $\boldsymbol{e} := \mathrm{id}_X \in (\mathrm{END}(X))(1)$, and define $\mu_n^{(k)} : \mathrm{Hom}(X^k, X) \times \mathrm{Hom}(X^n, X)^k = \mathrm{Hom}(X^k, X) \times \mathrm{Hom}(X^n, X^k) \to \mathrm{Hom}(X^n, X)$ to be the composition of maps: $(f, g) \mapsto g \circ f$. It is easy to check that this indeed defines an algebraic monad $\mathrm{END}(X)$, which acts on X in a canonical way. Furthermore, giving an action of an algebraic monad Σ on set X turns out to be the same thing as giving algebraic monad homomorphism $\Sigma \to \mathrm{END}(X)$, since a Σ-action on X is essentially a rule that transforms "formal operations" $t \in \Sigma(n)$ into their "values" $[t]_X : X^n \to X$.

We transfer some classical terminology to our case. For example, a Σ-module X is said to be *exact* or *faithful* if corresponding homomorphism $\Sigma \to \mathrm{END}(X)$ is injective.

0.4.23. (Presentations of algebraic monads and algebraic systems.) Let $\Sigma = \langle S \mid E \rangle$ be a presentation of an algebraic monad Σ. Clearly, algebraic monad homomorphisms $\Sigma \to \Sigma'$ are in one-to-one correspondence to graded maps $\varphi : S \to \|\Sigma'\|$, such that the images under φ of the relations from E hold in Σ'. Applying this to $\Sigma' = \mathrm{END}(X)$, we see that *a Σ-module structure on a set X is the same thing as assignment of a map $[f]_X : X^r \to X$ to each generator $f \in S$ of arity $r = r(f)$, in such a way that all relations from E hold.* Since S is a set of operations with some arities, and E is a set of relations between expressions involving operations from S and free variables, we see that (S, E) is an algebraic system, and a Σ-module is just an "algebra" for this algebraic system (S, E). In other words, "algebras" under an algebraic system (S, E) are exactly the $\langle S \mid E \rangle$-modules. Since any algebraic monad admits a presentation, the converse is also true: the category of modules over an algebraic monad can be described as the category of "algebras" for some algebraic system.

In this way we see that algebraic systems are nothing else than presentations of algebraic monads, and algebraic monads are just a convenient categorical way of describing algebraic systems, i.e. *theory of algebraic monads is a category-theoretic exposition of universal algebra.*

The reader might think that algebraic monads are not so useful, because they are just algebraic systems in another guise. If this be the case, and the reader is still not convinced of the convenience of using algebraic monads, we suggest to do following two exercises:

- Describe the intersection of two algebraic submonads of an algebraic monads in terms of their presentations.
- Find a presentation of $\mathrm{END}(X)$, and prove the correspondence between homomorphisms $\Sigma \to \mathrm{END}(X)$ and Σ-module structures on X, in terms of presentations of these algebraic monads.

0.4.24. (Examples: $\mathbb{Z}$ and $\mathbb{F}_1$.) For example, we can write

$$\begin{aligned}\mathbb{Z} = \langle 0^{[0]}, -^{[1]}, +^{[2]} \mid 0 + x = x,\ (x+y)+z = x+(y+z), \\ x + y = y + x,\ x + (-x) = 0 \rangle \quad (0.4.24.1)\end{aligned}$$

Strictly speaking, we should write $++\{1\}\{2\}\{3\} = +\{1\}+\{2\}\{3\}$ instead of $(x+y)+z = x+(y+z)$,...

In any case the above equality actually means that the category $\mathbb{Z}$-*Mod* consists of sets X, endowed by a constant 0_X, a unary operation $-_X : X \to X$, and a binary operation $+_X : X^2 \to X$, satisfying the above relations, i.e. abelian groups.

On the other hand, this algebraic monad $\mathbb{Z} = \Sigma_{\mathbb{Z}}$ can be described explicitly, since $\mathbb{Z}(n) = \Sigma_{\mathbb{Z}}(n) = \mathbb{Z}^n$ is the free $\mathbb{Z}$-module of rank n.

Of course, we can remove the commutativity relation from the above presentation of $\mathbb{Z}$, and add relations $x + 0 = x$ and $(-x) + x = 0$ instead, thus obtaining an algebraic monad $\mathbb{G}$, such that $\mathbb{G}$-*Mod* is the category of groups. Sets $\mathbb{G}(n)$ are free groups in n generators $\{1\}, \ldots, \{n\}$, and we have a natural homomorphism $\mathbb{G} \to \mathbb{Z}$. We'll see later that algebraic monad $\mathbb{Z}$ is *commutative*, while $\mathbb{G}$ is not.

Another example: the "field with one element" $\mathbb{F}_1$ can be defined by

$$\mathbb{F}_1 = \mathbb{F}_\varnothing\langle 0^{[0]}\rangle \quad , \tag{0.4.24.2}$$

i.e. it is the free algebraic monad generated by one constant. We see that $\mathbb{F}_1$-*Mod* is the category of sets X with one pointed element $0_X \in X$, and $\mathbb{F}_1$-homomorphisms are just maps of pointed sets, i.e. maps $f : X \to Y$, such that $f(0_X) = 0_Y$.

On the other hand, we can describe sets $\mathbb{F}_1(n)$ explicitly: any such set consists of $n + 1$ elements, namely, n basis elements $\{k\}_{\mathbf{n}}$, and a constant 0.

We hope that these examples illustrate how algebraic systems and algebraic monads are related to each other.

0.4.25. (Modules over algebraic monads.) After discussing the properties of algebraic monads themselves, and their relation to algebraic systems, we consider some properties of the category Σ-*Mod* of modules over an algebraic monad Σ. These are actually properties of "algebras" under a fixed algebraic system; we prefer to prove these properties again, using the theory of algebraic monads, instead of referring to well-known properties of such "algebras", partly because we want to illustrate how the theory of modules over algebraic monads works. Anyway, our proofs are quite short, compared to those classical proofs given in terms of algebraic systems.

For example, we show that arbitrary projective and filtered inductive limits of Σ-modules exist and can be computed on the level of underlying sets. (Notice that the statement about filtered inductive limits uses algebraicity of monad Σ in an essential way.) Furthermore, arbitrary inductive limits of Σ-modules exist. Monomorphisms of Σ-modules are just injective Σ-homomorphisms, and *strict* epimorphisms are the surjective Σ-homomorphisms. Since we have a notion of free Σ-modules and free Σ-modules, we have reasonable notions of a system of generators and of a presentation of Σ-module M, and we can define *finitely generated* and *finitely presented* Σ-modules as the Σ-modules isomorphic to a strict quotient of some free module $\Sigma(n)$, resp. to cokernel of a couple of homomorphisms $\Sigma(m) \rightrightarrows \Sigma(n)$.

These notions seem to have all the usual properties we expect from them, e.g. most of the properties of left modules over a classical associative ring, with some notable exceptions:

- Not all epimorphisms are strict, i.e. surjective;
- Direct sums (i.e. coproducts) $M \oplus N$ are not isomorphic to direct products $M \times N$;
- Direct sum of two monomorphisms needn't be a monomorphism.

0.4.26. (Scalar restriction and scalar extension.) Given any algebraic monad homomorphism $\rho : \Sigma \to \Xi$, we obtain a *scalar restriction functor* $\rho_* : \Xi\text{-}Mod \to \Sigma\text{-}Mod$, such that $\Gamma_\Sigma \circ \rho_* = \Gamma_\Xi$, where Γ_Σ and Γ_Ξ are the forgetful functors into the category of sets. Conversely (cf. **3.3.21**), any functor $H : \Xi\text{-}Mod \to \Sigma\text{-}Mod$, such that $\Gamma_\Sigma \circ H = \Gamma_\Xi$, equals ρ_* for some algebraic monad homomorphism $\rho : \Sigma \to \Xi$. For example, the forgetful functor from the category $\mathbb{Z}$-*Mod* of abelian groups into the category $\mathbb{G}$-*Mod* of groups is the scalar restriction functor with respect to canonical homomorphism $\mathbb{G} \to \mathbb{Z}$.

Furthermore, scalar restriction functor ρ_* admits a left adjoint ρ^*, called the *scalar extension* or *base change* functor. When no confusion can arise, we denote it by $\Xi \otimes_\Sigma -$, $- \otimes_\Sigma \Xi$ or $-_{(\Xi)}$, even if it is not immediately related to any tensor product. For example, if G is a group, then $\mathbb{Z} \otimes_{\mathbb{G}} G = G^{ab} = G/[G, G]$. Another example: $\mathbb{R} \otimes_{\mathbb{Z}_\infty} -$ transforms a torsion-free $\mathbb{Z}_\infty$-module $A = (A_{\mathbb{Z}_\infty}, A_{\mathbb{R}})$ into real vector space $A_{(\mathbb{R})} = A_{\mathbb{R}}$. Finally, scalar extension functor with respect to $\mathbb{F}_\varnothing \to \Sigma$ is just the free Σ-module functor $L_\Sigma : Sets \to \Sigma\text{-}Mod$.

0.4.27. (Flatness and unarity.) Of course, these scalar restriction and extension functors retain almost all their classical properties, e.g. scalar restriction commutes with arbitrary projective limits (hence is left exact), and scalar extension commutes with arbitrary inductive limits (hence is right exact). Notice, however, that scalar restriction $\rho_* : \Xi\text{-}Mod \to \Sigma\text{-}Mod$ needn't preserve direct sums (i.e. coproducts), as illustrated by the case $\rho : \mathbb{F}_\varnothing \to \mathbb{Z}$. In fact, *$\rho_*$ commutes with finite direct sums iff ρ_* is right exact iff ρ is* **unary**. This notion of *unarity* is fundamental for the theory of generalized rings, but completely absent from the classical theory of rings, since *all homomorphisms of classical rings are unary.*

Similarly, we say that ρ is flat, or that Ξ is flat over Σ, if ρ^* is (left) exact.

0.4.28. (Algebraic monads over topoi.) Chapter **4** treats the topos case as well. There is a natural approach to algebraic monads over a topos $\mathcal{E}$, based on *algebraic inner endofunctors.* However, the category of algebraic inner endofunctors turns out to be canonically equivalent to $\mathcal{E}^{\mathbb{N}} = \mathit{Funct}(\mathbb{N}, \mathcal{E})$, i.e. an algebraic inner endofunctor Σ over $\mathcal{E}$ can be described as a sequence $\{\Sigma(n)\}_{n\geq 0}$ of objects of $\mathcal{E}$ together with some transition morphisms $\Sigma(\varphi) : \Sigma(n) \to \Sigma(m)$, defined for each $\varphi : \mathbf{n} \to \mathbf{m}$. Furthermore, the "elementary" description of algebraic monads recalled in **0.4.6** transfers verbatim to topos case, if we replace all sets by objects of topos $\mathcal{E}$.

Applying this to the topos $\mathcal{E}$ of sheaves of sets over a site $\mathcal{S}$ or a topological space X, we obtain a notion of a "sheaf of algebraic monads" over $\mathcal{S}$ or X: it is a collection of sheaves of sets $\{\Sigma(n)\}_{n\geq 0}$, together with maps $\Sigma(\varphi) : \Sigma(n) \to \Sigma(n)$, a unit section $\boldsymbol{e} \in \Gamma(X, \Sigma(1))$ and "evaluation maps" $\mu_n^{(k)} : \Sigma(k) \times \Sigma(n)^k \to \Sigma(n)$, such that for any open subset $U \subset X$ (or any object $U \in \operatorname{Ob}\mathcal{S}$) $\Gamma(U, \Sigma)$ becomes an algebraic monad (over *Sets*). Sheaves of modules over such Σ are defined similarly. We can also define generalized ringed spaces (ringed spaces with a sheaf of generalized rings), their morphisms and so on.

0.5. (Commutativity.) Chapter **5** is dedicated to another fundamental property we require from our generalized rings, namely, *commutativity.* In fact, we define a (commutative) generalized ring as a commutative algebraic monad. A classical ring is commutative as an algebraic monad iff it is commutative in the classical sense, so our terminology extends the classical one.

0.5.1. (Definition of commutativity.) Given an algebraic monad Σ, two operations $t \in \Sigma(n)$, $t' \in \Sigma(m)$, a Σ-module X, and a matrix $(x_{ij})_{1\leq i\leq m, 1\leq j\leq n}$, $x_{ij} \in X$, we can consider two elements x', $x'' \in X$, defined as follows. Let $x_{i.} := t(x_{i1}, \ldots, x_{in})$ be the element of X, obtained by applying t to i-th row of (x_{ij}), and $x_{.j} := t'(x_{1j}, \ldots, x_{mj})$ be the element, obtained by applying t' to j-th column of (x_{ij}). Then $x' := t'(x_{1.}, \ldots, x_{m.})$, and $x'' := t(x_{.1}, \ldots, x_{.n})$. Now we say that *$t$ and t' commute on X* if $x' = x''$ for any choice of $x_{ij} \in X$. We say that *t and t' commute (in Σ)* if they commute on any Σ-module X.

Notice that t and t' commute iff the commutativity relation is fulfilled for the "universal matrix" $x_{ij} := \{(i-1)n + j\}$ in $X = \Sigma(nm)$. In this way commutativity of t and t' is equivalent to one equation $x' = x''$ in $\Sigma(nm)$, which will be denoted by $[t, t']$.

Given any subset $S \subset \|\Sigma\|$, we can consider its *commutant* $S' \subset \|\Sigma\|$, i.e. the set of all operations of Σ commuting with all operations from S. This subset $S' \subset \|\Sigma\|$ turns out to be the underlying set of an algebraic submonad of Σ, which will be also denoted by S'. We have some classical formulas like $S''' = S'$, $S'' \supset S$ and so on.

Finally, the *center* of an algebraic monad Σ is the commutant of $\|\Sigma\|$ itself, i.e. the set of all operations commuting with each operation of Σ.

We say that Σ is *commutative*, or a *generalized ring*, if it coincides with its center, i.e. if any two operations of Σ commute. If $S \subset \|\Sigma\|$ is any system of generators of Σ, it suffices to require that any t, $t' \in S$ commute.

0.5.2. (Why this definition of commutativity?) We would like to present some argument in favour of the above definition of commutativity. Let Σ be an algebraic monad, M and N be two Σ-modules. Then the set $\mathrm{Hom}(M, N) = \mathrm{Hom}_{\mathit{Sets}}(M, N) = N^M$ of *all* maps $\varphi : M \to N$ admits a natural Σ-module structure, namely, the product Σ-structure on N^M. We can describe this structure by saying that *operations $t \in \Sigma(n)$ act on maps $\varphi : M \to N$ pointwise:*

$$t(\varphi_1, \ldots, \varphi_n) : x \mapsto t\bigl(\varphi_1(x), \ldots, \varphi_n(x)\bigr) \tag{0.5.2.1}$$

So far we have used only Σ-module structure on N, but M might be just an arbitrary set. Now if M is a Σ-module, we get a subset $\mathrm{Hom}_\Sigma(M, N) \subset \mathrm{Hom}(M, N) = N^M$, and we might ask whether this subset is a Σ-submodule.

Looking at the case of modules over classical associative ring R, we see that $\mathrm{Hom}_R(M, N) \subset \mathrm{Hom}(M, N) = N^M$ is an R-submodule for all R-modules M and N if and only if R is commutative, and in this case the induced R-module structure on $\mathrm{Hom}_R(M, N)$ is its usual R-module structure, known from commutative algebra.

Therefore, it makes sense to say that an algebraic monad Σ is commutative iff $\mathrm{Hom}_\Sigma(M, N) \subset \mathrm{Hom}(M, N) = N^M$ is a Σ-submodule for all choices of Σ-modules M and N. When we express this condition in terms of operations of Σ, taking (0.5.2.1) into account, we obtain exactly the definition of commutativity given above in **0.5.1**.

0.5.3. (Consequences: bilinear maps, tensor products and inner Homs.) Whenever M and N are two modules over a generalized ring (i.e. commutative algebraic monad) Σ, we denote by $\mathbf{Hom}_\Sigma(M, N)$, or simply by $\mathrm{Hom}_\Sigma(M, N)$, the set $\mathrm{Hom}_\Sigma(M, N)$, considered as a Σ-module with respect to the structure induced from $\mathrm{Hom}(M, N) = N^M$.

Looking at the case of classical commutative rings, we see that we might expect formulas like

$$\mathrm{Hom}_\Sigma(M \otimes_\Sigma N, P) \cong \mathrm{Bilin}_\Sigma(M, N; P) \cong \mathrm{Hom}_\Sigma(M, \mathbf{Hom}_\Sigma(N, P)) \tag{0.5.3.1}$$

This expectation turns out to be correct, i.e. there is a tensor product functor $\otimes_\Sigma$ on Σ-*Mod* with the above property, as well as a notion of Σ-*bilinear maps* $\Phi : M \times N \to P$, fitting into the above formula.

This notion of Σ-bilinear map is very much like the classical one: a map $\Phi : M \times N \to P$ is said to be *bilinear* if the map $s_\Phi(x) : N \to P$, $y \mapsto \Phi(x, y)$, is Σ-linear (i.e. a Σ-homomorphism) for each $x \in M$, and $d_\Phi(y) : M \to P$, $x \mapsto \Phi(x, y)$, is Σ-linear for each $y \in N$. Multilinear maps are defined similarly.

In this way we obtain an ACU $\otimes$-structure on Σ-*Mod*, which admits inner Homs $\mathbf{Hom}_\Sigma$. The unit object with respect to this tensor product is $|\Sigma| = L_\Sigma(1)$, the free Σ-module of rank one.

When we apply these definitions to a classical commutative ring R, we recover classical notions of R-bilinear maps, tensor product of R-modules and so on. On the other hand, when we apply them to $\mathbb{Z}_\infty$-*Mod*, we recover the definitions of Chapter **2**, originally based on properties of maximal compact submonoids and symmetric compact convex sets. This shows that the tensor product we consider is indeed a very natural one.

0.5.4. (Algebras over a generalized ring.) Let Λ be a generalized ring. We can define a (non-commutative) Λ-algebra as a generalized ring Σ together with a *central* homomorphism $f : \Lambda \to \Sigma$ (i.e. all operations from the image of f must lie in the center of Σ). Of course, if Σ is commutative, all homomorphisms $\Lambda \to \Sigma$ are central, i.e. *a (commutative) Λ-algebra is a generalized ring Σ with a homomorphism $\Lambda \to \Sigma$.*

0.5.5. (Variants of free algebras.) We have several notions of a free algebra, depending on whether we impose some commutativity relations or not:

- Let Λ be an algebraic monad, and consider the category of Λ-monads, i.e. algebraic monad homomorphisms $\Lambda \to \Sigma$ with source Λ. We have a (graded) underlying set functor $\Sigma \to \|\Sigma\|$, which admits a left adjoint $S \mapsto \Lambda\langle S\rangle$. Indeed, if $\Lambda = \langle T \mid E\rangle$ is any presentation of Λ, then $\Lambda\langle S\rangle$ can be constructed as $\langle S, T \mid E\rangle$. Notice that this $\Lambda\langle S\rangle$ is something like a "very non-commutative" polynomial algebra over S: not only the indeterminates from S are not required to commute between themselves, they are even not required to commute with operations from Λ!

- Let Λ be a generalized ring, i.e. a commutative algebraic monad, and consider the category of all Λ-algebras, commutative or not, i.e. central homomorphisms $\Lambda \to \Sigma$. The graded underlying set functor $\Sigma \mapsto \|\Sigma\|$ still admits a left adjoint, which will be denoted by $S \mapsto \Lambda\{S\}$. If $\Lambda = \langle T \mid E\rangle$, then $\Lambda\{S\} = \langle T, S \mid E, [S, T]\rangle$, where $[S, T]$ denotes the set of all commutativity relations $[s, t]$, $s \in S$, $t \in T$. In other words, now the indeterminates from S still don't commute among themselves, but at least commute with operations from Λ.

- Finally, we can consider the category of commutative algebras $\Lambda \to \Sigma$ over a generalized ring Λ, and repeat the above reasoning. We obtain free commutative algebras $\Lambda[S] = \langle T, S \mid E, [S, T], [S, S]\rangle$, which are quite similar to classical polynomial algebras, especially when all indeterminates from S are unary.

0.5.6. (Presentations of Λ-algebras.) In any of the three cases listed above we can impose some relations on free algebras, thus obtaining the notion of a presentation of a Λ-algebra. Consider for example the case of commutative Λ-algebras, with $\Lambda = \langle T \mid E \rangle$ any generalized ring. Then, given any $\mathbb{N}_0$-graded set S and any set of relations $R \subset \|\Lambda[S] \times \Lambda[S]\|$, we can construct the strict quotient

$$\Lambda[S \mid R] = \Lambda[S]/\langle R \rangle = \langle T, S \mid E,\ [S, T],\ [S, S],\ R \rangle \tag{0.5.6.1}$$

This strict quotient is easily seen to be a commutative Λ-algebra, and whenever we are given a Λ-algebra isomorphism $\Sigma \cong \Lambda[S \mid R]$, we say that (S, R) *is a presentation of (commutative) Λ-algebra Σ.*

0.5.7. (Finitely generated/presented algebras.) If S can be chosen to be finite, we say that Σ *is a finitely generated Λ-algebra*, or *a Λ-algebra of finite type*, and if both S and R can be chosen to be finite, we speak about *finitely presented* Λ-algebras. These notions are quite similar to their classical counterparts.

0.5.8. (Pre-unary and unary algebras.) Apart from notions of finite generation and presentation, which have well-known counterparts in classical commutative algebra, we obtain notions of *pre-unary* and *unary* Λ-algebras. Namely, we say that a Λ-algebra Σ is *pre-unary* if it admits a system of unary generators, and *unary* if it admits a presentation $\Sigma = \Lambda[S \mid R]$, consisting only of unary generators (i.e. $S \subset |\Sigma| = \Sigma(1)$) and unary relations (i.e. $R \subset |\Lambda[S] \times \Lambda[S]|$).

These notions are very important for the theory of generalized rings, but they don't have non-trivial counterparts in classical algebra, since any algebra over a classical ring is automatically unary.

0.5.9. (Tensor products, i.e. coproducts of Λ-algebras.) Since any commutative Λ-algebra Σ_i admits some presentation $\Lambda[S_i \mid E_i]$ over Λ, we can easily construct *coproducts* in the category of commutative Λ-algebras, called also *tensor products* of Λ-algebras, by

$$\Sigma_1 \otimes_\Lambda \Sigma_2 = \Lambda[S_1, S_2 \mid E_1, E_2] \tag{0.5.9.1}$$

Using existence of filtered inductive limits and arbitrary projective limits of generalized rings, which actually coincide with those computed in the category of algebraic monads, we conclude that *arbitrary inductive and projective limits exist in the category of generalized rings.*

0.5.10. (Unary algebras as algebras in $\otimes$-category Λ-*Mod*.) Notice that we might try to generalize another classical definition of a Λ-algebra and say that a Λ-algebra A is just an algebra in ACU $\otimes$-category Λ-*Mod*, with respect to our tensor product $\otimes = \otimes_\Lambda$, and define A-modules accordingly.

However, the category of such algebras in Λ-*Mod* turns out to be *equivalent* to the category of *unary* Λ-algebras, and we obtain same categories of modules under this correspondence.

This allows us to construct for example tensor, symmetric or exterior algebras of a Λ-module M first by applying the classical constructions inside $\otimes$-category Λ-*Mod*, and then taking corresponding unary Λ-algebras. Monoid and group algebras can be also constructed in this way; in particular, they are unary.

Unary algebras have some other equivalent descriptions, which correspond to some properties always fulfilled in the classical case. For example, Σ is a (commutative) unary Λ-algebra iff the scalar restriction functor ρ_* with respect to $\Lambda \xrightarrow{\rho} \Sigma$ commutes with finite direct sums iff it is exact iff it admits a right adjoint $\rho^!$ iff the "projection formula" $\rho_*(M \otimes_\Sigma \rho^* N) \cong \rho_* M \otimes_\Lambda N$ holds.

0.5.11. ("Affine base change" theorem.) Another important result of Chapter **5**, called by us the *affine base change theorem* (cf. **5.4.2**), essentially asserts the following. Suppose we are given two commutative Λ-algebras Λ' and Σ, and put $\Sigma' := \Lambda' \otimes_\Lambda \Sigma$. Then we can start from a Σ-module M, compute its scalar restriction to Λ, and afterwards extend scalars to Λ'. On the other hand, we can first extend scalars to Σ', and then restrict them to Λ'. The affine base change theorem claims that *the two Λ'-modules thus obtained are canonically isomorphic, provided either Σ is unary or Λ' is flat over Λ.*

We also consider an example, which shows that the statement is not true without additional assumptions on Σ or Λ'.

0.5.12. We have discussed at some length some consequences of commutativity for categories of modules and algebras over a generalized ring Σ. Now we would like to discuss the commutativity relations themselves. At first glance commutativity of two operations of arity ≥ 2 seems to be not too useful. We are going to demonstrate that in fact such commutativity relations are very powerful, and that they actually imply classical associativity, commutativity and distributivity laws, thus providing a common generalization of all such laws.

0.5.13. (Commutativity for operations of low arity.) Let's start with some simple cases. Here Σ is some fixed algebraic monad.

- Two constants c, $c' \in \Sigma(0)$ commute iff $c = c'$. In particular, *a generalized ring contains at most one constant.*

- Two unary operations u, $u' \in |\Sigma| = \Sigma(1)$ commute iff $uu'\{1\} = u'u\{1\}$, i.e. $uu' = u'u$ in monoid $|\Sigma|$. This is classical commutativity of multiplication.

- An n-ary operation t commutes with a constant c iff $t(c, c, \ldots, c) = c$. For example, a constant c and a unary operation u commute iff $uc = c$ in $\Sigma(0)$. A binary operation $*$ and constant c commute iff $c * c = c$.

- An n-ary operation t commutes with unary u iff $u \cdot t(\{1\}, \ldots, \{n\}) = t(u\{1\}, \ldots, u\{2\})$. For example, if t is a binary operation $*$, this means $u(\{1\} * \{2\}) = u\{1\} * u\{2\}$, or $u(x * y) = ux * uy$, i.e. classical distributivity relation.

- Two binary operations $+$ and $*$ commute iff $(x+y)*(z+w) = (x*z) + (y * w)$. Notice that, while any operation of arity ≤ 1 automatically commutes with itself, this is not true even for a binary operation $*$, since $(x*y)*(z*w) = (x*z)*(y*w)$ is already a non-trivial condition.

0.5.14. (Zero.) We say that a *zero* of an algebraic monad Σ is just a central constant $0 \in \Sigma(0)$. Since it commutes with any other constant c, we must have $c = 0$, i.e. a zero is automatically the only constant of Σ. Furthermore, we have $u0 = 0$ for any unary u, $0*0 = 0$ for any binary $*$, and $t(0, \ldots, 0) = 0$ for an n-ary t, i.e. all operations fix the zero.

Of course, a generalized ring either doesn't have any constants (then we say that it is a *generalized ring without zero*), or it has a zero (and then it is a *generalized ring with zero*). We have a universal generalized ring with zero $\mathbb{F}_1 = \mathbb{F}_\varnothing[0^{[0]}] = \mathbb{F}_\varnothing\langle 0^{[0]}\rangle = \langle 0^{[0]}\rangle$, called *the field with one element.* Furthermore, $\mathbb{F}_1$-algebras are exactly the algebraic monads with a central constant, i.e. with zero.

0.5.15. (Addition.) Let Σ be an algebraic monad with zero 0. We say that a binary operation $+^{[2]} \in \Sigma(2)$ is a *pseudoaddition* iff $\{1\} + 0 = \{1\} = 0 + \{1\}$. This can be also written as $\boldsymbol{e} + 0 = \boldsymbol{e} = 0 + \boldsymbol{e}$ or $x + 0 = x = 0 + x$. An *addition* is simply a central pseudoaddition. If Σ admits an addition $+$, any other pseudoaddition $*$ must coincide with $+$: indeed, we can put $y = z = 0$ in commutativity relation $(x + y) * (z + w) = (x * z) + (y * z)$, and obtain $x * w = (x + 0) * (0 + w) = (x * 0) + (0 * w) = x + w$.

So let Σ be an algebraic monad with zero 0 and addition $+$. Commutativity of $+$ and unary operations $u \in |\Sigma|$ means $u(x + y) = ux + uy$, i.e. distributivity. Now consider the commutativity condition for $+$ and itself:

$(x+y)+(z+w)=(x+z)+(y+w)$. Putting here $x=w=0$, we obtain $y+z=z+y$, i.e. classical commutativity of addition. Next, if we put $z=0$, we get $(x+y)+w=x+(y+w)$, i.e. classical associativity of addition. Therefore, classical commutativity, associativity and distributivity laws are just consequences of our "generalized" commutativity law.

0.5.16. (Semirings.) Given any (classical) semiring R, we can consider corresponding algebraic monad Σ_R, which will be an algebraic monad with addition (hence also with zero). Conversely, whenever Σ is an algebraic monad with addition $+$, we get an addition $+$ on monoid $|\Sigma|=\Sigma(1)$, commuting with the action of all unary operations, i.e. $R:=|\Sigma|$ is a semiring. One can check that $\Sigma=\Sigma_R$, i.e. classical semirings correspond exactly to algebraic monads with addition. For example, semirings $\mathbb{N}_0$ and tropical numbers $\mathbb{T}$ can be treated as generalized rings. This reasoning also shows that any (generalized) algebra Σ over a semiring Λ is also an algebraic monad with addition (since $\Lambda\to\Sigma$ is central, it maps the zero and addition of Λ into zero and addition of Σ), i.e. is also given by a classical semiring. Furthermore, any such Λ-algebra Σ is automatically unary.

Of course, we have a universal algebraic monad with addition, namely,

$$\begin{aligned}\mathbb{N}_0 =&\mathbb{F}_1[+^{[2]}\,|\,\boldsymbol{e}+0=\boldsymbol{e}=0+\boldsymbol{e}] &(0.5.16.1)\\ =&\mathbb{F}_\varnothing[0^{[0]},+^{[2]}\,|\,\boldsymbol{e}+0=\boldsymbol{e}=0+\boldsymbol{e}] &(0.5.16.2)\\ =&\langle 0^{[0]},+^{[2]}\,|\,[+,+],\,\boldsymbol{e}+0=\boldsymbol{e}=0+\boldsymbol{e}\rangle &(0.5.16.3)\end{aligned}$$

0.5.17. (Symmetry.) A *symmetry* in an algebraic monad Σ is simply a central unary operation $-$, such that $-^2=\boldsymbol{e}$ in monoid $|\Sigma|$, i.e. $-(-\boldsymbol{e})=\boldsymbol{e}$, or $-(-x)=x$. Centrality of $-$ actually means $-t(\{1\},\ldots,\{n\})=t(-\{1\},\ldots,-\{n\})$ for any n-ary operation t. In particular, $-c=c$ for any constant c.

We have a universal algebraic monad with zero and symmetry, namely,

$$\mathbb{F}_{\pm1}=\mathbb{F}_1[-^{[1]}\,|\,-(-\boldsymbol{e})=\boldsymbol{e}]=\mathbb{Z}_\infty\cap\mathbb{Z} \qquad (0.5.17.1)$$

Using the notion of symmetry, we obtain a finite presentation of $\mathbb{Z}$:

$$\begin{aligned}\mathbb{Z} =&\mathbb{F}_{\pm1}[+^{[2]}\,|\,x+0=x=0+x,\,x+(-x)=0] &(0.5.17.2)\\ =&\mathbb{F}_\varnothing[0^{[0]},-^{[1]},+^{[2]}\,|\,x+0=x=0+x,\,x+(-x)=0] &(0.5.17.3)\end{aligned}$$

Reasoning as in the case of semirings, we see that classical rings can be described as algebraic monads with addition $+$ and symmetry $-$, compatible in the sense that $x+(-x)=0$. For example, if Λ is a classical commutative ring, and Σ is a generalized Λ-algebra, then algebraic monad Σ admits a zero, a symmetry and an addition (equal to the images of those of Λ), compatible in the above sense, hence Σ is a classical ring, i.e. a classical Λ-algebra. In other words, we cannot obtain any "new" generalized algebras over a classical commutative ring.

0.5.18. (Examples of generalized rings.) Let's list some generalized rings.

- Classical commutative rings, e.g. $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$,...
- Classical commutative semirings: $\mathbb{N}_0$, $\mathbb{R}_{\geq 0}$, $\mathbb{T}$,...
- Algebraic submonads of generalized rings, e.g. $\mathbb{Z}_\infty \subset \mathbb{R}$, $\mathbb{Z}_{(\infty)} \subset \mathbb{Q}$, or $A_N := \mathbb{Z}_{(\infty)} \cap \mathbb{Z}[1/N] \subset \mathbb{Q}$.
- *Archimedian valuation rings* $V \subset K$, defined as follows. Let K be a classical field, equipped with an archimedian valuation $|\cdot|$. Then

$$V(n) = \{(\lambda_1, \ldots, \lambda_n) \in K^n \,:\, \sum_i |\lambda_i| \leq 1\} \tag{0.5.18.1}$$

  For example, $\mathbb{Z}_{(\infty)} \subset \mathbb{Q}$ and $\mathbb{Z}_\infty \subset \mathbb{R}$ are special cases of this construction.

- Strict quotients of generalized rings. Consider for example the three-point $\mathbb{Z}_\infty$-module Q, obtained by identifying all inner points of segment $|\mathbb{Z}_\infty| = [-1, 1]$ (we've denoted this Q by $\mathbb{F}_\infty$ in **0.2.16**), and denote by $\mathbb{F}_\infty$ the image of induced homomorphism $\mathbb{Z}_\infty \to \mathrm{END}(Q)$. Then $\mathbb{F}_\infty$ is commutative, i.e. a generalized ring, being a strict quotient of $\mathbb{Z}_\infty$. Furthermore, it is generated over $\mathbb{F}_{\pm 1}$ by one binary operation $*$, the image of $(1/2)\{1\} + (1/2)\{2\} \in \mathbb{Z}_\infty(2)$:

$$\mathbb{F}_\infty = \mathbb{F}_{\pm 1}[*^{[2]} \,|\, x*(-x) = 0,\, x*x = x,\, x*y = y*x,\, (x*y)*z = x*(y*z)] \tag{0.5.18.2}$$

- Cyclotomic extensions of $\mathbb{F}_1$:

$$\mathbb{F}_{1^n} = \mathbb{F}_1[\zeta^{[1]} \,|\, \zeta^n = \boldsymbol{e}] \tag{0.5.18.3}$$

  Modules over $\mathbb{F}_{1^n}$ are sets X with a marked element 0_X and a permutation $\zeta_X : X \to X$, such that $\zeta_X^n = \mathrm{id}_X$ and $\zeta_X(0_X) = 0_X$. This is applicable to $\mathbb{F}_{\pm 1} = \mathbb{F}_{1^2}$.

- Affine generalized rings $\mathrm{Aff}_R \subset R$: take any classical commutative (semi)ring R, and put $\mathrm{Aff}_R(n) := \{\lambda_1\{1\} + \cdots + \lambda_n\{n\} \in R^n \mid \sum_i \lambda_i = 1\}$. Then Aff_R-*Mod* is just the category of affine R-modules.

- Generalized ring $\Delta := \mathrm{Aff}_{\mathbb{R}} \cap \mathbb{R}_{\geq 0} \subset \mathbb{R}$. Set $\Delta(n)$ is the standard simplex in $\mathbb{R}^n$, and Δ-*Mod* consists of abstract convex sets.

0.5.19. (Some interesting tensor products.) We compute some interesting tensor products of generalized rings: $\mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}$, and similarly $\mathbb{Z} \otimes \mathbb{Z}_\infty = \mathbb{R}$, $\mathbb{Z} \otimes \mathbb{Z}_{(\infty)} = \mathbb{Q}$, if the tensor products are computed over $\mathbb{F}_{\pm 1}$, $\mathbb{F}_1$ or $\mathbb{F}_\varnothing$. Furthermore, $\mathbb{F}_\infty \otimes_{\mathbb{F}_{\pm 1}} \mathbb{F}_\infty = \mathbb{F}_\infty$ and $\mathbb{Z}_{(\infty)} \otimes_{\mathbb{F}_{\pm 1}} \mathbb{Z}_{(\infty)} = \mathbb{Z}_{(\infty)}$.

0.5.20. (Tensor and symmetric powers.) Given any module M over any generalized ring Λ, we can consider its tensor and symmetric powers $T^n(M) = M^{\otimes n}$ and $S^n(M) = T^n(M)/\mathfrak{S}_n$. They seem to retain all their usual properties, e.g. they commute with scalar extension, and a symmetric power of a free Λ-module M is again free, with appropriate products of base elements of M as a basis.

0.5.21. (Exterior powers, alternating generalized rings and determinants.) On the other hand, exterior powers $\wedge^n(M)$ are more tricky to define. First of all, M has to be a module over a commutative $\mathbb{F}_{\pm 1}$-algebra Σ, i.e. generalized ring Σ must admit a zero and a symmetry. Even if this condition is fulfilled, these exterior powers $\wedge^n(M)$ don't retain some of their classical properties, e.g. exterior powers of a free module are not necessarily free, and in particular $\wedge^n L_\Sigma(n) \ncong |\Sigma|$. In order to deal with these problems we introduce and study *alternating* generalized rings. Over them exterior powers of a free module are still free with the "correct" basis, e.g. if M is free with basis $e_1, \ldots, e_n$, then $\wedge^n(M)$ is freely generated by $e_1 \wedge \ldots \wedge e_n$, so we write $\det M := \wedge^n(M)$, and use it to define *determinants* $\det u \in |\Sigma|$ of endomorphisms $u \in \mathrm{End}_\Sigma(M)$. Then $\det(uv) = \det(u)\det(v)$, $\det(\mathrm{id}_M) = \boldsymbol{e}$, so any automorphism has invertible determinant. However, invertibility of $\det(u)$ doesn't imply bijectivity of u in general case. We need to introduce some additional conditions (DET), which assure that invertibility $\det(u)$ indeed implies invertibility of u.

Our general impression of this theory of exterior powers and determinants is that one should avoid using exterior powers in the generalized ring context, and use symmetric powers instead whenever possible. Then we don't have to impose any complicated conditions on the base generalized ring Λ.

0.5.22. (Topos case.) Of course, definitions of commutativity and alternativity of algebraic monads can be transferred to the topos case, so we have a reasonable notion of a sheaf of generalized rings over a topological space X or a site $\mathcal{S}$. Furthermore, usually we consider only generalized commutatively ringed spaces or sites, i.e. require the structural sheaf $\mathscr{O}$ of algebraic monads to be commutative. Then we get a tensor product $\otimes_{\mathscr{O}}$ of (sheaves of) $\mathscr{O}$-modules, inner Homs $\mathbf{Hom}_{\mathscr{O}}(\mathscr{F}, \mathscr{G})$, functors $i_!$, i^*, i_*, related to an open immersion $i : U \to X$, and even the classical description of the category of $\mathscr{O}$-modules over X in terms of categories of modules over an open subset $U \subset X$ and its closed complement $Y = X - U$.

0.6. (Localization and generalized schemes.) Chapter **6** is dedicated to *unary localization*, *localization theories*, *spectra of generalized rings* and *generalized schemes*. We transfer some of results of EGA I and II to our situation, e.g. notions of morphisms of finite presentation or of finite type, affine and projective morphisms, and show existence of fibered products of generalized schemes. We discuss some examples, e.g. projective spaces and projective bundles.

While this “generalized algebraic geometry”, i.e. theory of generalized schemes, seems to be quite similar to its classical counterpart, there is a considerable distinction related to closed subschemes. Namely, a strict epimorphism (i.e. surjective homomorphism) of generalized rings $f : A \to B$ doesn't define a *closed* map ${}^a f : \operatorname{Spec} B \to \operatorname{Spec} A$. In fact, even the diagonal of $\mathbb{A}^1_{\mathbb{F}_1} = \operatorname{Spec} \mathbb{F}_1[T^{[1]}]$ over $\operatorname{Spec} \mathbb{F}_1$ is not closed.

We partially avoid this problem by defining a “closed” immersion as an affine morphism $i : Y \to X$, such that $\mathscr{O}_X \to i_* \mathscr{O}_Y$ is a strict epimorphism, a “closed” (generalized) subscheme as a subobject in the category of generalized schemes, given by a “closed” immersion, and a “separated” morphism $X \to S$ by requiring $\Delta_{X/S} : X \to X \times_S X$ to be a “closed” immersion. In this way affine and projective spaces over $\mathbb{F}_1$ are “separated”, even if their diagonals are not topologically closed.

0.6.1. (Multiplicative systems and unary localizations.) Let A be a generalized ring, M be an A-module. Recall that $|A|$ has a natural (commutative) monoid structure, and the underlying set of M admits a canonical monoid action of $|A|$. Now a *multiplicative system* S in A is simply a submonoid of $|A|$. Given any subset $S \subset |A|$, we can consider the multiplicative system (i.e. submonoid) $\langle S \rangle \subset |A|$, generated by S. Next, we can define the *localization* $A[S^{-1}]$ *of* A as the universal commutative A-algebra $i_A^S : A \to A[S^{-1}]$, such that all elements from S become invertible in $A[S^{-1}]$. Similarly, we define the localization $M[S^{-1}]$ of A-module M by considering the universal A-module homomorphism $i_M^S : M \to M[S^{-1}]$, such that all $s \in S$ act on $M[S^{-1}]$ by bijections.

When S is a multiplicative system, we write $S^{-1}A$ and $S^{-1}M$ instead of $A[S^{-1}]$ and $M[S^{-1}]$. We can construct $S^{-1}M$ in the classical fashion, as the set of couples $(x, s) \in M \times S$, modulo equivalence relation $(x, s) \sim (y, t)$ iff $utx = usy$ for some $u \in S$. Sets $(S^{-1}A)(n) = S^{-1}(A(n))$ admit a similar description. Next, if S is not a multiplicative system, we can always replace S by $\langle S \rangle$, and apply the above constructions.

In this way (unary) localizations always exist and have almost all of the classical properties. For example, $M[S^{-1}]$ admits a natural $A[S^{-1}]$-module structure, and coincides with the scalar extension $M \otimes_A A[S^{-1}]$. Furthermore, $A[S^{-1}]$ is always a flat A-algebra, i.e. $M \mapsto M[S^{-1}]$ is exact, and $A[S^{-1}]$-*Mod* is identified by the scalar restriction functor with the full subcategory of A-*Mod*, consisting of all A-modules N, such that all $[s]_N : N \to N$, $s \in S$, are bijections.

The reason why (unary) localization theory extends so nicely to the generalized ring context is that almost all constructions and proofs in this theory are "multiplicative", i.e. involve only the multiplication of $|A|$ and the action of $|A|$ on M.

0.6.2. (Ideals.) Now consider the *ideals* $\mathfrak{a}$ of A, i.e. the A-submodules $\mathfrak{a} \subset |A|$. They constitute a lattice with respect to inclusion, with $\inf(\mathfrak{a}, \mathfrak{b})$ given by the intersection $\mathfrak{a} \cap \mathfrak{b}$, and $\sup(\mathfrak{a}, \mathfrak{b})$ equal to $\mathfrak{a} + \mathfrak{b} = (\mathfrak{a}, \mathfrak{b}) = \mathrm{Im}(\mathfrak{a} \oplus \mathfrak{b} \to |A|)$. The largest element of this lattice is the unit ideal $(1) = (\boldsymbol{e}) = |A|$, and the smallest element is the *initial ideal* $\varnothing_A = L_A(0) \subset L_A(1) = |A|$. If A admits a zero 0, then $\varnothing_A = (0)$ is called the *zero ideal*; if not, we say that $\varnothing_A$ is the *empty ideal*.

We have also a notion of *principal ideal* $(a) = a|A| \subset |A|$, for any $a \in |A|$, and of *product of ideals* $\mathfrak{ab}$, defined for example as $\mathrm{Im}(\mathfrak{a} \otimes_A \mathfrak{b} \to A)$; this is the ideal generated by all products ab, $a \in \mathfrak{a}$, $b \in \mathfrak{b}$.

So far ideals in A have all their properties known from commutative algebra. However, when we try to compute strict quotients $A/\mathfrak{a}$ (for example first as algebras in A-*Mod*, and then as unary algebras over A), the picture is not so nice any longer. For example, if A admits a zero (this is actually a prerequisite for defining $A/\mathfrak{a}$), then the preimage of zero under $A \to A/\mathfrak{a}$ might be strictly larger that $\mathfrak{a}$, i.e. *quotient algebras and modules lose some of their classical properties.*

0.6.3. (Prime spectra.) Next, we define a *prime ideal* $\mathfrak{p} \subset |A|$ as an ideal, such that $|A| - \mathfrak{p}$ is a multiplicative system. We denote by $\operatorname{Spec} A$ or $\operatorname{Spec}^p A$ the *prime spectrum* of A, defined as the set of all prime ideals in $|A|$. For any subset $S \subset |A|$ we denote by $V(S)$ the set of prime ideals $\mathfrak{p} \supset S$. Clearly, $V(S) = V(\mathfrak{a})$, where $\mathfrak{a} = (S)$ is the ideal generated by S, and one can show that $V((1)) = \varnothing$, $V(\varnothing) = \operatorname{Spec} A$, $V(\bigcup_\alpha S_\alpha) = \bigcap_\alpha V(S_\alpha)$ and $V(\mathfrak{ab}) = V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$. This means that these $V(S)$ are the closed sets for some topology on $\operatorname{Spec} A$, called the *Zariski topology*. Principal open subsets $D(f) := \operatorname{Spec} A - V(\{f\}) = \{\mathfrak{p} : \mathfrak{p} \not\ni f\}$ constitute a base of Zariski topology on $\operatorname{Spec} A$, and any A-module M defines a *sheaf* $\tilde{M}$ on $\operatorname{Spec} A$, characterized by $\Gamma(D(f), M) = M_f = S_f^{-1} M$, where S_f is the multiplicative system $\{1, f, f^2, \ldots\}$.

Furthermore, $\tilde{A}(n) = \widetilde{A(n)}$ also turn out to be sheaves, and their collection defines a sheaf of generalized rings $\tilde{A} = \mathscr{O}_{\operatorname{Spec} A}$, and all $\tilde{M}$ are $\mathscr{O}_{\operatorname{Spec} A}$-modules. In this way $\operatorname{Spec} A$ becomes a generalized ringed space, and any A-module M defines a sheaf of modules $\tilde{M}$ over $\operatorname{Spec} A$; such sheaves of modules are called *quasicoherent.*

This construction is contravariant in A in the usual manner, i.e. any homomorphism $\varphi : A \to B$ defines a continuous map ${}^a\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$, given by $\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$. This map naturally extends to a morphism of generalized ringed spaces, having all the elementary properties of EGA I, e.g. direct images of quasicoherent sheaves correspond to scalar restriction with respect to φ, and pullbacks of quasicoherent sheaves are given by scalar extension.

In this way we obtain a theory of prime spectra of generalized rings, i.e. *affine generalized schemes*, having almost all elementary properties of EGA I. Of course, when we apply our theory to a classical ring A, we recover classical prime spectra $\operatorname{Spec} A$.

0.6.4. (Drawbacks of prime spectra.) Despite all their advantages and similarity to classical case, prime spectra don't behave very nicely when applied to generalized rings without sufficiently many unary operations. For example, $|\operatorname{Aff}_{\mathbb{Z}}| = \{\boldsymbol{e}\}$, and the only prime ideal of $\operatorname{Aff}_{\mathbb{Z}}$ is the initial ideal $\varnothing_{\operatorname{Aff}_{\mathbb{Z}}}$, i.e. $\operatorname{Spec} \operatorname{Aff}_{\mathbb{Z}}$ is a one-element set. On the other hand, one would rather expect $\operatorname{Spec} \operatorname{Aff}_{\mathbb{Z}}$ to be similar to $\operatorname{Spec} \mathbb{Z}$, but with a different (smaller) structural sheaf.

0.6.5. (Solution: open pseudolocalizations and localization theories.) We try to deal with this problem as follows. We say that a generalized ring homomorphism $\rho : A \to A'$ is a *pseudolocalization* if A' is flat over A, and $\rho_* : A'\text{-}\mathit{Mod} \to A\text{-}\mathit{Mod}$ is fully faithful. For example, $A \to S^{-1}A$ is a pseudolocalization for any multiplicative system S. Next, we say that $\rho : A \to A'$ is an *open pseudolocalization* if it is a finitely presented pseudolocalization. For example, $A \to A_f = A[f^{-1}]$ is an open pseudolocalization for any $f \in |A|$. There might be more open pseudolocalizations even in the classical case; notice that $\operatorname{Aff}_{\mathbb{Z}} \to \operatorname{Aff}_{\mathbb{Z}[f^{-1}]}$ is an open pseudolocalization for any $f \in \mathbb{Z}$.

After that, we define a *localization theory* $\mathcal{T}^?$ as follows. For any generalized ring A we must have a collection $\mathcal{T}^?_A$ of open pseudolocalizations of A, considered as a full subcategory of the category of A-algebras. We require all unary open localizations $A \to A_f$ to be contained in $\mathcal{T}^?_A$, and we impose some natural restrictions, e.g. stability of open pseudolocalizations from $\mathcal{T}^?$ under composition and base change.

We have a *minimal* localization theory, namely, the *unary localization theory* $\mathcal{T}^u$, consisting only of unary localizations $A \to A_f$. We have a maximal localization theory, called the *total localization theory* $\mathcal{T}^t$, which contains all open pseudolocalizations. Any other localization theory is somewhere in between: $\mathcal{T}^u \subset \mathcal{T}^? \subset \mathcal{T}^t$.

0.6.6. (Spectra with respect to a localization theory.) Once we fix a localization theory $\mathcal{T}^?$, we can construct the $\mathcal{T}^?$-*spectrum* $\operatorname{Spec}^? A$ of a generalized ring A as follows. We denote by $\mathcal{S}_A^?$ the category, opposite to the category $\mathcal{T}_A^?$ of open pseudolocalizations $A \to A'$ from $\mathcal{T}^?$ with source A. Since all pseudolocalizations are epimorphisms of generalized rings, all morphisms in $\mathcal{S}_A^?$ are monomorphisms, and there is at most one morphism between any two objects of $\mathcal{S}_A^?$. In other words, $\mathcal{S}_A^?$ is equivalent to category defined by an ordered set.

For example, if we choose the unary localization theory $\mathcal{T}^u$, then $\mathcal{S}_A^?$ can be thought of as the ordered set of principal open subsets $D(f) \subset \operatorname{Spec} A$, since $A \to A_g$ factorizes through $A \to A_f$ iff $D(g) \subset D(f)$. In this way objects of $\mathcal{S}_A^?$ are something like "abstract principal open subsets" of $\operatorname{Spec}^? A$.

For any A-module M we define a presheaf of sets $\tilde{M} : (\mathcal{S}_A^?)^0 \to \mathit{Sets}$, given by $\tilde{M} : (A \to A') \mapsto M_{(A')}$. Furthermore, we have a "structural presheaf" of generalized rings $\mathscr{O} : (A \to A') \mapsto A'$, and $\tilde{M}$ admits a natural $\mathscr{O}$-module structure.

Since $\mathcal{T}^?$ is closed under pushouts and composition, $\mathcal{S}_A^?$ has fibered products, which can be thought of as "intersections of abstract principal open subsets". We introduce on $\mathcal{S}_A^?$ the finest Grothendieck topology, such that all $\tilde{M}$ become sheaves. In other words, a family of morphisms $(U_\alpha \to U)$ is a cover in $\mathcal{S}_A^?$ iff the following diagram

$$\tilde{M}(U) \longrightarrow \prod_\alpha \tilde{M}(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} \tilde{M}(U_\alpha \cap U_\beta) \tag{0.6.6.1}$$

is left exact for any A-module M.

For example, if we work with $\mathcal{T}^u$, then $\{D(f_\alpha) \to D(f)\}_\alpha$ is a cover in $\mathcal{S}_A^u$ (where $D(f)$ denotes the object of $\mathcal{S}_A^u$ corresponding to $A \to A_f$) iff $D(f) = \bigcup D(f_\alpha)$ in $\operatorname{Spec}^p A$.

In general we obtain a Grothendieck topology on $\mathcal{S}_A^?$, and define the *strong spectrum* $\operatorname{Spec}_s^? A$ as the corresponding topos, i.e. the category of sheaves of sets on $\mathcal{S}_A^?$. It is a generalized ringed topos with structural sheaf $\mathscr{O}$, and all $\tilde{M}$ are (sheaves of) $\mathscr{O}$-modules.

0.6.7. (Finite spectra.) In fact, it turns out to be more convenient to use *finite spectra* $\operatorname{Spec}^? A = \operatorname{Spec}_f^? A$, defined by the Grothendieck topology on $\mathcal{S}_A^?$, generated by all *finite* covers $\{U_\alpha \to U\}$ with the same property as above. For example, these finite spectra $\operatorname{Spec}^? A$ are always quasicompact, as well as their "abstract principal open subsets" (i.e. objects of $\mathcal{S}_A^?$), hence they are coherent, and in particular admit enough points by a general result of SGA 4. Since they are obviously generated by open objects (i.e. subobjects of the final object), namely, by the objects $U \in \operatorname{Ob} \mathcal{S}_A^?$, identified with corresponding representable sheaves $h_U : U' \mapsto \operatorname{Hom}(U', U)$, we can conclude that *topos* $\operatorname{Spec}^? A$ *is given by a uniquely determined topological space*, which will be denoted by $\operatorname{Spec}^? A$ as well.

In this way we obtain a generalized ringed space $\operatorname{Spec}^? A$, and can forget about sites and topoi, used during its construction. Unary spectrum $\operatorname{Spec}^u A$ always coincides with the prime spectrum $\operatorname{Spec}^p A$ constructed before, so this notion of $\mathcal{T}^?$-spectra indeed generalizes that of prime spectra.

Notice that, while classical rings A sometimes admit open pseudolocalizations not of the form $A \to A_f$ (in fact, $A \to A'$ is an open pseudolocalization of classical rings iff $\operatorname{Spec} A' \to \operatorname{Spec} A$ is an open embedding of classical schemes), i.e. categories $\mathcal{S}_A^?$ might be distinct from $\mathcal{S}_A^u$, arising topoi $\operatorname{Spec}^? A$ are always equivalent, and corresponding topological spaces are canonically homeomorphic, i.e. we can write $\operatorname{Spec}^? A = \operatorname{Spec}^u A = \operatorname{Spec} A$ for a classical ring A, regardless of the localization theory chosen.

This property is due to the fact that $\operatorname{Spec}^u A' \to \operatorname{Spec}^u A$ is an open map for any open pseudolocalization of classical rings $A \to A'$, since open pseudolocalizations are flat and finitely presented. Furthermore, if we have some localization theory $\mathcal{T}^?$, such that $\operatorname{Spec}^? A' \to \operatorname{Spec}^? A$ is open for any open pseudolocalization $A \to A'$, then generalized ringed space $\operatorname{Spec}^? A$ is automatically isomorphic to $\operatorname{Spec}^t A$. In this way, any "reasonable" localization theory $\mathcal{T}^?$, having the property that $\operatorname{Spec}^? B \to \operatorname{Spec}^? A$ is an open map for any finitely presented flat A-algebra B, yields the same spectra as the total localization theory, i.e. the *total localization theory* $\mathcal{T}^t$ is a very natural candidate for constructing "correct" spectra.

0.6.8. (Unary spectra vs. total spectra.) We see that unary spectra $\operatorname{Spec}^u A = \operatorname{Spec}^p A$ have the advantage of admitting a direct description in terms of prime ideals, similar to the classical case, while total spectra $\operatorname{Spec}^t A$ can be expected to have better properties for non-classical generalized rings, yielding the same results for classical rings. For example, $\operatorname{Spec}^t \mathbb{Z} = \operatorname{Spec}^u \mathbb{Z} = \operatorname{Spec} \mathbb{Z}$, $\operatorname{Spec}^u \operatorname{Aff}_{\mathbb{Z}}$ consists only of one point, but $\operatorname{Spec}^t \operatorname{Aff}_{\mathbb{Z}}$ must be much larger, since it contains "abstract principal open subsets" corresponding to open pseudolocalizations $\operatorname{Aff}_{\mathbb{Z}} \to \operatorname{Aff}_{\mathbb{Z}[f^{-1}]}$.

Nevertheless, we lack a direct description of points of the total spectrum $\operatorname{Spec}^t A$, so the reader might find more convenient to consider only unary spectra Spec^u, and skip remarks concerning other localization theories while reading Chapter **6** for the first time.

0.6.9. (Generalized schemes.) Once we have a notion of spectra $\operatorname{Spec} A =$

$\mathrm{Spec}^? A$ (for some fixed localization theory $\mathcal{T}^?$), we can define an *affine generalized scheme* as a generalized ringed space isomorphic to some $\mathrm{Spec}\, A$, and a *generalized scheme* as a generalized ringed space $(X, \mathscr{O}_X)$, which admits an open cover by affine generalized schemes.

Morphisms of generalized schemes can be defined either as local morphisms of generalized locally ringed spaces (notice that the notions of local generalized rings, and local homomorphisms of such, depend on the choice of $\mathcal{T}^?$), or as generalized ringed space morphisms $f = (f, \theta) : (Y, \mathscr{O}_Y) \to (X, \mathscr{O}_X)$, such that for any open affine $\mathrm{Spec}\, B = V \subset Y$ and $\mathrm{Spec}\, A = U \subset X$, such that $f(V) \subset U$, the restriction $f|_V : V \to U$ is induced by a generalized ring homomorphism $A \to B$.

0.6.10. (Basic properties of generalized schemes.) We show that most basic properties of schemes and sheaves of modules over schemes, known from EGA I, transfer to our case without much modifications. For example, fibered products of generalized schemes exist and can be constructed in the classical fashion, by considering appropriate affine covers and putting $\mathrm{Spec}\, A \times_{\mathrm{Spec}\, C} \mathrm{Spec}\, B := \mathrm{Spec}(A \otimes_C B)$ for affine generalized schemes. Quasicoherent sheaves (of $\mathscr{O}_X$-modules) constitute a full subcategory of $\mathscr{O}_X$-*Mod*, closed under finite projective and arbitrary inductive limits. Quasicoherence is a local property (this is not so trivial as it might seem). Sheaves of $\mathscr{O}_X$-modules of finite type and of finite presentation can be defined in a natural way, and have some reasonable properties, e.g. any finitely presented $\mathscr{F}$ is quasicoherent, and a quasicoherent $\mathscr{F}$ is finitely presented iff for any open affine $U \subset X$ the $\Gamma(U, \mathscr{O})$-module $\Gamma(U, \mathscr{F})$ is finitely presented iff this condition holds for all U from an affine open cover of X.

We also have some classical filtered inductive limit properties, for example, $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}, -)$ commutes with filtered inductive limits of quasicoherent $\mathscr{O}_X$-modules, provided $\mathscr{F}$ is a finitely presented $\mathscr{O}_X$-module, and X is quasicompact and quasiseparated.

Now let us mention some notable distinctions from the classical case.

0.6.11. (Quasicoherent $\mathscr{O}_X$-algebras, affine morphisms, unarity.) We have a notion of a *quasicoherent $\mathscr{O}_X$-algebra $\mathscr{A}$*. It is more tricky than one might expect: $\mathscr{A}$ has to be a sheaf of generalized rings on X, equipped with a homomorphism $\mathscr{O}_X \to \mathscr{A}$, such that all components $\mathscr{A}(n)$ are quasicoherent $\mathscr{O}_X$-modules. Then we can construct *relative spectra* $\mathrm{Spec}\, \mathscr{A} \to X$, and define *affine morphisms* accordingly. We obtain a notion of *unary affine morphisms*, corresponding to unary quasicoherent algebras. This notion has no counterpart in classical algebraic geometry, where all affine morphisms are unary.

0.6.12. ("Closed" immersions are not closed.) Another distinction is that generalized scheme morphisms $\operatorname{Spec} B \to \operatorname{Spec} A$, induced by strict epimorphisms (i.e. surjective generalized ring homomorphisms) $A \twoheadrightarrow B$ are not necessarily closed. Roughly speaking, if A admits a zero 0, and B is obtained from A by imposing several relations of the form $f_i = 0$, $f_i \in |A|$, then $\operatorname{Spec} B$ can be identified with the closed subset $V(f_1, \ldots, f_n)$ of $\operatorname{Spec} A$, the complement of the union of principal open subsets $D(f_i)$. However, in general B is obtained by imposing relations $f_i = g_i$, and we cannot replace such relations with $f_i - g_i = 0$, in contrast with the classical case.

We cope with this problem by defining "closed" immersions $i : Y \to X$ as affine morphisms, defined by quasicoherent strict quotients of $\mathscr{O}_X$, and "closed" subschemes $Y \subset X$ as subobjects of X in the category of generalized schemes, defined by a "closed" immersion. We say that $X \to S$ is "separated" if $\Delta_{X/S} : X \to X \times_S X$ is "closed". Surprisingly, these notions retain some of their classical properties, e.g. any affine or projective morphism is automatically "separated", even if its diagonal is not closed in the topological sense.

0.6.13. (Absence of residue fields, and non-injective monomorphisms.) Another notable distinction is the absence of a reasonable theory of "generalized residue fields" $\kappa(x)$ of points $x \in X$ of a generalized scheme X. Even when such "residue fields" can be constructed (e.g. $\mathbb{F}_\infty$ is a natural candidate for the residue field of $\operatorname{Spec} \mathbb{Z}_{(\infty)}$ at ∞), they lack their usual properties, and cannot be used to transfer classical proofs to our situation.

For example, one of the classical application of residue fields is the statement about surjectivity of $|X \times_S Y| \to |X| \times_{|S|} |Y|$, where X and Y are S-schemes, and $|Z|$ denotes the underlying set of Z. In particular, a monomorphism of classical schemes is always injective on underlying sets.

These properties fail for generalized schemes. For example, $\operatorname{Spec} \mathbb{Z} \to \operatorname{Spec} \mathbb{F}_1$ is a monomorphism of generalized schemes (since $\mathbb{Z} \otimes_{\mathbb{F}_1} \mathbb{Z} = \mathbb{Z}$, i.e. $\operatorname{Spec} \mathbb{Z} \times_{\operatorname{Spec} \mathbb{F}_1} \operatorname{Spec} \mathbb{Z} = \operatorname{Spec} \mathbb{Z}$), but it is obviously not injective, since $\operatorname{Spec}^p \mathbb{F}_1$ consists of one point.

0.6.14. (Classical schemes as generalized schemes.) This statement about monomorphicity of $\operatorname{Spec} \mathbb{Z} \to \operatorname{Spec} \mathbb{F}_1$ and $\operatorname{Spec} \mathbb{Z} \to \operatorname{Spec} \mathbb{F}_\varnothing$, while discarding all hopes to obtain a non-trivial "square of $\operatorname{Spec} \mathbb{Z}$", has some benign consequences as well. Namely, the category of classical schemes, which can be also described as generalized schemes over $\operatorname{Spec} \mathbb{Z}$, turns out to constitute a full subcategory of the category of all generalized schemes. In this way we can safely treat classical schemes as generalized schemes, since no new morphisms arise. Furthermore, product of two classical schemes X and Y, computed over $\operatorname{Spec} \mathbb{F}_1$ or $\operatorname{Spec} \mathbb{F}_\varnothing$, coincides with their product over $\operatorname{Spec} \mathbb{Z}$, so we can write $X \times Y$ without any risk of confusion.

0.6.15. (Projective geometry and graded generalized rings.) A considerable part of Chapter **6** is dedicated to the study of *graded generalized rings* and *modules*, *(relative) projective spectra*, and related notions, i.e. to "generalized projective geometry". We don't want to explain this theory in any detail right now, since it is quite parallel to its classical counterpart, with the usual exception of notions of unarity and pre-unarity. We would like, however, to present a motivation for our definition of a graded generalized ring.

Suppose we are given a generalized scheme X, a commutative monoid Δ, and a homomorphism $\varphi : \Delta \to \mathrm{Pic}(X)$. Let's denote by $\mathscr{O}[\lambda]$ the line bundle corresponding to $\lambda \in \Delta$, and suppose we have some canonical isomorphisms $\mathscr{O}[\lambda + \delta] = \mathscr{O}[\lambda] \otimes_{\mathscr{O}_X} \mathscr{O}[\delta]$. For example, we might start from one "ample" line bundle $\mathscr{L}$ on X, and put $\Delta = \mathbb{Z}$, $\mathscr{O}[n] := \mathscr{L}^{\otimes n}$.

In the classical situation we would define a Δ-graded ring $R_* = \Gamma_*(X, \varphi)$ simply by putting $R_\lambda := \Gamma(X, \mathscr{O}[\lambda])$, using isomorphisms $\mathscr{O}[\lambda] \otimes \mathscr{O}[\delta] \to \mathscr{O}[\lambda+\delta]$ to define multiplication on $R_* := \bigoplus R_\lambda$. In the generalized situation we should care about operations of higher arities as well.

Thus we put $R_\lambda(n) := \Gamma(X, L_{\mathscr{O}_X}(n) \otimes_{\mathscr{O}_X} \mathscr{O}[\lambda])$, where $L_{\mathscr{O}_X}(n) = \mathscr{O}_X^{(n)} = \mathscr{O}_X(n)$ is the free $\mathscr{O}_X$-module of rank n. Twisting the "composition maps" $\mathscr{O}_X(k) \times \mathscr{O}_X(n)^k \to \mathscr{O}_X(n)$ with the aid of $\mathscr{O}[\lambda]$ and $\mathscr{O}[\mu]$, and taking global sections, we arrive to some "graded composition maps" $R_\lambda(k) \times R_\mu(n)^k \to R_{\lambda+\mu}(n)$.

In this way a Δ-graded algebraic monad R may be defined as a collection of sets $R_\lambda(n)$, $n \geq 0$, $\lambda \in \Delta$, together with some transition maps $R_\lambda(\varphi) : R_\lambda(n) \to R_\lambda(m)$, defining an algebraic endofunctor R_λ, a "unit element" $\boldsymbol{e} \in R_0(1)$, and some "graded composition maps" $\mu_{n,\mu}^{(k,\lambda)} : R_\lambda(k) \times R_\mu(n)^k \to R_{\lambda+\mu}(n)$, satisfying some natural "graded" analogues of the axioms of an algebraic monad. Of course, commutativity also has its graded counterpart, so we can speak about commutative graded algebraic monads, i.e. graded generalized rings.

Informally, $R_\lambda(n)$ should be thought of as "the degree λ part of the free R-module of rank n".

0.6.16. (Sections of projective bundles.) Once this "generalized projective geometry" is constructed, we apply it to symmetric algebras $S_{\mathscr{O}_S}(\mathscr{F})$ of quasicoherent sheaves $\mathscr{F}$ (e.g. locally free $\mathscr{O}_S$-modules, i.e. vector bundles) on a generalized scheme S, thus defining *projective bundles* $\mathbb{P}_S(\mathscr{F}) = \operatorname{Proj} S_{\mathscr{O}_S}(\mathscr{F})$ over S.

These projective bundles seem to retain most of their classical properties. For example, sections of $\mathbb{P}_S(\mathscr{F})$ over S are in one-to-one correspondence to those strict quotients of $\mathscr{F}$, which are line bundles over S. We can apply this to compute A-valued points of $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{F}_\varnothing} = \operatorname{Proj} \mathbb{F}_\varnothing[T_0^{[1]}, \ldots, T_n^{[1]}]$ over any generalized ring A with $\mathrm{Pic}(A) = 0$: elements of $\mathbb{P}^n(A)$ correspond to surjective homomorphisms from free A-module $A(n+1)$ into $|A| = A(1)$, considered modulo multiplication by elements from $|A|^\times$. Since $\mathrm{Hom}_A(A(n+1), |A|) \cong |A|^{n+1}$, we see that $\mathbb{P}^n(A)$ consists of $(n+1)$-tuples $(X_0 : X_1 : \ldots : X_n)$ of elements of $|A|$, generating together the unit ideal of $|A|$ (i.e. "coprime"), modulo multiplication by invertible elements from $|A|^\times$, exactly as in the classical case.

For example, $\mathbb{P}^n(\mathbb{F}_1)$ consists of $2^{n+1}-1$ points $(X_0 : \ldots : X_n)$, with $X_i \in \{0,1\}$, and not all $X_i = 0$. In particular, $\mathbb{P}^1(\mathbb{F}_1) = \{(0:1), (1:1), (1:0)\} = \{0, 1, \infty\}$. On the other hand, the underlying topological space of $\mathbb{P}^1_{\mathbb{F}_1} = \mathrm{Proj}\,\mathbb{F}_1[X, Y]$ consists also of three points: two closed points, corresponding to graded ideals (X) and (Y), and one generic point ξ, corresponding to (0).

It is interesting to note that $\mathbb{F}_1$-points 0 and ∞ are supported at the closed points of $\mathbb{P}^1_{\mathbb{F}_1}$, while $1 = (1:1)$ *is supported at the generic point* $\xi \in \mathbb{P}^1_{\mathbb{F}_1}$. This is possible only because $\mathbb{P}^1_{\mathbb{F}_1}$ is "separated" over $\mathrm{Spec}\,\mathbb{F}_1$, but not separated in the classical sense: otherwise any section would have been a closed map.

0.7. (Applications to Arakelov geometry.) Chapter **7** deals with the applications of the theory of generalized rings and schemes to Arakelov geometry. In particular, we construct the "compact model" $\widehat{\mathrm{Spec}\,\mathbb{Z}}$ of $\mathbb{Q}$, show that any algebraic variety $X/\mathbb{Q}$ admits a finitely presented model $\mathscr{X}$ over $\widehat{\mathrm{Spec}\,\mathbb{Z}}$, prove an "archimedian valuative criterion of properness", use it to extend rational points $P \in X(\mathbb{Q})$ to uniquely determined sections σ_P of a model $\mathscr{X}/\widehat{\mathrm{Spec}\,\mathbb{Z}}$, provided X and $\mathscr{X}$ are projective, and finally relate the arithmetic degree of the pullback $\sigma_P^*\mathscr{O}_{\mathscr{X}}(1)$ of the ample line bundle on $\mathscr{X}$ to the logarithmic height of point P.

Notice, however, that the development of an intersection theory and a theory of Chern classes is postponed until Chapter **10**.

0.7.1. (Construction of $\widehat{\mathrm{Spec}\,\mathbb{Z}}^{(N)}$.) Fix any integer $N > 1$, and consider generalized rings $B_N := \mathbb{Z}[N^{-1}] \subset \mathbb{Q}$, $A_N := B_N \cap \mathbb{Z}_{(\infty)}$. One can show that $1/N \in |A_N|$, and that $A_N[(1/N)^{-1}] = B_N$, i.e. $\mathrm{Spec}\,B_N$ is isomorphic to principal open subsets both of $\mathrm{Spec}\,\mathbb{Z}$ and of $\mathrm{Spec}\,A_N$. Therefore, we can patch together $\mathrm{Spec}\,\mathbb{Z}$ and $\mathrm{Spec}\,A_N$ along their principal open subsets isomorphic to $\mathrm{Spec}\,B_N$, and obtain a generalized scheme, denoted by $\widehat{\mathrm{Spec}\,\mathbb{Z}}^{(N)}$.

0.7.2. (Structure of $\widehat{\mathrm{Spec}\,\mathbb{Z}}^{(N)}$.) One can use any localization theory $\mathcal{T}^?$ to construct $\widehat{\mathrm{Spec}\,\mathbb{Z}}^{(N)}$. Let's describe the structure of $\widehat{\mathrm{Spec}\,\mathbb{Z}}^{(N)}$, constructed with the aid of unary localization theory $\mathcal{T}^u$. Then $\widehat{\mathrm{Spec}\,\mathbb{Z}}^{(N)} = \mathrm{Spec}\,\mathbb{Z} \cup \{\infty\}$ as a set. The topology on $\widehat{\mathrm{Spec}\,\mathbb{Z}}^{(N)}$ is such that the generic point ξ of $\mathrm{Spec}\,\mathbb{Z}$ is still generic, ∞ is a closed point, as well as all points p with $p \mid N$, while points p with $p \nmid N$ are not closed. Their closure consists of p and ∞. Furthermore, while the local rings of $\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$ at points p are equal to $\mathbb{Z}_{(p)}$ as expected, the local ring at ∞ equals A_N, not $\mathbb{Z}_{(\infty)}$ as one expects from the "true" compactification of $\operatorname{Spec}\mathbb{Z}$.

In this way $\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$ is just a crude approximation of $\widehat{\operatorname{Spec}\mathbb{Z}}$, which depends on the choice of $N > 1$ to a considerable extent.

0.7.3. (Morphisms $f_N^{NM} : \widehat{\operatorname{Spec}\mathbb{Z}}^{(NM)} \to \widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$.) Given any integers $N > 1$ and $M \geq 1$, we construct a natural generalized scheme morphism $f_N^{NM} : \widehat{\operatorname{Spec}\mathbb{Z}}^{(NM)} \to \widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$, which is an isomorphism on the open subschemes of $\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$ and $\widehat{\operatorname{Spec}\mathbb{Z}}^{(NM)}$, isomorphic to $\operatorname{Spec}\mathbb{Z}$. In other words, f_N^{NM} changes something only over the archimedian point ∞, and we see later that f_N^{NM} is projective, i.e. it behaves like a sort of blow-up over the archimedian point ∞ of $\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$. Notice that f_N^{NM} is bijective, i.e. this is something like a blow-up of a cusp-like singularity on a projective curve.

0.7.4. (Compactification $\widehat{\operatorname{Spec}\mathbb{Z}}$.) Since these f_N^{NM} are "transitive" in an obvious sense, we get a projective system of generalized schemes $\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$ over the filtered set of integers $N > 1$, ordered by division. It is very natural to define the "true" compactification $\widehat{\operatorname{Spec}\mathbb{Z}}$ as "$\varprojlim$" $\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$.

Unfortunately, this projective limit doesn't exist in the category of generalized schemes. However, we can compute it either in the category of pro-generalized schemes, or in the category of generalized ringed spaces. When we are concerned only with finitely presented objects (sheaves, schemes,…) over $\widehat{\operatorname{Spec}\mathbb{Z}}$, these two approaches tend to yield equivalent answers, so we can freely choose the most convenient approach in each particular situation.

0.7.5. ($\widehat{\operatorname{Spec}\mathbb{Z}}$ as a generalized ringed space.) When we consider $\widehat{\operatorname{Spec}\mathbb{Z}}$ as a generalized ringed space, we obtain something very similar to one's idea of the "smooth" compactification of $\operatorname{Spec}\mathbb{Z}$. For example, $\widehat{\operatorname{Spec}\mathbb{Z}} = \operatorname{Spec}\mathbb{Z} \cup \{\infty\}$, but now all points except the generic one ξ are closed, and the local ring at ∞ is indeed equal to $\mathbb{Z}_{(\infty)}$, as one would expect. Therefore, we can say that our $\widehat{\operatorname{Spec}\mathbb{Z}}$ is the "true" compactification of $\operatorname{Spec}\mathbb{Z}$.

0.7.6. (Finitely presented sheaves on $\widehat{\operatorname{Spec}\mathbb{Z}}$.) Notice that we have an open embedding $i : \operatorname{Spec}\mathbb{Z} \to \widehat{\operatorname{Spec}\mathbb{Z}}$, and natural morphisms $\hat{\xi} : \operatorname{Spec}\mathbb{Q} \to \widehat{\operatorname{Spec}\mathbb{Z}}$ ("generic point") and $\hat{\eta} : \operatorname{Spec}\mathbb{Z}_{(\infty)} \to \widehat{\operatorname{Spec}\mathbb{Z}}$) ("archimedian point"), both in the pro-generalized scheme and in the generalized ringed space descriptions of $\widehat{\operatorname{Spec}\mathbb{Z}}$. Therefore, if we start with a finitely presented $\mathscr{O}_{\widehat{\operatorname{Spec}\mathbb{Z}}}$-module $\mathscr{F}$, compute its pullbacks with respect to these three maps, and take their global sections, we obtain a finitely presented $\mathbb{Z}$-module $F_{\mathbb{Z}}$, a finite-dimensional $\mathbb{Q}$-vector space $F_{\mathbb{Q}}$, and a finitely presented $\mathbb{Z}_{(\infty)}$-module F_∞. Furthermore, since $\hat{\xi}$ factorizes through both i and $\hat{\eta}$, we get $F_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} \cong F_{\mathbb{Q}} \cong F_\infty \otimes_{\mathbb{Z}_{(\infty)}} \mathbb{Q}$.

In this way we obtain a functor from the category of finitely presented $\mathscr{O}_{\widehat{\operatorname{Spec}\mathbb{Z}}}$-modules into the category $\mathcal{C}$ of triples $(M_{\mathbb{Z}}, M_\infty, \theta)$, consisting of a finitely generated $\mathbb{Z}$-module $M_{\mathbb{Z}}$, a finitely presented $\mathbb{Z}_{(\infty)}$-module M_∞, and an isomorphism $\theta : M_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z} \xrightarrow{\sim} M_\infty \otimes_{\mathbb{Z}_{(\infty)}} \mathbb{Q}$ between their scalar extensions to $\mathbb{Q}$. In fact, *this functor is an equivalence of categories.* Furthermore, *other categories of finitely presented objects over* $\widehat{\operatorname{Spec}\mathbb{Z}}$, *e.g. finitely presented schemes or finitely presented sheaves on finitely presented schemes, admit a similar description in terms of corresponding objects over* $\operatorname{Spec}\mathbb{Z}$, $\operatorname{Spec}\mathbb{Z}_{(\infty)}$, *and isomorphisms between their base change to* $\operatorname{Spec}\mathbb{Q}$.

0.7.7. (Implication for finitely presented models.) An immediate implication is that constructing a finitely presented model $\bar{\mathscr{X}}/\widehat{\operatorname{Spec}\mathbb{Z}}$ of an algebraic variety $X/\mathbb{Q}$ is equivalent to constructing finitely presented models $\mathscr{X}/\mathbb{Z}$ and $\mathscr{X}_\infty/\mathbb{Z}_{(\infty)}$ of this X. Since existence and properties of models over $\mathbb{Z}$ have been intensively studied for quite a long time, we have to concentrate our efforts on models over $\mathbb{Z}_{(\infty)}$.

Before studying models of algebraic varieties, we discuss some further properties of $\widehat{\operatorname{Spec}\mathbb{Z}}$ and $\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$:

0.7.8. (Finite presentation of $\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$.) Generalized scheme $\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$ is finitely presented over $\mathbb{F}_{\pm 1} = \mathbb{F}_1[-^{[1]} \mid -(-\boldsymbol{e}) = \boldsymbol{e}]$, hence also over $\mathbb{F}_1$ and $\mathbb{F}_\varnothing$. Indeed, since $\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)} = \operatorname{Spec}\mathbb{Z} \cup \operatorname{Spec} A_N$ and $\mathbb{Z} = \mathbb{F}_{\pm 1}[+^{[2]} \mid x + 0 = x = 0 + x,\ x + (-x) = 0]$, all we have to check is that A_N is a finitely presented $\mathbb{F}_{\pm 1}$-algebra. We show this by presenting an explicit presentation of A_N over $\mathbb{F}_{\pm 1}$. Namely, A_N is generated over $\mathbb{F}_{\pm 1}$ by $(s_p^{[p]})_{p|N}$, where p runs through all prime divisors of N, and s_n denotes the *averaging operation* $s_n := (1/n)\{1\} + \cdots + (1/n)\{n\}$, subject to following *idempotency*, *symmetry* and *cancellation* relations:

$$s_n(\{1\}, \{1\}, \ldots, \{1\}) = \{1\} \tag{0.7.8.1}$$
$$s_n(\{1\}, \{2\}, \ldots, \{n\}) = s_n\bigl(\{\sigma(1)\}, \ldots, \{\sigma(n)\}\bigr), \quad \forall \sigma \in \mathfrak{S}_n \tag{0.7.8.2}$$
$$s_n(\{1\}, \ldots, \{n-1\}, -\{n-1\}) = s_n(\{1\}, \ldots, \{n-2\}, 0, 0) \tag{0.7.8.3}$$

For example, $A_2 = \mathbb{F}_{\pm 1}[*^{[2]} \mid x * x = x,\ x * y = y * x,\ x * (-x) = 0]$ is generated by one binary operation $* := s_2$, similarly to $\mathbb{Z}$ and $\mathbb{F}_\infty$. There are some interesting matrices over A_2, which satisfy the braid relations; cf. **7.1.32** for more details.

0.7.9. (Line bundles over $\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$.) Another interesting result is that $\operatorname{Pic}(\operatorname{Spec} A_N) = 0$; since $\operatorname{Pic}(\operatorname{Spec}\mathbb{Z}) = 0$, we see that $\operatorname{Pic}(\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}) = \mathbb{Z}^\times \backslash B_N^\times / |A_N|^\times = B_N^\times/\{\pm 1\} = B_{N,+}^\times$. We prefer to write this group of positive invertible elements of $B_N = \mathbb{Z}[N^{-1}]$ in additive form:

$$\operatorname{Pic}(\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}) = \log B_{N,+}^\times = \bigoplus_{p|N} \mathbb{Z} \cdot \log p \tag{0.7.9.1}$$

For any $\lambda \in B_{N,+}^\times$ we denote by $\mathscr{O}(\log\lambda)$ the line bundle on $\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$ corresponding to $\log\lambda$ under this isomorphism. In this way $\mathscr{O}(0) = \mathscr{O}$ and $\mathscr{O}(\log\lambda + \log\mu) = \mathscr{O}(\log\lambda) \otimes_{\mathscr{O}} \mathscr{O}(\log\mu)$.

We see that $\operatorname{Pic}(\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)})$ is a free abelian group of rank r, generated by $\mathscr{O}(\log p_i)$, where $p_1, \ldots, p_r$ are the distinct prime divisors of N. In this respect $\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$ is somewhat similar to $P^r := (\mathbb{P}^1)^r$, which also has $\operatorname{Pic} \cong \mathbb{Z}^r$. Notice, however, that $\operatorname{Aut}(P^r)$ acts transitively on the canonical basis of $\operatorname{Pic}(P^r)$, i.e. all standard generators of $\operatorname{Pic}(P^r)$ have the same properties, while $\operatorname{Aut}(\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}) = 1$ acts trivially on generators $\mathscr{O}(\log p_i)$ of $\operatorname{Pic}(\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)})$.

Another interesting consequence is that we can compute $\operatorname{Pic}(\widehat{\operatorname{Spec}\mathbb{Z}}) = \operatorname{Pic}(\text{“}\varprojlim\text{”}\, \widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}) = \varinjlim_{N>1} \log B_{N,+}^\times = \log \mathbb{Q}_+^\times$. We see that $\operatorname{Pic}(\widehat{\operatorname{Spec}\mathbb{Z}})$ is a free abelian group with a countable system of generators $\mathscr{O}(\log p)$, $p \in \mathbb{P}$. If $\mathscr{L} \cong \mathscr{O}(\log\lambda)$, we say that $\log\lambda$ is the *(arithmetic) degree* of line bundle $\mathscr{L}$ over $\widehat{\operatorname{Spec}\mathbb{Z}}$.

0.7.10. (Ampleness of $\mathscr{O}(\log N)$ on $\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$.) Comparison with $(\mathbb{P}^1)^r$ leads us to believe that $\mathscr{O}(\log N)$, being the tensor product of the standard generators of the Picard group of $\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$, must be an ample line bundle. In order to check this we have to compute graded generalized ring $R_* := \Gamma(\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}, \mathscr{O}(\log N))$, as explained in **0.6.15**, and check whether $\operatorname{Proj} R_*$ is isomorphic to $\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$. This turns out to be the case, i.e. $\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$ *is a projective generalized scheme* (absolutely, i.e. over $\operatorname{Spec}\mathbb{F}_\varnothing$, as well as over $\operatorname{Spec}\mathbb{F}_1$ and $\operatorname{Spec}\mathbb{F}_{\pm 1}$).

0.7.11. ($\widehat{\operatorname{Spec}\mathbb{Z}}$ as an infinite resolution of singularities.) In this way we arrive to the following picture. Each $\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$ is a projective finitely presented scheme over $\mathbb{F}_{\pm 1}$, containing $\operatorname{Spec}\mathbb{Z}$ as an open subscheme. All $f_N^{NM} : \widehat{\operatorname{Spec}\mathbb{Z}}^{(NM)} \to \widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$ are projective finitely presented morphisms of generalized schemes as well, identical on open subsets isomorphic to $\operatorname{Spec}\mathbb{Z}$. We have already discussed that each individual f_N^{NM} looks like a blow-up of a complicated cusp-like singularity at ∞. Therefore, one might consider the "true" or "smooth" compactification $\widehat{\operatorname{Spec}\mathbb{Z}}$ as the result of an infinite resolution of singularities of any "non-smooth" compactification $\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$. Some further discussion may be found in **7.1.48**.

0.7.12. (Existence of models.) After discussing properties of $\widehat{\operatorname{Spec}\mathbb{Z}}$ and $\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$, we show existence of finitely presented models over $\widehat{\operatorname{Spec}\mathbb{Z}}$ or $\mathbb{Z}_{(\infty)}$ of affine or projective algebraic varieties $X/\mathbb{Q}$. Since a finitely presented model $\bar{\mathscr{X}}/\widehat{\operatorname{Spec}\mathbb{Z}}$ of $X/\mathbb{Q}$ is essentially the same thing as finitely presented models $\mathscr{X}/\mathbb{Z}$ and $\mathscr{X}_\infty/\mathbb{Z}_{(\infty)}$, we need to consider only models over $\mathbb{Z}_{(\infty)}$. Then we have two basic ways of constructing models:

- If $X = \operatorname{Spec}\mathbb{Q}[T_1, \ldots, T_n]/(f_1, \ldots, f_m)$ for some polynomicals $f_i \neq 0$, we may multiply f_i by some non-zero rational numbers so as to have $f_i \in |\mathbb{Z}_{(\infty)}[T_1^{[1]}, \ldots, T_n^{[1]}]|$, and put $\mathscr{X}_\infty := \operatorname{Spec}\mathbb{Z}_{(\infty)}[T_1^{[1]}, \ldots, T_n^{[1]} \,|\, f_1 = 0, \ldots, f_m = 0]$. This is obviously a finitely presented $\mathbb{Z}_{(\infty)}$-model of X, and the case of a projective X is dealt with similarly, writing $X = \operatorname{Proj}\mathbb{Q}[T_1, \ldots, T_n]/(F_1, \ldots, F_m)$ for some homogeneous polynomials F_i.

- If X is a closed subvariety of some algebraic variety P, known to admit a $\mathbb{Z}_{(\infty)}$-model $\mathscr{P}_\infty$ (e.g. $\mathbb{A}^n$ or $\mathbb{P}^n$), then we can take for $\mathscr{X}_\infty$ the "scheme-theoretical closure" of $X \subset P \subset \mathscr{P}_\infty$ in $\mathscr{P}_\infty$.

The first of these two approaches always produces finitely presented models, which, however, usually have some $\mathbb{Z}_{(\infty)}$-torsion, i.e. "torsion in the archimedian fiber", which may be thought of some sort of "embedded analytic torsion". On the other hand, the second approach always produces models without $\mathbb{Z}_{(\infty)}$-torsion, but they are usually not finitely presented, but only of finite type. In particular, such models cannot be immediately extended to models over $\widehat{\operatorname{Spec}\mathbb{Z}}$.

All we can do is represent such a model as a projective limit of finitely presented models $\mathscr{X}_\infty^{(\alpha)}$, extend each of them to a finitely presented model $\bar{\mathscr{X}}^{(\alpha)}$ over $\widehat{\operatorname{Spec}\mathbb{Z}}$, and compute the projective limit $\bar{\mathscr{X}} :=$ "$\varprojlim$" $\bar{\mathscr{X}}^{(\alpha)}$ in the category of pro-generalized schemes. In this way we can always obtain a torsion-free pro-finitely presented model $\bar{\mathscr{X}}/\widehat{\operatorname{Spec}\mathbb{Z}}$ in the category of pro-generalized schemes over $\widehat{\operatorname{Spec}\mathbb{Z}}$.

This construction is quite similar to "non-archimedian Arakelov theory" constructions of [BGS] and [GS], where the projective system of all models of a fixed variety X/K is considered, instead of choosing fixed models of algebraic varieties X, Y, ..., which might be poorly adapted to interesting morphisms $f : X \to Y$.

0.7.13. (Application: heights of points on projective varieties.) We give an interesting application of the above constructions. Namely, let $X \subset \mathbb{P}^n_{\mathbb{Q}}$ be a projective variety over $\mathbb{Q}$, embedded into $\mathbb{P}^n_{\mathbb{Q}}$ as a closed subvariety. Denote by $\mathscr{X}$ the "scheme-theoretical closure" of X in $\mathbb{P}^n_{\widehat{\operatorname{Spec}\mathbb{Z}}}$; it is a torsion-free model of X of finite type, but in general not of finite presentation. Next, let $\mathscr{O}_{\mathscr{X}}(1)$ be the natural ample line bundle on $\mathscr{X}$, i.e. the pullback of the Serre line bundle on $\mathscr{P} := \mathbb{P}^n_{\widehat{\operatorname{Spec}\mathbb{Z}}}$ with respect to "closed" embedding $\mathscr{X} \to \mathscr{P}$.

We show that any rational point $P \in X(\mathbb{Q})$ extends to a unique section $\sigma_P : \widehat{\operatorname{Spec}\mathbb{Z}} \to \mathscr{X}$, using an "archimedian valuative criterion" for this. Furthermore, we compute the arithmetic degree of line bundle $\sigma_P^*\mathscr{O}_{\mathscr{X}}(1)$ over $\widehat{\operatorname{Spec}\mathbb{Z}}$: it turns out to be equal to the *logarithmic height* of point $P \in X(\mathbb{Q}) \subset \mathbb{P}^n(\mathbb{Q})$, defined by the classical formula

$$h(P) = \log \prod_{v \in \mathbb{P} \cup \{\infty\}} \max\bigl(|X_0|_v, |X_1|_v, \ldots, |X_n|_v\bigr) \qquad (0.7.13.1)$$

If we choose homogeneous coordinates $P = (X_0 : X_1 : \cdots : X_n)$ of P to be coprime integers, then $h(P) = \log\max(|X_0|, \ldots, |X_n|)$.

This theorem seems to be a nice counterpart of the classical theorem of Arakelov geometry, relating heights of rational points to arithmetic degrees of pullbacks of Serre bundles.

0.8. (Homological and homotopic algebra.) Chapter **8** opens the "homological", or rather "homotopic" part of this work, consisting of the three last chapters. It begins with a detailed introduction or "plan" (cf. **8.0**) for this chapter, and for the remaining "homotopic" chapters as well. Therefore, we won't repeat that introduction here; instead, we would like to explain how homotopic algebra replaces homological algebra, for non-additive categories like Σ-*Mod*, where Σ is some generalized ring. This is especially useful because homotopic algebra is not widely known among specialists in arithmetic geometry.

0.8.1. (Model categories.) One of the most fundamental notions of homotopic algebra is that of a *model category*, due to Quillen (cf. [Quillen]; cf. also [DwyerSpalinski] for a short introduction to model categories). A *model category* $\mathcal{C}$ is simply a category $\mathcal{C}$, with arbitrary inductive and projective limits (actually Quillen originally required only existence of finite limits), together with three distinguished classes of morphisms, called *fibrations*, *cofibrations* and *weak equivalences*. Furthermore, if a morphism is both a weak equivalence and a fibration (resp. cofibration), it is said to be an *acyclic fibration* (resp. *acyclic cofibration*).

These three distinguished classes of morphisms are subject to certain axioms. For example, each of them is stable under composition and retracts, and weak equivalences satisfy the *2-out-of-3 axiom:* "if two of $g \circ f$, f and g are weak equivalences, so is the third". Furthermore, there is an important *factorization axiom*, which claims that any morphism f can be factorized into a cofibration, followed by an acyclic fibration, as well as into an acycloc cofibation, followed by a fibration (usually one can even choose such factorizations functorially in f). This axiom is especially useful for constructions in model categories, e.g. for constructing derived functors.

Finally, the last important model category axiom says that acyclic cofibrations have the *LLP (left lifting property)* with respect to fibrations, or equivalently, fibrations have the *RLP (right lifting property)* with respect to acyclic cofibrations, and that cofibrations have the LLP with respect to acyclic fibrations. Recall that a morphism $i : A \to B$ is said to have the LLP with respect to $p : X \to Y$, or equivalently, p is said to have the RLP with respect to i, if for any two morphisms $u : A \to X$ and $v : B \to Y$, such that $pu = vi$, one can find a "lifting" $h : B \to X$, such that $hi = u$ and $ph = v$:

$$\begin{tikzcd} A \arrow[r, dashed, "u"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, dashed, "v"] \arrow[ur, dotted, "\exists h"] & Y \end{tikzcd} \tag{0.8.1.1}$$

0.8.2. (Example: non-negative chain complexes.) Let us make an example to illustrate the above definition. Let $\mathscr{A}$ be any abelian category with sufficiently many projective objects, e.g. $\mathcal{A} = R\text{-}Mod$ for a classical ring R. Then the category $\mathcal{C} := \mathrm{Ch}(\mathcal{A})$ of *non-negative* chain complexes $K = (\cdots \xrightarrow{\partial} K_1 \xrightarrow{\partial} K_0 \to 0)$ admits a natural model category structure. Weak equivalences for this model category structure are just the quasi-isomorphisms, i.e. chain maps $f : K \to L$, such that all $H_n(f) : H_n(K) \to H_n(L)$ are isomorphisms in $\mathcal{A}$. A chain map $f : K \to L$ is a fibration iff all its components $f_n : K_n \to L_n$ with $n > 0$ are epimorphisms (surjective R-linear maps, if $\mathcal{A} = R\text{-}Mod$). Finally, $f : K \to L$ is a cofibration iff all its components $f_n : K_n \to L_n$ are monomorphisms with projective cokernels.

Then the lifting axiom becomes a reformulation of a well-known statement from homological algebra about chain maps from complexes of projective modules into acyclic complexes.

0.8.3. (Fibrant and cofibrant objects and replacements.) Let $\mathcal{C}$ be a model category, and denote by $\varnothing_{\mathcal{C}}$ and $e_{\mathcal{C}}$ its initial and final objects. An object $X \in \operatorname{Ob}\mathcal{C}$ is said to be *cofibrant* if $\varnothing_{\mathcal{C}} \to X$ is a cofibration, and *fibrant* if $X \to e_{\mathcal{C}}$ is a fibration. We denote by $\mathcal{C}_c$ (resp. $\mathcal{C}_f$, $\mathcal{C}_{cf}$) full subcategories of $\mathcal{C}$, consisting of cofibrant (resp. fibrant, resp. fibrant and cofibrant) objects. Finally, a *cofibrant replacement* of an object $X \in \mathrm{Ob}\,\mathcal{C}$ is a weak equivalence $P \to X$ with $P \in \mathrm{Ob}\,\mathcal{C}_c$ (sometimes we consider *strong* cofibrant replacements, i.e. require $P \to X$ to be an acyclic fibration), and similarly a *fibrant replacement* is a weak equivalence $X \to Q$ with fibrant target. Factorization axiom of model categories, when applied to $\varnothing_{\mathcal{C}} \to X$ and $X \to e_{\mathcal{C}}$, shows existence of (co)fibrant replacements.

For example, if $\mathcal{C} = \mathrm{Ch}(\mathcal{A})$ is the category of non-negatively graded chain complexes as above, then $\mathcal{C}_f = \mathcal{C}$, but $\mathcal{C}_c$ consists of complexes of projective objects, and a cofibrant replacement $P \to X$ is a quasi-isomorphism from a complex P consisting of projective objects to given complex X, i.e. a *projective resolution.*

This explains how cofibrant replacements are used in general: under some additional restrictions, we can construct left derived functor $\mathbb{L}F$ of a functor $F : \mathcal{C} \to \mathcal{D}$ between model categories by putting $\mathbb{L}F(X) := F(P)$, where $P \to X$ is any cofibrant replacement of X.

One can also say that *fibrations and cofibrations, and the factorization axiom are used for constructions in model categories.*

0.8.4. (Homotopic categories and derived functors.) Whenever we have a model category $\mathcal{C}$, or just any category $\mathcal{C}$ with a distinguished class of weak equivalences, we denote by $\mathrm{Ho}\,\mathcal{C}$ the *homotopic category* of $\mathcal{C}$, equal by definition to the localization of $\mathcal{C}$ with respect to the set of weak equivalences. Canonical functor $\mathcal{C} \to \mathrm{Ho}\,\mathcal{C}$ is denoted by $\gamma_{\mathcal{C}}$ or simply by γ. One of its fundamental properties is that f *is a weak equivalence in model category* $\mathcal{C}$ *iff* $\gamma(f)$ *is an isomorphism in* $\mathrm{Ho}\,\mathcal{C}$.

Given a functor $F : \mathcal{C} \to \mathcal{D}$ between two model categories, we define its *left derived functor* $\mathbb{L}F : \mathrm{Ho}\,\mathcal{C} \to \mathrm{Ho}\,\mathcal{D}$ as the functor, which is as close from the left to making the obvious diagram commutative as possible. More precisely, we must have a natural transformation $\xi : \mathbb{L}F \circ \gamma_C \to \gamma_D \circ F$, such that for any functor $H : \mathrm{Ho}\,\mathcal{C} \to \mathrm{Ho}\,\mathcal{D}$ and any natural transformation $\eta : H \circ \gamma_C \to \gamma_D \circ F$ there is a unique natural transformation $\zeta : H \to \mathbb{L}F$, such that $\eta = \xi \circ (\zeta \star \gamma_{\mathcal{C}})$.

For example, if we have a functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories with enough projectives, and denote by the same letter F its extension $F : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{B})$ to non-negative chain complexes, then $\mathrm{Ho}\,\mathrm{Ch}(\mathcal{A})$ is equivalent to full subcategory $\mathcal{D}^{\leq 0}(\mathcal{A})$ of the derived category of $\mathcal{A}$, and similarly for $\mathrm{Ho}\,\mathrm{Ch}(\mathcal{B})$, and then $\mathbb{L}F$ corresponds to a left derived functor in the sense of Verdier.

We see that the homotopic category $\mathrm{Ho}\,\mathcal{C}$ plays the role of the derived category (or at least of its non-positive part). It depends only on weak equivalences of $\mathcal{C}$, hence the same is true for derived functors as well. In fact, one can have different model structures on the same category $\mathcal{C}$, having the same weak equivalences; in such situations we get same $\operatorname{Ho}\mathcal{C}$ and $\mathbb{L}F$.

However, the correct choice of (co)fibrations is crucial for showing existence of derived functors, because of the following statement, due to Quillen: *if a functor $F : \mathcal{C} \to \mathcal{D}$ between two model categories transforms acyclic cofibrations between cofibrant objects into weak equivalences, then $\mathbb{L}F : \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{D}$ exists and can be computed with the aid of cofibrant replacements, i.e. $(\mathbb{L}F)(\gamma X) \cong \gamma F(P)$ for any cofibrant replacement $P \to X$.*

This corresponds to the classical computation of derived functors via projective resolutions.

0.8.5. (Simplicial objects and Dold–Kan correspondence.) We see that in order to obtain "homological algebra" over a generalized ring Σ we have to do two things: (a) construct a counterpart of $\mathrm{Ch}(\Sigma\text{-}\mathit{Mod})$; this is necessary since the "non-negative chain complex" description makes sense only over abelian categories, and (b) introduce a reasonable model category structure on this replacement for $\mathrm{Ch}(\Sigma\text{-}\mathit{Mod})$; we need this to be able to define and construct homotopic categories and derived functors.

Of course, the principal requirement for these constructions is that we should recover $\mathrm{Ch}(\Sigma\text{-}\mathit{Mod})$ with its model category structure explained above whenever Σ is a classical ring.

It turns out that the natural replacement for $\mathrm{Ch}(\mathcal{A})$ for a non-abelian category $\mathcal{A}$ (e.g. $\mathcal{A} = \Sigma\text{-}\mathit{Mod}$) is the category $s\mathcal{A} = \mathit{Funct}(\mathbf{\Delta}^0, \mathcal{A})$ of *simplicial objects* over $\mathcal{A}$. This is due to the following fact: *whenever $\mathcal{A}$ is an abelian category, there are natural equivalences $K : \mathrm{Ch}(\mathcal{A}) \leftrightarrows s\mathcal{A} : N$ between the category of non-negative chain complexes over $\mathcal{A}$ and the category of simplicial objects over $\mathcal{A}$.* This correspondence between chain complexes and simplicial objects is called *Dold–Kan correspondence.*

0.8.6. (Simplicial objects.) Since the simplicial objects play a fundamental role in the homotopic algebra parts of our work, we would like to fix some notations, mostly consistent with those of [GZ].

We denote by $[n]$ the *standard finite ordered set* $\{0, 1, \ldots, n\}$, endowed with its natural linear order. Notice that $[n]$ is an $n+1$-element ordered set, while $\mathbf{n} = \{1, 2, \ldots, n\}$ is an n-element unordered set. One shouldn't confuse these two notations.

Next, we denote by $\mathbf{\Delta}$ the category of all standard finite ordered sets $[n]$, $n \geq 0$, considered as a full subcategory of the category of ordered sets (i.e. morphisms $\varphi : [n] \to [m]$ are the non-decreasing maps). We denote by $\partial_n^i : [n-1] \to [n]$, $0 \leq i \leq n > 0$, the *i-th face map*, defined as increasing injection not taking value i, and by $\sigma_n^i : [n+1] \to [n]$, $0 \leq i \leq n$, the *i-th* *degeneracy map*, i.e. the non-decreasing surjection, which takes value i twice.

Now a *simplicial object* over a category $\mathcal{A}$ is simply a contravariant functor $X : \mathbf{\Delta}^0 \to \mathcal{A}$. We usually write X_n instead of $X([n])$, $n \geq 0$, and put $d_i^{n,X} := X(\partial_n^i) : X_n \to X_{n-1}$, $s_i^{n,X} := X(\sigma_n^i) : X_n \to X_{n+1}$. Since any morphism in $\mathbf{\Delta}$ can be written as a product of face and degeneracy maps, a simplicial object can be described in terms of a sequence of objects $(X_n)_{n\geq 0}$ and a collection of face and degeneracy morphisms $d_i^{n,X} : X_n \to X_{n-1}$, $s_i^{n,X} : X_n \to X_{n+1}$, $0 \leq i \leq n$, subject to certain relations.

0.8.7. (Simplicial sets.) Our next step is to introduce a model category structure on $s\Sigma$-*Mod*, compatible via Dold–Kan correspondence with one we had on $\mathrm{Ch}(\Sigma\text{-}Mod)$ whenever Σ is a classical ring. We consider first the "basic" case $\Sigma = \mathbb{F}_\varnothing$, i.e. we need a model structure on the category $s\mathbb{F}_\varnothing$-*Mod* $=$ *sSets* of simplicial sets.

This category has indeed a classical model category structure, which can be described as follows. Denote by $\Delta(n) \in \mathrm{Ob}\, sSets$ the "standard n-dimensional simplex", i.e. the contravariant functor $\mathbf{\Delta}^0 \to Sets$, $[k] \mapsto \mathrm{Hom}_{\mathbf{\Delta}}([k],[n])$, represented by $[n]$. We denote by $\dot{\Delta}(n) \subset \Delta(n)$ the "boundary" of $\Delta(n)$; actually $\dot{\Delta}(n)_k$ can be described as the set of all non-decreasing non-surjective maps $\varphi : [k] \to [n]$.

We denote by $\Lambda_k(n) \subset \dot{\Delta}(n)$, $0 \leq k \leq n$, the simplicial set, obtained from $\dot{\Delta}(n)$ by removing its k-th $(n-1)$-dimensional face. Set $\Lambda_k(n)_m$ consists of all order-preserving maps $\varphi : [m] \to [n]$, such that $[n] - \varphi([m])$ is distinct from both $\varnothing$ and $\{k\}$.

Now consider the set of *standard cofibrations* $I := \{\dot{\Delta}(n) \to \Delta(n)\}_{n\geq 0}$ and the set of *standard acyclic cofibrations* $J := \{\Lambda_k(n) \to \Delta(n)\}_{0\leq k\leq n>0}$. We *define* acyclic fibrations in *sSets* as the set of morphisms, which have the RLP with respect to all standard cofibrations from I, and fibrations as the set of morphisms having the RLP with respect to J. After that cofibrations (resp. acyclic cofibrations) are defined as morphisms having the LLP with respect to all acyclic fibrations (resp. fibrations). In particular, all morphisms from I (resp. J) are indeed cofibrations (resp. acyclic cofibrations). Finally, we define *weak equivalences* in *sSets* as morphisms, which can be decomposed into an acyclic cofibration, followed by an acyclic fibration.

It is a theorem of Quillen that the cofibrations, fibrations and weak equivalences thus defined do satisfy the model category axioms, so we obtain a natural model category structure on *sSets*. This model category structure admits a "topological" description: if $|X|$ denotes the geometric realization of simplicial set X, then $f : X \to Y$ is a weak equivalence iff $|f| : |X| \to |Y|$ is a weak equivalence in topological sense, i.e. iff all $\pi_n(|f|, x) : \pi_n(|X|; x) \to \pi_n(|Y|; f(x))$ are bijective, for all $n \geq 0$ and all choices of base point $x \in |X|$.

Furthermore, cofibrations in *sSets* admit a very simple description: *$f : X \to Y$ is a cofibration of simplicial sets iff f is a monomorphism, i.e. iff all $f_n : X_n \to Y_n$ are injective.*

0.8.8. (Model category structure on $s\Sigma$-*Mod*.) Now we are ready to describe our model category structure on $s\Sigma$-*Mod*, for any algebraic monad Σ. First of all, notice that we have adjoint *free Σ-module* and *forgetful* functors $L_\Sigma : sSets \leftrightarrows s\Sigma\text{-}Mod : \Gamma_\Sigma$, equal to simplicial extensions of corresponding functors $Sets \leftrightarrows \Sigma\text{-}Mod$. It is extremely natural to expect that L_Σ preserves cofibrations and acyclic cofibrations, especially if we hope to construct a left derived $\mathbb{L}L_\Sigma$ afterwards. In particular, we can apply L_Σ to "standard generators" I and J of the model structure on *sSets*: we see that all morphisms from $L_\Sigma(I)$ (resp. $L_\Sigma(J)$) must be cofibrations (resp. acyclic cofibrations) in $s\Sigma$-*Mod*.

A natural next step is to consider the model structure on $s\Sigma$-*Mod*, generated by these sets $L_\Sigma(I)$ and $L_\Sigma(J)$ in the same way as the model category structure on *sSets* was generated by I and J. This means that we *define* fibrations in $s\Sigma$-*Mod* as morphisms with the RLP with respect to all morphisms from $L_\Sigma(J)$, and so on.

We apply a theorem of Quillen ([Quillen, 2.4], th. 4) to show that we indeed obtain a model category structure on $s\Sigma$-*Mod* in this way. Adjointness of L_Σ and Γ_Σ shows that *$f : X \to Y$ is a fibration (resp. acyclic fibration) in $s\Sigma$-Mod iff $\Gamma_\Sigma(f)$ is one in sSets.* Furthermore, it turns our that *$f : X \to Y$ is a weak equivalence in $s\Sigma$-Mod iff $\Gamma_\Sigma(f)$ is one in sSets.* Since cofibrations are characterized by their LLP with respect to acyclic fibrations, this completely determines the model category structure of $s\Sigma$-*Mod*.

0.8.9. (Derived category of Λ-modules.) One can check that the model category structure just defined on $s\Lambda$-*Mod* indeed corresponds to the natural model category structure on $\mathrm{Ch}(\Lambda\text{-}Mod)$ under Dold–Kan correspondence whenever Λ is a classical ring. Therefore, it is extremely natural to put $\mathcal{D}^{\leq 0}(\Lambda) = \mathcal{D}^{\leq 0}(\Lambda\text{-}Mod) := \mathrm{Ho}\, s\Lambda\text{-}Mod$. If Λ admits a zero (i.e. is an $\mathbb{F}_1$-algebra), then we have a natural *suspension functor* $\Sigma : s\Lambda\text{-}Mod \to s\Lambda\text{-}Mod$, which has a left derived functor $\Sigma = L\Sigma : \mathcal{D}^{\leq 0}(\Lambda) \to \mathcal{D}^{\leq 0}(\Lambda)$. This suspension functor is actually a generalization of the degree translation functor T on $\mathrm{Ch}(\Lambda)$. Therefore, it is natural to define $\mathcal{D}^-(\Lambda)$ as the *stable homotopic category* of $s\Lambda$-*Mod*, obtained by formally inverting functor $L\Sigma$ on $\mathrm{Ho}\, s\Lambda\text{-}Mod$.

0.8.10. (Derived category of $\mathbb{F}_1$-modules.) It is interesting to note that $s\mathbb{F}_1$-*Mod* is exactly the category of pointed simplicial sets, with its model category structure extensively used in algebraic topology. Therefore, $\mathcal{D}^{\leq 0}(\mathbb{F}_1) = \mathrm{Ho}\, s\mathbb{F}_1\text{-}Mod$ is the topologists' homotopic category, and $\mathcal{D}^-(\mathbb{F}_1)$ is the *stable* *homotopic category*. In this way a lot of statements in algebraic topology become statements about homological algebra over $\mathbb{F}_1$.

0.8.11. (Derived scalar extension.) Given any algebraic monad homomorphism $\rho : \Sigma \to \Xi$, we have a scalar extension functor $\rho^* = \Xi \otimes_\Sigma - : \Sigma\text{-}\mathcal{M}od \to \Xi\text{-}\mathcal{M}od$, which can be extended componentwise to a functor $\rho^* = s\rho^* : s\Sigma\text{-}\mathcal{M}od \to s\Xi\text{-}\mathcal{M}od$. This functor happens to transform generating cofibrations $L_\Sigma(I)$ of $s\Sigma\text{-}\mathcal{M}od$ into cofibrations $L_\Xi(I)$ in $s\Xi\text{-}\mathcal{M}od$, and similarly for $L_\Sigma(J)$. Starting from this fact, we show that ρ^* preserves cofibrations and acyclic cofibrations, hence by Quillen's theorem it admits a left derived $\mathbb{L}\rho^* : \mathcal{D}^{\leq 0}(\Sigma) \to \mathcal{D}^{\leq 0}(\Xi)$, which can be computed with the aid of cofibrant replacements: $\mathbb{L}\rho^* X = \rho^* P$ for any cofibrant replacement $P \to X$. If Σ and Ξ are classical rings, this $\mathbb{L}\rho^*$ corresponds to the usual left derived functor, since $P \to X$ corresponds via Dold–Kan to a projective resolution of a chain complex of Σ-modules. On the other hand, if Σ (hence also Ξ) is an algebraic monad with zero, $\mathbb{L}\rho^*$ commutes with suspension functors, hence it induces a “stable left derived functor” $\mathbb{L}\rho^* : \mathcal{D}^-(\Sigma) \to \mathcal{D}^-(\Xi)$, which also corresponds via Dold–Kan to the classical left derived functor whenever Σ and Ξ are classical rings.

0.8.12. (Derived tensor product.) Now let Λ be a generalized ring, i.e. a commutative algebraic monad. Then we have a tensor product $\otimes = \otimes_\Lambda : \Lambda\text{-}\mathcal{M}od \times \Lambda\text{-}\mathcal{M}od \to \Lambda\text{-}\mathcal{M}od$, and we want to construct a *derived tensor product* $\underline{\underline{\otimes}} : \mathcal{D}^{\leq 0}(\Lambda) \times \mathcal{D}^{\leq 0}(\Lambda) \to \mathcal{D}^{\leq 0}(\Lambda)$. We show that such a derived tensor product exists, and can be computed as follows. We extend $\otimes$ to simplicial Λ-modules by putting $(X \otimes_\Lambda Y)_n := X_n \otimes_\Lambda Y_n$ for all $n \geq 0$. After that we define $X \underline{\underline{\otimes}} Y := P \otimes_\Lambda Q$, where $P \to X$ and $Q \to Y$ are arbitrary cofibrant replacements.

If Λ is a classical commutative ring, this construction yields the usual derived product, because of *Eilenberg–Zilber theorem*, which says that taking the diagonal of a bisimplicial object over an abelian category corresponds via Dold–Kan (up to a homotopy equivalence) to computing the total complex of a bicomplex. Notice, however, that in the classical situation it is enough to replace *one* of the arguments X and Y by any its projective resolution, while in the generalized context this usually does not suffice: we *must* take cofibrant replacements of both X and Y.

If Λ admits a zero, this derived tensor product $\underline{\underline{\otimes}}$ commutes with suspension in each variable, hence it extends to the stable homotopic category, yielding a functor $\underline{\underline{\otimes}} : \mathcal{D}^-(\Lambda) \times \mathcal{D}^-(\Lambda) \to \mathcal{D}^-(\Lambda)$.

An important remark is that $\underline{\underline{\otimes}}$ defines an ACU $\otimes$-structure on $\mathcal{D}^{\leq 0}(\Lambda)$ (or on $\mathcal{D}^-(\Lambda)$, when Λ admits a zero), which has inner Homs $\mathbb{R}\,\mathrm{Hom}$; these inner Homs are indeed constructed as some right derived functors.

0.9. (Homotopic algebra over topoi.) The aim of Chapter **9** is to transfer the previous results to the case of modules over a generalized ringed topos $\mathcal{E} = (\mathcal{E}, \mathscr{O})$, e.g. topos of sheaves of sets over a site $\mathcal{S}$ or a topological space X. In particular, results of this chapter are applicable to generalized ringed spaces, e.g. generalized schemes.

0.9.1. (Stacks.) We start by recalling some general properties of stacks, which can be found for example in [Giraud]. Let us explain here why we need to use stacks at all. The main idea is that a stack (over a site or a topos) is the "correct" counterpart of a sheaf of categories.

Consider the following example. Let $\mathcal{S}$ be a site with fibered products (e.g. the category of open subsets $U \subset X$ of a topological space U.) Then a presheaf of sets $\mathscr{F}$ over $\mathcal{S}$ is simply a contravariant functor $\mathcal{S}^0 \to \mathit{Sets}$, i.e. for any $U \in \operatorname{Ob} \mathcal{S}$ we have a set $\mathscr{F}(U)$, and for any $V \to U$ we have a "restriction map" $\rho_U^V : \mathscr{F}(U) \to \mathscr{F}(V)$, often denoted by $x \mapsto x|_V$. Of course, these restriction maps must be "transitive".

Now the sheaf condition for $\mathscr{F}$ can be expressed as follows. Suppose we have a property P of sections of $\mathscr{F}$ (i.e. for any U and any $x \in \mathscr{F}(U)$ we can determine whether x has this property). Suppose that P is compatible with restrictions (i.e. if x has P, then so has any $x|_V$), that a section $x \in \mathscr{F}(U)$ with property P is uniquely determined (i.e. if both x and $x' \in \mathscr{F}(U)$ have property P for some U, then $x = x'$), and finally that a section x with property P exists "locally" over some U, i.e. there is some cover $(U_\alpha \to U)$ in $\mathcal{S}$, such that in each $\mathscr{F}(U_\alpha)$ there is a (necessarily unique) element x_α with property P. The sheaf condition then says that there is a unique element $x \in \mathscr{F}(U)$, restricting to x_α on each U_α. In other words, *the sheaf condition means that from local existence and uniqueness of a section with some property we can deduce global existence and uniqueness.*

Let's try to generalize this to "sheaves of categories". First of all, a "presheaf of categories" $\mathcal{C}$ over $\mathcal{S}$ must be a collection of categories $\mathcal{C}_U = \mathcal{C}(U)$, defined for each $U \in \operatorname{Ob} \mathcal{S}$, together with some "pullback" or "restriction" functors $\varphi^* : \mathcal{C}(U) \to \mathcal{C}(V)$, defined for each $V \xrightarrow{\varphi} U$ (we usually denote $\varphi^* X$ by $X|_V$). However, these pullback functors are usually defined only up to a canonical isomorphism, hence $(\varphi \circ \psi)^*$ might be not equal to $\psi^* \circ \varphi^*$, but must be canonically isomorphic to it. Of course, these isomorphisms $(\varphi \circ \psi)^* \cong \psi^* \circ \varphi^*$ must be compatible on triple compositions $\chi \circ \varphi \circ \psi$.

There is a better equivalent way of describing "presheaves of categories", which doesn't require to fix choices of all pullback functors φ^*. Namely, one considers the category $\mathcal{C}$, with objects given by $\operatorname{Ob} \mathcal{C} := \bigsqcup_{U \in \operatorname{Ob} \mathcal{S}} (\mathcal{C}(U))$, and morphisms given by $\operatorname{Hom}_{\mathcal{C}}((V, Y), (U, X)) := \bigsqcup_{\varphi \in \operatorname{Hom}_{\mathcal{S}}(V,U)} \operatorname{Hom}_{\mathcal{C}(V)}(Y, \varphi^* X)$, together with the natural projection functor $p : \mathcal{C} \to \mathcal{S}$. This $\mathcal{S}$-category

$\mathcal{C} \xrightarrow{p} \mathcal{S}$ turns out to contain all data we need, e.g. $\mathcal{C}(U) = \mathcal{C}_U$ is the pre-image of $U \subset \operatorname{Ob}\mathcal{S}$, considered as a point subcategory of $\mathcal{S}$, but determines φ^* only up to isomorphism, i.e. we don't have to fix any choice of φ^*. Transitivity conditions for φ^* translate into some conditions for $\mathcal{C} \xrightarrow{p} \mathcal{S}$, and when these conditions hold, we say that $\mathcal{C}$ *is a fibered category over* $\mathcal{S}$. (cf. SGA 1 or [Giraud] for more details). However, we often use the "naïve" description in terms of categories $\mathcal{C}(U)$ and pullback functors φ^*, just to convey the main ideas of the constructions we perform.

Next step is to define "sheaves of categories". Let $\mathcal{C}$ be a "presheaf of categories", i.e. a fibered category $\mathcal{C} \to \mathcal{S}$. Let P be a property of objects of $\mathcal{C}$, compatible with restrictions, and determining an object of fiber $\mathcal{C}(U) = \mathcal{C}_U$ uniquely up to a unique isomorphism. Notice that we cannot expect objects of categories to be singled out by some category-theoretic property, since any isomorphic object must have the same property; anyway, the restriction functors are always defined up to isomorphism, so compatibility with restrictions makes sense only for such properties. Now suppose that objects with property P exist "locally" over some $U \in \operatorname{Ob}\mathcal{S}$, i.e. one can find a cover $(U_\alpha \to U)$ in $\mathcal{S}$, and objects $X_\alpha \in \operatorname{Ob}\mathcal{C}(U_\alpha)$, having property P. However, uniqueness up to isomorphism doesn't imply that the restrictions of X_α and X_β to $U_{\alpha\beta} := U_\alpha \times_U U_\beta$ (i.e. to $U_\alpha \cap U_\beta$, if we work with open subsets of a topological space) coincide; they rather imply existence of isomorphisms $\theta_{\alpha\beta}$ between these two restrictions, and uniqueness of these isomorphisms translates into "transitivity" $\theta_{\alpha\gamma} = \theta_{\beta\gamma} \circ \theta_{\alpha\beta}$ for the restrictions of these isomorphisms to $U_{\alpha\beta\gamma}$.

In this way we obtain a *descent datum* for $\mathcal{C}$ over $U_\alpha \to U$. We expect it to be *efficient*, i.e. to determine an object $X \in \operatorname{Ob}\mathcal{C}(U)$, unique up to a unique isomorphism, such that $X|_{U_\alpha} \cong X_\alpha$. Replacing "presheaves of categories" with technically more convenient "fibered categories over $\mathcal{S}$", we arrive to the correct definition: a stack over a site $\mathcal{S}$ is a fibered category $\mathcal{C} \xrightarrow{p} \mathcal{S}$, such that any descent datum for $\mathcal{C}$ over any cover of $\mathcal{S}$ is efficient (cf. [Giraud]).

0.9.2. (Kripke–Joyal semantics.) After discussing some basic properties of stacks, we outline a variant of the so-called *Kripke–Joyal semantics*, adapted for proving statements about stacks. The key idea is that one can transfer classical proofs of statements about sets, maps and categories to the topos case (i.e. to the case of sheaves over a site $\mathcal{S}$), by replacing in a regular fashion sets by sheaves of sets, i.e. objects of corresponding topos $\mathcal{E} = \tilde{\mathcal{S}}$, maps of sets by morphisms in $\mathcal{E}$, categories by stacks over $\mathcal{S}$, and functors between categories by *cartesian* functors between stacks. However, there is one important restriction: the proof being transferred must be *intuitionistic*, i.e. it cannot involve the logical law of excluded middle, and any methods of proofs such as *reductio ad absurdum*, based on this logical law.

For example, most of elementary algebra (ring and module theory) is intuitionistic, and that's the reason why most statements about rings and modules can be generalized to statements about sheaves of rings and sheaves of modules over them.

0.9.3. (Model stacks.) Then we use Kripke–Joyal semantics to transfer model category axioms to the topos case, thus obtaining an interesting notion of a *model stack* $\mathcal{C}$ over a site $\mathcal{S}$ or a topos $\mathcal{E}$. It is interesting to note that individual fibers $\mathcal{C}(U)$ of a model stack $\mathcal{C}$ are not necessarily model categories, since the lifting properties hold in $\mathcal{C}$ only locally (i.e. after passing to a suitable cover).

We study this new notion of model stack, and prove some counterparts of Quillen's results about model categories.

0.9.4. (Pseudomodel stacks.) However, when we try to construct a model structure on stack $\mathbf{\mathfrak{s}SETS}$, given by $(\mathbf{\mathfrak{s}SETS})(U) := s\widetilde{\mathcal{S}_{/U}}$ (simplicial sheaves of sets over U), we fail to transfer Quillen's proof. The problem is that in the case of model categories we knew that arbitrary direct sums (coproducts) of (acyclic) cofibrations are (acyclic) cofibrations again, since they could be characterized by the LLP with respect to acyclic or all fibrations. However, in model stacks cofibrations are characterized by *local* LLP with respect to acyclic fibrations. This is sufficient to conclude only that *finite* direct sums of cofibrations are cofibrations. On the other hand, Quillen's "small object argument", used to prove the factorization axiom in *sSets*, expects the class of cofibrations to be closed under arbitrary direct sums, (possibly infinite) sequential compositions and retracts. Therefore, we are unable to prove the factorization axiom for $\mathbf{\mathfrak{s}SETS}$.

The only way to deal with this problem is to weaken either the factorization or the lifting axiom of model stacks. We decided to weaken the lifting axiom, which is used to obtain "homotopic" description of derived (i.e. homotopic) categories, but preserve the factorization axiom, because it is crucial for proving existence of derived functors.

In this way we arrive to the axioms of a *pseudomodel stack.* We construct a very natural structure of pseudomodel stack on $\mathbf{\mathfrak{s}SETS}$. Fibrations and acyclic fibrations are still characterized by their *local* RLP with respect to "standard generators" $\underline{J}$ and $\underline{I}$ (considered here as maps of *constant* simplicial sheaves over $\mathcal{S}$), while cofibrations and acyclic cofibrations are defined as the smallest local classes of morphisms, containing $\underline{I}$ (resp. $\underline{J}$) and stable under arbitrary direct sums, pushouts, sequential composition and retracts.

This notion of pseudomodel stack seems to be sufficient for our purpose. At least, it leads to a reasonable definition of derived categories and derived functors, and we are able to transfer and apply Quillen's results about existence of derived functors.

0.9.5. (Pseudomodel category structure on $\mathfrak{s}\mathscr{O}$-**MOD**.) Now let $\mathcal{E} = (\mathcal{E}, \mathscr{O})$ be a generalized ringed topos. Consider the stack $\mathcal{C} := \mathfrak{s}\mathscr{O}$-**MOD**, characterized by $\mathcal{C}(U) := s\mathscr{O}|_U$-*Mod*. We manage to construct a pseudomodel category structure on this stack, generated by $L_{\mathscr{O}}(\underline{I})$ and $L_{\mathscr{O}}(\underline{J})$ in a similar way to what we did for $\mathfrak{s}$**SETS**.

If topos $\mathcal{E}$ has enough points (e.g. it is given by some topological space X), then $f : X \to Y$ *is a fibration (resp. acyclic fibration, weak equivalence) in* $\mathcal{C}(U) = s\mathscr{O}|_U$*-Mod iff* $f_p : X_p \to Y_p$ *has this property in* $s\mathscr{O}_p$*-Mod for any point* p *of* $\mathcal{E}_{/U}$. We can define *pointwise cofibrations* and *pointwise acyclic cofibrations* in a similar fashion. Any cofibration is a pointwise cofibration, but not the other way around. However, we can use these pointwise (acyclic) cofibrations, together with the above (pointwise) fibrations, acyclic fibrations and weak equivalences to define another pseudomodel structure on stack $\mathcal{C}$, called the *pointwise pseudomodel structure*.

Since the weak equivalences are the same, these two pseudomodel structures lead to the same derived categories and derived functors.

0.9.6. (Derived categories and functors.) If $\mathcal{C}$ is a pseudomodel stack, we have a class of weak equivalences in its "fiber category" $\mathcal{C}(e_{\mathcal{E}})$ over the final object of $\mathcal{E}$ (if $\mathcal{E}$ is given by a topological space X, this corresponds to taking $\mathcal{C}(X) = \mathcal{C}_X$). Therefore, we can define *homotopic categories* $\operatorname{Ho}\mathcal{C}(e_{\mathcal{E}})$ as the localization of $\mathcal{C}(e_{\mathcal{E}})$ with respect to weak equivalences, and define left or right derived functors accordingly.

For example, if $\mathcal{C} = \mathfrak{s}\mathscr{O}$-**MOD**, then $\mathcal{C}(e_{\mathcal{E}}) = s\mathscr{O}$-*Mod* is the category of simplicial sheaves of $\mathscr{O}$-modules. Using the pointwise description of weak equivalences, we see that for a classical sheaf of rings $\mathscr{O}$ our weak equivalences in $s\mathscr{O}$-*Mod* still correspond to quasi-isomorphisms of chain complexes, hence we can still write $\mathcal{D}^{\leq 0}(\mathscr{O}) := \operatorname{Ho}\mathcal{C}(e_{\mathcal{E}})$. If $\mathscr{O}$ admits a zero, we obtain a "stable" homotopic category $\mathcal{D}^{-}(\mathscr{O})$ as well.

0.9.7. (Derived pullbacks and tensor products.) We manage to generalize our results on existence of derived scalar extensions and tensor products to the topos case, thus proving existence of left derived pullbacks with respect to morphisms of generalized ringed spaces, and of derived tensor products. When sheaves of generalized rings involved are classical, we recover Verdier's derived pullbacks and derived tensor products.

0.9.8. (Derived symmetric powers.) The last very important thing done in Chapter **9** is the proof of existence of *derived symmetric powers* $\mathbb{L}S^n_{\mathscr{O}}$, which turn out to admit a description in terms of cofibrant replacements, as usual.

The proof is quite long and technical. (It would be complicated even in the classical situation, since $S^n_{\mathscr{O}}$ are non-additive functors even then.) In order to avoid some topos-related technicalities (such as defining a *constructible* pseudomodel structure on $\mathfrak{s}\mathscr{O}$**-MOD**), we give a complete proof only for the *pointwise* pseudomodel structure, thus assuming the base topos to have enough points.

Nevertheless, this result is very important for the next chapter, and it seems to be new even for classical rings. Notice that we obtain only functors $\mathbb{L}S^n_{\mathscr{O}} : \mathcal{D}^{\leq 0}(\mathscr{O}) \to \mathcal{D}^{\leq 0}(\mathscr{O})$, but no "stable" versions $\mathcal{D}^{-}(\mathscr{O}) \to \mathcal{D}^{-}(\mathscr{O})$. This is sufficient for our applications.

0.10. (Perfect cofibrations and intersection theory.) In the final Chapter **10** we combine the previous results to obtain a reasonable notion of perfect simplicial sheaf of modules and perfect cofibrations, then construct K_0 of perfect cofibrations (and briefly discuss higher K-theory), and finally transfer Grothendieck's construction of Chow rings and Chern classes to our case.

This chapter (and also this work) ends with an application of this intersection theory to the "compactified $\widehat{\operatorname{Spec}\mathbb{Z}}$", constructed in Chapter **7**. We manage to construct the moduli spaces of vector bundles on $\widehat{\operatorname{Spec}\mathbb{Z}}$, compute K_0 of vector bundles and perfect cofibrations, and use these results to compute the Chow ring of $\widehat{\operatorname{Spec}\mathbb{Z}}$.

0.10.1. (Cofibrations as presentations of their cokernels.) Recall that in the classical case of modules over a classical ring Λ cofibrations in $s\Lambda\text{-}Mod \cong \operatorname{Ch}(\Lambda\text{-}Mod)$ correspond to monomorphisms of non-negative chain complexes $u : A \to B$, with $P = \operatorname{Coker} u$ a complex of projective modules. Therefore, one might think of $u : A \to B$ as a "presentation" of this complex P.

This language is quite convenient. For example, if u' is a pushout of u, then $\operatorname{Coker} u' \cong \operatorname{Coker} u$, i.e. "$u'$ and u represent the same complex P". This suggests that we might use u as a representative of P in some K_0-group, and require $[u'] = [u]$ for any pushout u' of u. Any composable sequence of cofibrations $A \xrightarrow{u} B \xrightarrow{v} C$ defines a short exact sequence $0 \to \operatorname{Coker} v \to \operatorname{Coker} vu \to \operatorname{Coker} u \to 0$, i.e. *composition of cofibrations corresponds to short exact sequences*. Therefore, we might expect to have $[vu] = [u] + [v]$ in some K_0-group. Similarly, longer finite composable sequences of cofibrations correspond to finite filtrations on the quotient complex, and sequential compositions of cofibrations correspond to (infinite) discrete exhausitive filtrations.

It is convenient to extend this way of thinking about cofibrations to the case of cofibrations $u : A \to B$ of simplicial modules over any generalized ring (or even algebraic monad) Λ. Actually this is already helpful in the two previous chapters. For example, for any two morphisms $i : K \to L$ and

$u : A \to B$ in $s\Lambda$-*Mod* we construct a new morphism $u \,\square\, i \colon ? \to B \otimes L$, such that $\operatorname{Coker} u \,\square\, i = \operatorname{Coker} u \otimes \operatorname{Coker} i$ in classical situation. In this way this "box product" of morphisms may be thought of as corresponding to the tensor product of their "virtual cokernels".

The proof of existence of derived tensor products, given in Chapter **8**, actually involves the following statement: *$u \,\square\, i$ is a cofibration whenever u and i are cofibrations; if at least one of u or i is acyclic, then $u \,\square\, i$ is also acyclic.* Similarly, our proof of existence of derived symmetric powers in Chapter **9** involves some "symmetric powers" $\rho_n(u) :? \to S^n(B)$, having property $\operatorname{Coker} \rho_n(u) \cong S^n(\operatorname{Coker}(u))$ in classical situation.

0.10.2. (Perfect cofibrations.) Let $\mathcal{E} = (\mathcal{E}, \mathscr{O})$ be any generalized ringed topos. Recall that we've defined in Chapter **9** the cofibrations in $s\mathscr{O}$-*Mod* as the smallest *local* class of morphisms, containing "standard generators" from $L_{\mathscr{O}}(\underline{I})$, and closed under some operations, such as pushouts, retracts and finite or infinite sequential compositions.

This leads us to define *perfect* cofibrations as the set of morphisms $u : A \to B$ in $s\mathscr{O}$-*Mod*, which can be *locally* obtained from morphisms of $L_{\mathscr{O}}(\underline{I})$ by means of pushouts, finite compositions and retracts. In classical situation perfect cofibrations u correspond to injective maps of complexes with perfect cokernel.

Next, we say that X is a perfect simplicial object if $\varnothing \to X$ is a perfect cofibration. This is again compatible with classical terminology.

0.10.3. (K_0 of perfect cofibrations.) After that we define K_0 of perfect cofibrations between perfect objects as the free abelian group, generated by such cofibrations $[u]$, modulo certain relations. Namely, we impose relations $[\mathrm{id}_A] = 0$, $[u'] = [u]$ for any pushout u' of u, and $[vu] = [u] + [v]$ for any composable u and v. Another important relation is $[u'] = [u]$ if u' is isomorphic to u in the derived category $\mathcal{D}^{\leq 0}(\mathcal{E}, \mathscr{O})$.

We denote by $[A]$ the class of $[\varnothing \to A]$ in K_0. Since $[u] = [B] - [A]$ for any perfect cofibration $u : A \to B$ between perfect simplicial objects, our K_0 is generated by classes $[A]$ of perfect simplicial objects.

Under some restrictions (such as quasicompactness of $\mathcal{E}$) we are able to show that K_0 is already generated by *constant* perfect simplicial objects, i.e. by *vector bundles*, similarly to the classical case.

The last important remark is that operations $\square$ and ρ_n preserve perfect cofibrations and perfect cofibrations between perfect objects, and are compatible with the relations of K_0 (notice that we have to apply here our results on existence of derived tensor products and symmetric powers). In this way we obtain a *multiplication* on K_0, given by $[u] \cdot [v] = [u \,\square\, v]$, or equivalently, $[A] \cdot [B] = [A \otimes B]$, and *symmetric power operations* $s^n : [u] \mapsto [\rho^n(u)]$,

$[A] \mapsto [S^n(A)]$. This multiplication and symmetric power operations determine a *pre-λ-ring structure* on K_0, since the generating function $\lambda_t(x) := \sum_{n\geq 0} \lambda^n(x)t^n$ for the "external power operations" λ^n can be expressed in terms of symmetric powers by $\lambda_t(x) := s_{-t}(x)^{-1}$.

Notice that we never need to require $\mathscr{O}$ to be an alternating sheaf of $\mathbb{F}_{\pm 1}$-algebras, since we use symmetric powers to define λ-structure on K_0, which always have good properties, even when $\mathscr{O}$ doesn't have symmetry or zero.

0.10.4. (Waldhausen's construction.) Our construction of K_0 turns out to be a modification of Waldhausen's construction of K_0, adapted to the case of categories with cofibrations and weak equivalences, but without a zero object. Since Waldhausen's construction yields a definition of higher K-groups as well, we can expect higher algebraic K-theory of generalized rings to admit a description in terms of a suitable modification of Waldhausen's construction as well.

0.10.5. (Chow rings and Chern classes.) Once we obtain a pre-λ-ring $K^0 = K_0(\mathcal{E}, \mathscr{O})$, we can apply Grothendieck's reasoning: construct γ-operations and γ-filtration, define the Chow ring $CH(\mathcal{E})_{\mathbb{Q}}$ as the associated graded of $K^0_{\mathbb{Q}}$ with respect to the γ-filtration, and define Chern classes of elements of K^0 (and in particular, of vector bundles) with the aid of γ-operations again.

The only complication here is that we don't prove that pre-λ-ring $K^0 = K_0(\mathcal{E}, \mathscr{O})$ is a λ-ring. Instead, we replace K^0 by its largest quotient K^0_λ, which is a λ-ring.

0.10.6. (Intersection theory of $\widehat{\operatorname{Spec}\mathbb{Z}}$.) Chapter **10**, hence also this work, ends with the application of the above constructions to $\widehat{\operatorname{Spec}\mathbb{Z}}$, already constructed in Chapter **7**. We show that any finitely generated projective $\mathbb{Z}_{(\infty)}$-module is free, and use this result to classify vector bundles and constant perfect cofibrations between vector bundles over $\widehat{\operatorname{Spec}\mathbb{Z}}$. We use this classification to compute $K^0 = K^0(\widehat{\operatorname{Spec}\mathbb{Z}})$, and the Chow ring $CH(\widehat{\operatorname{Spec}\mathbb{Z}}, \mathbb{Q})$.

The answer is that $K^0(\widehat{\operatorname{Spec}\mathbb{Z}}) = \mathbb{Z} \oplus \log \mathbb{Q}^*_+$. The isomorphism is given here by $\mathscr{E} \mapsto (\operatorname{rank}\mathscr{E}, \deg\mathscr{E})$, where $\operatorname{rank}\mathscr{E}$ denotes the rank of a vector bundle $\mathscr{E}$, and $\deg\mathscr{E} = \deg\det(\mathscr{E})$ is its "arithmetic degree" in $\operatorname{Pic}(\widehat{\operatorname{Spec}\mathbb{Z}}) \cong \log \mathbb{Q}^*_+$. The Chow ring $CH(\widehat{\operatorname{Spec}\mathbb{Z}})$, i.e. the associated graded of $K^0(\widehat{\operatorname{Spec}\mathbb{Z}})$ with respect to γ-filtration, also turns out to be $\mathbb{Z} \oplus \log \mathbb{Q}^*_+$, with graded components $CH^0(\widehat{\operatorname{Spec}\mathbb{Z}}) = \mathbb{Z}$ and $CH^1(\widehat{\operatorname{Spec}\mathbb{Z}}) = \log \mathbb{Q}^*_+ = \operatorname{Pic}(\widehat{\operatorname{Spec}\mathbb{Z}})$. The only non-trivial Chern class $c_1 : K^0 \to CH^1(\widehat{\operatorname{Spec}\mathbb{Z}})$ turns out to be the arithmetic degree map.

This result illustrates that the intersection theory we constructed yields very natural results when applied to the simplest "Arakelov variety" $\widehat{\operatorname{Spec}\mathbb{Z}}$, leading us to believe that it will work equally well in other situations, related to Arakelov geometry or not.

1 Motivation: Looking for a compactification of $\operatorname{Spec}\mathbb{Z}$

1.1. (Original motivation.) Arakelov geometry appeared as an attempt to transfer some proofs valid for algebraic varieties (especially curves and abelian varieties) over a functional base field K to the number field case. For example, people wanted to prove statements like the Shafarevich conjecture or the Mordell conjecture.

1.1.1. Suppose we are given some functional field K. This means that K is finitely generated of transcendence degree one over some base field k, usually assumed to be either finite or algebraically closed. Then there is a unique smooth projective curve C over k such that $K = k(C)$; in other words, $\operatorname{Spec} K$ is the generic point ξ of C.

1.1.2. Given an algebraic variety X/K (usually supposed to be smooth and projective), one might look for *models* $\mathscr{X} \to C$ of X: by definition, these are flat projective schemes with generic fiber $\mathscr{X}_\xi$ isomorphic to X. Now, $\mathscr{X}$ is a (projective) algebraic variety over k. For example, if X/K is a curve, $\mathscr{X}/k$ will be a projective surface. Now we can use intersection theory on $\mathscr{X}$ to obtain some interesting bounds and estimates. For example, any rational point $P \in X(K)$ lifts to a section $\sigma_P : C \to \mathscr{X}$ by the valuative criterion of properness. Hence for any two rational points P, $Q \in X(K)$ of a projective curve X/K one can compute the *intersection numbers* $(P \cdot Q) = (\sigma_P(C) \cdot \sigma_Q(C))$ and systematically use them, e.g. to obtain some bounds for the total number of rational points of X/K.

1.1.3. Observe that for such reasoning we need $\mathscr{X}$ to be proper over k, so the properness of our model C of K over k is very important. If we replace C by an affine curve $C^\circ \subset C$, no argument of this sort will be possible.

1.1.4. Note that any functional field K contains a subfield $K' = k(t)$ isomorphic to the field of rational functions over k, and K/K' is a finite extension. Hence, once a smooth proper model C' of K'/k is constructed, one can take the normalization of C' in K as a smooth proper model of K. This means that it is sufficient to consider functional fields of form $K = k(T)$ to learn how to construct smooth proper models of *all* functional fields.

Similarly, any number field K is by definition a finite extension of $K' = \mathbb{Q}$, so it should be sufficient to construct (in some sense) a "smooth proper model of $\mathbb{Q}$". Such a model is usually thought of as a "compactification of $\operatorname{Spec}\mathbb{Z}$".

1.1.5. Consider first the functional case, i.e. $K = k(T)$. The first candidate for a model C of K over k is the affine line $\mathbb{A}^1_k = \operatorname{Spec} k[T]$: it is a smooth (hence regular) curve (i.e. one-dimensional scheme), and has the field of rational functions equal to $K = k(T)$. However, it is not proper, so one has to consider the projective line $C := \mathbb{P}^1_k$. The affine line $\mathbb{A}^1_k$ is an open subscheme of $\mathbb{P}^1_k$; its complement is exactly one point – the *point at infinity* ∞.

One sees that each closed point p of $C = \mathbb{P}^1_k$ defines a discrete valuation $v_p : K \to \mathbb{Z} \cup \{+\infty\}$ with the valuation ring equal to the local ring $\mathscr{O}_{C,p}$. This also defines a norm $|\cdot|_p$ on the field K by the rule $|x|_p := \rho^{-v_p(x)}$ where ρ is any real number greater than one (one can take $\rho = e$ or $\rho = e^{[\kappa(p):k]}$). Conversely, all norms on K which are trivial on k (condition automatically fulfilled when k is algebraic over a finite field) come from points of $\mathbb{P}^1_k$, and this correspondence is one-to-one provided we identify equivalent norms (i.e. norms such that $|\cdot|_1 = |\cdot|_2^\alpha$ for some positive α).

If we consider just the affine line $\mathbb{A}^1_k = C^\circ \subset C$, we do not have any point corresponding to the valuation $|\cdot|_\infty$ given by $|f(T)/g(T)|_\infty = \rho^{\deg f - \deg g}$, that corresponds to the infinite point on the projective line. Note that it is the only valuation $|\cdot|$ of K trivial on k, for which $|T|_\infty > 1$. For all other valuations we have $|f|_p \leq 1$ for any polynomial $f \in k[T]$.

Another way to see that C° is not complete ($=$ proper over k) is this. On a complete curve C the degree of the divisor $\operatorname{div}(f)$ of a non-zero rational function $f \in k(C) = K$ is zero. In other words, we have

$$\sum_{P \in C, P \neq \xi} v_p(f) \cdot [k(P) : k] = 0 \tag{1.1.5.1}$$

After elevating some $\rho > 1$ into this power we obtain the “product formula”:

$$\prod_{P \in C, P \neq \xi} |f|_P = 1 \tag{1.1.5.2}$$

Of course, once we omit the point ∞, both these formulas cease to be true.

1.1.6. The first obvious candidate for a smooth model of $\mathbb{Q}$ is $\operatorname{Spec}\mathbb{Z}$. Indeed, this is a one-dimensional regular scheme with the field of rational functions equal to $\mathbb{Q}$. However, $C^\circ = \operatorname{Spec}\mathbb{Z}$ is the counterpart of the affine line $\mathbb{A}^1_k = \operatorname{Spec} k[T]$, not of the projective line $\mathbb{P}^1_k$. For example, both $\operatorname{Spec}\mathbb{Z}$ and $\mathbb{A}^1_k = \operatorname{Spec} k[T]$ are affine, spectra of principal ideal domains. Closed points of $\mathbb{A}^1_k$ correspond to irreducible unitary polynomials $\pi \in k[T]$, and those of $\operatorname{Spec}\mathbb{Z}$ – to prime numbers $p \in \mathbb{Z}$. In both cases these closed points correspond to all valuations of the field of rational functions but one. For $\mathbb{A}^1_k$ the omitted valuation is the valuation $|\cdot|_\infty$ given by the degrees of polynomials. For $\operatorname{Spec}\mathbb{Z}$ it is the only archimedian valuation $|\cdot|_\infty$ defined by the usual absolute value of rational (or real) numbers.

If we normalize the p-adic norms on $\mathbb{Q}$ by requiring $|p|_p = p^{-1}$, we have a product formula for any $f \in \mathbb{Q}^*$, *provided we take into account the remaining archimedean norm* $|\cdot|_\infty$:

$$|f|_\infty \cdot \prod_{(p)\in \operatorname{Spec}\mathbb{Z},(p)\neq(0)} |f|_p = 1 \tag{1.1.6.1}$$

All this means that $C^\circ = \operatorname{Spec}\mathbb{Z}$ is a counterpart of $\mathbb{A}^1_k$, not of $\mathbb{P}^1_k$, and to construct the "compactification" C of $\operatorname{Spec}\mathbb{Z}$ one has to take into account the archimedean valuation $|\cdot|_\infty$, which should be thought of as corresponding to some "point at infinity" ∞, which is the complement of C° in C.

1.1.7. Up to now this picture has been just a fancy way of describing the essential ideas of Arakelov geometry. In one of subsequent sections we will indeed construct the compactification $\widehat{\operatorname{Spec}\mathbb{Z}}$ of $\operatorname{Spec}\mathbb{Z}$ as a "generalized scheme". It happens to be a topological space with a local sheaf of "generalized rings". **We shall assign precise mathematical meaning to these notions in the remaining part of this work.**

1.2. Let's describe now how one could describe a smooth projective curve C (say, $C = \mathbb{P}^1_k$) in terms of the complement C° of a point ∞ in C and the valuation $|\cdot|_\infty$ on $K = k(C)$ corresponding to the omitted point.

1.2.1. Let's treat the functional field case first. Then $|\cdot|_\infty$ is non-archimedean, so we can reconstruct the local ring $\mathscr{O}_{C,\infty} = \{f \in K : |f|_\infty \leq 1\}$ as well as its maximal ideal $\mathfrak{m}_{C,\infty} = \{f \in K : |f|_\infty < 1\}$. Then one observes that $C^\circ \to C$ and $C_\infty := \operatorname{Spec}\mathscr{O}_{C,\infty} \to C$ form a covering for the flat topology (fpqc). Indeed, they are flat and cover the whole of C.

So we can apply faithfully flat descent here: We assign to an object $\mathscr{X}$ (say, a quasi-projective scheme, or a vector bundle, or a quasi-coherent sheaf on such a scheme) over C a triple $(\mathscr{X}^\circ, \mathscr{X}_\infty, \sigma)$, where $\mathscr{X}^\circ$ and $\mathscr{X}_\infty$ are objects of the same sort over C° and C_∞, respectively (the pullbacks of $\mathscr{X}$), and $\sigma : \mathscr{X}^\circ_{(K)} \xrightarrow{\sim} (\mathscr{X}_\infty)_{(K)}$ is an isomorphism of the pullbacks of these objects onto $C^\circ \times_C C_\infty = \operatorname{Spec} K$ (this is the canonical isomorphism for a triple constructed from some $\mathscr{X}$ over C).

Then faithfully flat descent assures us that for some kinds of objects (e.g. quasi-projective schemes or quasi-coherent sheaves on such schemes) the functor thus constructed from the category of objects of this kind on C into the category of triples described above is an equivalence of categories.

1.2.2. So, for example, to describe a quasi-projective scheme $\mathscr{X}$ over C we can use a quasi-projective scheme $\mathscr{X}^\circ$ over C° together with a "$\mathscr{O}_{C,\infty}$-structure" on its generic fiber $X := \mathscr{X}^\circ_\xi = \mathscr{X}^\circ_{(K)}$, i.e. a quasi-projective scheme $\mathscr{X}_\infty$ over $\mathscr{O}_{C,\infty}$ equipped by an isomorphism of K-schemes $\sigma : X \xrightarrow{\sim} (\mathscr{X}_\infty)_{(K)}$.

Similarly, a vector bundle $\mathscr{E}$ on C is essentially the same thing as a vector bundle $\mathscr{E}^\circ$ on C° equipped by a $\mathscr{O}_{C,\infty}$-structure on its generic fiber $\mathscr{E}_\xi$, which is a finite-dimensional K-vector space.

1.2.3. Note that we have used here the fact that $C^\circ \times_C C^\circ \cong C^\circ$ and $C_\infty \times_C C_\infty \cong C_\infty$: otherwise we would also need some descent data on these fibered products as well.

1.2.4. Geometrically, C° is C without one closed point ∞, $C_\infty = \operatorname{Spec} \mathscr{O}_{C,\infty}$ consists of two points $\{\infty, \xi\}$, and their fibered product over C is $\operatorname{Spec} K = \{\xi\}$. So we just glue C° and C_∞ by their generic points to reconstruct C:

$$\begin{array}{ccc} \operatorname{Spec} K = \{\xi\} & \longrightarrow & C_\infty = \{\xi, \infty\} \\ \downarrow & & \downarrow \\ C^\circ = C - \{\infty\} & \longrightarrow & C \end{array} \tag{1.2.4.1}$$

1.3. Sometimes it is more convenient to work with the completions $\hat{C}_\infty = \operatorname{Spec} \hat{\mathscr{O}}_{C,\infty}$ and $C^\circ \times_C \hat{C}_\infty = \operatorname{Spec} \hat{K}$, where $\hat{K} = \hat{\mathscr{O}}_{C,\infty} \otimes_{\mathscr{O}_{C,\infty}} K$ is the completion of K with respect to $|\cdot|_\infty$. In this case two morphisms $C^\circ \to C$ and $\hat{C}_\infty \to C$ form a faithfully flat family, so we might have used faithfully flat descent here. However, $\hat{C} \times_C \hat{C} = \operatorname{Spec} \hat{\mathscr{O}}_{C,\infty} \otimes_{\mathscr{O}_{C,\infty}} \hat{\mathscr{O}}_{C,\infty}$ is *not* isomorphic to $\hat{C}$, so in this case our descent data would include an isomorphism of the two pullbacks to this scheme of the object $\hat{\mathscr{X}}_\infty$ defined over $\hat{C}_\infty$.

1.3.1. Since this scheme usually is not noetherian and not too much can be said about its structure, people usually proceed in the naïve way. They map an object $\mathscr{X}$ over C into a triple $(\mathscr{X}^\circ, \hat{\mathscr{X}}_\infty, \sigma)$, where $\mathscr{X}^\circ$ is an object of the same sort over C°, $\hat{\mathscr{X}}_\infty$ is over $\hat{C}_\infty$, and σ is an isomorphism of the pullbacks of these two objects on $C^\circ \times_C \hat{C}_\infty = \operatorname{Spec} \hat{K}$. In other words, some of descent data is omitted.

1.3.2. So the functor that maps an object $\mathscr{X}$ over C into a triple as above is *not* necessarily an equivalence of categories now. However, it is still faithful, and it often turns out to be fully faithful or even an equivalence of categories, when restricted to objects $\mathscr{X}$ with some additional finiteness or flatness conditions over the infinite point ∞.

1.3.3. Let's show how this works for vector bundles $\mathscr{E}$ over C. We map such a vector bundle into a vector bundle $\mathscr{E}^\circ$ over C° together a $\hat{\mathscr{O}}_{C,\infty}$-structure $\hat{E}_\infty$ on the finite dimensional $\hat{K}$-vector space $\hat{E} := \mathscr{E}^\circ_{(\hat{K})} = E \otimes_K \hat{K}$, where $E := \mathscr{E}^\circ_\xi$ is the generic fiber of $\mathscr{E}^\circ$. Now observe that the corresponding $\mathscr{O}_{C,\infty}$-structure

E_∞ on E is completely determined by this data since $E_\infty = \hat{E}_\infty \cap E \subset \hat{E}$, so we get some descent datum for $C^\circ, C_\infty \to C$, that uniquely determines the original vector bundle $\mathscr{E}$.

Note that any choice of a $\hat{\mathscr{O}}_{C,\infty}$-structure $\hat{E}_\infty$ on $\hat{E}$ corresponds to a unique $\mathscr{O}_{C,\infty}$-structure on E since the intersection $\hat{E}_\infty \cap E$ necessarily has a rank equal to $\dim_K E$. The real reason behind this is that, apart from general faithfully flat descent theory, in this case we also have $\mathscr{O}_{C,\infty} = \hat{\mathscr{O}}_{C,\infty} \cap K \subset \hat{K}$. In other words, the following square is cartesian, hence bicartesian in the category of rings:

$$\begin{tikzcd} \mathscr{O}_{C,\infty} \arrow[r] \arrow[d] & K \arrow[d] \\ \hat{\mathscr{O}}_{C,\infty} \arrow[r] & \hat{\mathscr{O}}_{C,\infty} \otimes_{\mathscr{O}_{C,\infty}} K = \hat{K} \end{tikzcd} \tag{1.3.3.1}$$

This means that our functor is still an equivalence of categories, at least for the categories of vector bundles. In other words, *any choice of a $\hat{\mathscr{O}}_{C,\infty}$-structure on the completed generic fiber of $\mathscr{E}$ is algebraic.*

1.4. Let's try to transport these arguments to the case $K = \mathbb{Q}$, $C^\circ = \operatorname{Spec} Z$. So, a flat quasi-projective scheme $\mathscr{X}$ over the compactification $C = \widehat{\operatorname{Spec}\mathbb{Z}}$ should correspond to a flat quasi-projective scheme $\mathscr{X}^\circ$ over $C^\circ = \operatorname{Spec} Z$ together with a "$\mathbb{Z}_\infty$-structure" on its completed generic fiber $X_\mathbb{R} := \mathscr{X}_\xi \otimes_\mathbb{Q} \mathbb{R}$, which is an algebraic scheme over the completed field of rational functions $\hat{K} = \mathbb{Q}_\infty = \mathbb{R}$. In other words, to construct a model over $\widehat{\operatorname{Spec}\mathbb{Z}}$ of some (flat) variety $X/\mathbb{Q}$ we need to construct its model $\mathscr{X}^\circ$ over $\operatorname{Spec}\mathbb{Z}$, and to construct a $\mathbb{Z}_\infty$-model $\hat{\mathscr{X}}_\infty$ of $X_\mathbb{R} := X_{(\mathbb{R})}$.

Since there is not much mystery about models over $\operatorname{Spec}\mathbb{Z}$ (they are just some schemes in the usual sense), we will be mostly concerned with describing models over $\mathbb{Z}_\infty$.

1.4.1. Here we denote by $\mathbb{Z}_\infty \subset \mathbb{Q}_\infty = \mathbb{R}$ the conjectural counterpart of the completed localization $\hat{\mathscr{O}}_{C,\infty}$, and by $\mathbb{Z}_{(\infty)} = \mathbb{Z}_\infty \cap \mathbb{Q} \subset \mathbb{Q}$ the counterpart of the (non-completed) localization $\mathscr{O}_{C,\infty}$. Maybe $\hat{\mathbb{Z}}_\infty$ and $\mathbb{Z}_\infty$ would be a better choice of notation; our choice is motivated by the p-adic analogy: $\mathbb{Z}_p \subset \mathbb{Q}_p$ is the completed localization of $\mathbb{Z}$ at a prime p, while the usual localization $\mathbb{Z}_p \cap \mathbb{Q}$ is denoted by $\mathbb{Z}_{(p)}$.

Of course, for now $\mathbb{Z}_\infty$ and $\mathbb{Z}_{(\infty)}$ are just some formal symbols that we use in our considerations. On the contrary, our analogue of $\hat{K}$ is $\mathbb{Z}_\infty \otimes_\mathbb{Z} \mathbb{Q} = \mathbb{Q}_\infty = \mathbb{R}$, so it has a well-known mathematical meaning. So we will try to describe different objects over $\mathbb{Z}_\infty$ as corresponding objects over its "fraction field" $\mathbb{R}$ together with some additional structure.

1.4.2. Actually it would be better to study the non-completed localization $\mathbb{Z}_{(\infty)}$ instead, since it should give a theory capable of handling objects with "torsion over ∞", and without any additional "algebraicity" restrictions (i.e. any descent datum would be efficient).

We will return to this question later.

1.4.3. We introduce some more fancy notation: we denote by $\bar{\mathbb{Z}}_\infty$ the "algebraic closure of $\mathbb{Z}_\infty$ in $\mathbb{C}$". Then, by Galois descent a $\mathbb{Z}_\infty$-structure on, say, some algebraic variety $X_\mathbb{R}$ over $\mathbb{R}$ should be essentially the same thing as a $\bar{\mathbb{Z}}_\infty$-structure on its complexification $X_\mathbb{C}$, invariant under complex conjugation.

1.5. (Vector bundles over $\widehat{\operatorname{Spec}\mathbb{Z}}$.) Let's discuss first how a vector bundle $\mathscr{E}$ over $\widehat{\operatorname{Spec}\mathbb{Z}}$ looks like. Firstly, we need some vector bundle $\mathscr{E}^\circ$ over $\operatorname{Spec}\mathbb{Z}$. Vector bundles over $\mathbb{Z}$ correspond simply to free $\mathbb{Z}$-modules of finite rank; so $\mathscr{E}^\circ$ corresponds to some $E_\mathbb{Z}$, that can be identified with a lattice in $E := E_\mathbb{Z}\otimes\mathbb{Q} \cong \mathscr{E}_\xi$. Secondly, we need to define a "$\mathbb{Z}_\infty$-structure" or a "$\mathbb{Z}_\infty$-lattice" in the real vector space $E_\mathbb{R} = E_{(\mathbb{R})}$. So we have to discuss what this might mean.

1.5.1. This question is usually answered as follows. A $\mathbb{Z}_\infty$-lattice (resp. $\bar{\mathbb{Z}}_\infty$-lattice) in some finite-dimensional real (resp. complex) vector space E is just a positive-definite quadratic (resp. hermitian) form on this space. *Our approach will be slightly different, but let's adopt this point of view for now.*

1.5.2. Here are some reasons for such a definition. Given a finite-dimensional $\mathbb{Q}_p$-vector space, any $\mathbb{Z}_p$-lattice $\Lambda \subset E$ defines a maximal compact (for the p-adic topology) subgroup $G_\Lambda := GL_{\mathbb{Z}_p}(\Lambda) = \{g \in GL(E) : g(\Lambda) = \Lambda\}$ in the locally compact group $GL(E) = GL_{\mathbb{Q}_p}(E)$. Conversely, all maximal compact subgroups in $GL(E)$ are of this form, and $G_\Lambda = G_{\Lambda'}$ iff Λ and Λ' are similar, i.e. $\Lambda' = c\Lambda$ for some $c \in \mathbb{Q}_p^*$. This means that $\mathbb{Z}_p$-lattices considered up to similitude are in one-to-one correspondence with maximal compact subgroups of $GL(E)$.

Now, if E is a real vector space, $GL(E)$ is again a locally compact group, and its maximal compact subgroups are the orthogonal groups $G_Q := O(Q)$, where Q is any positive-definite quadratic form on E. Again, $G_Q = G_{Q'}$ iff $Q' = cQ$ for some $c > 0$, so it seems quite natural to assume that $\mathbb{Z}_\infty$-lattices in E correspond to positive-definite quadratic forms.

Similarly, if E is a (finite-dimensional) complex vector space, the maximal compact subgroups of $GL(E) = GL_\mathbb{C}(E)$ are the unitary groups of positive-definite hermitian forms, so $\bar{\mathbb{Z}}_\infty$-lattices in E should correspond to positive-definite hermitian forms.

1.5.3. Hence a vector bundle $\mathscr{E}$ over $\widehat{\operatorname{Spec}\mathbb{Z}}$ corresponds to a free $\mathbb{Z}$-module $E_{\mathbb{Z}}$ of finite rank, together with a positive definite quadratic form on the real vector space $E_{\mathbb{R}} := E_{\mathbb{Z}} \otimes \mathbb{R}$. Vector bundles over $\hat{C}_\infty$, i.e. free $\mathbb{Z}_\infty$-modules of finite rank, correspond to quadratic real vector spaces. Note that these categories are *not* additive (they don't have neither direct products nor coproducts), for any reasonable choice of morphisms.

1.6. (Usual description of Arakelov varieties.) Now let's try to describe more complicated objects. Suppose we start with a *complete* (i.e. proper over $\mathbb{C}$) algebraic variety $X/\mathbb{C}$, and we are given some proper model $\mathscr{X}/\bar{\mathbb{Z}}_\infty$, whatever this might mean. Now we would like to describe a vector bundle $\mathscr{E}$ on $\mathscr{X}$.

1.6.1. First of all, the generic fiber $E = \mathscr{E}_\xi = \mathscr{E}_{(\mathbb{C})}$ of $\mathscr{E}$ is just a vector bundle on X, and we know what this means. Next, if we believe that $\bar{\mathbb{Z}}_\infty$ is something like a valuation ring, then "by the valuative criterion of properness" any $\mathbb{C}$-point $P \in X(\mathbb{C})$ should lift to a unique section $\sigma_P : \operatorname{Spec}\bar{\mathbb{Z}}_\infty \to \mathscr{X}$, so we can consider the pullback $\mathscr{E}_P := \sigma_P^*(\mathscr{E})$. This is a vector bundle on $\operatorname{Spec}\bar{\mathbb{Z}}_\infty$ with generic fiber $\mathscr{E}_P \otimes \mathbb{C} \cong E_P$, i.e. we get a $\bar{\mathbb{Z}}_\infty$-lattice in each fiber E_P. By our convention this means that we have a hermitian form on each E_P, and it is quite natural to assume that these metrics depend continuously or even smoothly on the point P.

In this way we see that a vector bundle $\mathscr{E}$ on our model $\mathscr{X}$ gives rise to an (algebraic) vector bundle E on X equipped by a smooth hermitian metric, when considered as a holomorphic vector bundle over the complex analytic variety $X(\mathbb{C})$.

1.6.2. If we start with a complete algebraic variety $X/\mathbb{R}$ and a vector bundle $\mathscr{E}$ on its proper model $\mathscr{X}/\mathbb{Z}_\infty$, we can extend these data to an algebraic variety $X_{(\mathbb{C})}$ over $\mathbb{C}$ together with a vector bundle $\mathscr{E}_{(\bar{\mathbb{Z}}_\infty)}$ on its proper model $\mathscr{X}_{(\bar{\mathbb{Z}}_\infty)}$. Then, after reasoning as above, we end up with an (algebraic) vector bundle E on X and a hermitian metric h_E on the corresponding holomorphic vector bundle on complex analytic variety $X(\mathbb{C})$, compatible with complex conjugation on $X(\mathbb{C})$.

1.6.3. If we are lucky, it may turn out that this collection of data completely determines the original vector bundle $\mathscr{E}$. Of course, we cannot expect any such pair (E, h_E) to be "algebraic", i.e. to correspond to a vector bundle $\mathscr{E}$ on $\mathscr{X}$. The metrics expected to be algebraic are usually called *admissible*.

1.6.4. Suppose that $X/\mathbb{R}$ is complete and *smooth*, and that it has a proper and smooth model $\mathscr{X}/\mathbb{Z}_\infty$. Then the sheaf of relative Kähler differentials $\Omega^1_{\mathscr{X}/\mathbb{Z}_\infty}$ is locally free, i.e. it is a vector bundle. Hence the above arguments can be applied to $\mathscr{E} = \Omega^1_{\mathscr{X}/\mathbb{Z}_\infty}$. The generic fiber E of $\mathscr{E}$ in this case is $\Omega^1_{X/\mathbb{R}}$, i.e. it is the cotangent bundle T^*X of X.

In this way a smooth proper model $\mathscr{X}$ of a complete smooth algebraic variety $X/\mathbb{R}$ gives us a hermitian metric on $T^*X(\mathbb{C})$, invariant under complex conjugation, i.e. a cometric on $X(\mathbb{C})$. Since this cometric is supposed to be non-degenerate, it corresponds by duality to a hermitian metric on $X(\mathbb{C})$ (i.e. on $TX(\mathbb{C})$), compatible with complex conjugation.

1.6.5. Note that in this reasoning it is important that not only X is supposed to be smooth, but $\mathscr{X}$ as well. One might expect an $\mathscr{X}$ with some singularities in the special fiber to give a singular (co)metric on $X(\mathbb{C})$.

1.6.6. Again, we can hope that this data (X plus a hermitian metric h_X on $X(\mathbb{C})$, compatible with complex conjugation) completely determines $\mathscr{X}$. At least, this is what people usually assume while doing Arakelov geometry.

So an arithmetic (or Arakelov) variety over $\operatorname{Spec}\mathbb{Z}$ is usually *defined* to be a flat proper scheme $\mathscr{X}/\mathbb{Z}$ with smooth generic fiber $X = \mathscr{X}_\xi/\mathbb{Q}$, equipped by a conjugation-invariant hermitian metric h_X on $X(\mathbb{C})$. Some additional "admissibility" requirements are usually imposed on h_X, the most common of them being the requirement to be a Kähler metric.

1.6.7. Of course, such an approach cannot work for arithmetic varieties not supposed to be generically smooth and proper. We are going to provide a more direct approach and define an arithmetic variety $\mathscr{X}$ as some "generalized scheme". In particular, it will be a topological space, so the "fiber over infinity" will receive a very concrete meaning, and the informal considerations of this chapter will get a rigorous interpretation.

2 $\mathbb{Z}_\infty$-Lattices and flat $\mathbb{Z}_\infty$-modules

In this chapter we are going to study in more detail $\mathbb{Z}_\infty$-lattices and $\mathbb{Z}_\infty$-structures on real vector spaces. At first we restrict ourselves to $\mathbb{Z}_\infty$-lattices in finite-dimensional vector spaces. We resume the considerations of **1.5.1** and **1.5.2**, and try to extend them so as to obtain a notion of a $\mathbb{Z}_\infty$-structure on a finite $\mathbb{R}$-algebra A. This leads to a considerable change of point of view with respect to one recalled in **1.5.1**.

2.1. (Lattices stable under multiplication.) Recall that our currently adopted approach consists in describing sublattices A of a finite-dimensional (real or p-adic) vector space E in terms of maximal compact subgroups $G_A = \{g : g(A) = A\}$ of $GL(E)$; cf. **1.5.2**.

Suppose we are given an algebra structure on E; we want to describe $\mathbb{Z}_p$-structures on E, i.e. those sublattices A of E which are stable under multiplication of E. Of course, we would like to obtain a description in terms of corresponding subgroups G_A.

2.1.1. Let's try the p-adic case first. So E is a finite $\mathbb{Q}_p$-algebra, and we are looking for finite flat $\mathbb{Z}_p$-algebras A, such that $A_{(\mathbb{Q}_p)} \cong E$; in other words, A must be a sublattice of E stable under multiplication: $A \cdot A = A$.

2.1.2. From the abstract point of view, a $\mathbb{Z}_p$-algebra A is a $\mathbb{Z}_p$-module A equipped by two morphisms $\eta_A : \mathbb{Z}_p \to A$ (the unit) and $\mu_A : A \otimes A \to A$ (the multiplication), subject to some conditions. If $A_{(\mathbb{Q}_p)} \cong E$ as a $\mathbb{Q}_p$-algebra, then we must have $(\eta_A)_{(\mathbb{Q}_p)} = \eta_E$ and similarly for the multiplication. Hence, if we start from several automorphisms $\gamma_1, \ldots, \gamma_n \in GL(E)$ that preserve A, and construct from them some new element γ of $GL(E)$ using only tensor products and maps μ_E and η_E, the resulting element must also preserve A, since the same construction can be done first in the category of $\mathbb{Z}_p$-modules, yielding some $\gamma' : A \to A$, and then $\gamma = \gamma'_{(\mathbb{Q}_p)}$ will preserve A.

2.1.3. Proceeding in this way, for any two $\gamma_1, \gamma_2 : E \to E$ we define two new maps $\gamma_1 \llcorner \gamma_2$ and $\gamma_1 \lrcorner \gamma_2$ from E to E:

$$\gamma_1 \llcorner \gamma_2 : E \xrightarrow{1_E \otimes \eta_E} E \otimes E \xrightarrow{\gamma_1 \otimes \gamma_2} E \otimes E \xrightarrow{\mu_E} E \tag{2.1.3.1}$$

$$\gamma_1 \lrcorner \gamma_2 : E \xrightarrow{\eta_E \otimes 1_E} E \otimes E \xrightarrow{\gamma_1 \otimes \gamma_2} E \otimes E \xrightarrow{\mu_E} E \tag{2.1.3.2}$$

In other words, $(\gamma_1 \llcorner \gamma_2)(x) = \gamma_1(x) \cdot \gamma_2(1)$ and $(\gamma_1 \lrcorner \gamma_2)(x) = \gamma_1(1) \cdot \gamma_2(x)$.

2.1.4. Clearly, these are examples of constructions considered in **2.1.2**, i.e. if both γ_1 and γ_2 preserve some $\mathbb{Z}_p$-subalgebra $A \subset E$, the same is true for $\gamma_1 \llcorner \gamma_2$ and $\gamma_1 \lrcorner \gamma_2$.

So we are tempted to describe sublattices $A \subset E$ that are $\mathbb{Z}_p$-subalgebras by requiring $\gamma_1 \llcorner \gamma_2$, $\gamma_1 \lrcorner \gamma_2 \in G_A$ for all γ_1, $\gamma_2 \in G_A$.

However, these two new elements need not be invertible, i.e. in some cases they are not in $G = GL(E)$. Even when they lie in G, all we can say about them is that $(\gamma_1 \llcorner \gamma_2)(A) \subset A$ and similarly for $\gamma_1 \lrcorner \gamma_2$, condition insufficient to conclude $\gamma_1 \llcorner \gamma_2 \in G_A$.

Observe that we can avoid both these problems by considering from the very beginning the *compact submonoids* $M_A := \{g \in \mathrm{End}(E) : g(A) \subset A\} \cong \mathrm{End}_{\mathbb{Z}_p}(A)$ of $\mathrm{End}(E)$.

Then conditions $\gamma_1, \gamma_2 \in M_A$ imply $\gamma_1 \llcorner \gamma_2$, $\gamma_1 \lrcorner \gamma_2 \in M_A$ whenever the sublattice $A \subset E$ is a $\mathbb{Z}_p$-algebra.

2.2. Before proceeding further we would like to check whether these M_A still determine sublattices $A \subset E$ up to similitude, and whether they can be described as maximal compact submonoids of $\mathrm{End}(E)$. The answer to both these questions is *positive,* so the situation is completely similar to one we had before when we considered maximal compact subgroups of $GL(E)$:

Theorem. *Let E be a finite-dimensional $\mathbb{Q}_p$-vector space. For any $\mathbb{Z}_p$-sublattice $A \subset E$ denote by M_A the submonoid of* $\mathrm{End}(E)$ *consisting of endomorphisms of E that preserve A, i.e. $M_A = \{g \in \mathrm{End}(E) : g(A) \subset A\}$. Then:*

a) The M_A are maximal compact submonoids in $\mathrm{End}(E)$ *(with respect to the p-adic topology);*

b) All maximal compact submonoids of $\mathrm{End}(E)$ *are of this form;*

c) $M_A = M_{A'}$ iff A and A' are similar, i.e. iff $A' = c \cdot A$ for some $c \in \mathbb{Q}_p^*$.

2.2.1. Let us check first that any compact submonoid $M \subset \mathrm{End}(E)$ stabilizes some lattice A in E, i.e. that $M \subset M_A$. For this take any sublattice $A_0 \subset E$ and consider the $\mathbb{Z}_p$-submodule A of E generated by $S := M \cdot A_0 = \{g(x) : g \in M, x \in A_0\}$. Note that both M and A_0 are compact, hence S is compact, hence S and A are contained in a compact sublattice $p^{-n}A_0$ for some $n > 0$, hence the closure $\bar{A}$ is a compact $\mathbb{Z}_p$-submodule of E. On the other hand, it contains A_0, hence it is open, and all open compact $\mathbb{Z}_p$-submodules in E are lattices. So we have constructed a sublattice $\bar{A}$ in E, stable under all elements $g \in M$, since $gS = gM \cdot A_0 \subset M \cdot A_0 = S$. In other words, we have $M \subset M_{\bar{A}}$.

2.2.2. Clearly, if A and A' are similar, then $M_A = M_{A'}$. Let us prove that $M_A \subset M_{A'}$ implies that A and A' are similar, hence $M_A = M_{A'}$; this will complete the proof of the theorem.

So suppose that $M_A \subset M_{A'}$. Case $E = 0$ is trivial, so assume $E \neq 0$. After rescaling A', we can assume that A' is contained in A, but not in pA. Choose any element $u \in A' - pA \subset A - pA$. According to Lemma **2.2.3** below, for any element $v \in A$ we can find $g \in M_A$, for which $g(u) = v$. Then, since $u \in A'$ and $g \in M_A \subset M_{A'}$, we have $v = g(u) \in A'$. So we have proved $A \subset A'$, hence $A = A'$.

Lemma 2.2.3 *If A is a sublattice in E, then for any $u \in A - pA$, $v \in A$ there is an element $g \in M_A \cong \mathrm{End}_{\mathbb{Z}_p}(A)$, such that $g(u) = v$. In other words, for any $u \in A - pA$ we have $A = M_A \cdot u = \{g(u) : g \in M_A\}$.*

Proof. Indeed, A is a free $\mathbb{Z}_p$-module of finite rank, so $u \in A - pA$ implies that we can complete u to some base $e_1 = u$, e_2, …, e_n of A. Then we can define an element $g \in \mathrm{End}_{\mathbb{Z}_p}(A)$ with required properties by putting $g(e_1) := v$, $g(e_k) := 0$ for $k > 1$.

2.2.4. Now we would like to check whether the conditions $\gamma_1 \llcorner \gamma_2$, $\gamma_1 \lrcorner \gamma_2 \in M_A$ for all γ_1, $\gamma_2 \in M_A$, necessary for a sublattice A in a finite-dimensional $\mathbb{Q}_p$-algebra E to be a $\mathbb{Z}_p$-subalgebra, are in fact sufficient.

The answer is again positive, modulo some rescaling:

Theorem. *Let A be a $\mathbb{Z}_p$-sublattice in a finite-dimensional $\mathbb{Q}_p$-algebra E. Suppose that $1 \in A - pA$ (any lattice is similar to exactly one lattice of this sort). The following conditions are equivalent:*

(i) A is a $\mathbb{Z}_p$-subalgebra in E;

(ii) For any γ_1 and γ_2 in M_A the elements $\gamma_1 \llcorner \gamma_2$ and $\gamma_1 \lrcorner \gamma_2$ defined in **2.1.3** *also belong to M_A;*

(iii) For any $\gamma \in M_A$ we have $1_E \llcorner \gamma \in M_A$, where 1_E denotes the identity of End(E).

Proof. (i)⇒(ii) has been explained in **2.1.4**, and (ii)⇒(iii) is evident since $1_E \in M_A$. Let's prove (iii)⇒(i). Let u and v be two elements of A; we want to prove that $uv \in A$. According to Lemma **2.2.3**, we can find an element $\gamma \in M_A$, such that $\gamma(1) = v$. By assumption $1_E \llcorner \gamma \in M_A$, hence $uv = 1_E(u) \cdot \gamma(1) = (1_E \llcorner \gamma)(u)$ belongs to A, q.e.d.

2.3. (Maximal compact submonoids of End(E).) Now it is natural to discuss the archimedian case. Let E be a finite-dimensional real vector space. Any positive-definite quadratic form Q on E defines a maximal compact submonoid $M_Q := \{g \in \mathrm{End}(E) : \forall x, Q(g(x)) \leq Q(x)\}$. Is it true that all maximal compact submonoids of End(E) are of this form?

Contrary to what we might have expected after looking to the p-adic case, the answer is *negative*. There are many other maximal compact submonoids of $\mathrm{End}(E)$. Now we are going to describe them.

2.3.1. Recall that a subset A of a (not necessarily finite-dimensional) real vector space E is said to be *convex*, if $\lambda x_1 + (1-\lambda)x_2 \in A$ whenever x_1, $x_2 \in A$ and $0 \leq \lambda \leq 1$. This condition is equivalent to A being closed under baricentric combinations of its points, i.e. $\lambda_1 x_1 + \cdots + \lambda_n x_n \in A$ whenever $x_1, \ldots, x_n \in A$, all $\lambda_k \geq 0$ and $\sum_k \lambda_k = 1$.

For any subset $S \subset E$ there is a smallest convex subset $\mathrm{conv}(S) \subset E$, containing S. This subset is called the *convex hull* of S; note that it can be described as the set of all baricentric combinations of elements of S.

We say that A is *(central) symmetric*, if $A = -A$, i.e. if it is stable under multiplication by -1.

Finally, we say that A is a *convex body*, if it is convex, and if its affine span (i.e. the smallest affine subspace of E containing A) is equal to E. For finite-dimensional E this is equivalent to the non-emptiness of the interior of A.

A symmetric convex set A is a convex body iff its linear span $\mathbb{R} \cdot A = \bigcup_{\lambda \geq 0} \lambda A$ is equal to the whole of E. Subsets $A \subset E$ with the latter property are also called *absorbent*. In the finite-dimensional case this is equivalent to 0 being an interior point of A, i.e. to A being a neighborhood of the origin.

In the infinite-dimensional case a similar description of convex bodies can be obtained, provided we equip E with its "algebraic" topology, i.e. with its finest locally convex topology (cf. [EVT], ch. II, §4, n. 2).

2.3.2. Note that compact symmetric convex bodies A in a finite-dimensional real space E are in one-to-one correspondence with *norms* on E, i.e. maps $\|\cdot\| : E \to \mathbb{R}_{\geq 0}$ satisfying following conditions:

1. $\|\lambda x\| = |\lambda| \cdot \|x\|$ for any $\lambda \in \mathbb{R}$, $x \in E$;

2. $\|x + y\| \leq \|x\| + \|y\|$ for any $x, y \in E$;

3. $\|x\| = 0$ iff $x = 0$.

Indeed, any such norm defines a compact symmetric convex body, namely $A_{\|\cdot\|} := \{x \in E : \|x\| \leq 1\}$. Conversely, any such body A defines a norm $\|\cdot\|_A$, namely $\|x\|_A := \inf\{\lambda > 0 : \lambda^{-1}x \in A\}$.

Recall that this is a one-to-one correspondence, and that all these norms define the same topology on E, hence all of them are continuous.

Now we are in position to state an analogue of Theorem **2.2**:

Theorem 2.3.3 *Let E be a finite-dimensional real vector space. For any symmetric compact convex body $A \subset E$ we denote by M_A the compact submonoid of* $\mathrm{End}(E)$ *defined by $M_A := \{g \in \mathrm{End}(E) : g(A) \subset A\}$. Then:*

a) The M_A are maximal compact submonoids of $\mathrm{End}(E)$ *(with respect to the real topology);*

b) All maximal compact submonoids of $\mathrm{End}(E)$ *are of this form;*

c) $M_A = M_{A'}$ iff A and A' are similar, i.e. iff $A' = c \cdot A$ for some $c \in \mathbb{R}^*$.

2.3.4. Let us check that any compact submonoid $M \subset \mathrm{End}(E)$ stabilizes some symmetric compact convex body A, i.e. that $M \subset M_A$. For this take any such body $A_0 \subset E$ (e.g. the closed unit ball with respect to some metric on E) and consider the convex hull A of the set $S := M \cdot A_0 = \{g(x) : g \in M, x \in A_0\}$. Note that S is compact since both M and A_0 are compact, hence S is bounded, hence the same is true for its convex hull A. Now $\bar{A}$ is a compact convex subset; it is symmetric since $S = -S$, and it is a neighborhood of the origin since $\bar{A} \supset A_0$. So we have constructed a symmetric compact convex body $\bar{A}$, clearly stable under M since $gS = gMA_0 \subset MA_0 = S$ for any $g \in M$.

2.3.5. Clearly, if A and A' are similar, then $M_A = M_{A'}$. Let us prove that $M_A \subset M_{A'}$ implies that A and A' are similar, hence $M_A = M_{A'}$; this will complete the proof of the theorem.

So suppose that $M_A \subset M_{A'}$. Case $E = 0$ is trivial, so we assume $E \neq 0$. Put $\lambda := \sup_{x \in A'} \|x\|_A$. Since A' is compact and $\|\cdot\|_A$ continuous, this supremum is actually achieved at some point $u \in A'$, hence $\lambda = \|u\|_A$ is finite and > 0. After rescaling A', we can assume $\lambda = 1$, i.e. $A' \subset A$, and $\|u\|_A = \lambda = 1$. Now, according to Lemma **2.3.6** below, for any element $v \in A$ we can find some $g \in M_A$, for which $g(u) = v$. Then, since $u \in A'$ and $g \in M_A \subset M_{A'}$, we have $v = g(u) \in A'$. So we obtain $A \subset A'$, hence $A = A'$.

Lemma 2.3.6 *If A is a symmetric convex body in E, then for any two u, $v \in A$, such that $\|u\|_A = 1$, there is an endomorphism $g \in M_A \subset \mathrm{End}(E)$, such that $g(u) = v$.*

Proof. Indeed, our condition for u means that u belongs to the boundary ∂A of A, i.e. that it doesn't belong to the interior A^0 of A. Hence by Hahn–Banach theorem (cf. [EVT], ch. II, §3, cor. 2 of th. 1) there is a linear form $\varphi : E \to \mathbb{R}$, such that $\varphi(x) \leq \|x\|_A$ for all $x \in E$ and $\varphi(u) = \|u\|_A = 1$, hence $-1 \leq \varphi(A) \leq 1 = \varphi(u)$. We can define our endomorphism g by $g(x) := \varphi(x) \cdot v$. The image of A under g is $[-1, 1] \cdot v \subset A$, hence $g \in M_A$, and by construction $g(u) = v$.

2.4. (Category of $\mathbb{Z}_\infty$-lattices.) Thus we are led to define a $\mathbb{Z}_\infty$-*lattice* in a finite-dimensional real vector space E as a symmetric compact convex body A in E:

Definition 2.4.1 *We define the category of $\mathbb{Z}_\infty$-lattices $\mathbb{Z}_\infty$-Lat as follows. Its objects are pairs $A = (A_{\mathbb{Z}_\infty}, A_{\mathbb{R}})$, consisting of a finite-dimensional real vector space $A_{\mathbb{R}}$ and a compact symmetric convex body $A_{\mathbb{Z}_\infty} \subset A_{\mathbb{R}}$; when no confusion can arise, we denote $A_{\mathbb{Z}_\infty}$ by the same letter A as the whole pair.*

The morphisms from $A = (A_{\mathbb{Z}_\infty}, A_{\mathbb{R}})$ to $B = (B_{\mathbb{Z}_\infty}, B_{\mathbb{R}})$ are pairs $f = (f_{\mathbb{Z}_\infty}, f_{\mathbb{R}})$, where $f_{\mathbb{R}} : A_{\mathbb{R}} \to B_{\mathbb{R}}$ is an $\mathbb{R}$-linear map, and $f_{\mathbb{Z}_\infty} : A_{\mathbb{Z}_\infty} \to B_{\mathbb{Z}_\infty}$ is required to coincide with the restriction of $f_{\mathbb{R}}$ to $A_{\mathbb{Z}_\infty}$. When no confusion can arise, we denote $f_{\mathbb{Z}_\infty}$ simply by f.

Composition of morphisms is defined in the natural way.

Note that a morphism $f = (f_{\mathbb{Z}_\infty}, f_{\mathbb{R}})$ is determined by any of its components, since $f_{\mathbb{Z}_\infty}$ is the restriction of $f_{\mathbb{R}}$ to $A_{\mathbb{Z}_\infty}$; on the other hand, $A_{\mathbb{Z}_\infty}$ is absorbent, i.e. any element of $A_{\mathbb{R}}$ is of form λx for some $\lambda > 0$, $x \in A_{\mathbb{Z}_\infty}$, hence $f_{\mathbb{R}}(\lambda x) = \lambda f_{\mathbb{Z}_\infty}(x)$.

We could also describe $\mathrm{Hom}_{\mathbb{Z}_\infty\text{-}Lat}(A, B)$ as the set of $\mathbb{R}$-linear maps $f : A_{\mathbb{R}} \to B_{\mathbb{R}}$, such that $f(A) \subset B$. Another possible description is this: $\mathbb{Z}_\infty$-*Lat* is the category of finite-dimensional normed real spaces, and the morphisms $f : A \to B$ are the $\mathbb{R}$-linear maps $A_{\mathbb{R}} \to B_{\mathbb{R}}$ of norm ≤ 1, i.e. we require $\|f(x)\|_B \leq \|x\|_A$ for all $x \in A_{\mathbb{R}}$.

This choice of morphisms is motivated by the p-adic case, in which we had $M_A \cong \mathrm{End}_{\mathbb{Z}_p}(A)$ for any $\mathbb{Z}_p$-lattice A in a $\mathbb{Q}_p$-vector space. In our case we also have $\mathrm{End}_{\mathbb{Z}_\infty\text{-}Lat}(A) \cong M_A$, where M_A is the corresponding maximal compact submonoid in $\mathrm{End}(A_{\mathbb{R}})$ (cf. **2.3.3**).

2.4.2. We are going to embed $\mathbb{Z}_\infty$-*Lat* into the category of "flat $\mathbb{Z}_\infty$-modules" $\mathbb{Z}_\infty$-*Fl.Mod* as a full subcategory, and later we will embed this latter into the category of (all) $\mathbb{Z}_\infty$-modules $\mathbb{Z}_\infty$-*Mod*, again as a full subcategory. That's why we will say that morphisms $f : A \to B$ of $\mathbb{Z}_\infty$-*Lat* are $\mathbb{Z}_\infty$-*linear maps* or $\mathbb{Z}_\infty$-*homomorphisms*, and will denote $\mathrm{Hom}_{\mathbb{Z}_\infty\text{-}Lat}(A, B)$ simply by $\mathrm{Hom}_{\mathbb{Z}_\infty}(A, B)$.

2.4.3. Note that $\mathbb{Z}_\infty$-*Lat* is already a nicer category than the category of quadratic vector spaces. At least, it has finite inductive and projective limits, and in particular finite direct sums (i.e. coproducts) and products. For example, the product $A \times B$ is equal to $(A_{\mathbb{Z}_\infty} \times B_{\mathbb{Z}_\infty}, A_{\mathbb{R}} \times B_{\mathbb{R}})$, and the direct sum $A \oplus B$ is given by $(\mathrm{conv}(A_{\mathbb{Z}_\infty} \cup B_{\mathbb{Z}_\infty}), A_{\mathbb{R}} \oplus B_{\mathbb{R}})$. Note that both $\mathbb{R}$-vector spaces $(A \times B)_{\mathbb{R}}$ and $(A \oplus B)_{\mathbb{R}}$ can be identified with $A_{\mathbb{R}} \oplus B_{\mathbb{R}}$, while the corresponding convex subsets are different: all we can say is $(A \oplus B)_{\mathbb{Z}_\infty} \subset (A \times B)_{\mathbb{Z}_\infty}$, i.e. we have a canonical *monomorphism* $A \oplus B \to A \times B$. The fact that this is a monomorphism will be explained later as a manifestation of *hypoadditivity* of $\mathbb{Z}_\infty$. Note that in an additive category it would have been an isomorphism, hence $\mathbb{Z}_\infty$-*Lat* is not additive.

In the language of normed vector spaces the direct sum corresponds to norm $\|x+y\|_{A\oplus B} = \|x\|_A + \|y\|_B$, and the direct product to norm $\|x+y\|_{A\times B} = \sup(\|x\|_A, \|y\|_B)$ on the same vector space $A_\mathbb{R} \oplus B_\mathbb{R}$, where we take $x \in A_\mathbb{R}$, $y \in B_\mathbb{R}$.

2.5. Before proceeding further let us study the $\mathbb{Z}_\infty$-lattices A which are stable under multiplication in a given finite-dimensional real algebra E. This should give us an understanding of finite flat $\mathbb{Z}_\infty$-algebras. Here is an analogue of Theorem **2.2.4**:

Theorem. *Let A be a $\mathbb{Z}_\infty$-sublattice in a finite-dimensional $\mathbb{R}$-algebra E, i.e. $A \subset E$ is a symmetric compact convex body. Suppose that $1 \in \partial A$, i.e. $\|1\|_A = 1$ (any $\mathbb{Z}_\infty$-lattice in E is similar to exactly one $\mathbb{Z}_\infty$-lattice of this sort). The following conditions are equivalent:*

(i) A is closed under multiplication of E, i.e. $\mu_E(A \times A) \subset A$, or, in other words, A is a multiplicative submonoid of E;

(ii) For any γ_1 and γ_2 in M_A the elements $\gamma_1 \llcorner \gamma_2$ and $\gamma_1 \lrcorner \gamma_2$ defined in **2.1.3** *also belong to M_A;*

(iii) For any $\gamma \in M_A$ we have $1_E \llcorner \gamma \in M_A$, where 1_E denotes the identity of $\mathrm{End}(E)$.

(iv) E is a Banach algebra with respect to $\|\cdot\|_A$, i.e. $\|1\|_A = 1$ and $\|xy\|_A \leq \|x\|_A \cdot \|y\|_A$ for all $x, y \in E$.

Proof. (i)$\Rightarrow$(ii): Clear from definitions (cf. **2.1.3**); note that we use $1 \in A$ here. (ii)$\Rightarrow$(iii): Evident since $1_E \in M_A$. (iii)$\Rightarrow$(i): Same as in the proof of **2.2.4**, with Lemma **2.2.3** replaced by **2.3.6**. (iv)$\Rightarrow$(i): Follows from $A = \{x : \|x\|_A \leq 1\}$. (i)$\Rightarrow$(iv): If one of x and y is zero, the statement is trivial; assume $x \neq 0$, $y \neq 0$. Put $x' := x/\|x\|_A$, $y' := y/\|y\|_A$. Then $\|x'\|_A = \|y'\|_A = 1$, hence $x', y' \in A$, so by (i) we get $x'y' \in A$, i.e. $\|x'y'\|_A \leq 1$, hence $\|xy\|_A = \|x\|_A \cdot \|y\|_A \cdot \|x'y'\|_A \leq \|x\|_A \cdot \|y\|_A$.

In other words, the $\mathbb{Z}_\infty$-lattices A of E, which are at the same time "finite $\mathbb{Z}_\infty$-subalgebras", are exactly the symmetric compact convex absorbent (multiplicative) submonoids of E.

2.5.1. In particular, $\mathbb{R}$ contains exactly one such "$\mathbb{Z}_\infty$-subalgebra", namely, $\mathbb{Z}_\infty := ([-1,1], \mathbb{R})$. This is our first definition of $\mathbb{Z}_\infty$ itself; later it will be improved.

2.5.2. Note that multiplicative submonoids appeared again in the statement of **2.5**. This suggests some connection to maximal compact submonoids M_A considered in **2.3.3**. Indeed, for any $\mathbb{Z}_\infty$-lattice A in E, the corresponding maximal compact submonoid M_A is easily seen to be symmetric and convex, hence it satisfies the conditions of **2.5** for $\operatorname{End}(E)$, i.e. $M_A \cong \operatorname{End}_{\mathbb{Z}_\infty}(A)$ is a "finite $\mathbb{Z}_\infty$-subalgebra" of $\operatorname{End}(E)$, as one might have expected.

2.5.3. If the finite-dimensional $\mathbb{R}$-algebra E is commutative and semisimple, all the "finite $\mathbb{Z}_\infty$-subalgebras" in E are contained in a maximal one A_{max}; it is natural to call this A_{max} the *integral closure of* $\mathbb{Z}_\infty$ *in* E. To construct this "integral closure" consider all embeddings $\sigma_i : E \to \mathbb{C}$; then $A_{max} = \{x \in E : \forall i, |\sigma_i(x)| \leq 1\}$. Indeed, A_{max} is a closed symmetric convex body, stable under multiplication; it is compact since $\bigcap_i \operatorname{Ker} \sigma_i = 0$, so it is a "finite $\mathbb{Z}_\infty$-subalgebra" of E. If A is another one, then by compactness of A, there is some $C > 0$, such that $|\sigma_i(x)| \leq C$ for all $x \in A$ and i. Since A is closed under multiplication, we get $|\sigma_i(x^n)| = |\sigma_i(x)|^n \leq C$ for any $n \geq 1$, hence $|\sigma_i(x)| \leq 1$ for each i, i.e. $A \subset A_{max}$.

2.5.4. In particular, we can apply this to $E = \mathbb{C}$. The integral closure of $\mathbb{Z}_\infty$ in $\mathbb{C}$ will be denoted by $\bar{\mathbb{Z}}_\infty$ (cf. **1.4.3**). So we have $\bar{\mathbb{Z}}_\infty = (\{z \in \mathbb{C} : |z| \leq 1\}, \mathbb{C})$, with the multiplication induced by that of $\mathbb{C}$.

2.5.5. Any commutative semisimple $\mathbb{R}$-algebra E is isomorphic to $\mathbb{R}^r \times \mathbb{C}^s$ for some $r, s \geq 0$. The considerations of **2.5.3** show that the integral closure of $\mathbb{Z}_\infty$ in E coincides with $\mathbb{Z}_\infty^r \times \bar{\mathbb{Z}}_\infty^s$, as one might have expected.

2.5.6. Resuming the considerations of **1.5.3** in our new context, we see that a vector bundle $\mathscr{E}$ on $\widehat{\operatorname{Spec}\mathbb{Z}}$ corresponds to a free $\mathbb{Z}$-module $E_{\mathbb{Z}}$ of finite rank, together with a $\mathbb{Z}_\infty$-lattice, i.e. a symmetric compact convex body A, in $E_{\mathbb{R}} := E_{\mathbb{Z}} \otimes \mathbb{R}$. In other words, we are given both a $\mathbb{Z}$-lattice $E_{\mathbb{Z}}$ and a symmetric convex body A in the same real vector space $E_{\mathbb{R}}$. The "global sections" of $\mathscr{E}$ correspond to the intersection $E_{\mathbb{Z}} \cap A$; this is a finite set ("an $\mathbb{F}_1$-module"), and there are a lot of interesting theorems offering bounds for the number of lattice points inside a convex set, starting with Minkowski theorem. So one might think of these theorems as some "Riemann–Roch theorems" for $\widehat{\operatorname{Spec}\mathbb{Z}}$.

2.5.7. In particular, we have the trivial line bundle $\mathscr{O}_{\widehat{\operatorname{Spec}\mathbb{Z}}}$, defined by $\mathbb{Z}$ and $\mathbb{Z}_\infty$ in $\mathbb{R}$. If $K/\mathbb{Q}$ is a number field, we can consider the normalization of $\widehat{\operatorname{Spec}\mathbb{Z}}$ in K; as a vector bundle over $\widehat{\operatorname{Spec}\mathbb{Z}}$, it will be given by the integral closure $\mathscr{O}_K$ of $\mathbb{Z}$ in K, and by the integral closure $\mathscr{O}_{K,\infty}$ of $\mathbb{Z}_\infty$ in $K \otimes_{\mathbb{Q}} \mathbb{R}$. We have a description of $\mathscr{O}_{K,\infty}$ in **2.5.3**; when we compute the "global sections" of our vector bundle over $\widehat{\operatorname{Spec}\mathbb{Z}}$, we get $\mathscr{O}_K \cap \mathscr{O}_{K,\infty} = \{x \in \mathscr{O}_K : |\sigma(x)| \leq 1$

for any $\sigma : K \to \mathbb{C}\}$. By classical algebraic number theory this set is known to be the union of zero and the subgroup of roots of unity in $\mathscr{O}_K$, which is a finite cyclic group.

2.5.8. What about a complex analogue of **2.3.3**? In other words, what are the maximal compact submonoids of $\mathrm{End}_{\mathbb{C}}(E)$, when E is a finite-dimensional $\mathbb{C}$-vector space? One can check that an analogue of **2.3.3** still holds, provided we consider *balanced* compact convex bodies $A \subset E$, i.e. we require $\lambda A \subset A$ for any $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$. In other words, $\bar{\mathbb{Z}}_\infty \cdot A \subset A$, i.e. A is a "$\bar{\mathbb{Z}}_\infty$-module".

Note that balanced compact convex bodies $A \subset E$ are in one-to-one correspondence with the *complex* norms $\|\cdot\|$ on E, i.e. norms on E, such that $\|\lambda x\| = |\lambda| \cdot \|x\|$ for any $\lambda \in \mathbb{C}$, $x \in E$.

Then we define $M_A := \{g \in \mathrm{End}_{\mathbb{C}}(E) : g(A) \subset A\}$ as before, and these are precisely the maximal compact submonoids of $\mathrm{End}_{\mathbb{C}}(E)$. The proof is essentially that of **2.3.3**, but we have to apply complex Hahn–Banach in the lemma (cf. [EVT], ch. II, §8, n. 3, cor. 1 of th. 1).

We do not develop this point of view further since these "$\bar{\mathbb{Z}}_\infty$-lattices" will be later described as $\mathbb{Z}_\infty$-lattices equipped with a $\bar{\mathbb{Z}}_\infty$-module structure, hence their theory will be a particular case of the theory of modules over a $\mathbb{Z}_\infty$-algebra.

2.5.9. We have already observed (cf. **2.5.3**), that any *commutative* semisimple $\mathbb{R}$-algebra E contains a unique maximal compact submonoid (note that it automatically will be symmetric and convex, otherwise we would replace it by the closure of its symmetric convex hull). What about non-commutative semisimple $\mathbb{R}$-algebras? Clearly, it is enough to consider the case of a simple $\mathbb{R}$-algebra E, and there are three kinds of these. If $E = M(n, \mathbb{R}) = \mathrm{End}_{\mathbb{R}}(\mathbb{R}^n)$, its maximal compact submonoids correspond to $\mathbb{Z}_\infty$-lattices in $\mathbb{R}^n$ (cf. **2.3.3**), and the maximal compact submonoids of $E = M(n, \mathbb{C}) = \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$ correspond to $\bar{\mathbb{Z}}_\infty$-lattices in $\mathbb{C}^n$ (cf. **2.5.8**).

So only the quaternionic case $E = M(n, \mathbb{H})$ remains. Note that there is the largest compact submonoid (hence a $\mathbb{Z}_\infty$-subalgebra) $\mathbb{Z}_\infty^{\mathbb{H}} := \{q \in \mathbb{H} : |q| \leq 1\}$ in $\mathbb{H}$, where $|q| := \sqrt{q\bar{q}}$ is the quaternionic absolute value. Then it seems quite plausible that maximal compact submonoids of $M(n, \mathbb{H}) \cong \mathrm{End}_{\mathbb{H}}(\mathbb{H}^n)$ are exactly the stabilizers M_A of those $\mathbb{Z}_\infty$-lattices $A \subset \mathbb{H}^n$ which are $\mathbb{Z}_\infty^{\mathbb{H}}$-submodules, i.e. $\mathbb{Z}_\infty^{\mathbb{H}} \cdot A \subset A$. In the language of norms they correspond to quaternionic norms $\|\cdot\|$ on $\mathbb{H}^n$, i.e. we require $\|\lambda x\| = |\lambda| \cdot \|x\|$ for any $\lambda \in \mathbb{H}$, $x \in \mathbb{H}^n$. Actually the proof of **2.3.3** still works, provided we have a quaternionic Hahn–Banach theorem.

2.6. When A is a $\mathbb{Z}_p$-lattice in a finite-dimensional $\mathbb{Q}_p$-algebra E, stable under multiplication $\mu_E : E \times E \to E$ of E, then A is a $\mathbb{Z}_p$-algebra, with the multiplication $\mu_A : A \times A \to A$ induced by that of E. Note that μ_A, being the restriction of μ_E, is automatically a $\mathbb{Z}_p$-bilinear map. Conversely, if we are given a $\mathbb{Z}_p$-bilinear map $\mu_A : A \times A \to A$ on some $\mathbb{Z}_p$-lattice A, subject to some additional conditions, we obtain a $\mathbb{Z}_p$-algebra structure on A.

In other words, a finite flat $\mathbb{Z}_p$-algebra can be roughly described as a $\mathbb{Z}_p$-lattice A equipped by a $\mathbb{Z}_p$-bilinear map μ_A. We would like to obtain a similar description of finite flat $\mathbb{Z}_\infty$-algebras. For this we have to define $\mathbb{Z}_\infty$-bilinear maps as follows:

Definition 2.6.1 *Let M, N, P be three $\mathbb{Z}_\infty$-lattices (i.e. three objects of $\mathbb{Z}_\infty$-Lat). A* $\mathbb{Z}_\infty$-bilinear map *$M \times N \to P$ is by definition a pair $\Phi = (\Phi_{\mathbb{Z}_\infty}, \Phi_{\mathbb{R}})$, consisting of an $\mathbb{R}$-bilinear map $\Phi_{\mathbb{R}} : M_{\mathbb{R}} \times N_{\mathbb{R}} \to P_{\mathbb{R}}$, and a map of sets $\Phi_{\mathbb{Z}_\infty} : M_{\mathbb{Z}_\infty} \times N_{\mathbb{Z}_\infty} \to P_{\mathbb{Z}_\infty}$, required to coincide with the restriction of $\Phi_{\mathbb{R}}$. When no confusion can arise, we denote $\Phi_{\mathbb{Z}_\infty}$ by the same letter Φ. We denote by* $\mathrm{Bilin}_{\mathbb{Z}_\infty}(M, N; P)$ *the set of all $\mathbb{Z}_\infty$-bilinear maps $M \times N \to P$.*

2.6.2. One can make for $\mathbb{Z}_\infty$-bilinear map remarks similar to those made for $\mathbb{Z}_\infty$-linear maps (i.e. morphisms of $\mathbb{Z}_\infty$-*Lat*) after **2.4.1**. In particular, $\Phi = (\Phi_{\mathbb{Z}_\infty}, \Phi_{\mathbb{R}})$ is completely determined by either of its components. It can be described as an $\mathbb{R}$-linear map $\Phi_{\mathbb{R}} : M_{\mathbb{R}} \times N_{\mathbb{R}} \to P_{\mathbb{R}}$, such that $\Phi_{\mathbb{R}}(M_{\mathbb{Z}_\infty} \times N_{\mathbb{Z}_\infty}) \subset P_{\mathbb{Z}_\infty}$. In the language of normed vector spaces this means that we consider $\mathbb{R}$-bilinear maps of norm ≤ 1, i.e. $\|\Phi_{\mathbb{R}}(x, y)\|_P \leq \|x\|_M \cdot \|y\|_N$ for all $x \in M_{\mathbb{R}}$, $y \in N_{\mathbb{R}}$.

Note that, similar to what we have for modules over a commutative ring, a $\mathbb{Z}_\infty$-bilinear form can be described as a map $\Phi : M \times N \to P$ (i.e. $M_{\mathbb{Z}_\infty} \times N_{\mathbb{Z}_\infty} \to P_{\mathbb{Z}_\infty}$), such that for any $x \in M$ the map $s_\Phi(x) : y \mapsto \Phi(x, y)$ is $\mathbb{Z}_\infty$-linear (i.e. extends to a morphism $N \to P$ in $\mathbb{Z}_\infty$-*Lat*), as well as the map $d_\Phi(y) : x \mapsto \Phi(x, y)$ for any $y \in N$.

Proposition 2.6.3 *(Existence of tensor products.) For any two $\mathbb{Z}_\infty$-lattices M and N the functor* $\mathrm{Bilin}_{\mathbb{Z}_\infty}(M, N; -)$ *is representable by some $\mathbb{Z}_\infty$-lattice $M \otimes_{\mathbb{Z}_\infty} N$, i.e. we have* $\mathrm{Bilin}_{\mathbb{Z}_\infty}(M, N; P) \cong \mathrm{Hom}_{\mathbb{Z}_\infty}(M \otimes_{\mathbb{Z}_\infty} N, P)$, *functorially in P. In other words, we have a universal $\mathbb{Z}_\infty$-bilinear map $\Phi_0 : M \times N \to M \otimes_{\mathbb{Z}_\infty} N$, such that for any other $\mathbb{Z}_\infty$-bilinear map $\Psi : M \times N \to P$ there is a unique $\mathbb{Z}_\infty$-linear map $\psi : M \otimes_{\mathbb{Z}_\infty} N \to P$ satisfying $\Psi = \psi \circ \Phi_0$.*

Proof. We define our tensor product $M \otimes N$ in $\mathbb{Z}_\infty$-*Lat* by $(M \otimes M)_{\mathbb{R}} := M_{\mathbb{R}} \otimes_{\mathbb{R}} N_{\mathbb{R}}$, $(M \otimes N)_{\mathbb{Z}_\infty} := \mathrm{conv}(S)$, where $S := \{x \otimes y : x \in M_{\mathbb{Z}_\infty}, y \in N_{\mathbb{Z}_\infty}\}$. Clearly, $(M \otimes N)_{\mathbb{Z}_\infty}$ is a symmetric convex body in $(M \otimes N)_{\mathbb{R}}$; it is compact, since it is the convex hull of a compact subset S of a finite-dimensional real vector space (this is a simple corollary of the Carathéodory theorem, which asserts that each point of $\mathrm{conv}(S)$ can be written as a baricentric combination of at most $\dim(M \otimes N)_\mathbb{R} + 1$ points of S). Now the required universal property follows immediately from that of $M_\mathbb{R} \times N_\mathbb{R} \to (M \otimes N)_\mathbb{R}$, and from the definition of $(M \otimes N)_{\mathbb{Z}_\infty}$.

2.6.4. Sets of $\mathbb{Z}_\infty$-polylinear maps $\mathrm{Polylin}_{\mathbb{Z}_\infty}(M_1, M_2, \ldots, M_n; P)$ and multiple tensor products $M_1 \otimes M_2 \otimes \cdots \otimes M_n$ are defined essentially in the same way. The universal property of multiple tensor products gives us immediately a chain of isomorphisms $M_1 \otimes (M_2 \otimes M_3) \cong M_1 \otimes M_2 \otimes M_3 \cong (M_1 \otimes M_2) \otimes M_3$, i.e. we have an associativity constraint for our tensor product. It clearly satisfies the pentagon axiom, since all vertices of the pentagon diagram are canonically isomorphic to a quadruple tensor product. Similarly, the natural commutativity isomorphisms $M_1 \otimes M_2 \cong M_2 \otimes M_1$ obviously satisfy the hexagon axiom. Finally, $\mathbb{Z}_\infty$ is a unit for this tensor product, i.e. $\mathbb{Z}_\infty \otimes M \cong M$: this can be seen either from the explicit construction of $\mathbb{Z}_\infty \otimes M$ given in **2.6.3**, or from its universal property: one has then to check $\mathrm{Bilin}_{\mathbb{Z}_\infty}(\mathbb{Z}_\infty, M; N) \cong \mathrm{Hom}_{\mathbb{Z}_\infty}(M, N)$.

Therefore, $\mathbb{Z}_\infty$-*Lat* is an ACU (associative, commutative, with unity) $\otimes$-category with respect to the tensor product introduced above. We could proceed and define $\mathbb{Z}_\infty$-algebras and modules over these algebras in terms of this $\otimes$-structure; however, it will be more convenient to do this after extending $\mathbb{Z}_\infty$-*Lat* to a larger category. It will be sufficient for now to remark that this definition of algebras in $\mathbb{Z}_\infty$-*Lat* is equivalent to one we had in **2.5**.

2.6.5. Note that *inner Homs exist in* $\mathbb{Z}_\infty$-*Lat*. In other words, for any two $\mathbb{Z}_\infty$-lattices N and P the functor $\mathrm{Hom}_{\mathbb{Z}_\infty}(- \otimes N, P) \cong \mathrm{Bilin}_{\mathbb{Z}_\infty}(-, N; P)$ is representable by some $\mathbb{Z}_\infty$-lattice $\mathbf{Hom}_{\mathbb{Z}_\infty}(N, P)$. This "inner Hom" $H := \mathbf{Hom}_{\mathbb{Z}_\infty}(N, P)$ can be constructed as follows. We put $H_\mathbb{R} := \mathrm{Hom}_\mathbb{R}(N_\mathbb{R}, P_\mathbb{R})$ and $H_{\mathbb{Z}_\infty} := \{\varphi : N_\mathbb{R} \to P_\mathbb{R} : \varphi(N_{\mathbb{Z}_\infty}) \subset P_{\mathbb{Z}_\infty}\}$. The corresponding norm $\|\cdot\|_H$ on $H_\mathbb{R}$ is defined by the usual rule for norms of linear maps between normed vector spaces: $\|\varphi\|_H = \sup_{\|x\|_N \leq 1} \|\varphi(x)\|_P$. Since $H_{\mathbb{Z}_\infty} = \{\varphi : \|\varphi\|_H \leq 1\}$, this also shows that $H_{\mathbb{Z}_\infty}$ is a compact symmetric convex body in $H_\mathbb{R}$, i.e. $H = (H_{\mathbb{Z}_\infty}, H_\mathbb{R})$ is indeed an object of $\mathbb{Z}_\infty$-*Lat*. Now the verification of the required property for $\mathbf{Hom}(N, P)$ is straightforward.

2.6.6. This gives us a lot of canonical morphisms and isomorphisms, valid in any ACU $\otimes$-category with inner Homs. For example,

$$\mathbf{Hom}(M, \mathbf{Hom}(N, P)) \cong \mathbf{Hom}(M \otimes N, P) \cong \mathbf{Hom}(N, \mathbf{Hom}(M, P)) \tag{2.6.6.1}$$

$$\Gamma(\mathbf{Hom}(M, N)) \cong \mathrm{Hom}_{\mathbb{Z}_\infty}(M, N), \quad \mathbf{Hom}(\mathbb{Z}_\infty, M) \cong M \tag{2.6.6.2}$$

Here we denote by $\Gamma = \Gamma_{\mathbb{Z}_\infty}$ the "functor of global sections" in $\mathbb{Z}_\infty$-*Lat*, given by $\Gamma(-) := \mathrm{Hom}(\mathbb{Z}_\infty, -)$. Note that in our situation $\Gamma(M) = M_{\mathbb{Z}_\infty}$ for any $M = (M_{\mathbb{Z}_\infty}, M_\mathbb{R})$ in $\mathbb{Z}_\infty$-*Lat*.

2.6.7. We can define "inner Bilins" as well: $\mathbf{Bilin}(M, N; P) := \mathbf{Hom}(M \otimes N, P)$. Clearly, $\Gamma(\mathbf{Bilin}(M, N; P)) = \mathrm{Bilin}(M, N; P)$. In the language of normed vector spaces $B := \mathbf{Bilin}(M, N; P)$ corresponds to the space $B_\mathbb{R}$ of $\mathbb{R}$-bilinear maps $\Phi : M_\mathbb{R} \times N_\mathbb{R} \to P_\mathbb{R}$, equipped by the usual norm: $\|\Phi\|_B = \sup_{\|x\|_M \leq 1,\, \|y\|_N \leq 1} \|\Phi(x, y)\|_P$.

2.6.8. Once we have the unit object and inner Homs, we can define *dual objects* by $M^* := \mathbf{Hom}_{\mathbb{Z}_\infty}(M, \mathbb{Z}_\infty)$. In our situation we get the dual convex body $M^*_{\mathbb{Z}_\infty}$ to $M_{\mathbb{Z}_\infty}$ inside the dual space $(M^*)_\mathbb{R} = (M_\mathbb{R})^*$. This corresponds to the usual norm $\|\varphi\|_{M^*} = \sup_{\|x\|_M \leq 1} |\varphi(x)|$ on the space of linear functionals $M^*_\mathbb{R}$.

We also have canonical $\mathbb{Z}_\infty$-homomorphisms $M \to M^{**}$, which in our situation turn out to be isomorphisms, i.e. all $\mathbb{Z}_\infty$-lattices are *reflexive*. In fact, this is an immediate consequence of the Hahn–Banach theorem, which says that the dual to the dual norm is the original one.

2.6.9. Is the $\otimes$-category $\mathbb{Z}_\infty$-*Lat rigid?* It already has inner Homs, and all its objects are reflexive. However, the remaining property fails: the map $M^* \otimes N \to \mathbf{Hom}(M, N)$ is a monomorphism, but usually not an isomorphism. To see this take $M = N = \mathbb{Z}_\infty \oplus \mathbb{Z}_\infty$; then the element $1_M \in \mathrm{Hom}(M, M) = \Gamma(\mathbf{Hom}(M, M))$ doesn't come from any element of $\Gamma(M^* \otimes M)$, since any f from the image of this set has $|\operatorname{tr} f| \leq 1$.

2.6.10. We are going to embed $\mathbb{Z}_\infty$-*Lat* into larger categories. However, $\mathbb{Z}_\infty$-*Lat* can be always retrieved as the full subcategory consisting of the reflexive objects of any of these larger categories, subject to some additional finiteness conditions.

2.7. (Torsion-free $\mathbb{Z}_\infty$-modules.) Now we want to construct the category of "flat" or "torsion-free" $\mathbb{Z}_\infty$-modules $\mathbb{Z}_\infty$-*Fl.Mod*, which will contain $\mathbb{Z}_\infty$-*Lat* as a full subcategory.

2.7.1. As usual, we consider the p-adic case first. How can one reconstruct the category of flat $\mathbb{Z}_p$-modules $\mathbb{Z}_p$-*Fl.Mod*, starting from its full subcategory of $\mathbb{Z}_p$-lattices (i.e. free modules of finite rank) $\mathbb{Z}_p$-*Lat*? Observe that $\mathbb{Z}_p$-*Fl.Mod* has arbitrary inductive limits, hence the inclusion $\mathbb{Z}_p$-*Lat* $\to \mathbb{Z}_p$-*Fl.Mod* induces a functor $\mathrm{Ind}(\mathbb{Z}_p\text{-}Lat) \to \mathbb{Z}_p$-*Fl.Mod* from the category of ind-objects over $\mathbb{Z}_p$-*Lat* (i.e. the category of filtered inductive systems of objects of $\mathbb{Z}_p$-*Lat*; we refer to SGA 4 I for a more detailed discussion of ind-objects) to $\mathbb{Z}_p$-*Fl.Mod*. This functor is essentially surjective, since any torsion-free $\mathbb{Z}_p$-module can be represented as a filtered inductive limit of its finitely generated submodules, which are automatically $\mathbb{Z}_p$-lattices. It is quite easy to check that it is in fact an equivalence of categories, i.e. $\mathbb{Z}_p$-*Fl.Mod* $\cong \mathrm{Ind}(\mathbb{Z}_p\text{-}Lat)$.

2.7.2. So we are tempted to construct $\mathbb{Z}_\infty$-*Fl.Mod* as $\mathrm{Ind}(\mathbb{Z}_\infty\text{-}\mathit{Lat})$. However, we would like to have a more explicit description of this category, so we give a direct definition of a category $\mathbb{Z}_\infty$-*Fl.Mod* that contains $\mathbb{Z}_\infty$-*Lat* as a full subcategory and is closed under arbitrary inductive limits. This will give a functor $\mathrm{Ind}(\mathbb{Z}_\infty\text{-}\mathit{Lat}) \to \mathbb{Z}_\infty\text{-}\mathit{Fl.Mod}$. The reader is invited to check later, if so inclined, that this functor is indeed an equivalence of categories, to convince himself that the definition of $\mathbb{Z}_\infty$-*Fl.Mod* given below is the "correct" one.

Definition 2.7.3 *We define the category of flat (or torsion-free) $\mathbb{Z}_\infty$-modules $\mathbb{Z}_\infty$-Fl.Mod as follows. Its objects are pairs $A = (A_{\mathbb{Z}_\infty}, A_{\mathbb{R}})$, consisting of a real vector space $A_{\mathbb{R}}$ and a symmetric convex body $A_{\mathbb{Z}_\infty} \subset A_{\mathbb{R}}$ (in particular, $A_{\mathbb{Z}_\infty}$ is required to be absorbent; cf.* **2.3.1***). When no confusion can arise, we denote $A_{\mathbb{Z}_\infty}$ by the same letter A as the whole pair.*

The morphisms $f = (f_{\mathbb{Z}_\infty}, f_{\mathbb{R}})$ of $\mathbb{Z}_\infty$-Fl.Mod and their composition are defined in the same way as in **2.4.1**.

2.7.4. In contrast with **2.4.1**, now we do not require $A_{\mathbb{R}}$ to be finite-dimensional, and we do not require $A_{\mathbb{Z}_\infty}$ to be compact or even closed (with respect to the "algebraic topology" on $A_{\mathbb{R}}$ in the infinite-dimensional case).

2.7.5. Any flat $\mathbb{Z}_\infty$-module A defines a *seminorm* $\|\cdot\|_A : A_{\mathbb{R}} \to \mathbb{R}_{\geq 0}$ by the same formula as before: $\|x\|_A = \inf\{\lambda > 0 : \lambda^{-1}x \in A_{\mathbb{Z}_\infty}\}$. However, in general this will be just a seminorm, i.e. it has only properties 1. and 2. of **2.3.2**. This seminorm is still continuous with respect to the algebraic topology on $A_{\mathbb{R}}$. However, it doesn't determine $A_{\mathbb{Z}_\infty}$ uniquely, unless we know $A_{\mathbb{Z}_\infty}$ to be closed: all we can say is $\{x : \|x\|_A < 1\} \subset A_{\mathbb{Z}_\infty} \subset \{x : \|x\|_A \leq 1\}$ (cf. [EVT], ch. II, §2, prop. 22).

2.8. *Arbitrary projective limits exist in $\mathbb{Z}_\infty$-Fl.Mod.* Indeed, any such limit $L = \varprojlim_{\alpha\in I} F_\alpha$ can be computed as follows. (Here we consider generalized projective limits, i.e. limits in the sense of [MacLane]; so I is actually a small category, and $F : \alpha \mapsto F_\alpha$ is a functor $I \to \mathbb{Z}_\infty$-*Fl.Mod*; however, for simplicity we write generalized projective limits in the same way as the "classical" projective limits along an ordered set.) First of all, compute $V := \varprojlim_\alpha F_{\alpha,\mathbb{R}}$ in the category $\mathbb{R}$-*Vect* of $\mathbb{R}$-vector spaces, and denote by $\pi_\alpha : V \to F_{\alpha,\mathbb{R}}$ the canonical projections. Then put $L_{\mathbb{Z}_\infty} := \bigcap_\alpha \pi_\alpha^{-1}(F_{\alpha,\mathbb{Z}_\infty})$. This is clearly a symmetric convex subset in V. It need not be absorbent, so we take $L_{\mathbb{R}} := \mathbb{R} \cdot L_{\mathbb{Z}_\infty} \subset V$ and put $L := (L_{\mathbb{Z}_\infty}, L_{\mathbb{R}})$. The projections $\pi'_\alpha : L \to F_\alpha$ are given by the restrictions of π_α onto $L_{\mathbb{R}}$; they are well-defined (i.e. they define morphisms in $\mathbb{Z}_\infty$-*Fl.Mod*) since by construction $\pi_\alpha(L_{\mathbb{Z}_\infty}) \subset F_{\alpha,\mathbb{Z}_\infty}$.

We have to check that L with $\{\pi'_\alpha : L \to F_\alpha\}$ have the universal property required for projective limits. So suppose $\{\rho_\alpha : M \to F_\alpha\}$ is any compatible system of morphisms from some object M into our projective system. By definition of V we have a unique $\mathbb{R}$-linear map $\varphi : M_\mathbb{R} \to V$, such that $\pi_\alpha \circ \varphi = \rho_{\alpha,\mathbb{R}}$ for all α. Since $\rho_{\alpha,\mathbb{R}}(M_{\mathbb{Z}_\infty}) \subset F_{\alpha,\mathbb{Z}_\infty}$, we see that $\varphi(M_{\mathbb{Z}_\infty})$ is contained in each $\pi_\alpha^{-1}(F_{\alpha,\mathbb{Z}_\infty})$, hence in their intersection $L_{\mathbb{Z}_\infty}$. Now $M_\mathbb{R} = \mathbb{R} \cdot M_{\mathbb{Z}_\infty}$, hence $\varphi(M_\mathbb{R}) \subset \mathbb{R} \cdot L_{\mathbb{Z}_\infty} = L_\mathbb{R} \subset V$. This gives us a $\mathbb{Z}_\infty$-homomorphism $\varphi' : M \to L$, such that $\pi'_\alpha \circ \varphi' = \rho_\alpha$ for all α; its uniqueness is immediate.

This construction can be expressed in the language of norms, at least if all $F_{\alpha,\mathbb{Z}_\infty}$ are closed, as follows. We introduce on the projective limit V of given normed vector spaces the sup-norm by $\|x\|_{\sup} := \sup_\alpha \|\pi_\alpha(x)\|_{F_\alpha}$. Then we consider the subspace $L_\mathbb{R} := \{x \in V : \|x\|_{\sup} < +\infty\}$ of elements with finite sup-norm. Then $L_\mathbb{R}$ is a normed vector space with respect to the restriction of the sup-norm, and $L_{\mathbb{Z}_\infty} = \{x : \|x\|_{\sup} \leq 1\}$.

2.8.1. Note that, if I is finite, then $L_\mathbb{R} = V$ in the above notations. In other words, functor $\rho^* : X \mapsto X_\mathbb{R}$ commutes with finite projective limits, i.e. it is left exact. To show this take any $x \in V$ and observe that its sup-norm is finite (being the supremum of a finite set of real numbers), hence we can find some $\lambda > \|x\|_{\sup}$. Now it follows from definitions that $\pi_\alpha(\lambda^{-1}x)$ belongs to $F_{\alpha,\mathbb{Z}_\infty}$ for each α, hence $\lambda^{-1}x \in L_{\mathbb{Z}_\infty}$ and $x \in \mathbb{R} \cdot L_{\mathbb{Z}_\infty} = L_\mathbb{R}$.

2.8.2. In particular, finite products in $\mathbb{Z}_\infty$-*Fl.Mod* are computed componentwise: $A \times B = (A_{\mathbb{Z}_\infty} \times B_{\mathbb{Z}_\infty}, A_\mathbb{R} \times B_\mathbb{R})$; cf. **2.4.3**.

2.8.3. Another consequence is this: *a morphism $f = (f_{\mathbb{Z}_\infty}, f_\mathbb{R})$ is a monomorphism (in $\mathbb{Z}_\infty$-Fl.Mod) iff $f_\mathbb{R}$ is injective iff $f_{\mathbb{Z}_\infty}$ is injective.* Indeed, we have just seen that $X \mapsto X_\mathbb{R}$ is left exact, hence transforms monomorphisms into monomorphisms. So if f is monic, then $f_\mathbb{R}$ is monic in $\mathbb{R}$-*Vect*, i.e. injective. Since $f_{\mathbb{Z}_\infty}$ is a restriction of $f_\mathbb{R}$, injectivity of $f_\mathbb{R}$ implies injectivity of $f_{\mathbb{Z}_\infty}$. Finally, this last injectivity clearly implies that f is monic.

2.8.4. This means that *subobjects* $M = (M_{\mathbb{Z}_\infty}, M_\mathbb{R})$ of a fixed object L can be characterized by requiring $M_\mathbb{R}$ to be an $\mathbb{R}$-linear subspace of $L_\mathbb{R}$, and $M_{\mathbb{Z}_\infty}$ to be contained in $L_{\mathbb{Z}_\infty}$.

For example, the subobjects (i.e. "ideals") of $\mathbb{Z}_\infty = ([-1, 1], \mathbb{R})$ are $0 = (0, 0)$, $\lambda\mathbb{Z}_\infty = ([-\lambda, \lambda], \mathbb{R})$ and $\lambda\mathfrak{m}_\infty = ((-\lambda, \lambda), \mathbb{R})$ for $0 < \lambda \leq 1$. Of these 0 and $\lambda\mathbb{Z}_\infty$ lie in $\mathbb{Z}_\infty$-*Lat*, and $\lambda\mathfrak{m}_\infty$ does not, since the open interval $(-\lambda, \lambda)$ is not compact in $\mathbb{R}$. Note that among all ideals $\neq \mathbb{Z}_\infty$ there is a maximal one, namely, $\mathfrak{m}_\infty$, so $\mathbb{Z}_\infty$ is something like a local ring.

2.9. *Arbitrary inductive limits exist in $\mathbb{Z}_\infty$-Fl.Mod.* Let us compute such a limit $L = \varinjlim_\alpha F_\alpha$. First of all, put $L_\mathbb{R} := \varinjlim_\alpha F_{\alpha,\mathbb{R}}$ in $\mathbb{R}$-*Vect*. Denote by $\sigma_{\alpha,\mathbb{R}} : F_{\alpha,\mathbb{R}} \to L_\mathbb{R}$ the canonical maps into the inductive limit, and put $L_{\mathbb{Z}_\infty} := \operatorname{conv}\bigl(\bigcup_\alpha \sigma_{\alpha,\mathbb{R}}(F_{\alpha,\mathbb{Z}_\infty})\bigr)$. Clearly, $L_{\mathbb{Z}_\infty}$ is a symmetric and convex subset of $L_\mathbb{R}$. To check that it is absorbent observe that $L_\mathbb{R}$ is generated as an $\mathbb{R}$-vector space by the union of $\sigma_{\alpha,\mathbb{R}}(F_{\alpha,\mathbb{R}})$, and each $F_{\alpha,\mathbb{R}}$ is generated by $F_{\alpha,\mathbb{Z}_\infty}$. Hence the union of $\sigma_{\alpha,\mathbb{R}}(F_{\alpha,\mathbb{Z}_\infty})$ generates $L_\mathbb{R}$; *a fortiori* this is true for $L_{\mathbb{Z}_\infty}$.

In this way we get an object $L = (L_{\mathbb{Z}_\infty}, L_\mathbb{R})$ of $\mathbb{Z}_\infty$-*Fl.Mod*. Clearly, $\sigma_{\alpha,\mathbb{R}}(F_{\alpha,\mathbb{Z}_\infty}) \subset L_{\mathbb{Z}_\infty}$, so we get a compatible system of morphisms $\sigma_\alpha : F_\alpha \to L$. Let's check the universal property of this system. If $\tau_\alpha : F_\alpha \to M$ is another compatible system of morphisms, we get a unique $\mathbb{R}$-linear map $\varphi_\mathbb{R} : L_\mathbb{R} \to M_\mathbb{R}$, such that $\tau_{\alpha,\mathbb{R}} = \varphi_\mathbb{R} \circ \sigma_{\alpha,\mathbb{R}}$, since $L_\mathbb{R}$ is an inductive limit of $F_{\alpha,\mathbb{R}}$. Now $\tau_{\alpha,\mathbb{R}}(F_{\alpha,\mathbb{Z}_\infty})$ is contained in $M_{\mathbb{Z}_\infty}$, hence each $\sigma_{\alpha,\mathbb{R}}(F_{\alpha,\mathbb{Z}_\infty})$ is contained in the convex set $\varphi_\mathbb{R}^{-1}(M_{\mathbb{Z}_\infty})$, therefore, the convex hull $L_{\mathbb{Z}_\infty}$ of their union is contained in this convex set as well. This means that $\varphi_\mathbb{R}$ does indeed induce a morphism $\varphi : L \to M$ in $\mathbb{Z}_\infty$-*Fl.Mod*, such that $\tau_\alpha = \varphi \circ \sigma_\alpha$, and this φ is clearly unique, q.e.d.

2.9.1. Note that *filtered* inductive limits of $\mathbb{Z}_\infty$-*Fl.Mod* can be computed componentwise in the category of sets, i.e. in this case we have $L_{\mathbb{Z}_\infty} = \varinjlim F_{\alpha,\mathbb{Z}_\infty}$ and $L_\mathbb{R} = \varinjlim F_{\alpha,\mathbb{R}}$ in the category of sets. This can be either checked directly (define $L := (L_{\mathbb{Z}_\infty}, L_\mathbb{R})$ and check that it lies in $\mathbb{Z}_\infty$-*Fl.Mod*), or deduced from the description given in **2.9**.

2.9.2. (Infinite direct sums and norms.) In particular, arbitrary (not necessarily finite) direct sums (i.e. coproducts) $L = \bigoplus_\alpha F_\alpha$ exist in $\mathbb{Z}_\infty$-*Fl.Mod*. We see that $L_\mathbb{R} = \bigoplus_\alpha F_{\alpha,\mathbb{R}}$, and $L_{\mathbb{Z}_\infty} = \operatorname{conv}\bigl(\bigcup_\alpha F_{\alpha,\mathbb{Z}_\infty}\bigr)$, where the union and the convex hull are computed inside $L_\mathbb{R}$. This convex hull can be clearly described as the set of "octahedral" combinations $\sum_\alpha \lambda_\alpha x_\alpha$, where all $x_\alpha \in F_{\alpha,\mathbb{Z}_\infty}$, almost all $\lambda_\alpha \in \mathbb{R}$ are equal to zero and $\sum_\alpha |\lambda_\alpha| \leq 1$. (Actually, baricentric combinations would suffice here.) The corresponding norm $\|\cdot\|_L$ is the "L_1-norm" $\|\sum_\alpha x_\alpha\|_L = \sum_\alpha \|x_\alpha\|_{F_\alpha}$. One checks that if all F_α are closed (i.e. each $F_{\alpha,\mathbb{Z}_\infty}$ is closed in $F_{\alpha,\mathbb{R}}$ with respect to the algebraic topology) and do not contain any non-trivial real vector spaces (i.e. each $F_{\alpha,\mathbb{Z}_\infty}$ is given by a *norm* on $F_{\alpha,\mathbb{R}}$, and not just a seminorm), then the same is true for L, hence $L_{\mathbb{Z}_\infty} = \{x : \|x\|_L \leq 1\}$.

2.9.3. (Right exactness of $X \mapsto X_\mathbb{R}$.) Our construction of inductive limits shows that the functor $\rho^* : X \mapsto X_\mathbb{R}$ commutes with arbitrary inductive limits, and, in particular, it is right exact, hence exact (cf. **2.8.1**). This leads us to believe that it might have a right adjoint $\rho_* : \mathbb{R}$-*Vect* $\to \mathbb{Z}_\infty$-*Fl.Mod*. This adjoint indeed exists and can be defined by $\rho_*(V) := (V, V)$ for any $\mathbb{R}$-vector space V, i.e. any $\mathbb{R}$-vector space can be considered as a flat $\mathbb{Z}_\infty$-module. Note that ρ_* is fully faithful, so $\mathbb{R}$-*Vect* can be identified with a full subcategory of $\mathbb{Z}_\infty$-*Fl.Mod*. It is also interesting to remark that in the language of (semi)norms ρ^*V corresponds to V together with the seminorm which is identically zero; i.e. we equip V with its coarsest (non-separated) topology.

2.9.4. (Strict epimorphisms and quotients.) Another immediate consequence is this: *a morphism $f = (f_{\mathbb{Z}_\infty}, f_\mathbb{R}) : M \to P$ is a strict epimorphism iff both $f_{\mathbb{Z}_\infty}$ and $f_\mathbb{R}$ are subjective iff $f_\mathbb{R} : M_\mathbb{R} \to P_\mathbb{R}$ is surjective and $P_{\mathbb{Z}_\infty} = f_\mathbb{R}(M_{\mathbb{Z}_\infty})$.* By definition, f is a strict epimorphism iff it is a cokernel of its kernel pair $M \times_P M \rightrightarrows M$. The construction of inductive limits given above implies that if $f : M \to P$ is cokernel of a pair of morphisms, then $P_\mathbb{R}$ is the corresponding quotient in $\mathbb{R}$-*Vect*, and $P_{\mathbb{Z}_\infty} = \mathrm{conv}\bigl(f(M_{\mathbb{Z}_\infty})\bigr) = f(M_{\mathbb{Z}_\infty})$. Conversely, if this is the case, then clearly f is the cokernel of its kernel pair.

Thus we see that strict quotients P of a fixed object M are in one-to-one correspondence with quotients $\pi_\mathbb{R} : M_\mathbb{R} \to P_\mathbb{R}$ of the corresponding $\mathbb{R}$-vector space $M_\mathbb{R}$. For each of these strict quotients we have $P_{\mathbb{Z}_\infty} = \pi_\mathbb{R}(M_{\mathbb{Z}_\infty})$.

In the language of (semi)norms strict quotients correspond to quotient norms: $\|y\|_P = \inf_{x \in \pi_\mathbb{R}^{-1}(y)} \|x\|_M$.

2.9.5. (Strict monomorphisms and subobjects.) Dually, $f : N \to M$ is a strict monomorphism, if it is the kernel of some pair of morphisms, and then it is the kernel of its cokernel pair. The description of projective limits given in **2.8** shows immediately that *$f : N \to M$ is a strict monomorphism iff $f_\mathbb{R}$ is injective and $N_{\mathbb{Z}_\infty} = f_\mathbb{R}^{-1}(M_{\mathbb{Z}_\infty})$.* Hence strict subobjects of M are given by $N = (N_{\mathbb{Z}_\infty}, N_\mathbb{R})$, where $N_\mathbb{R} \subset M_\mathbb{R}$ is an $\mathbb{R}$-linear subspace, and $N_{\mathbb{Z}_\infty} = M_{\mathbb{Z}_\infty} \cap N_\mathbb{R}$. We see that strict subobjects of M are in one-to-one correspondence with $\mathbb{R}$-subspaces of $M_\mathbb{R}$. In the language of (semi)norms this means that we consider the restriction of $\|\cdot\|_M$ to $N_\mathbb{R} \subset M_\mathbb{R}$.

2.9.6. (Coimages.) We see that any morphism $f : M \to N$ decomposes uniquely into a strict epimorphism $\pi : M \to Q$ (namely, the cokernel of the kernel pair $M \times_N M \rightrightarrows M$ of f), followed by a monomorphism $i : Q \to N$. Uniqueness of such a decomposition is a general fact. To show existence one has to check that the morphism $i : Q \to N$ induced by f is a monomorphism. We know that injectivity of $i_\mathbb{R}$ suffices for this; now we apply exact functor ρ^* and see that $i_\mathbb{R} : \mathrm{Coim}\, f_\mathbb{R} \to N_\mathbb{R}$ is indeed a monomorphism in $\mathbb{R}$-*Vect*. We will say that Q is the *image* of M in N under f (even if it would be more accurate to say that it is the *coimage* of f).

2.9.7. (Inductive limits in $\mathbb{Z}_\infty$-*Mod* are different.) We are going to embed $\mathbb{Z}_\infty$-*Fl.Mod* into a larger category $\mathbb{Z}_\infty$-*Mod* later. This inclusion functor will have a left adjoint; hence it commutes with arbitrary projective limits, i.e. the projective limits computed in **2.8** will remain such when considered in $\mathbb{Z}_\infty$-*Mod*. This is in general *not* true for arbitrary inductive limits, even if we consider only finite inductive limits. For example, if we compute the cokernel of the inclusion and of the zero morphisms $\mathfrak{m}_\infty \rightrightarrows \mathbb{Z}_\infty$, we obtain 0 in $\mathbb{Z}_\infty$-*Fl.Mod*, but in $\mathbb{Z}_\infty$-*Mod* this cokernel "$\mathbb{Z}_\infty/\mathfrak{m}_\infty$" will be a non-trivial torsion $\mathbb{Z}_\infty$-module (cf. **2.14.13**).

However, *filtered* inductive limits will be the same in $\mathbb{Z}_\infty$-*Mod* and in $\mathbb{Z}_\infty$-*Fl.Mod*, because in both categories they can be described as in **2.9.1**.

2.9.8. (Zero object.) Our category $\mathbb{Z}_\infty$-*Fl.Mod* has a *zero object* $0 := (0, 0)$, i.e. an object that is both initial and final at the same time. Hence we have a pointed element in each set of morphisms $\mathrm{Hom}_{\mathbb{Z}_\infty}(M, N)$, namely, the *zero morphism* $0_{MN} : M \to 0 \to N$. Note that 0 lies actually in $\mathbb{Z}_\infty$-*Lat*, so the same is true in this smaller category.

2.10. (Multilinear algebra.) Bilinear and polylinear $\mathbb{Z}_\infty$-maps between objects of $\mathbb{Z}_\infty$-*Fl.Mod* are defined exactly in the same way as it has been done in **2.6.1** and **2.6.4**, once we replace "$\mathbb{Z}_\infty$-lattices" with "flat $\mathbb{Z}_\infty$-modules" in these definitions. Moreover, double and multiple tensor products can be still constructed in the same way as in **2.6.3**. For example, $M \otimes N = M \otimes_{\mathbb{Z}_\infty} N$ is defined to be $L = (L_{\mathbb{Z}_\infty}, L_\mathbb{R})$, where $L_\mathbb{R} = M_\mathbb{R} \otimes_\mathbb{R} N_\mathbb{R}$, and $L_{\mathbb{Z}_\infty}$ is the convex hull of the set of all products $\{x \otimes y : x \in M_{\mathbb{Z}_\infty}, y \in N_{\mathbb{Z}_\infty}\}$. Clearly, $L_{\mathbb{Z}_\infty}$ generates $L_\mathbb{R}$ as an $\mathbb{R}$-vector space, so this is indeed an object of $\mathbb{Z}_\infty$-*Fl.Mod*.

2.10.0. (Relation to Grothendieck's tensor product of seminorms.) Notice that the tensor product $L = M \otimes N$ just discussed (cf. also **2.6.3**) admits a direct description in terms of seminorms on $L_\mathbb{R} = M_\mathbb{R} \otimes N_\mathbb{R}$, $M_\mathbb{R}$ and $N_\mathbb{R}$. Namely, if $\|\cdot\|$, $\|\cdot\|_1$ and $\|\cdot\|_2$ denote the seminorms defined by L, M and N, respectively, then

$$\|z\| = \inf_{z=\sum_{i=1}^n x_i \otimes y_i} \sum_{i=1}^n \|x_i\|_1 \cdot \|y_i\|_2, \quad \text{for any } z \in L_\mathbb{R}. \tag{2.10.0.1}$$

This is nothing else than the projective tensor product of seminorms defined by Grothendieck in [Gr0]. However, this doesn't describe our tensor product completely since $L_{\mathbb{Z}_\infty}$ isn't completely determined by $\|\cdot\|$.

2.10.1. (ACU $\otimes$-structure on $\mathbb{Z}_\infty$-*Fl.Mod*.) In this way we get an ACU $\otimes$-structure on $\mathbb{Z}_\infty$-*Fl.Mod*, compatible with one introduced on $\mathbb{Z}_\infty$-*Lat* before. In particular, $\mathbb{Z}_\infty$ is the unit object with respect to this tensor product, so we still have a "global sections functor" $\Gamma = \mathrm{Hom}_{\mathbb{Z}_\infty}(\mathbb{Z}_\infty, -) : \mathbb{Z}_\infty\text{-}\mathit{Fl.Mod} \to$ *Sets*, that maps a flat $\mathbb{Z}_\infty$-module $M = (M_{\mathbb{Z}_\infty}, M_\mathbb{R})$ into its "underlying set" $M_{\mathbb{Z}_\infty} \cong \mathrm{Hom}_{\mathbb{Z}_\infty}(\mathbb{Z}_\infty, M)$. This terminology is again motivated by the p-adic case, where $\mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p, -)$ maps a flat $\mathbb{Z}_p$-module M into its underlying set.

2.10.2. *Inner Homs* $\mathbf{Hom}_{\mathbb{Z}_\infty}(N, P)$ *exist in* $\mathbb{Z}_\infty$-*Fl.Mod*. We construct such an inner Hom $H = \mathbf{Hom}(N, P)$ as follows. Put $H' := \mathrm{Hom}_\mathbb{R}(N_\mathbb{R}, P_\mathbb{R})$, then

$H_{\mathbb{Z}_\infty} := \{f \in H' : f(N_{\mathbb{Z}_\infty}) \subset P_{\mathbb{Z}_\infty}\}$, $H_{\mathbb{R}} := \mathbb{R} \cdot H_{\mathbb{Z}_\infty}$ and $H := (H_{\mathbb{Z}_\infty}, H_{\mathbb{R}})$. Clearly, $H_{\mathbb{Z}_\infty}$ is a symmetric convex subset of H', hence it is a symmetric convex body in the real vector subspace $H_{\mathbb{R}} \subset H'$, so H is indeed an object of $\mathbb{Z}_\infty$-*Fl.Mod*. Required universal property $\mathrm{Hom}(M, H) \cong \mathrm{Hom}(M \otimes N, P) \cong \mathrm{Bilin}(M, N; P)$ can be deduced now from the similar property of H' in $\mathbb{R}$-*Vect* in the same manner as it has been done for projective limits in **2.8**.

2.10.3. Existence of inner Homs implies that *tensor products in $\mathbb{Z}_\infty$-Fl.Mod commute with arbitrary inductive limits in each variable.* In particular, $(M' \oplus M'') \otimes N \cong (M' \otimes N) \oplus (M'' \otimes N)$.

2.10.4. Note that $\Gamma(\mathbf{Hom}_{\mathbb{Z}_\infty}(N, P)) \cong \mathrm{Hom}_{\mathbb{Z}_\infty}(N, P)$, as one expects for general reasons. Another interesting observation is this: For any $M = (M_{\mathbb{Z}_\infty}, M_{\mathbb{R}})$ we have $M \otimes_{\mathbb{Z}_\infty} \mathbb{R} = (M_{\mathbb{R}}, M_{\mathbb{R}}) = M_{\mathbb{R}}$, if we identify $\mathbb{R}$-*Vect* with a full subcategory of $\mathbb{Z}_\infty$-*Fl.Mod* as in **2.9.3**. So we can write $M \otimes_{\mathbb{Z}_\infty} \mathbb{R}$ or $M_{(\mathbb{R})}$ instead of $\rho^* M$ or $M_{\mathbb{R}}$. This leads us to believe that $\rho_* : \mathbb{R}\text{-}\mathit{Vect} \to \mathbb{Z}_\infty\text{-}\mathit{Fl.Mod}$ must have a right adjoint $\rho^! : \mathbb{Z}_\infty\text{-}\mathit{Fl.Mod} \to \mathbb{R}\text{-}\mathit{Vect}$, given by $\rho^!(M) := \mathbf{Hom}_{\mathbb{Z}_\infty}(\mathbb{R}, M)$, and this is indeed the case. This $\mathbb{R}$-vector space $\rho^! M$ can be described more explicitly as the largest $\mathbb{R}$-vector subspace of $M_{\mathbb{R}}$ contained in $M_{\mathbb{Z}_\infty}$.

2.10.5. (Duals.) Of course, we define the *dual* M^* of some flat $\mathbb{Z}_\infty$-module M by $M^* := \mathbf{Hom}_{\mathbb{Z}_\infty}(M, \mathbb{Z}_\infty)$. We have a lot of canonical maps like $M^* \otimes N \to \mathbf{Hom}(M, N)$ and $M \to M^{**}$, coming from the general theory of $\otimes$-categories. One can check that M is *reflexive*, i.e. $M \to M^{**}$ is an isomorphism iff $M_{\mathbb{R}}$ is a Banach space with respect to $\|\cdot\|_M$, and $M_{\mathbb{Z}_\infty}$ is closed, i.e. $M_{\mathbb{Z}_\infty} = \{x : \|x\|_M \leq 1\}$. For any M its double dual M^{**} is something like the *completion* of M: $(M^{**})_{\mathbb{R}}$ is the completion of $M_{\mathbb{R}}$ with respect to $\|\cdot\|_M$, and $(M^{**})_{\mathbb{Z}_\infty}$ is the closure of the image of $M_{\mathbb{Z}_\infty}$ in this completed space.

One should be careful with these definitions. For example, $\mathbb{Z}_\infty^{**} = \mathbb{Z}_\infty$, but $\mathbb{R}^{**} = 0$.

2.10.6. (Hilbert spaces.) We have seen that Banach spaces have an interior characterization in $\mathbb{Z}_\infty$-*Fl.Mod*. Hilbert spaces have such a characterization as well: they correspond to flat $\mathbb{Z}_\infty$-modules H equipped by a *perfect* symmetric pairing $\varphi : H \times H \to \mathbb{Z}_\infty$, i.e. φ is a symmetric $\mathbb{Z}_\infty$-bilinear form, such that the induced map $\tilde{\varphi} : H \to H^*$ is an isomorphism. Actually, to obtain Hilbert spaces we must also require φ to be positive definite, otherwise $-\varphi$ would also do.

2.10.7. (Free $\mathbb{Z}_\infty$-modules.) For any flat $\mathbb{Z}_\infty$-module M and any set S we can consider the *product* M^S and the *direct sum* $M^{(S)}$ of the constant family with value M and index set S. If S is a standard finite set $\mathbf{n} = \{1, 2, \ldots, n\}$, $n \geq 0$, we write $M^{(n)}$ and M^n instead of $M^{(\mathbf{n})}$ and $M^{\mathbf{n}}$. Note that we have a canonical *monomorphism* $M^{(S)} \to M^S$, which usually is *not* an isomorphism, even for a finite S. By definition, we have

$$\mathrm{Hom}_{\mathbb{Z}_\infty}(N, M^S) \cong \mathrm{Hom}_{\mathbb{Z}_\infty}(N, M)^S \cong \mathrm{Hom}_{Sets}\big(S, \mathrm{Hom}_{\mathbb{Z}_\infty}(N, M)\big) \tag{2.10.7.1}$$

$$\mathrm{Hom}_{\mathbb{Z}_\infty}(M^{(S)}, N) \cong \mathrm{Hom}_{\mathbb{Z}_\infty}(M, N)^S \cong \mathrm{Hom}_{Sets}\big(S, \mathrm{Hom}_{\mathbb{Z}_\infty}(M, N)\big) \tag{2.10.7.2}$$

From these formulas we deduce a lot of other canonical isomorphisms, for example, $P \otimes_{\mathbb{Z}_\infty} M^{(S)} \cong (P \otimes_{\mathbb{Z}_\infty} M)^{(S)}$, $\mathbf{Hom}_{\mathbb{Z}_\infty}(M^{(S)}, N) \cong \mathbf{Hom}_{\mathbb{Z}_\infty}(M, N)^S$, $(M^{(S)})^* \cong (M^*)^S$ and so on. We can specialize these constructions to the case $M = \mathbb{Z}_\infty$. We obtain $M^{(S)} \cong \mathbb{Z}_\infty^{(S)} \otimes M$, $\mathbb{Z}_\infty^{(S)} \otimes \mathbb{Z}_\infty^{(T)} \cong \mathbb{Z}_\infty^{(S\times T)}$, $M^S \cong \mathbf{Hom}(\mathbb{Z}_\infty^{(S)}, M)$, $(\mathbb{Z}_\infty^{(S)})^* \cong \mathbb{Z}_\infty^S$, and in particular

$$\mathrm{Hom}_{\mathbb{Z}_\infty}(\mathbb{Z}_\infty^{(S)}, M) \cong \Gamma(M)^S \cong \mathrm{Hom}_{Sets}(S, \Gamma(M)) \tag{2.10.7.3}$$

This last equality means that the "forgetful functor" $\Gamma : \mathbb{Z}_\infty$-*Fl.Mod* $\to$ *Sets* has a left adjoint $L_{\mathbb{Z}_\infty} :$ *Sets* $\to \mathbb{Z}_\infty$-*Fl.Mod*, given by $S \mapsto \mathbb{Z}_\infty^{(S)}$. That's the reason why we say that any (flat) $\mathbb{Z}_\infty$-module M, isomorphic to some $\mathbb{Z}_\infty^{(S)}$, is *a free $\mathbb{Z}_\infty$-module (of rank* $\mathrm{card}\, S$*)*. The existence of this left adjoint to Γ will be crucial for our definition of $\mathbb{Z}_\infty$-*Mod*.

2.10.8. Since free $\mathbb{Z}_\infty$-modules $\mathbb{Z}_\infty^{(S)}$ will be very important to us later, let's describe them more explicitly. According to **2.9.2**, $\mathbb{Z}_\infty^{(S)} = (\Sigma_\infty(S), \mathbb{R}^{(S)})$, where $\Sigma_\infty(S)$ is the convex hull of all $\pm\{s\}$, $s \in S$. Here we denote by $\{s\}$ the basis element of $\mathbb{R}^{(S)}$ corresponding to some $s \in S$. Hence $\mathbb{R}^{(S)}$ is the set of formal linear combinations $\sum_s \lambda_s\{s\}$ of elements of S (i.e. all but finitely many $\lambda_s \in \mathbb{R}$ are equal to zero), and $\Sigma_\infty(S)$ can be described as the set of all formal *octahedral combinations* $\sum_s \lambda_s\{s\}$ of elements of S, i.e. we require in addition that $\sum_s |\lambda_s| \leq 1$. This means that $\Sigma_\infty(S)$ is closed in $\mathbb{R}^{(S)}$, and the corresponding norm is the L_1-norm: $\big\|\sum_s \lambda_s\{s\}\big\| = \sum_s |\lambda_s|$.

If S is infinite, $\mathbb{R}^{(S)}$ is not complete with respect to this norm, hence $\mathbb{Z}_\infty^{(S)}$ is not reflexive. On the other hand, if S is finite, e.g. $S = \mathbf{n} = \{1, 2, \ldots, n\}$, then $\mathbb{Z}_\infty^{(n)} = (\Sigma_\infty(n), \mathbb{R}^n)$ lies in $\mathbb{Z}_\infty$-*Lat*, hence is reflexive. In this case $\Sigma_\infty(n)$ is the convex hull of $\pm e_i$, where $\{e_i\}_{i=1}^n$ is the standard base of $\mathbb{R}^n$. In other words, $\Sigma_\infty(n)$ is the *standard n-dimensional octahedron* with vertices $\pm e_i$.

2.10.9. (Finitely generated modules.) We say that a flat $\mathbb{Z}_\infty$-module M is *finitely generated*, or is *of finite type*, if it is a strict quotient of some free $\mathbb{Z}_\infty$-module $\mathbb{Z}_\infty^{(n)}$ of finite rank. The description of strict quotients given in **2.9.4** combined with the description of $\mathbb{Z}_\infty^{(n)}$ given in **2.10.8**, yields the following characterization of finitely generated flat $\mathbb{Z}_\infty$-modules: M is finitely generated iff $M_\mathbb{R}$ is finite-dimensional, and $M_{\mathbb{Z}_\infty}$ is the convex hull of a finite number of vectors $\pm x_i$ in $M_\mathbb{R}$. In other words, $M_{\mathbb{Z}_\infty}$ has to be a symmetric convex polyhedron in a finite-dimensional real vector space $M_\mathbb{R}$. This implies that all flat $\mathbb{Z}_\infty$-modules of finite type are $\mathbb{Z}_\infty$-lattices, but not conversely. In particular, $\bar{\mathbb{Z}}_\infty$ is a $\mathbb{Z}_\infty$-lattice, but it is not finitely generated. However, any $\mathbb{Z}_\infty$-lattice, as well as any flat $\mathbb{Z}_\infty$-module, is a filtered inductive limit of its finitely generated submodules.

2.11. (Category of torsion-free algebras and modules.) Our ACU $\otimes$-structure on category $\mathbb{Z}_\infty$-*Fl.Mod* allows us to define flat $\mathbb{Z}_\infty$-algebras and ($\mathbb{Z}_\infty$-flat) modules over these algebras as algebras and modules for this $\otimes$-structure.

2.11.1. For example, an (associative) algebra A in $\mathbb{Z}_\infty$-*Fl.Mod* is a flat $\mathbb{Z}_\infty$-module A, together with multiplication $\mu = \mu_A : A \otimes_{\mathbb{Z}_\infty} A \to A$ and unit $\eta = \eta_A : \mathbb{Z}_\infty \to A$, subject to associativity and unit relations: $\mu \circ (\mu \otimes 1_A) = \mu \circ (1_A \otimes \mu)$, $\mu \circ (\eta \otimes 1_A) = 1_A = \mu \circ (1_A \otimes \eta)$.

We can give an alternative description of this structure. Observe for this that μ_A corresponds to a $\mathbb{Z}_\infty$-bilinear map $\mu' = \mu'_A : A \times A \to A$, and $\eta_A \in \mathrm{Hom}_{\mathbb{Z}_\infty}(\mathbb{Z}_\infty, A) = \Gamma(A) \cong A_{\mathbb{Z}_\infty}$ corresponds to some element $\eta' = \eta'_A \in A_{\mathbb{Z}_\infty} \subset A_\mathbb{R}$. Then $\mu'_\mathbb{R}$ defines on $A_\mathbb{R}$ the structure of an $\mathbb{R}$-algebra with unity η', and $A_{\mathbb{Z}_\infty} \subset A_\mathbb{R}$ is a symmetric convex absorbent multiplicative *submonoid* of $A_\mathbb{R}$, containing the unity η'. Clearly, this is an equivalent description of flat $\mathbb{Z}_\infty$-algebras; for $\mathbb{Z}_\infty$-lattices it coincides with one we had in **2.5**.

2.11.2. For example, $\mathbb{Z}_\infty$, $\bar{\mathbb{Z}}_\infty$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Z}_\infty^n$ are $\mathbb{Z}_\infty$-algebras; this is not the case for $\mathbb{Z}_\infty^{(n)} \subset \mathbb{Z}_\infty^n$, since it does not contain the unity. Note that the tensor product of two flat $\mathbb{Z}_\infty$-algebras is an $\mathbb{Z}_\infty$-algebra again. In particular, for any flat $\mathbb{Z}_\infty$-algebra A we obtain that $A \otimes_{\mathbb{Z}_\infty} \mathbb{R} = A_{(\mathbb{R})} \cong A_\mathbb{R}$ is a $\mathbb{Z}_\infty$-algebra, and even an $\mathbb{R}$-algebra; we have already seen this before.

2.11.3. Another example of a (non-commutative) flat $\mathbb{Z}_\infty$-algebra is given by the "inner Ends" $A := \mathbf{End}_{\mathbb{Z}_\infty}(M) := \mathbf{Hom}_{\mathbb{Z}_\infty}(M, M)$, with the multiplication given by composition of endomorphisms; more formally, we consider canonical morphism $\mathbf{Hom}(M', M'') \otimes \mathbf{Hom}(M, M') \to \mathbf{Hom}(M, M'')$, existing in any ACU $\otimes$-category with inner Homs, for $M'' = M' = M$.

2.11.4. A *(left) A-module structure* on a flat $\mathbb{Z}_\infty$-module M is by definition a morphism $\alpha = \alpha_M : A \otimes M \to M$, such that $\alpha \circ (1_A \otimes \alpha) = \alpha \circ (\mu \otimes 1_M)$ and $\alpha \circ (\eta \otimes 1_M) = 1_M$. Of course, α corresponds to a $\mathbb{Z}_\infty$-bilinear map $\alpha' : A \times M \to M$. It induces an $A_\mathbb{R}$-module structure $\alpha'_\mathbb{R}$ on the real vector space $M_\mathbb{R}$, such that $A_{\mathbb{Z}_\infty} \cdot M_{\mathbb{Z}_\infty} = M_{\mathbb{Z}_\infty}$. In other words, $M_{\mathbb{Z}_\infty}$ is a symmetric convex body in the $A_\mathbb{R}$-module $M_\mathbb{R}$, stable under action of $A_{\mathbb{Z}_\infty} \subset A_\mathbb{R}$. Note that in this way we obtain an action of monoid $A_{\mathbb{Z}_\infty}$ on the set $M_{\mathbb{Z}_\infty}$.

2.11.5. Both these descriptions of A-modules are useful. For example, the first of them, combined with **2.10.3**, shows us that the direct sum of two A-modules is again an A-module, while the second one shows that the product of two A-modules is an A-module.

2.11.6. Another possible description of an A-module is this. Note that $\alpha_M : A \otimes M \to M$ corresponds by adjointness to some $\gamma : A \to \mathbf{End}(M)$, and α_M is an A-module structure on M iff γ is a $\mathbb{Z}_\infty$-algebra homomorphism with respect to the $\mathbb{Z}_\infty$-structure on $\mathbf{End}(M)$ considered in **2.11.3**.

2.11.7. Of course, (left) A-module homomorphisms, or A-linear maps $f : M = (M, \alpha_M) \to N = (N, \alpha_N)$ are defined to be $\mathbb{Z}_\infty$-linear maps $f : M \to N$, such that $\alpha_N \circ (1_A \otimes f) = f \circ \alpha_M$. In our situation this means that $f_\mathbb{R} : M_\mathbb{R} \to N_\mathbb{R}$ is an $A_\mathbb{R}$-linear map, and $f_\mathbb{R}(M_{\mathbb{Z}_\infty}) \subset N_{\mathbb{Z}_\infty}$.

Now we can define the category *A-Fl.Mod* of *(left) A-modules which are flat over* $\mathbb{Z}_\infty$. Any $\mathbb{Z}_\infty$-algebra homomorphism $f : A \to B$ gives us a "scalar restriction" functor $f_* :$ *B-Fl.Mod* $\to$ *A-Fl.Mod*, which turns out to have both left and right adjoints $f^*, f^! :$ *A-Fl.Mod* $\to$ *B-Fl.Mod*. If A is commutative, we obtain a natural notion of A-bilinear maps, hence of tensor products $\otimes_A$ and inner Homs $\mathbf{Hom}_A$. We do not develop these notions here, because we would like to remove the $\mathbb{Z}_\infty$-flatness restriction first. We will return to these questions later, in the context of generalized rings.

2.11.8. Given any flat $\mathbb{Z}_\infty$-module M, we can construct its *tensor algebra* $T(M)$ in the usual way. Namely, we consider the tensor powers $T^n(M) = T^n_{\mathbb{Z}_\infty}(M) := M \otimes M \otimes \cdots \otimes M$ (tensor product of n copies of M; $T^1(M) = M$ and $T^0(M) = \mathbb{Z}_\infty$), and put $T(M) := \bigoplus_{n\geq 0} T^n(M)$. Canonical maps $T^n(M) \otimes T^m(M) \to T^{n+m}(M)$ induce the multiplication on $T(M)$; thus $T(M)$ becomes a graded associative flat $\mathbb{Z}_\infty$-algebra. We have a canonical embedding $M \to T(M)$ that maps M into $T^1(M) \cong M$. The tensor algebra together with this embedding have the usual universal property with respect to $\mathbb{Z}_\infty$-linear maps $\varphi : M \to A$ from M into associative flat $\mathbb{Z}_\infty$-algebras A: any such map lifts to a unique homomorphism of $\mathbb{Z}_\infty$-algebras $\varphi^\sharp : T(M) \to A$.

Note that tensor products of free modules are free, so, if M is a free $\mathbb{Z}_\infty$-module, then all $T^n(M)$ and $T(M)$ are also free.

2.11.9. The *symmetric algebra* $S(M)$ can be defined by its universal property with respect to $\mathbb{Z}_\infty$-linear maps of M into *commutative* algebras: any such map $\varphi : M \to A$ should lift to a unique $\mathbb{Z}_\infty$-algebra homomorphism $\varphi^\sharp : S(M) \to A$. This symmetric algebra does exist, and it is a commutative graded $\mathbb{Z}_\infty$-algebra: $S(M) = \bigoplus_{n\geq 0} S^n(M)$. We have a canonical $\mathbb{Z}_\infty$-algebra map $\pi : T(M) \to S(M)$. It is a strict epimorphism in

$\mathbb{Z}_\infty$-*Fl.Mod*, and it respects the grading; the induced maps $\pi_n : T^n(M) \to S^n(M)$ are also strict epimorphisms. The *symmetric powers* $S^n(M)$ have the usual universal property with respect to symmetric polylinear maps from M^n into a variable $\mathbb{Z}_\infty$-module N. They can be constructed as the image of $T^n_{\mathbb{Z}_\infty}(M) \subset T^n_{\mathbb{R}}(M_{\mathbb{R}})$ under the canonical surjection $\pi_{n,\mathbb{R}} : T^n_{\mathbb{R}}(M_{\mathbb{R}}) \to S^n_{\mathbb{R}}(M_{\mathbb{R}})$, i.e. $S^n(M) = (S^n(M)_{\mathbb{Z}_\infty}, S^n(M)_{\mathbb{R}})$, where $S^n(M)_{\mathbb{R}} = S^n_{\mathbb{R}}(M_{\mathbb{R}})$, and $S^n(M)_{\mathbb{Z}_\infty} = \pi_{n,\mathbb{R}}(T^n(M)_{\mathbb{Z}_\infty})$ (recall the description of strict quotients given in **2.9.4**).

2.11.10. The *exterior algebra* $\Lambda(M)$ is defined similarly, but we consider $\mathbb{Z}_\infty$-linear maps $\varphi : M \to A$ into associative algebras A, subject to additional condition $\varphi(x)^2 = 0$ for any x in $M_{\mathbb{Z}_\infty}$ (or in $M_{\mathbb{R}}$). Again, $\Lambda(M) = \bigoplus_{n\geq 0} \Lambda^n(M)$ is graded supercommutative, and it is a strict quotient of $T(M)$, and any individual graded component $\Lambda^n(M)$ is strict quotient of $T^n(M)$; together with property $(\Lambda^n(M))_{\mathbb{R}} = \Lambda^n_{\mathbb{R}}(M_{\mathbb{R}})$ these conditions determine $\Lambda^n(M)$ and $\Lambda(M)$ uniquely. Finally, all the $\Lambda^n(M)$ have the usual universal property with respect to alternating polylinear maps from M^n to a variable $\mathbb{Z}_\infty$-module N.

2.11.11. *Monoid algebras* give us another example of $\mathbb{Z}_\infty$-algebras. Given any monoid M with unity e, we define the *monoid algebra* $\mathbb{Z}_\infty[M]$ to be the free $\mathbb{Z}_\infty$-module $\mathbb{Z}_\infty^{(M)}$, with unity $\{e\}$ and multiplication induced by that of M by means of the canonical isomorphism $\mathbb{Z}_\infty^{(M)} \otimes \mathbb{Z}_\infty^{(M)} \cong \mathbb{Z}_\infty^{(M\times M)}$. In other words, $\mathbb{Z}_\infty[M]$ consists of formal octahedral combinations of its basis elements $\{m\}$, and the multiplication is defined by the usual requirement $\{m\} \cdot \{m'\} = \{mm'\}$. According to **2.10.8**, we have $\mathbb{Z}_\infty[M]_{\mathbb{R}} = \mathbb{R}[M]$, and the corresponding symmetric convex body in $\mathbb{R}[M]$ is given by the L_1-norm $\left\|\sum_m \lambda_m\{m\}\right\| = \sum_m |\lambda_m|$.

Note that $\mathbb{Z}_\infty[M]$ has the usual universal property of monoid algebras, namely, $\mathbb{Z}_\infty$-algebra homomorphisms $f : \mathbb{Z}_\infty[M] \to A$ are in one-to-one correspondence with monoid homomorphisms $f^\flat : M \to A^\times$, where $A^\times$ denotes $A_{\mathbb{Z}_\infty}$, considered as a monoid under multiplication.

2.11.12. Now we can combine together several of the above constructions and construct the *polynomial algebras*. Given any set X, we define the polynomial algebra $\mathbb{Z}_\infty[X]$ in variables from X to be a commutative algebra, such that $\mathrm{Hom}_{\mathit{Sets}}(X, \Gamma(A)) \cong \mathrm{Hom}_{\mathbb{Z}_\infty-alg}(\mathbb{Z}_\infty[X], A)$ for any *commutative* (flat) $\mathbb{Z}_\infty$-algebra A. If X is finite, say $X = \mathbf{n} = \{1, 2, \dots, n\}$, we denote the corresponding elements of $\mathbb{Z}_\infty[X]$ by T_i, and write $\mathbb{Z}_\infty[T_1, T_2, \dots, T_n]$ instead of $\mathbb{Z}_\infty[\mathbf{n}]$.

We have two ways of constructing polynomial algebras:
a) We can construct $\mathbb{Z}_\infty[X]$ as the symmetric algebra of the free module $\mathbb{Z}_\infty^{(X)}$. Indeed, for any commutative $\mathbb{Z}_\infty$-algebra we have $\mathrm{Hom}_{\mathbb{Z}_\infty-alg}(S(\mathbb{Z}_\infty^{(X)}), A) \cong$

$\mathrm{Hom}_{\mathbb{Z}_\infty}(\mathbb{Z}_\infty^{(X)}, A) \cong \mathrm{Hom}_{Sets}(X, \Gamma(A)) = \Gamma(A)^X$. In particular, we obtain a grading on $\mathbb{Z}_\infty[X]$.

b) We can construct first the free commutative monoid $N(X)$ generated by X, i.e. the set of monomials in X, and then define $\mathbb{Z}_\infty[X]$ to be the monoid algebra $\mathbb{Z}_\infty[N(X)]$. Indeed, for any commutative algebra A we have $\mathrm{Hom}_{\mathbb{Z}_\infty-alg}(\mathbb{Z}_\infty[N(X)], A) \cong \mathrm{Hom}_{Mon}(N(X), A^\times) \cong \mathrm{Hom}_{Sets}(X, \Gamma(A)) = \Gamma(A)^X$. In this way we see that $\mathbb{Z}_\infty[X]$ is free as a $\mathbb{Z}_\infty$-module, i.e. the symmetric algebra of a free module is free. This is actually true for any individual symmetric power $S^n(\mathbb{Z}_\infty^{(X)})$, since it admits the set $N_n(X) \subset N(X)$ of all monomials of degree n as a basis.

The corresponding norm on $\mathbb{Z}_\infty[X]_{\mathbb{R}} = \mathbb{R}[X]$ is again the L_1-norm with respect to the basis given by the monomials $N(X)$.

2.12. (Arakelov affine line.) We would like to consider an interesting example – namely, the *Arakelov affine line* $\mathbb{A}^1_{\mathbb{Z}_\infty}$. Of course, $\mathbb{A}^1_{\mathbb{Z}_\infty} = \mathrm{Spec}\,\mathbb{Z}_\infty[T]$. However, we haven't yet developed the theory of "generalized rings" like $\mathbb{Z}_\infty$ and $\mathbb{Z}_\infty[T]$ and their spectra, so we will try to study this affine line in terms of $\mathbb{Z}_\infty$-sections and norms, as described in **1.6**.

2.12.1. First of all, let's take some $\mathbb{R}$-rational point P_λ in the generic fiber $\mathbb{A}^1_{\mathbb{R}} = \mathrm{Spec}\,\mathbb{R}[T] = \mathrm{Spec}\,\mathbb{Z}_\infty[T]_{(\mathbb{R})}$. It is given by some real number λ, and the corresponding section $\sigma'_\lambda : \mathrm{Spec}\,\mathbb{R} \to \mathrm{Spec}\,\mathbb{R}[T]$ is given by the evaluation at λ map $\pi_\lambda : \mathbb{R}[T] \to \mathbb{R}$, $F(T) \mapsto F(\lambda)$. Now we want to compute the scheme-theoretic closure of $\{P_\lambda\}$ in $\mathbb{A}^1_{\mathbb{Z}_\infty}$. The p-adic case tells us that we have to consider $\mathrm{Spec}\,C_\lambda$ for this, where C_λ is the image of $\mathbb{Z}_\infty[T]$ in $\mathbb{R}$ under π_λ. Since $\mathbb{Z}_\infty[T]$ is the convex hull of $\pm T^k$ in $\mathbb{R}[T]$, we see that $C_\lambda = (C_\lambda, \mathbb{R})$ is given by $C_\lambda = \mathrm{conv}\{\pm\lambda^k : k \geq 0\}$. If $|\lambda| > 1$, we get $C_\lambda = \mathbb{R}$, i.e. P_λ is closed in $\mathbb{A}^1_{\mathbb{Z}_\infty}$. If $|\lambda| \leq 1$, then $C_\lambda = \mathbb{Z}_\infty$, so P_λ lifts to a $\mathbb{Z}_\infty$-section σ_λ of $\mathbb{A}^1_{\mathbb{Z}_\infty}$, in perfect accordance with the p-adic case.

2.12.2. Now we would like to extract some infinitesimal data at P_λ when P_λ lifts to a $\mathbb{Z}_\infty$-section, i.e. when $|\lambda| \leq 1$. More precisely, we want to obtain some cometric at these points.

In the usual situation we would have $\sigma^*_\lambda(\Omega^1) \cong \mathfrak{p}_\lambda/\mathfrak{p}^2_\lambda$, where $\mathfrak{p}_\lambda$ is the ideal defining the image of this section, i.e. the kernel of $\mathbb{Z}_\infty[T] \to C_\lambda = \mathbb{Z}_\infty$. Clearly, we must have $(\mathfrak{p}_\lambda)_{(\mathbb{R})} = \mathfrak{p}'_\lambda = \mathrm{Ker}\,\pi_\lambda = (T - \lambda) \cdot \mathbb{R}[T]$, so let's take $\mathfrak{p}_\lambda := \mathbb{Z}_\infty[T] \cap \mathfrak{p}'_\lambda$, and let's assume that $\mathfrak{p}_\lambda \to E = \mathfrak{p}_\lambda/\mathfrak{p}^2_\lambda$ is a strict epimorphism, and that $E_{\mathbb{R}} = \mathfrak{p}'_\lambda/{\mathfrak{p}'_\lambda}^2 \cong \mathbb{R}$, where the isomorphism between $E_{\mathbb{R}}$ and $\mathbb{R}$ maps the class of $T-\lambda$ into 1. Note that $T-\lambda$ usually corresponds to dT, evaluated at P_λ, so we have chosen the value of dT at P_λ as our basis for $E_{\mathbb{R}} = \sigma^*_\lambda\Omega^1_{\mathbb{A}^1/\mathbb{R}}$.

2.12.3. Let's compute the induced $\mathbb{Z}_\infty$-structure. Roughly speaking, we have to compute the quotient norm $\|\cdot\|_\lambda$ of the restriction of the L_1-norm of $\mathbb{R}[T]$ onto hyperplane $\mathfrak{p}'_\lambda = \{F : F(\lambda) = 0\}$, and the quotient is computed with respect to the map $\mathfrak{p}'_\lambda \to E_\mathbb{R} = \mathbb{R}$ given by $F \mapsto F'(\lambda)$. We are actually interested in the value of our cometric on $(dT)_\lambda \in E_\mathbb{R}$, which has been identified with $1 \in \mathbb{R}$.

2.12.4. The result of this computation is the following. For each $n \geq 1$ denote by α_n the only positive root of polynomial $\lambda^{n+1} + (n+1)\lambda - n$, i.e. the only positive λ for which $\lambda^n = n\lambda^{-1} - (n+1)$, and put $\alpha_0 := 0$. Then we have $0 = \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < \cdots < 1$. The claim is that for $\alpha_{n-1} \leq \lambda \leq \alpha_n$ we have $\|dT\|_\lambda = (1+\lambda^n)/(n\lambda^{n-1}) = \inf_{k\geq 1}(1+\lambda^k)/(k\lambda^{k-1})$ and $E = (n\lambda^{n-1})/(1+\lambda^n)\cdot\mathbb{Z}_\infty$. For $-1 < \lambda \leq 0$ we have $\|\cdot\|_{-\lambda} = \|\cdot\|_\lambda$, and for $|\lambda| \geq 1$ we have $\|\cdot\|_\lambda = 0$.

2.12.5. Let's check first the properties of α_n that have been implicitly stated above. For any integer $n \geq 1$ put $G_n(\lambda) := \lambda^{n+1} + (n+1)\lambda - n$. Since $G_n(0) = -n < 0$ and $G_n(1) = 2 > 0$, this polynomial has a root α_n between 0 and 1. On the other hand, $G'_n(\lambda) > 0$ for $\lambda > 0$, so α_n is the only positive root of G_n. We have just seen that $0 < \alpha_n < 1$; let's check $\alpha_{n-1} < \alpha_n$ for $n \geq 2$. For this note that $\lambda G_{n-1}(\lambda) - G_n(\lambda) = n\lambda^2 - (n-1)\lambda - (n+1)\lambda + n = n(\lambda-1)^2 > 0$ for $\lambda < 1$. Apply this for $\lambda = \alpha_{n-1} < 1$: then $G_{n-1}(\lambda) = 0$, so we obtain $G_n(\alpha_{n-1}) < 0$. Since G_n is strictly increasing, we get $\alpha_{n-1} < \alpha_n$.

We have shown that $0 = \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < \cdots < 1$. Put $\varphi_k(\lambda) := (1+\lambda^k)/(k\lambda^{k-1})$ for any $\lambda > 0$ and any integer $k \geq 1$, and put $\varphi(\lambda) := \inf_{k\geq 1}\varphi_k(\lambda)$. It has been implicitly stated that for any $0 < \lambda < 1$ we can find an integer $n \geq 1$, for which $\alpha_{n-1} \leq \lambda \leq \alpha_n$, and then $\varphi(\lambda) = \varphi_n(\lambda)$, i.e. $\varphi_k(\lambda) \geq \varphi_n(\lambda)$ for all $k \geq 1$.

To show existence of n we have to check that $\sup_n \alpha_n = 1$. Let's prove that $n/(n+2) < \alpha_n < 1$. We know that $G_n(1) = 2 > 0$, and for $\lambda = n/(n+2)$ we have $G_n(\lambda) = (n+1+\lambda^n)\lambda - n < (n+2)\lambda - n = 0$, hence α_n is indeed between $n/(n+2)$ and 1.

Finally, let's show that for $0 < \lambda \leq \alpha_n$ we have $\varphi_n(\lambda) \leq \varphi_{n+1}(\lambda)$, and for $\lambda \geq \alpha_n$ we have $\varphi_n(\lambda) \geq \varphi_{n+1}(\lambda)$. Then for $\alpha_{n-1} \leq \lambda \leq \alpha_n$ we deduce by induction $\varphi_n(\lambda) \leq \varphi_{n+1}(\lambda) \leq \varphi_{n+2}(\lambda) \leq \cdots$ and $\varphi_n(\lambda) \leq \varphi_{n-1}(\lambda) \leq \cdots \leq \varphi_1(\lambda)$, hence $\varphi_k(\lambda) \geq \varphi_n(\lambda)$ for any integer $k \geq 1$ and $\varphi(\lambda) = \varphi_n(\lambda)$.

Let's compute $\varphi_n(\lambda) - \varphi_{n+1}(\lambda) = \frac{1+\lambda^n}{n\lambda^{n-1}} - \frac{1+\lambda^{n+1}}{(n+1)\lambda^n} = \bigl(n(n+1)\lambda^n\bigr)^{-1}\cdot\bigl((n+1)\lambda(1+\lambda^n) - n(1+\lambda^{n+1})\bigr) = \bigl(n(n+1)\lambda^n\bigr)^{-1}\cdot G_n(\lambda)$, so the sign of this expression coincides with that of $G_n(\lambda)$, and G_n is strictly increasing for $\lambda > 0$. This proves our inequality.

2.12.6. Recall that we have to restrict the L_1-norm of $\mathbb{R}[T]$ onto $\mathfrak{p}'_\lambda$, and compute the quotient norm of this restriction with respect to $\mathfrak{p}'_\lambda \to \mathfrak{p}'_\lambda/\mathfrak{p}'^2_\lambda =$

$E_{\mathbb{R}} \cong \mathbb{R}$. Instead of doing this, we can first compute the quotient norm on $\mathbb{R}[T]/\mathfrak{p}_\lambda'^2 \cong \mathbb{R}^2$, and then restrict it to $E_{\mathbb{R}}$. This is clear in the language of norms; since they do not define completely the corresponding $\mathbb{Z}_\infty$-structure, we would like to prove this statement directly for $\mathbb{Z}_\infty$-modules.

Lemma 2.12.7 *Let $i : N \to M$ be a strict monomorphism and $\pi : M \to M'$ be a strict epimorphism in $\mathbb{Z}_\infty$-Fl.Mod. Suppose we are given a decomposition $\pi \circ i : N \xrightarrow{\sigma} N' \xrightarrow{j} M'$ of $\pi \circ i$ into an epimorphism σ, followed by a monomorphism j (cf.* **2.9.6***, where existence of such decompositions with a strict σ is shown). Suppose that the square, obtained by applying $\rho^* : \mathbb{Z}_\infty$-Fl.Mod $\to \mathbb{R}$-Vect to the commutative square* (2.12.7.1) *below, is bicartesian in $\mathbb{R}$-Vect (since the vertical arrows are epic and horizontal arrows are monic, this condition is equivalent to* $\operatorname{Im} i_{\mathbb{R}} \supset \operatorname{Ker} \pi_{\mathbb{R}}$*):*

$$\begin{tikzcd} N \arrow[r, "i"] \arrow[d, "\sigma"'] & M \arrow[d, "\pi"] \\ N' \arrow[r, "j"] & M' \end{tikzcd} \tag{2.12.7.1}$$

Then $j : N' \to M'$ is a strict monomorphism, $\sigma : N \to N'$ is a strict epimorphism, and this square is bicartesian in $\mathbb{Z}_\infty$-Fl.Mod. The decomposition $j \circ \sigma$ of $\pi \circ i$ with the above properties is unique.

Proof. a) Let's check that the square is cartesian. The explicit construction of projective limits given in **2.8** shows that this is equivalent to proving that it becomes cartesian after applying ρ^*, something that we know already, and that $N_{\mathbb{Z}_\infty} = i_{\mathbb{R}}^{-1}(M_{\mathbb{Z}_\infty}) \cap \sigma_{\mathbb{R}}^{-1}(N'_{\mathbb{Z}_\infty})$. This is clear, since $\sigma_{\mathbb{R}}^{-1}(N'_{\mathbb{Z}_\infty})$ contains $N_{\mathbb{Z}_\infty}$ (this is true for any morphism), and $i_{\mathbb{R}}^{-1}(M_{\mathbb{Z}_\infty}) = N_{\mathbb{Z}_\infty}$ since i is a strict monomorphism (cf. **2.9.5**).

b) Let's check that the square is cocartesian. Since we know already that it becomes cocartesian after applying ρ^*, this amounts to check $M'_{\mathbb{Z}_\infty} = \operatorname{conv}\big(j_{\mathbb{R}}(N'_{\mathbb{Z}_\infty}) \cup \pi_{\mathbb{R}}(M_{\mathbb{Z}_\infty})\big)$. Again, $j_{\mathbb{R}}(N'_{\mathbb{Z}_\infty}) \subset M'_{\mathbb{Z}_\infty}$ for general reasons, and $\pi_{\mathbb{R}}(M_{\mathbb{Z}_\infty}) = M'_{\mathbb{Z}_\infty}$ because π is a strict epimorphism (cf. **2.9.4**).

c) By assumption, j is a monomorphism, i.e. $j_{\mathbb{R}}$ is injective. We have to show that j is a strict monomorphism, i.e. that $j_{\mathbb{R}}^{-1}(M'_{\mathbb{Z}_\infty}) \subset N'_{\mathbb{Z}_\infty}$, the opposite inclusion being evident. So let's take some $y' \in N'_{\mathbb{R}}$, such that $x' := j_{\mathbb{R}}(y') \in M'_{\mathbb{Z}_\infty}$. Since π is a strict epimorphism, $\pi_{\mathbb{R}}(M_{\mathbb{Z}_\infty}) = M'_{\mathbb{Z}_\infty}$, so we can find some $x \in M_{\mathbb{Z}_\infty}$, such that $\pi_{\mathbb{R}}(x) = x'$. We know that after application of ρ^* our square becomes cartesian in $\mathbb{R}$-*Vect*, hence in *Sets*, so $N_{\mathbb{R}} \cong M_{\mathbb{R}} \times_{M'_{\mathbb{R}}} N'_{\mathbb{R}}$, and, since x and y' have same image in $M'_{\mathbb{R}}$, we can find some element $y \in N_{\mathbb{R}}$, such that $i_{\mathbb{R}}(y) = x$ and $\sigma_{\mathbb{R}}(y) = y'$. Since i is a strict monomorphism and $i_{\mathbb{R}}(y) = x \in M_{\mathbb{Z}_\infty}$, we see that $y \in N_{\mathbb{Z}_\infty}$. Since

$\sigma_\mathbb{R}(N_{\mathbb{Z}_\infty}) \subset N'_{\mathbb{Z}_\infty}$, we see that $y' = \sigma_\mathbb{R}(y) \in \sigma_\mathbb{R}(N_{\mathbb{Z}_\infty}) \subset N'_{\mathbb{Z}_\infty}$. This reasoning is valid for any $y' \in j_\mathbb{R}^{-1}(M'_{\mathbb{Z}_\infty}) \supset N'_{\mathbb{Z}_\infty}$, so we see that both $j_\mathbb{R}^{-1}(M'_{\mathbb{Z}_\infty})$ and $\sigma_\mathbb{R}(N_{\mathbb{Z}_\infty})$ coincide with $N'_{\mathbb{Z}_\infty}$, i.e. j is a strict monomorphism, and σ is a strict epimorphism, q.e.d.

2.12.8. An immediate consequence of this lemma is that a $\mathbb{Z}_\infty$-structure $V_{\mathbb{Z}_\infty}$ on an $\mathbb{R}$-vector space V induces a canonical $\mathbb{Z}_\infty$-structure on any its subquotient V'/V'', $V'' \subset V' \subset V$, which can be defined either by first restricting the original $\mathbb{Z}_\infty$-structure to V' ($V'_{\mathbb{Z}_\infty} := V_{\mathbb{Z}_\infty} \cap V'$), and then taking its quotient ($(V'/V'')_{\mathbb{Z}_\infty} := \mathrm{Im}(V'_{\mathbb{Z}_\infty} \to V'/V'')$), or by taking first the quotient V/V'', and then restricting the quotient $\mathbb{Z}_\infty$-structure to $V'/V'' \subset V/V''$. To see this one has just to apply the lemma for $N = V'$, $M = V$, $N' = V'/V''$, $M' = V/V''$ with the induced $\mathbb{Z}_\infty$-structures described above; for V'/V'' we take the quotient of that of V', and obtain that $V'/V'' \to V/V''$ is strict, i.e. this $\mathbb{Z}_\infty$-structure of V'/V'' coincides with that induced from V/V''.

2.12.9. In our situation we apply this lemma for $M = \mathbb{Z}_\infty[T]$, $N = \mathfrak{p}_\lambda = \mathfrak{p}'_\lambda \cap \mathbb{Z}_\infty[T]$, $N' = \mathfrak{p}_\lambda/\mathfrak{p}_\lambda^2 = E$ and $M' = D := \mathbb{Z}_\infty[T]/\mathfrak{p}_\lambda^2$. The latter notation is understood as the strict quotient D of $\mathbb{Z}_\infty[T]$, such that $D_\mathbb{R} = \mathbb{R}[T]/{\mathfrak{p}'_\lambda}^2$. Note that $D_\mathbb{R}$ can be identified with $\mathbb{R}^2$, and the projection $\pi_\mathbb{R} : \mathbb{R}[T] \to D_\mathbb{R}$ with the map $F(T) \mapsto (F(\lambda), F'(\lambda))$. Then $E_\mathbb{R} = \mathbb{R}$ is identified with the coordinate axis $0 \times \mathbb{R}$ in $D_\mathbb{R}$.

Now we have to compute the strict quotient $\mathbb{Z}_\infty$-structure on $D_\mathbb{R}$, given by $D_{\mathbb{Z}_\infty} = \pi_\mathbb{R}(\mathbb{Z}_\infty[T])$. Since $\mathbb{Z}_\infty[T] = \mathrm{conv}(\{\pm T^n\}_{n \geq 0})$, we see that

$$D_{\mathbb{Z}_\infty} = \mathrm{conv}\bigl(\{\pm(1,0)\} \cup \{\pm(\lambda^n, n\lambda^{n-1})\}_{n \geq 1}\bigr). \tag{2.12.9.1}$$

After computing this convex hull we will have $E_{\mathbb{Z}_\infty} = D_{\mathbb{Z}_\infty} \cap E_\mathbb{R}$, since $D \to E$ is a strict monomorphism according to the above lemma.

The following theorem is quite helpful for computing infinite convex hulls like (2.12.9.1):

Theorem 2.12.10 *("octahedral Carathéodory theorem") Let S be a subset of an n-dimensional real vector space V. Denote by M the $\mathbb{Z}_\infty$-submodule of V, generated by S, i.e. the smallest $\mathbb{Z}_\infty$-submodule containing S (i.e. such that $S \subset M_{\mathbb{Z}_\infty}$); in other words, $M = (M_{\mathbb{Z}_\infty}, M_\mathbb{R})$, where $M_{\mathbb{Z}_\infty} =$* $\mathrm{conv}(S \cup -S)$ *is the symmetric convex hull of S and $M_\mathbb{R} = \mathbb{R} \cdot M_{\mathbb{Z}_\infty}$ is the $\mathbb{R}$-span of S.*

Then any point $x \in M_{\mathbb{Z}_\infty}$ can be written as an octahedral combination of at most $\dim M_\mathbb{R} \leq \dim V = n$ *linearly independent elements of S:*

$$x = \sum_{i=1}^{k} \lambda_i x_i, \quad x_i \in S, \quad \sum_{i=1}^{k} |\lambda_i| \leq 1. \tag{2.12.10.1}$$

Proof. Clearly, any $x \in M_{\mathbb{Z}_\infty}$ can be written as an octahedral combination of a finite subset of S, i.e. it has a representation (2.12.10.1) for some $k \geq 0$. Fix some x and choose such a representation with minimal k. We are going to show that the x_i in such a minimal representation must be linearly independent; since they lie in $M_\mathbb{R}$, this will imply that $k \leq \dim M_\mathbb{R} \leq n$. Indeed, suppose we have a non-trivial linear relation $\sum_{i=1}^k \mu_i x_i = 0$. After renumbering the x_i, λ_i and μ_i, we can assume that $\lambda_i > 0$ for $1 \leq i \leq r$ and $\lambda_i < 0$ for $r+1 \leq i \leq k$ for some $0 \leq r \leq k$ (note that $\lambda_i = 0$ would contradict minimality of k). Changing the signs of all μ_i if necessary, we can assume that $\sum_{i=1}^r \mu_i \geq \sum_{i=r+1}^k \mu_i$.

Observe that this choice of signs guarantees that the set of indices i for which $\lambda_i\mu_i > 0$ is not empty: otherwise we would have $\mu_i \leq 0$ for $i \leq r$ and $\mu_i \geq 0$ for $i > r$, hence $\sum_{i=1}^r \mu_i \leq 0 \leq \sum_{i=r+1}^k \mu_i$; together with the opposite inequality this implies $\mu_i = 0$ for all i. This contradicts the non-triviality of chosen linear relation between the x_i.

Now put $\rho := \min_{\lambda_i\mu_i>0} \lambda_i/\mu_i$. We have just checked that this minimum is computed along a non-empty set, hence ρ is well-defined; clearly, $\rho > 0$. Put $\lambda'_i := \lambda_i - \rho\mu_i$. Then $x = \sum_{i=1}^k \lambda'_i x_i$, and at least one of λ'_i is zero by the choice of ρ. Let's check that this combination is still octahedral, i.e. $\sum_{i=1}^k |\lambda'_i| \leq 1$; this will give a contradiction with the choice of k, and thus it will prove linear independence of x_i.

From the choice of ρ we see that all λ'_i, if non-zero, have the same signs as λ_i: $\lambda'_i \geq 0$ for $i \leq r$ and $\lambda'_i \leq 0$ for $i > r$. Hence $\sum_i |\lambda'_i| = \sum_{i\leq r} \lambda'_i - \sum_{i>r} \lambda'_i = \sum_i |\lambda_i| - \rho\delta$, where $\delta := \sum_{i\leq r} \mu_i - \sum_{i>r} \mu_i \geq 0$ according to our choice of signs of μ_i. We also have $\rho > 0$; hence $\sum_i |\lambda'_i| \leq \sum_i |\lambda_i| \leq 1$, i.e. our new expression $x = \sum_i \lambda'_i x_i$ is an octahedral combination with smaller number of non-zero terms, q.e.d.

2.12.11. This theorem has an obvious p-adic counterpart. Namely, if M is the $\mathbb{Z}_p$-submodule generated by an arbitrary subset S of a finite-dimensional $\mathbb{Q}_p$-vector space V, then any element of M can be written as a $\mathbb{Z}_p$-linear combination of several $\mathbb{Q}_p$-linearly independent elements of S; hence the total number of these elements doesn't exceed $\dim_{\mathbb{Q}_p} M_{(\mathbb{Q}_p)} \leq \dim_{\mathbb{Q}_p} V$.

2.12.12. Let's apply this theorem to compute $D_{\mathbb{Z}_\infty} = \operatorname{conv}(\pm(\lambda^n, n\lambda^{n-1})) \subset D_\mathbb{R} = \mathbb{R}^2$. We see that $D_{\mathbb{Z}_\infty} = \bigcup_{k,l\geq 0} A_{kl}$, where $A_{kl} := \operatorname{conv}\bigl(\pm(\lambda^k, k\lambda^{k-1}), \pm(\lambda^l, l\lambda^{l-1})\bigr)$ is the $\mathbb{Z}_\infty$-sublattice generated by these two generators of $D_{\mathbb{Z}_\infty}$. Therefore, $E_{\mathbb{Z}_\infty} = D_{\mathbb{Z}_\infty} \cap (0 \times \mathbb{R}) = \bigcup_{k,l\geq 0} B_{kl}$, where $B_{kl} := A_{kl} \cap (0 \times \mathbb{R})$ is a compact symmetric convex subset of $0 \times \mathbb{R} \cong \mathbb{R}$, hence $B_{kl} = 0 \times [-b_{kl}, +b_{kl}]$ for some real number $b_{kl} \geq 0$.

2.12.13. If $\lambda = 0$, then $D_{\mathbb{Z}_\infty} = \operatorname{conv}((\pm 1, 0), (0, \pm 1)) = \mathbb{Z}_\infty^{(2)}$ is the standard two-dimensional octahedron, and $E_{\mathbb{Z}_\infty} = \mathbb{Z}_\infty$ as stated in **2.12.4**. Case $\lambda < 0$ is reduced to $\lambda > 0$ by means of the automorphism $T \mapsto -T$ of $\mathbb{R}[T]$ that preserves $\mathbb{Z}_\infty[T]$, maps λ into $-\lambda$ and dT into $-dT$, hence $\|dT\|_\lambda = \|-dT\|_{-\lambda} = \|dT\|_{-\lambda}$. So we can assume $\lambda > 0$.

2.12.14. Let's compute the numbers $b_{kl} \geq 0$, always assuming $\lambda > 0$. Clearly, $b_{kk} = 0$ for any $k \geq 0$ and $b_{kl} = b_{lk}$, so we can assume $k > l \geq 0$. Now, $y \in B_{kl}$ is equivalent to $(0, y) \in A_{kl}$, i.e. to the existence of $u, v \in \mathbb{R}$ with $|u| + |v| \leq 1$, such that $(0, y) = u(\lambda^k, k\lambda^{k-1}) + v(\lambda^l, l\lambda^{l-1})$. This is equivalent to $\lambda^k u + \lambda^l v = 0$ and $y = k\lambda^{k-1}u + l\lambda^{l-1}v$. Since both λ^k and λ^l are non-zero, the first of these conditions is equivalent to $u = w\lambda^l$, $v = -w\lambda^k$ for some real w. Then $|u| + |v| \leq 1$ can be rewritten as $|w| \leq (\lambda^k + \lambda^l)^{-1}$, and our equation for y becomes $y = (k-l)\lambda^{k+l-1}w$, so the maximal possible value of y is $b_{kl} = (k-l)\lambda^{k+l-1}/(\lambda^k + \lambda^l) = (k-l)\lambda^{k-1}/(\lambda^{k-l} + 1)$.

2.12.15. If $\lambda \geq 1$, then for $l = 1$ and $k \gg 1$ we have $b_{k1} > k/3$, hence $E_{\mathbb{Z}_\infty}$, being the union of all $[-b_{kl}, b_{kl}]$, coincides with the whole $\mathbb{R}$; the corresponding seminorm on $E_{\mathbb{R}} = \mathbb{R}$ is identically zero, as stated in **2.12.4**. So we restrict ourselves to the case $0 < \lambda < 1$.

In this case $b_{kl} = (k-l)\lambda^{k-1}/(1 + \lambda^{k-l}) \leq b_{k0} = k\lambda^{k-1}/(1 + \lambda^k) = \varphi_k(\lambda)^{-1}$ for any $0 \leq l < k$, hence $E_{\mathbb{Z}_\infty}$ is equal to the union of $[-b_{k0}, b_{k0}] = [-\varphi_k(\lambda)^{-1}, \varphi_k(\lambda)^{-1}]$ for all integer $k \geq 1$. We know that for any $0 < \lambda < 1$ the sequence $\varphi_k(\lambda)$ has a minimal element $\varphi(\lambda) = \varphi_n(\lambda)$, where n is determined by $\alpha_{n-1} \leq \lambda \leq \alpha_n$ (cf. **2.12.5**). Therefore, $E_{\mathbb{Z}_\infty} = [-\varphi(\lambda)^{-1}, \varphi(\lambda)^{-1}] = \varphi(\lambda)^{-1} \cdot \mathbb{Z}_\infty$, and $\|dT\|_\lambda = \|1\|_E = \varphi(\lambda)$. This finishes the proof of **2.12.4**.

2.12.16. Thus we have obtained a singular cometric $\|\cdot\|_\lambda$ on the real affine line, given by $\|dT\|_\lambda = \varphi(|\lambda|)$ for $|\lambda| < 1$ and $\|\cdot\|_\lambda = 0$ for $|\lambda| \geq 1$. We see that this cometric is continuous and piecewise smooth for $|\lambda| < 1$; we want to study its asymptotic behavior as $\lambda \to 1-$. We will show that $\|dT\|_\lambda = \varphi(\lambda) = e^\kappa(1-\lambda) + O((1-\lambda)^2)$ as $\lambda \to 1-$, for some positive constant $\kappa > 0$. This will imply the continuity of $\|\cdot\|_\lambda$ at all $\lambda \in \mathbb{R}$.

Denote by κ the only real solution of $\kappa = 1 + e^{-\kappa}$. Numerically, $\kappa = 1,2784645\ldots$. We will need the asymptotic behavior of α_n as $n \to +\infty$. Notice for this that for any $\lambda > 0$ we have $G'_{n-1}(\lambda) = n\lambda^{n-1} + n > n$, hence by the mean value theorem we get $|\lambda - \alpha_{n-1}| \leq n^{-1}|G_{n-1}(\lambda)|$ for any $\lambda > 0$. Now put $\lambda := 1 - \kappa/n$; then $n \log \lambda = -\kappa + O(n^{-1})$, hence $\lambda^n = e^{-\kappa} + O(n^{-1})$ and $G_{n-1}(\lambda) = \lambda^n + n\lambda - (n-1) = e^{-\kappa} + O(n^{-1}) + (n-\kappa) - (n-1) = O(n^{-1})$ according to the choice of κ, hence $\bigl|(1 - \kappa/n) - \alpha_{n-1}\bigr| = O(n^{-2})$ and $\alpha_{n-1} = 1 - \kappa/n + O(n^{-2})$. Replacing n by $n+1$, we get $\alpha_n = 1 - \kappa/n + O(n^{-2})$ as well, so $\alpha_{n-1} \leq \lambda \leq \alpha_n$ implies $\lambda = 1 - \kappa/n + O(n^{-2})$, hence $n = \kappa(1-\lambda)^{-1} + O(1)$.

Now let's estimate $\varphi(\lambda) = \varphi_n(\lambda) = (1 + \lambda^n)/(n\lambda^{n-1}) = (\lambda/n)(1 + \lambda^{-n})$. We know that $\lambda = 1 - \kappa/n + O(n^{-2})$, hence $\lambda^n = e^{-\kappa} + O(n^{-1})$ as before, hence $\varphi(\lambda) = n^{-1}(1+e^\kappa+O(n^{-1})) = (1+e^\kappa)n^{-1}+O(n^{-2})$. Now we combine this with $1 + e^\kappa = \kappa e^\kappa$ and with the asymptotic expression for n. We get $\varphi(\lambda) = e^\kappa(1-\lambda) + O((1-\lambda)^2)$ for $\lambda \to 1-$, as claimed above.

2.12.17. (Complex Arakelov affine line.) Of course, we can consider the *complex Arakelov affine line* $\mathbb{A}^1_{\bar{\mathbb{Z}}_\infty} = \mathbb{A}^1_{\mathbb{Z}_\infty} \otimes_{\mathbb{Z}_\infty} \bar{\mathbb{Z}}_\infty = \operatorname{Spec} \bar{\mathbb{Z}}_\infty[T]$, where $\bar{\mathbb{Z}}_\infty[T] = \bar{\mathbb{Z}}_\infty \otimes_{\mathbb{Z}_\infty} \mathbb{Z}_\infty[T]$ is the polynomial algebra over $\bar{\mathbb{Z}}_\infty$ in one variable T, easily seen to satisfy the usual universal property in the category of $\bar{\mathbb{Z}}_\infty$-algebras with one pointed element.

Then we might ask about the induced cometrics $\|\cdot\|_\lambda$ for all $\lambda \in \mathbb{A}^1(\mathbb{C}) = \mathbb{C}$. However, the answer remains essentially the same as before: $\|dT\|_\lambda = \varphi(|\lambda|)$ for $|\lambda| < 1$, and $\|dT\|_\lambda = 0$ for $|\lambda| \geq 1$. To prove this we reduce first to the case $\lambda \geq 0$, by observing that for any complex ε with $|\varepsilon| = 1$ (e.g. $|\lambda|/\lambda$) the map $T \mapsto \varepsilon T$ induces an automorphism of $\bar{\mathbb{Z}}_\infty[T]$, hence of $\mathbb{A}^1_{\bar{\mathbb{Z}}_\infty}$, so we must have $\|dT\|_\lambda = \|\varepsilon\, dT\|_{\varepsilon\lambda} = \|dT\|_{\varepsilon\lambda} = \|dT\|_{|\lambda|}$ if we take $\varepsilon := |\lambda|/\lambda$.

Now we want to say that our previous computations for a real λ and the corresponding point $P_\lambda \in \mathbb{A}^1(\mathbb{R})$ remain valid after extending everything to $\mathbb{C}$ (or, more precisely, to $\bar{\mathbb{Z}}_\infty$). To prove this we consider again the strict epimorphism $\mathbb{Z}_\infty[T] \to F = \mathbb{Z}_\infty[T]/\mathfrak{p}_\lambda^2$ and the strict monomorphism $E = \mathfrak{p}_\lambda/\mathfrak{p}_\lambda^2 \to F$, and apply our complexification functor $\bar{\mathbb{Z}}_\infty \otimes_{\mathbb{Z}_\infty} -$ to them. Clearly, this functor preserves strict epimorphisms (i.e. cokernels of pairs of morphisms), being right exact. If we check that $E_{(\bar{\mathbb{Z}}_\infty)} \to F_{(\bar{\mathbb{Z}}_\infty)}$ remains a strict monomorphism, we'll obtain that the subquotient $\bar{\mathbb{Z}}_\infty$-structure on $E_\mathbb{C}$ coincides with $E_{(\bar{\mathbb{Z}}_\infty)}$; since $E_{\mathbb{Z}_\infty} = \varphi(\lambda)^{-1} \cdot \mathbb{Z}_\infty$, we get $E_{\bar{\mathbb{Z}}_\infty} = \varphi(\lambda)^{-1} \cdot \bar{\mathbb{Z}}_\infty$ and $\|z\, dT\|_\lambda = |z| \cdot \varphi(|\lambda|)$ as claimed above.

So we want to check that $j : E \to F$ remains a strict monomorphism after tensor multiplication with $\bar{\mathbb{Z}}_\infty$. For this we show that j admits a left inverse $p : F \to E$. Recall that $E_\mathbb{R} \cong \mathbb{R}$ and that $E_{\mathbb{Z}_\infty}$ is closed, hence $E \cong \mathbb{R}$ or $E \cong \mathbb{Z}_\infty$. In the first case existence of a left inverse is evident; in the second we apply Hahn–Banach theorem to linear form $1_E : E_\mathbb{R} \to E_\mathbb{R} = \mathbb{R}$ and extend it to a linear form $p : F_\mathbb{R} \to \mathbb{R} = E_\mathbb{R}$ with norm ≤ 1, i.e. to a morphism $p : F \to E$. Once existence of p is proved, we observe that $j \otimes 1_P$ also admits a left inverse $p \otimes 1_P$ for any P (e.g. $P = \bar{\mathbb{Z}}_\infty$), hence it is a strict monomorphism, being the kernel of the pair $1_{E\otimes P}$ and $(jp) \otimes 1_P : E \otimes P \rightrightarrows E \otimes P$.

2.12.18. (Projective line.) Of course, now we can take two copies of $\mathbb{A}^1_{\mathbb{Z}_\infty}$ and glue them along open subsets $\mathbb{G}_{m,\mathbb{Z}_\infty} = \operatorname{Spec} \mathbb{Z}_\infty[T, T^{-1}]$ by means of the map given by $T \mapsto T^{-1}$ into an *Arakelov projective line* $\mathbb{P}^1_{\mathbb{Z}_\infty}$. The corresponding cometric $\|\cdot\|_\lambda$ for $|\lambda| < 1$ will be the same, since these points lift to $\mathbb{Z}_\infty$-sections already in $\mathbb{A}^1_{\mathbb{Z}_\infty}$. For $|\lambda| > 1$ we observe that $T \mapsto T^{-1}$ defines an automorphism of $\mathbb{P}^1$, hence $\|dT\|_\lambda = \|d(T^{-1})\|_{\lambda^{-1}} = |\lambda|^2 \cdot \|dT\|_{\lambda^{-1}} =$

$|\lambda|^2\varphi(|\lambda|^{-1})$. In particular, for $\lambda \to 1+$ we have $\|dT\|_\lambda = e^\kappa(\lambda - 1) + O((\lambda - 1)^2)$. We cannot make a detailed analysis for $|\lambda| = 1$ now, but it seems natural to assume "by continuity" that $\|dT\|_\lambda = 0$ for $|\lambda| = 1$.

So we have obtained a continuous singular cometric on $\mathbb{P}^1(\mathbb{C})$ with singularities along the unit circle. The corresponding metric will have a "simple pole" along this circle. Note that the singular locus is not Zariski closed in this situation; it is actually Zariski dense.

2.12.19. (Hyperbolic plane?) Once we have a metric on $\mathbb{C}$, we can define the length $l(\gamma)$ of a piecewise C^1-curve $\gamma : [a, b] \to \mathbb{C}$ by the usual formula $l(\gamma) = \int_a^b \|\gamma'(t)\|_{\gamma(t)} dt$. Then we can define the distance between two points of $\mathbb{C}$ as the infimum of lengths of curves connecting these two points, thus obtaining a metric space structure on $\mathbb{C}$ and on $\mathbb{P}^1(\mathbb{C})$. Note that the unit circle is at infinite distance from any point either inside or outside this circle. When $\lambda \to 1-$, the distance between 0 and λ is $\rho(0, \lambda) = e^{-\kappa} \log(1-\lambda)+O(1)$. In this respect the interior of the unit circle, equipped with this metric, behaves like the usual Poincaré model of the hyperbolic plane inside the unit circle. Of course, our metric is only piecewise smooth.

2.12.20. (Origin of singularities.) Let's give some indications why Arakelov affine and projective lines constructed here turned out to have singular metrics. Later we will construct projective spaces $\mathbb{P}_{\mathbb{Z}_\infty}(V) = \operatorname{Proj} S_{\mathbb{Z}_\infty}(V)$ for any $\mathbb{Z}_\infty$-lattice V, and in particular we will get some (co)metrics on the complex points of $\mathbb{P}(V_\mathbb{R})$.

The $\mathbb{Z}_\infty$-sections of $\mathbb{P}(V)$ will correspond to strict quotients of V, isomorphic to $\mathbb{Z}_\infty$, i.e. to strict epimorphisms $V \to \mathbb{Z}_\infty$ modulo ± 1. By duality, they correspond to strict monomorphisms $\mathbb{Z}_\infty \to V^*$ modulo sign, i.e. to the boundary points $\partial V^* = V^* - \mathfrak{m}_\infty V^*$, again modulo sign.

Now, the projective line constructed in this section corresponds to $V = \mathbb{Z}_\infty^{(2)}$, hence $V^* = \mathbb{Z}_\infty^2$ is the square with vertices $(\pm 1, \pm 1)$. Its boundary is smooth outside these vertices, which correspond exactly to $\lambda = \pm 1$. So we can think of singularity of our cometric at these points as a consequence of non-smoothness of ∂V^* at corresponding points. Our other critical points α_n, where the (co)metric is not smooth, condense at $|\lambda| = 1$, so they could be thought of as "echoes" of this singularity.

So we are inclined to think that if V^* has a smooth boundary, then the corresponding (co)metric on the projective space will be everywhere smooth and non-singular. Some evidence to this is given by the quadratic and hermitian V's (i.e. $V_{\mathbb{Z}_\infty} = \{x : Q(x) \leq 1\}$, where Q is a positive definite quadratic or hermitian form on $V_\mathbb{R}$ or $V_\mathbb{C}$), which will be shown later to give rise to Fubiny–Study metrics on corresponding projective spaces.

2.13. (Spectra of flat $\mathbb{Z}_\infty$-algebras.) We will obtain later the theory of spectra of (arbitrary) $\mathbb{Z}_\infty$-algebras as a part of theory of spectra of generalized rings. For several reasons this seems to be a more natural approach than to develop the theory only for one special case. However, we would like to show the impatient reader a glimpse of this special case of the theory. The exposition will be somewhat sketchy here, since we are going to give the rigorous proofs later in a more general context.

2.13.1. It will be more convenient to think of some object $M = (M_{\mathbb{Z}_\infty}, M_{\mathbb{R}})$ of $\mathbb{Z}_\infty$-*Fl.Mod* as a set $M = M_{\mathbb{Z}_\infty}$, equipped by a flat $\mathbb{Z}_\infty$-module structure. By definition, a (flat) $\mathbb{Z}_\infty$-module structure on some set M is simply an embedding $j_M : M \to M_{\mathbb{R}}$ of M into some real vector space $M_{\mathbb{R}}$, such that $j_M(M)$ becomes a symmetric convex body in $M_{\mathbb{R}}$. We say that a map of sets $f : M \to N$ is compatible with given $\mathbb{Z}_\infty$-structures on them if f extends to a $\mathbb{R}$-linear map $f_{\mathbb{R}} : M_{\mathbb{R}} \to N_{\mathbb{R}}$, necessarily unique.

Of course, this is just another description of $\mathbb{Z}_\infty$-*Fl.Mod*; we have only switched our focus of interest to $M_{\mathbb{Z}_\infty}$. In particular, a (commutative flat) $\mathbb{Z}_\infty$-algebra A can be now described as a (commutative) monoid, equipped by a flat $\mathbb{Z}_\infty$-module structure, with respect to which the multiplication becomes $\mathbb{Z}_\infty$-bilinear.

2.13.2. The notions of a multiplicative system $S \subset A$ and an invertible element $s \in A$ involve only the multiplicative (i.e. monoid) structure of A, hence they are well-defined in our situation.

2.13.3. (Ideals.) An *ideal* of A is by definition an A-submodule $\mathfrak{a} \subset A$, i.e. a $\mathbb{Z}_\infty$-submodule, stable under the multiplication of A: $A \cdot \mathfrak{a} \subset A$. The *intersection* of two ideals $\mathfrak{a} \cap \mathfrak{b}$ is understood in the usual way. The *sum* $\mathfrak{a} + \mathfrak{b} = (\mathfrak{a}, \mathfrak{b})$ is understood as the ideal generated by $\mathfrak{a} \cup \mathfrak{b}$, i.e. the smallest ideal containing both $\mathfrak{a}$ and $\mathfrak{b}$; it consists of all octahedral combinations of elements of $\mathfrak{a}$ and $\mathfrak{b}$, i.e. $\mathfrak{a} + \mathfrak{b} = \{\lambda x + \mu y : |\lambda| + |\mu| \leq 1,\ x \in \mathfrak{a},\ y \in \mathfrak{b}\}$. Finally, the *product* $\mathfrak{a}\mathfrak{b}$ is by definition the ideal generated by all products xy, $x \in \mathfrak{a}$, $y \in \mathfrak{b}$. It can be also described as the image of $\mathfrak{a} \otimes_{\mathbb{Z}_\infty} \mathfrak{b} \to A$.

An ideal $\mathfrak{p}$ in A is *prime* if its complement is a multiplicative system, i.e. if $1 \notin \mathfrak{p}$, and $s, t \notin \mathfrak{p}$ implies $st \notin \mathfrak{p}$. We say that $\mathfrak{m}$ is a *maximal ideal* if it $\neq A$ and if it is not contained in any ideal $\neq \mathfrak{m}$ and $\neq A$. Finally, we say that A is *local* if it has exactly one maximal ideal.

Note that *any maximal ideal* $\mathfrak{m}$ *is prime.* Indeed, if $s, t \notin \mathfrak{m}$, then $(\mathfrak{m}, s) = (\mathfrak{m}, t) = A$ by maximality, hence $(\mathfrak{m}, st) \supset (\mathfrak{m}^2, s\mathfrak{m}, t\mathfrak{m}, st) = (\mathfrak{m}, s)(\mathfrak{m}, t) = A$ and $st \notin \mathfrak{m}$.

Also note that the Krull theorem still holds: *Any ideal* $\mathfrak{a} \neq A$ *is contained in some maximal ideal* $\mathfrak{m}$. For this we apply the Zorn lemma to the set of ideals of A, containing $\mathfrak{a}$ and distinct from A, and observe that filtered unions of ideals are ideals.

Some corollaries of these theorems also hold. For example, the union of all maximal ideals of A coincides with the complement of the set of invertible elements of A.

Another interesting statement is that $\mathfrak{ab} \subset \mathfrak{p}$ *iff* $\mathfrak{a} \subset \mathfrak{p}$ *or* $\mathfrak{b} \subset \mathfrak{p}$, *for a prime ideal* $\mathfrak{p}$.

2.13.4. (Localization.) Given a multiplicative subset $S \subset A$ and an A-module M, we can construct their *localizations* $S^{-1}A$ and $S^{-1}M$ as follows. As a set, $S^{-1}M = M \times S/\sim$, where $(x, s) \sim (y, t)$ iff $utx = usy$ for some $u \in S$, and similarly for $S^{-1}A$, so this is the usual definition of localizations. The multiplication of $S^{-1}A$ and the action of $S^{-1}A$ on $S^{-1}M$ are defined in the usual way. To obtain $\mathbb{Z}_\infty$-structures on $S^{-1}M$ and $S^{-1}A$, we write them down as filtered inductive limits: $S^{-1}M = \varinjlim_{s \in \mathcal{S}} M_{[s]}$. Here the index category $\mathcal{S}$ is defined as follows: $\operatorname{Ob}\mathcal{S} = S$, and morphisms $[s] \to [s']$ are given by elements $t \in S$, such that $s' = ts$. Our inductive system is given by $M_{[s]} = M$ for all s, and the transition maps $M_{[s]} \to M_{[st]}$ are given by multiplication by t. They are clearly $\mathbb{Z}_\infty$-linear, and filtered inductive limits in $\mathbb{Z}_\infty$-*Fl.Mod* coincide with those computed in *Sets* (cf. **2.9.1**), so we indeed get a flat $\mathbb{Z}_\infty$-structure on $S^{-1}M$ and $S^{-1}A$.

Then $S^{-1}A$ becomes a $\mathbb{Z}_\infty$-algebra and even an A-algebra, and $S^{-1}M$ is an $S^{-1}A$-module; from their inductive limit description one can obtain their usual universal properties: $S^{-1}A$ is the universal A-algebra in which all elements of S become invertible, and $M \to S^{-1}M$ is the universal A-linear map of M into a module, on which all elements of S act bijectively.

Of course, we introduce the usual notations $M_\mathfrak{p}$ and M_f if S is the complement of a prime ideal $\mathfrak{p}$, resp. if $S = S_f = \{1, f, f^2, \ldots\}$. If g divides some power of f, we get canonical homomorphisms $M_g \to M_f$.

In short, this notion of localization seems to possess all usual properties of localizations of commutative rings and modules over them.

2.13.5. (Definition of $\operatorname{Spec} A$.) Now we define the *(prime) spectrum* $\operatorname{Spec} A$ to be the set of all prime ideals of A. For any $M \subset A$ we define $V(M) \subset \operatorname{Spec} A$ by $V(M) := \{\mathfrak{p} : \mathfrak{p} \supset M\}$. We have the usual properties: $V(0) = \operatorname{Spec} A$, $V(1) = \varnothing$, $V(\bigcup M_\alpha) = \bigcap V(M_\alpha)$, $V(M) = V(\langle M \rangle)$, where $\langle M \rangle$ is the ideal generated by M, and finally $V(\mathfrak{ab}) = V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$. These properties show that the $\{V(M)\}$ are the closed subsets of $X := \operatorname{Spec} A$ for some topology on X, called the *Zariski topology*. Clearly, the principal open subsets $D(f) := X - V(\{f\}) = \{\mathfrak{p} : f \notin \mathfrak{p}\}$ form a base of this topology. Note that $\bigcup D(f_\alpha) = X$ iff the f_α generate the unit ideal A, hence X is quasicompact.

Any $\mathbb{Z}_\infty$-algebra homomorphism $\varphi : A \to B$ induces a continuous map ${}^a\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ by the usual rule $\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$. If $B = S^{-1}A$, $\operatorname{Spec} B$

can be identified by means of this map with the subset $\{\mathfrak{p} : S \cap \mathfrak{p} = \varnothing\} \subset X$, equipped with the induced topology. In particular, $\operatorname{Spec} A_f$ is homeomorphic to $D(f) \subset X$.

2.13.6. (Quasicoherent sheaves.) Any A-module M defines a presheaf of $\mathbb{Z}_\infty$-modules $\widetilde{M}$ on $X = \operatorname{Spec} A$ with respect to the base $\{D(f)\}$, given by $\Gamma(D(f), \widetilde{M}) := M_f$. These presheaves are actually sheaves, something that will be checked later in a more general context. In particular, we get the *structure sheaf* $\mathscr{O}_X := \widetilde{A}$. It is a sheaf of $\mathbb{Z}_\infty$-algebras, and each $\widetilde{M}$ is a sheaf of $\mathscr{O}_X$-modules. Of course, we have the usual functoriality with respect to $\mathbb{Z}_\infty$-algebra homomorphisms $\varphi : A \to B$; for quasicoherent sheaves (i.e. those isomorphic to some $\widetilde{M}$) this functoriality involves tensor products over A which haven't been explained in detail yet (cf. **2.11.7**).

2.13.7. (Flat $\mathbb{Z}_\infty$-schemes.) Now we define *flat $\mathbb{Z}_\infty$-schemes* to be topological spaces equipped with a sheaf of $\mathbb{Z}_\infty$-algebras, each point of which has a neighborhood isomorphic to $\operatorname{Spec} A$ for some flat $\mathbb{Z}_\infty$-algebra A. Flat $\mathbb{Z}_\infty$-schemes, isomorphic to some $\operatorname{Spec} A$, are called *affine*. *Morphisms* of flat $\mathbb{Z}_\infty$-schemes are just *local* morphisms of topological spaces equipped by local sheaves of flat $\mathbb{Z}_\infty$-algebras. As usual, morphisms $X \to \operatorname{Spec} A$ are in one-to-one correspondence with $\mathbb{Z}_\infty$-algebra homomorphisms $A \to \Gamma(X, \mathscr{O}_X)$. Note that the category of flat $\mathbb{Z}_\infty$-schemes has fibered products, the existence of which is shown in the same way as in EGA I, once we have the tensor products $\otimes_A$.

2.13.8. (Schemes over $\operatorname{Spec} \mathbb{R}$.) Any $\mathbb{R}$-algebra A has a natural flat $\mathbb{Z}_\infty$-algebra structure; $\operatorname{Spec} A$ in this case coincides with the usual spectrum of an affine ring. In other words, the category of $\mathbb{R}$-schemes can be identified with a full subcategory of the category of flat $\mathbb{Z}_\infty$-schemes.

2.13.9. For example, $S := \operatorname{Spec} \mathbb{Z}_\infty = \{0, \mathfrak{m}_\infty\}$, and the only non-trivial open subset is $\{0\}$, so $\operatorname{Spec} \mathbb{Z}_\infty$ looks exactly like $\operatorname{Spec} \mathbb{Z}_p$. The structural sheaf is given by $\Gamma(S, \mathscr{O}_S) = \mathbb{Z}_\infty$, $\Gamma(\{0\}, \mathscr{O}_S) = \mathbb{R}$ and $\Gamma(\varnothing, \mathscr{O}_S) = 0$. Spectrum of $\bar{\mathbb{Z}}_\infty$ gives the same topological space, but of course with a different structure sheaf.

For any flat $\mathbb{Z}_\infty$-algebra A we have a canonical homomorphism $\mathbb{Z}_\infty \to A$, hence there is a unique morphism from any flat $\mathbb{Z}_\infty$-scheme X into $\operatorname{Spec} \mathbb{Z}_\infty$, i.e. $\operatorname{Spec} \mathbb{Z}_\infty$ is the final object of this category. The generic fiber $X_\xi = X_{\mathbb{R}}$, i.e. the pullback of the open subset $\{0\} \cong \operatorname{Spec} \mathbb{R}$ with respect to this morphism $X \to \operatorname{Spec} \mathbb{Z}_\infty$, is an $\mathbb{R}$-scheme in the usual sense. For example, the generic (or open) fiber of $\operatorname{Spec} A$ is $\operatorname{Spec} A_{\mathbb{R}}$. Not so much can be said about the closed fiber.

2.13.10. In this way we see that any prime ideal $\mathfrak{p}$ of A falls into one of two subcategories:

a) Ideals over 0, i.e. $\mathfrak{p} \cap \mathbb{Z}_\infty = 0$. These are in one-to-one correspondence with (usual) prime ideals $\mathfrak{p}'$ in the $\mathbb{R}$-algebra A. This correspondence is given by $\mathfrak{p} \mapsto \mathfrak{p}_\mathbb{R}$, $\mathfrak{p}' \mapsto \mathfrak{p}' \cap A$; in other words, we consider on $\mathfrak{p}'$ the $\mathbb{Z}_\infty$-structure induced from $A_\mathbb{R}$, i.e. $\mathfrak{p}$ is the strict subobject of A, corresponding to $\mathfrak{p}' \subset A_\mathbb{R}$.

b) Ideals over $\mathfrak{m}_\infty$, i.e. $\mathfrak{p} \cap \mathbb{Z}_\infty = \mathfrak{m}_\infty$, hence $\mathfrak{p} \supset \mathfrak{m}_\infty$ and $\mathfrak{p} \supset \mathfrak{m}_\infty A$. In general we cannot say too much about these ideals. One of the reasons is that we cannot describe them as prime ideals in $A/\mathfrak{m}_\infty A$, since this quotient is zero in $\mathbb{Z}_\infty$-*Fl.Mod*, and is $\mathbb{Z}_\infty$-torsion in the category $\mathbb{Z}_\infty$-*Mod* that will be constructed later.

2.13.11. Let's consider $X = \operatorname{Spec} A$ for $A = \mathbb{Z}_\infty^2 = \mathbb{Z}_\infty \times \mathbb{Z}_\infty$. Contrary to what one might expect, $X \neq \operatorname{Spec}\mathbb{Z}_\infty \sqcup \operatorname{Spec}\mathbb{Z}_\infty$, because we can have prime ideals that contain both idempotents $e_1 = (1,0)$ and $e_2 = (0,1)$. In our case the open fiber consists of two points (closed in this fiber), given by ideals $0 \times \mathbb{Z}_\infty$ and $\mathbb{Z}_\infty \times 0$. However, the closed fiber consists of *three* points $0 \times \mathfrak{m}_\infty$, $\mathfrak{m}_\infty \times 0$ and $\mathfrak{m} := A - \{(\pm 1, \pm 1)\}$. The latter is the only maximal ideal of A, i.e. A is local, and $\mathfrak{m}$ defines the only closed point of X. The complement of this closed point is an open subscheme, isomorphic to $\operatorname{Spec}\mathbb{Z}_\infty \sqcup \operatorname{Spec}\mathbb{Z}_\infty$. We'll give an explanation for this phenomenon in **5.3.14**.

2.13.12. Now we would like to consider the affine line $\mathbb{A}^1_{\mathbb{Z}_\infty} = \operatorname{Spec}\mathbb{Z}_\infty[T]$ from this point of view. The prime ideals $\mathfrak{p}$ in the open fiber are in one-to-one correspondence with $\mathfrak{p}' \in \operatorname{Spec}\mathbb{R}[T]$, so we obtain immediately their complete description: there is the zero ideal (0), ideals $\mathfrak{p}_\lambda := \mathbb{Z}_\infty[T] \cap (T - \lambda)\mathbb{R}[T] = \{F(T) \in \mathbb{Z}_\infty[T] : F(\lambda) = 0\}$ for each real $\lambda \in \mathbb{R}$, and $\mathfrak{p}_z := \mathbb{Z}_\infty[T] \cap (T - z)(T - \bar{z})\mathbb{R}[T] = \{F(T) \in \mathbb{Z}_\infty[T] : F(z) = 0\}$ for each complex z with $\operatorname{Im} z > 0$. All these ideals are closed in this fiber, with the only exception of the zero ideal.

The closed fiber, i.e. prime ideals $\mathfrak{p} \subset \mathbb{Z}_\infty[T]$ containing $\mathfrak{m}_\infty \cdot \mathbb{Z}_\infty[T]$, seem to be more complicated to describe. Here are some examples of them:

- The only maximal ideal $\mathfrak{m} := \mathbb{Z}_\infty[T] - \{\pm 1\}$; i.e. $\mathbb{Z}_\infty[T]$ is *local*.
- Prime ideal $\mathfrak{q} := \mathbb{Z}_\infty[T] - \{\pm T^k\}_{k \geq 0}$.
- Prime ideals $\mathfrak{q}_n := \{F(T) \in \mathbb{Z}_\infty[T] : |F(\zeta_n)| < 1\}$, where ζ_n is a primitive root of unity of degree n. In particular, we have $\mathfrak{q}_+ := \mathfrak{q}_1 = \{F(T) : |F(1)| < 1\}$ and $\mathfrak{q}_- := \mathfrak{q}_2$. Note that $\mathfrak{m}$ can be defined in the same way with $\zeta_0 := 0$. For odd n the complement of $\mathfrak{q}_n$ can be described as the set of polynomials $F(T) = \sum_k a_k T^k$ with $a_{qn+r} \neq 0$ only for at most one value of remainder $0 \leq r < n$, and with all a_{qn+r}

of same sign and of sum equal to ± 1. When n is even, the description is slightly more complicated: we require $a_{qn/2+r} \neq 0$ for at most one value of $0 \leq r < n/2$, and all $(-1)^q a_{qn/2+r}$ of same sign and of sum equal to ± 1 for this value of r.

Note that $\mathfrak{q}_n \subset \mathfrak{q}_{nn'}$ for all integer n and odd n'. This means that the minimal between these ideals are the $\mathfrak{q}_{2^k}$ for $k \geq 0$; so they should correspond to irreducible components of the special fiber, if we admit that this list of prime ideals exhausts the whole of $\operatorname{Spec} \mathbb{Z}_\infty[T]$. Note that the closed fiber has infinite Krull dimension in any case.

2.13.13. (Projective $\mathbb{Z}_\infty$-schemes.) We can also define *graded* flat $\mathbb{Z}_\infty$-algebras $A = \bigoplus_{n\geq 0} A_n$, either in terms of such direct sum decompositions, understood as in **2.9.2**, subject to usual conditions $A_n \cdot A_m \subset A_{n+m}$ and $1 \in A_0$, or in terms of sequences $(A_n)_{n\geq 0}$ of flat $\mathbb{Z}_\infty$-modules and $\mathbb{Z}_\infty$-bilinear multiplication maps $\mu_{nm} : A_n \times A_m \to A_{n+m}$, subject to usual associativity and commutativity relations. Both descriptions are easily seen to be equivalent, and graded $\mathbb{Z}_\infty$-flat A-modules M can be described in a similar way. If $S \subset A$ is a multiplicative system, consisting only of homogeneous elements, we can define a natural grading on $S^{-1}A$ and $S^{-1}M$, and consider their degree zero part. Another possible construction of $(S^{-1}M)_0$ is by means of the inductive limit $\varinjlim_{s\in\mathcal{S}} M_{[s]}$, where this time we take $M_{[s]} := M_{\deg s}$ (cf. **2.13.4**). If $f \in A$ is homogeneous of positive degree, we denote the degree zero part of M_f by $M_{(f)}$, and similarly for A_f.

Now we can define the *projective spectrum* $\operatorname{Proj} A$ to be the set of homogeneous prime ideals $\mathfrak{p} \subset A$, not containing the ideal $A^+ := \bigoplus_{n>0} A_n$, with the topology induced by that of $\operatorname{Spec} A \supset \operatorname{Proj} A$. Next, the reasoning of EGA II 2 shows us that the open subsets $D_+(f) := D(f) \cap \operatorname{Proj} A$ for homogeneous $f \in A^+$ form a base of topology on $\operatorname{Proj} A$, and that each $D_+(f)$ is homeomorphic to $\operatorname{Spec} A_{(f)}$. Modulo some compatibility issues, checked in the same way as in *loc. cit.*, we obtain in this way a flat $\mathbb{Z}_\infty$-scheme structure on $\operatorname{Proj} A$.

2.13.14. (Projective spaces.) In particular, for any $\mathbb{Z}_\infty$-lattice V we consider the projective space $\mathbb{P}_{\mathbb{Z}_\infty}(V) := \operatorname{Proj} S_{\mathbb{Z}_\infty}(V)$, and similarly over $\bar{\mathbb{Z}}_\infty$. Note that the $\mathbb{Z}_\infty$ or $\bar{\mathbb{Z}}_\infty$-structure of these projective spaces give rise to metrics on their complex-valued points in the same way as in **2.12.2**, at least if we believe in the "valuative criterion of properness" in our situation. Locally this means considering the subquotient $\mathbb{Z}_\infty$-structure on $\mathfrak{m}_P/\mathfrak{m}_P^2$, where $\mathfrak{m}_P$ is the maximal ideal corresponding to some closed point of the open fiber.

If we start with a hermitian $\bar{\mathbb{Z}}_\infty$-module V, we end up with a metric on the complex projective space $\mathbb{P}(V_{(\mathbb{C})}) \cong \mathbb{P}^{n-1}(\mathbb{C})$, clearly equivariant under the action of $\mathrm{Aut}_{\bar{\mathbb{Z}}_\infty}(V) = U(n)$. There is only one such metric, up to multiplication by a positive constant, namely, the Fubini–Study metric on the projective space.

We will use similar arguments later to show that any smooth projective variety with Kähler metric induced by the Fubini–Study metric on an ambient projective space admits a description in terms of flat $\mathbb{Z}_\infty$-schemes.

2.13.15. However, for several reasons we do not want to develop further the theory of (flat) $\mathbb{Z}_\infty$-schemes right now. We are going to develop the theory of generalized rings first, and then we shall construct a theory of their spectra and generalized schemes, which will contain the theory sketched above as a special case. We would like to list some reasons for this.

- Up to now we have considered only flat $\mathbb{Z}_\infty$-modules and flat $\mathbb{Z}_\infty$-schemes. However, even if we are interested only in ($\mathbb{Z}_\infty$-flat) models of algebraic varieties over $\mathbb{R}$, their closed subschemes, required for example to construct a reasonable intersection theory, in general need not be $\mathbb{Z}_\infty$-flat. On the other hand, the theory of (abstract) $\mathbb{Z}_\infty$-modules and $\mathbb{Z}_\infty$-schemes doesn't seem to be more simple than that of modules and schemes over an arbitrary (hypoadditive) generalized ring, so we don't lose much if we do all constructions in the more general setting.

- The reader may have observed that the localization theory of flat $\mathbb{Z}_\infty$-algebras and modules seems to enjoy almost all the usual properties known from commutative algebra. However, this is not the case with the other important construction of commutative algebra, namely, the theory of quotient modules and rings. Even if we consider the category of all (abstract) $\mathbb{Z}_\infty$-modules, we get $Q := \mathbb{Z}_\infty/\frac{1}{2}\mathbb{Z}_\infty \cong \mathbb{Z}_\infty/\mathfrak{m}_\infty$, so the kernel of $\mathbb{Z}_\infty \to Q$ is equal to $\mathfrak{m}_\infty$, and not to $\frac{1}{2}\mathbb{Z}_\infty$ (cf. **2.14.13**); another similar phenomenon is given by $\mathbb{R}/\mathbb{Z}_\infty$, which is equal to zero even in $\mathbb{Z}_\infty$-*Mod*. However, if we consider strict quotients of a $\mathbb{Z}_\infty$-algebra in the larger category of generalized rings, the situation becomes more natural. In this way we are led to consider closed (generalized) subschemes of a flat $\mathbb{Z}_\infty$-scheme that are not flat themselves.

- If we develop separately the theory of $\mathbb{Z}_\infty$-schemes and the usual theory of (Grothendieck) schemes, we will still be forced to describe objects (say, schemes) over $\widehat{\mathrm{Spec}\,\mathbb{Z}}$ as corresponding objects over $\mathrm{Spec}\,\mathbb{Z}$ and $\mathrm{Spec}\,\mathbb{Z}_\infty$, the pullbacks of which to $\mathrm{Spec}\,\mathbb{R}$ agree (cf. **1.4**). On the other hand, the theory of generalized rings and schemes enables us to construct $\widehat{\mathrm{Spec}\,\mathbb{Z}}$ and, say, models $\mathscr{X} \to \widehat{\mathrm{Spec}\,\mathbb{Z}}$ as generalized schemes, since both the theory of commutative rings and the theory of $\mathbb{Z}_\infty$-algebras are special cases of this general theory.

- Furthermore, we will be able to consider $\mathscr{X}$ as a generalized scheme over $\mathbb{F}_{\pm 1} := \Gamma(\widehat{\operatorname{Spec}\mathbb{Z}}, \mathscr{O}_{\widehat{\operatorname{Spec}\mathbb{Z}}}) = \mathbb{Z} \cap \mathbb{Z}_\infty := \mathbb{Z} \times_{\mathbb{R}} \mathbb{Z}_\infty$, where this intersection (i.e. fibered product) is computed in the category of generalized rings. Among other things, this will lead to a reasonable construction of intersection theory on $\mathscr{X}$, with $\mathbb{F}_{\pm 1}$ for the base ring.

2.14. (Abstract $\mathbb{Z}_\infty$-modules.) Now we are going to use our knowledge of the category $\mathbb{Z}_\infty$-*Fl.Mod* of flat (or torsion-free) $\mathbb{Z}_\infty$-modules to construct the category $\mathbb{Z}_\infty$-*Mod* of all (abstract) $\mathbb{Z}_\infty$-modules. We shall also need for this the definition and some basic properties of monads; they can be found either in the next chapter or in [MacLane].

2.14.1. Recall that we have constructed a pair of adjoint functors: the *forgetful functor* $\Gamma_{\mathbb{Z}_\infty} : \mathbb{Z}_\infty\text{-}\mathit{Fl.Mod} \to \mathit{Sets}$, $A \mapsto A_{\mathbb{Z}_\infty} \cong \operatorname{Hom}_{\mathbb{Z}_\infty}(\mathbb{Z}_\infty, A)$, and its left adjoint $L_{\mathbb{Z}_\infty} : \mathit{Sets} \to \mathbb{Z}_\infty\text{-}\mathit{Fl.Mod}$, $S \mapsto \mathbb{Z}_\infty^{(S)}$ (cf. **2.10.7**). Denote by $\xi : \mathrm{Id}_{\mathit{Sets}} \to \Gamma_{\mathbb{Z}_\infty} L_{\mathbb{Z}_\infty}$ and $\eta : L_{\mathbb{Z}_\infty}\Gamma_{\mathbb{Z}_\infty} \to \mathrm{Id}_{\mathbb{Z}_\infty\text{-}\mathit{Fl.Mod}}$ the natural transformations defining adjointness between $L_{\mathbb{Z}_\infty}$ and $\Gamma_{\mathbb{Z}_\infty}$.

By general theory of monads this pair of adjoint functors gives us a monad $\Sigma_\infty = \Sigma_{\mathbb{Z}_\infty} = (\Sigma_\infty, \mu, \varepsilon)$ on *Sets*, where $\Sigma_\infty := \Gamma_{\mathbb{Z}_\infty} L_{\mathbb{Z}_\infty}$ is an endofunctor on *Sets*, $\varepsilon := \xi : \mathrm{Id}_{\mathit{Sets}} \to \Sigma_\infty$ is the unit, and $\mu := \Gamma_{\mathbb{Z}_\infty} * \eta * L_{\mathbb{Z}_\infty} : \Sigma_\infty^2 \to \Sigma_\infty$ is the multiplication of this monad.

2.14.2. Once we have a monad Σ_∞ on the category of sets, we can consider the category of Σ_∞-algebras or Σ_∞-sets, usually denoted by $\mathit{Sets}^{\Sigma_\infty}$. By definition, its objects are sets X, equipped with a Σ_∞-structure $\alpha : \Sigma_\infty(X) \to X$, i.e. a map of sets subject to conditions $\alpha \circ \varepsilon_X = \mathrm{id}_X$ and $\alpha \circ \mu_X = \alpha \circ \Sigma_\infty(\alpha) : \Sigma_\infty^2(X) \to X$. Morphisms $f : (X, \alpha_X) \to (Y, \alpha_Y)$ are maps of sets $f : X \to Y$, compatible with given Σ_∞-structures, i.e. $f \circ \alpha_X = \alpha_Y \circ \Sigma_\infty(f)$.

Clearly, we have a forgetful functor $\Gamma' : \mathit{Sets}^{\Sigma_\infty} \to \mathit{Sets}$, $(X, \alpha) \mapsto X$. It has a left adjoint L', given by $L'(S) := (\Sigma_\infty(S), \mu_S)$, such that $\Gamma' L'$ is still equal to Σ_∞.

Finally, $\Gamma_{\mathbb{Z}_\infty} : \mathbb{Z}_\infty\text{-}\mathit{Fl.Mod} \to \mathit{Sets}$ canonically factorizes through Γ', yielding a functor $I : \mathbb{Z}_\infty\text{-}\mathit{Fl.Mod} \to \mathit{Sets}^{\Sigma_\infty}$, such that $\Gamma' I = \Gamma_{\mathbb{Z}_\infty}$; recall that $I : A \mapsto \bigl(\Gamma_{\mathbb{Z}_\infty}(A), \Gamma_{\mathbb{Z}_\infty}(\eta_A)\bigr)$. Now, by definition Γ would be a *monadic functor* if I had been an equivalence of categories; in this case we would be able to identify $\mathbb{Z}_\infty$-*Fl.Mod* with $\mathit{Sets}^{\Sigma_\infty}$, $\Gamma_{\mathbb{Z}_\infty}$ with Γ', and $L_{\mathbb{Z}_\infty}$ with L'.

2.14.3. However, this is not the case: $I : \mathbb{Z}_\infty\text{-}\mathit{Fl.Mod} \to \mathit{Sets}^{\Sigma_\infty}$ *is fully faithful, but not an equivalence of categories.* We will prove this statement later; let's discuss its consequences first.

Definition 2.14.4 *We define the category $\mathbb{Z}_\infty$-Mod of (abstract) $\mathbb{Z}_\infty$-modules to be the category $\mathit{Sets}^{\Sigma_\infty}$ of sets equipped with a Σ_∞-structure. We* *identify $\mathbb{Z}_\infty$-Fl.Mod with a full subcategory of $\mathbb{Z}_\infty$-Mod by means of functor I constructed above. A $\mathbb{Z}_\infty$-*structure *α on some set X is by definition the same thing as a Σ_∞-structure, i.e. a map $\alpha : \Sigma_\infty(X) \to X$, subject to two conditions recalled in* **2.14.2**. *A map $f : X \to Y$ between two $\mathbb{Z}_\infty$-modules is $\mathbb{Z}_\infty$-*linear*, or a $\mathbb{Z}_\infty$-*homomorphism*, if it respects the $\mathbb{Z}_\infty$-structures, i.e. if it defines a morphism in $\mathbb{Z}_\infty$-Mod.*

2.14.5. Note that this definition is more "algebraic" than **2.4.1** and **2.7.3**: instead of considering pairs $(A_{\mathbb{Z}_\infty}, A_\mathbb{R})$, consisting of a symmetric convex body in an ambient real space, we consider sets $A = A_{\mathbb{Z}_\infty}$, equipped with some $\mathbb{Z}_\infty$-structure, i.e. Σ_∞-structure.

2.14.6. The reason for this definition is the following. If we start from any commutative ring K (say, $K = \mathbb{Z}_p$), and consider the category K-*Mod* of K-modules together with the forgetful functor $\Gamma_K : K\text{-}Mod \to Sets$, then this functor turns out to be monadic, i.e. it has a left adjoint $L_K : S \mapsto K^{(S)}$, hence it defines a monad $\Sigma_K := \Gamma_K L_K$ on *Sets*, and induced functor $I_K : K\text{-}Mod \to Sets^{\Sigma_K}$ is an equivalence.

However, if we start from the category $\mathbb{Z}_p$-*Fl.Mod* of flat $\mathbb{Z}_p$-modules, we will end up with the same monad $\Sigma_{\mathbb{Z}_p}$, but induced functor $\mathbb{Z}_p\text{-}Fl.Mod \to Sets^{\Sigma_{\mathbb{Z}_p}} \cong \mathbb{Z}_p\text{-}Mod$ won't be an equivalence, but just a fully faithful functor. In this way we can reconstruct first $\Sigma_{\mathbb{Z}_p}$, and then $\mathbb{Z}_p\text{-}Mod \cong Sets^{\Sigma_{\mathbb{Z}_p}}$, starting from the category $\mathbb{Z}_p$-*Fl.Mod* and the forgetful functor $\mathbb{Z}_p\text{-}Fl.Mod \to Sets$. This is exactly what we have done in **2.14.4** for $\mathbb{Z}_\infty$.

2.14.7. Recall the explicit description of $\Sigma_\infty(S) = \Gamma_{\mathbb{Z}_\infty}(\mathbb{Z}_\infty^{(S)})$ given in **2.10.8**: $\Sigma_\infty(S) \subset \mathbb{R}^{(S)}$ consists of all formal octahedral combinations $\sum_{s\in S} \lambda_s\{s\}$ of elements of S, where almost all $\lambda_s \in \mathbb{R}$ are equal to zero and $\sum_s |\lambda_s| \leq 1$. Hence a Σ_∞-structure α on some set S is a map $\alpha : \Sigma_\infty(S) \to S$, i.e. a way to evaluate formal octahedral combinations of elements of S. We denote $\alpha\bigl(\sum_i \lambda_i\{s_i\}\bigr)$ by $\bigl\langle\sum_i \lambda_i s_i\bigr\rangle_\alpha$, or even by $\sum_i \lambda_i s_i$, when no confusion can arise. Here $s_i \in S$, almost all $\lambda_i \in \mathbb{R}$ are zero, and $\sum_i |\lambda_i| \leq 1$, as usual.

If we start from an object $A = (A_{\mathbb{Z}_\infty}, A_\mathbb{R})$ of $\mathbb{Z}_\infty$-*Fl.Mod* and construct a Σ_∞-structure α_A on $A_{\mathbb{Z}_\infty}$ as explained in **2.14.2** (this corresponds to considering $I(A)$), then $\alpha_A = \Gamma(\eta_A) : \Sigma_\infty(A_{\mathbb{Z}_\infty}) \to A_{\mathbb{Z}_\infty}$, and we see immediately that the image in $A_\mathbb{R}$ of $\langle\sum_i \lambda_i s_i\rangle_{\alpha_A} \in A_{\mathbb{Z}_\infty}$ coincides with the corresponding linear combination computed in $A_\mathbb{R}$; this explains our notation.

2.14.8. In particular, we can apply this observation to $A := \mathbb{Z}_\infty^{(S)}$. In this case it follows from definitions that $\alpha_A = \mu_S$, so we obtain an explicit description of $\mu_S : \Sigma_\infty^2(S) \to \Sigma_\infty(S)$. Namely, $\mu_S\bigl(\sum_i \lambda_i\{\sum_j \mu_{ij}\{s_j\}\}\bigr)$ equals $\sum_{i,j} \lambda_i\mu_{ij}\{s_j\}$. An explicit description of $\varepsilon_S : S \to \Sigma_\infty(S)$ is even easier to obtain: ε_S maps any $s \in S$ into corresponding basis element

$\{s\} \in \Sigma_\infty(S) \subset \mathbb{R}^{(S)}$.

Now, our conditions for a map $\alpha : \Sigma(S) \to S$ to be a Σ_∞-structure are $\alpha \circ \varepsilon_S = 1_S$ and $\alpha \circ \Sigma_\infty(\alpha) = \alpha \circ \mu_S$. The first condition translates into $\alpha(\{s\}) = s$, i.e. $\langle s \rangle_\alpha = s$, for any $s \in S$; this requirement seems to be quite natural.

The second condition translates into

$$\Bigl\langle \sum_{i=1}^n \lambda_i \bigl\langle \sum_{j=1}^m \mu_{ij} s_j \bigr\rangle_\alpha \Bigr\rangle_\alpha = \Bigl\langle \sum_{j=1}^m \bigl(\sum_{i=1}^n \lambda_i \mu_{ij} \bigr) s_j \Bigr\rangle_\alpha \quad , \tag{2.14.8.1}$$

where $s_j \in S$, $\sum_i |\lambda_i| \leq 1$ and $\sum_j |\mu_{ij}| \leq 1$ for any i. If we remove the angular brackets, the equation we obtain looks like the usual distributivity relation, written for arbitrary *octahedral* combination of octahedral combinations of some $s_j \in S$.

2.14.9. Since any formal octahedral combination involves only finitely many elements of a set X, we see that a Σ_∞-structure α on X is completely determined by the family of maps $\alpha_n : \Sigma(n) \times X^n \to X$, $n \geq 0$, given by $\alpha_n(\lambda_1, \ldots, \lambda_n; x_1, \ldots, x_n) := \langle \lambda_1 x_1 + \cdots + \lambda_n x_n \rangle_\alpha = \alpha(\lambda_1\{x_1\} + \cdots + \lambda_n\{x_n\})$. Here $\Sigma(n) = \{(\lambda_1, \ldots, \lambda_n) : \sum_i |\lambda_i| \leq 1\}$ is the standard octahedron in $\mathbb{R}^n$; cf. **2.10.8**.

Of course, these maps α_n, $n \geq 0$, have to satisfy some compatibility relations (e.g. invariance under all permutations of arguments σ) in order to define together some $\alpha : \Sigma_\infty(X) \to X$:

$$\alpha_n(\lambda_1, \ldots, \lambda_n; x_1, \ldots, x_n) = \alpha_n(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(n)}; x_{\sigma(1)}, \ldots, x_{\sigma(n)}),\ \forall \sigma; \tag{2.14.9.1}$$

$$\alpha_{n+1}(\lambda_1, \ldots, \lambda_n, 0; x_1, \ldots, x_n, x_{n+1}) = \alpha_n(\lambda_1, \ldots, \lambda_n; x_1, \ldots, x_n); \tag{2.14.9.2}$$

$$\alpha_{n+1}(\lambda_1, \ldots, \lambda_n, \lambda_{n+1}; x_1, \ldots, x_n, x_n) = \alpha_n(\lambda_1, \ldots, \lambda_n + \lambda_{n+1}; x_1, \ldots, x_n) \tag{2.14.9.3}$$

Resulting α is a Σ_∞-structure iff the unit and associativity relations are fulfilled:

$$\alpha_1(1; x) = x \tag{2.14.9.4}$$

$$\begin{aligned} &\alpha_n\bigl(\lambda_1, \ldots, \lambda_n;\ \alpha_m(\mu_{11}, \ldots, \mu_{1m}; x_1, \ldots, x_m), \ldots, \alpha_m(\mu_{n1}, \ldots, \mu_{nm}; x_1, \ldots)\bigr) \\ &= \alpha_m\bigl(\sum_{i=1}^n \lambda_i \mu_{i1}, \ldots, \sum_{i=1}^n \lambda_i \mu_{im};\ x_1, \ldots, x_m\bigr) \end{aligned} \tag{2.14.9.5}$$

We have thus obtained a description of $\mathbb{Z}_\infty$-modules in terms of a countable family of operations α_n.

2.14.10. Note that *all α_n, hence also α, are completely determined by* $\alpha_2 : \Sigma_\infty(2) \times X^2 \to X$. Indeed, for α_0 and α_1 we have $\alpha_1(\lambda; x) = \alpha_2(\lambda, 0; x, x)$ and $\alpha_0 = \alpha_2(0, 0; x, x)$ for any $x \in X$ (note that X has to be non-empty, otherwise we won't be able to retrieve its zero element α_0). For α_n, $n \geq 3$, we prove this statement by induction, using identity

$$\alpha_n(\lambda_1, \ldots, \lambda_{n-2}, \lambda_{n-1}, \lambda_n;\, x_1, \ldots, x_{n-2},\, x_{n-1}, x_n) = \\ \alpha_{n-1}(\lambda_1, \ldots, \lambda_{n-2}, \mu;\, x_1, \ldots, x_{n-2}, \alpha_2(\lambda_{n-1}/\mu, \lambda_n/\mu;\, x_{n-1}, x_n)) \quad (2.14.10.1)$$

when $\mu := |\lambda_{n-1}| + |\lambda_n| \neq 0$; if it is zero, we use (2.14.9.2) instead.

We see that we might describe a $\mathbb{Z}_\infty$-module X as a triple (X, α_0, α_2), where X is a set, $\alpha_0 \in X$ is its marked element ("zero"), and $\alpha_2 : \Sigma_\infty(2) \times X^2 \to X$ is the binary octahedral combination evaluation map discussed above. These data have to satisfy several axioms, which do not seem to be especially enlightening, so we don't list them here. The most interesting among them is the associativity/distributivity axiom for α_2:

$$\alpha_2(\lambda'\nu', \mu;\, x, \alpha_2(\nu, \rho;\, y, z)) = \alpha_2(\lambda', \mu\rho;\, \alpha_2(\nu', \rho';\, x, y), z), \qquad (2.14.10.2)$$

for any $x, y, z \in X$, whenever $\mu\nu = \lambda'\rho'$, $|\lambda'\nu'| + |\mu| \leq 1$, $|\lambda'| + |\mu\rho| \leq 1$, $|\nu| + |\rho| \leq 1$ and $|\nu'| + |\rho'| \leq 1$.

In this form the definition of (abstract) $\mathbb{Z}_\infty$-modules is very similar to that of abstract convex sets, which in fact can be obtained in the same way starting from the monad $\mathbf{\Delta}$ that maps any set S into the simplex with vertices $\{s\}$, i.e. the convex hull of all basis elements $\{s\}$ of $\mathbb{R}^{(S)}$.

2.14.11. We see that any $\mathbb{Z}_\infty$-module X has a marked element, namely, $\alpha_0 := \alpha_X(0)$; we denote it by 0_X or 0, and call it the *zero element of X*. We have also a map $\alpha_1 : \mathbb{Z}_\infty \times X \to X$, which defines an action of $\mathbb{Z}_\infty$, considered as a multiplicative monoid, on X, i.e. $\alpha_1(\lambda, \alpha_1(\mu, x)) = \alpha_1(\lambda\mu, x)$ for any $x \in X$ and $|\lambda|, |\mu| \leq 1$. Of course, $\alpha_1(\lambda, x)$ is usually written as $\lambda \cdot x$ or λx, so the above formula can be written simply as $\lambda(\mu x) = (\lambda\mu)x$. Note that 0_X is a fixed point for this action, i.e. $\lambda \cdot 0_X = 0_X$ for any $|\lambda| \leq 1$, and $0 \cdot x = 0_X$ for any $x \in X$. However, this structure of a set with marked point, equipped by an action of monoid $\mathbb{Z}_\infty = [-1, 1]$, doesn't determine uniquely α_2, i.e. the structure of a $\mathbb{Z}_\infty$-module on X.

2.14.12. We see that a $\mathbb{Z}_\infty$-module X can be described as an algebraic system $X = (X, \alpha_0, \alpha_2)$, consisting of a set X, its element α_0 and a map $\alpha_2 : \Sigma_\infty(2) \times X^2 \to X$, subject to a finite set of axioms. Therefore, the category of $\mathbb{Z}_\infty$-modules has all the usual properties of categories defined by algebraic systems. We list some of these properties below without proof; most of them can be easily checked directly, and all of them will be proved later in **4.6** for the categories of modules over arbitrary generalized rings.

- Arbitrary projective limits exist in $\mathbb{Z}_\infty$-*Mod*; they can be essentially computed in *Sets* (i.e. $\Gamma_{\mathbb{Z}_\infty} : \mathbb{Z}_\infty\text{-}\mathit{Mod} \to \mathit{Sets}$ commutes with arbitrary projective limits).

- Arbitrary inductive limits exist in $\mathbb{Z}_\infty$-*Mod* (this is the only complicated statement in this list).

- Filtered inductive limits exist in $\mathbb{Z}_\infty$-*Mod*; they can be computed in *Sets*, i.e. $\Gamma_{\mathbb{Z}_\infty}$ commutes with filtered inductive limits.

- A $\mathbb{Z}_\infty$-linear map $f : M \to N$ is a monomorphism in $\mathbb{Z}_\infty$-*Mod* iff it is injective as a map of sets (i.e. iff $\Gamma_{\mathbb{Z}_\infty}(f)$ is injective).

- A $\mathbb{Z}_\infty$-linear map $f : M \to N$ is a strict epimorphism iff it is surjective as a map of sets.

- Given an injective map of sets $f : N \to M$ and a $\mathbb{Z}_\infty$-module structure on M, there is at most one $\mathbb{Z}_\infty$-module structure (called the *induced structure*) on N, compatible with f. Therefore, subobjects (i.e. $\mathbb{Z}_\infty$-submodules) of M are in one-to-one correspondence with those subsets N of M, which admit an induced $\mathbb{Z}_\infty$-module structure.

- Similarly, given a surjective map of sets $f : M \to Q$ and a $\mathbb{Z}_\infty$-module structure on M, there is at most one $\mathbb{Z}_\infty$-module structure on Q, compatible with f. Strict quotients of M are in one-to-one correspondence with certain quotient sets Q of M.

- Any $\mathbb{Z}_\infty$-homomorphism $f : M \to N$ decomposes into a strict epimorphism $p : M \to I$, followed by a monomorphism $i : I \to N$. As a set, I coincides with $f(M)$; its $\mathbb{Z}_\infty$-module structure can be described either as one induced from N or as one induced from M. We denote I by $f(M)$ or $\operatorname{Im} f$ and say that it is the *image* of f.

2.14.13. Consider the $\mathbb{Z}_\infty$-module $Q = \mathbb{F}_\infty$, constructed as follows. Its underlying set consists of three elements, denoted 0, $+1$ and -1. The Σ_∞-structure $\alpha = \alpha_Q$ on $Q = \{0, +1, -1\}$ is given by

$$\alpha_Q\bigl(\lambda_{-1}\{-1\} + \lambda_0\{0\} + \lambda_1\{+1\}\bigr) = \begin{cases} +1 & \text{if } \lambda_1 - \lambda_{-1} = 1 \\ -1 & \text{if } \lambda_1 - \lambda_{-1} = -1 \\ 0 & \text{if } |\lambda_1 - \lambda_{-1}| < 1 \end{cases} \tag{2.14.13.1}$$

Here $|\lambda_{-1}| + |\lambda_0| + |\lambda_1| \leq 1$, so in any case $|\lambda_1 - \lambda_{-1}| \leq 1$, hence the cases listed above exhaust all possibilities. We have to check that this α_Q satisfies the axioms for a Σ_∞-structure; for this we consider the surjective map $\varphi : \mathbb{Z}_\infty \to Q$, which maps $\pm 1 \in \mathbb{Z}_\infty = [-1,1]$ into $\pm 1 \in Q$, and all other elements of $\mathbb{Z}_\infty$ into 0. It is easy to check that $\varphi \circ \alpha_{\mathbb{Z}_\infty} = \alpha_Q \circ \Sigma_\infty(\varphi)$; since φ is surjective, this implies that α_Q is indeed a $\mathbb{Z}_\infty$-structure on Q, and that $\varphi : \mathbb{Z}_\infty \to Q$ is a surjective $\mathbb{Z}_\infty$-linear map, i.e. a strict epimorphism in $\mathbb{Z}_\infty$-*Mod*.

We see that Q is obtained from $\mathbb{Z}_\infty$ by identifying all elements of $\mathfrak{m}_\infty$ with zero; this explains our alternative notation $\mathbb{F}_\infty$ and $\mathbb{Z}_\infty/\mathfrak{m}_\infty$ for Q.

Note that for any non-zero object $A = (A_{\mathbb{Z}_\infty}, A_{\mathbb{R}})$ of $\mathbb{Z}_\infty$-*Fl.Mod* the underlying set of corresponding object $I(A)$ of $\mathbb{Z}_\infty$-*Mod* is equal to $A_{\mathbb{Z}_\infty}$, hence it is infinite. Since $Q \neq 0$ and it is finite, we see that it cannot be isomorphic to any $I(A)$, i.e. it doesn't lie in the essential image of $I : \mathbb{Z}_\infty\text{-}Fl.Mod \to \mathbb{Z}_\infty\text{-}Mod$, hence I cannot be an equivalence of categories.

2.14.14. Clearly, $I : \mathbb{Z}_\infty\text{-}Fl.Mod \to \mathbb{Z}_\infty\text{-}Mod$ is faithful, since $\Gamma' \circ I = \Gamma := \Gamma_{\mathbb{Z}_\infty}$ is faithful; in order to prove **2.14.3** we have to check that I is fully faithful, i.e. that *for any two flat $\mathbb{Z}_\infty$-modules A and B, any map of underlying sets $f : \Gamma(A) \to \Gamma(B)$ compatible with induced Σ_∞-structures actually defines a morphism $f' = (f'_{\mathbb{Z}_\infty}, f'_{\mathbb{R}}) : A \to B$ in $\mathbb{Z}_\infty$-Fl.Mod with $f'_{\mathbb{Z}_\infty} = f$ (cf.* **2.7.3**).

Recall that the Σ_∞-structure on $\Gamma(A)$ is given by $\alpha_A := \Gamma(\eta_A) : \Gamma(\mathbb{Z}_\infty^{(A)}) = \Sigma_\infty\Gamma(A) \to \Gamma(A)$, and similarly for B; so the compatibility of f with these Σ_∞-structures means commutativity of the following square:

$$\begin{tikzcd} \Gamma(\mathbb{Z}_\infty^{(A)}) \arrow[r, "\Sigma_\infty(f)"] \arrow[d, "\Gamma(\eta_A)"] & \Gamma(\mathbb{Z}_\infty^{(B)}) \arrow[d, "\Gamma(\eta_B)"] \\ \Gamma(A) \arrow[r, "f"] & \Gamma(B) \end{tikzcd} \tag{2.14.14.1}$$

Now consider the following diagram with right exact rows:

$$\begin{tikzcd} R_A \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "R_f"'] & \mathbb{Z}_\infty^{(A)} \arrow[r, "\eta_A"] \arrow[d, "\mathbb{Z}_\infty^{(f)}"'] & A \arrow[d, dashed, "f'"'] \\ R_B \arrow[r, shift left] \arrow[r, shift right] & \mathbb{Z}_\infty^{(B)} \arrow[r, "\eta_B"] & B \end{tikzcd} \tag{2.14.14.2}$$

Note that η_A is a strict epimorphism in $\mathbb{Z}_\infty$-*Fl.Mod*, hence it is cokernel of its kernel pair $R_A \rightrightarrows \mathbb{Z}_\infty^{(A)}$, and similarly for η_B; therefore, once we construct the two left vertical arrows in $\mathbb{Z}_\infty$-*Fl.Mod*, we'll obtain immediately an $f' : A \to B$ in $\mathbb{Z}_\infty$-*Fl.Mod* completing this diagram; since $\Gamma(\eta_A)$ is surjective and (2.14.14.1) is commutative, necessarily $\Gamma(f') = f$ as required. Now, the middle vertical arrow $L_{\mathbb{Z}_\infty}(f) = \mathbb{Z}_\infty^{(f)}$ exists for any map of sets f; to show existence of R_f we observe that $\mathbb{Z}_\infty^{(f)} \times \mathbb{Z}_\infty^{(f)}$ maps the underlying set of strict subobject R_A of $\mathbb{Z}_\infty^{(A)} \times \mathbb{Z}_\infty^{(A)}$ into the underlying set of $R_B \subset \mathbb{Z}_\infty^{(B)} \times \mathbb{Z}_\infty^{(B)}$ because of commutativity of (2.14.14.1), hence it induces some $R_f : R_A \to R_B$ with required properties.

This finishes the proof of **2.14.3**.

2.14.15. Henceforth we shall identify $\mathbb{Z}_\infty$-*Fl.Mod* with a full subcategory of $\mathbb{Z}_\infty$-*Mod* by means of the functor I constructed in **2.14.2**. Recall that the category of $\mathbb{R}$-vector spaces $\mathbb{R}$-*Vect* has been identified in **2.9.3** with a full subcategory of $\mathbb{Z}_\infty$-*Fl.Mod*, hence it is now identified with a full subcategory of $\mathbb{Z}_\infty$-*Mod*. In other words, we have a canonical $\mathbb{Z}_\infty$-structure α_V on any $\mathbb{R}$-vector space V; it is easy to see that $\alpha_V : \Sigma_\infty(V) \to V$ simply evaluates any given formal octahedral combination of elements of V in the natural way by means of the $\mathbb{R}$-vector space structure on V.

Let us denote the functor just constructed by $\rho_* : \mathbb{R}\text{-}\mathit{Vect} \to \mathbb{Z}_\infty\text{-}\mathit{Mod}$. It is not very hard to prove that it admits both a left adjoint ρ^* and a right adjoint $\rho^!$, similarly to what we had before for flat $\mathbb{Z}_\infty$-modules. We will see later that formulas $\rho^* M = \mathbb{R} \otimes_{\mathbb{Z}_\infty} M$ and $\rho^! M = \mathbf{Hom}_{\mathbb{Z}_\infty}(\mathbb{R}, M)$ are still valid in this context, so we can denote $\rho^* M$ also by $M_{(\mathbb{R})}$ or even $M_\mathbb{R}$.

2.14.16. Existence of $M_{(\mathbb{R})}$ and of a $\mathbb{Z}_\infty$-linear map $i_M : M \to M_{(\mathbb{R})}$, universal with respect to $\mathbb{Z}_\infty$-linear maps from M into $\mathbb{R}$-vector spaces, can be shown directly in several different ways. For example, we can construct $M_{(\mathbb{R})}$ as the quotient of free $\mathbb{R}$-vector space $\mathbb{R}^{(M)}$ by the $\mathbb{R}$-vector subspace generated by all elements of the form $\{\lambda_1 x_1 + \cdots + \lambda_n x_n\} - \lambda_1\{x_1\} - \cdots - \lambda_n\{x_n\}$, for all $n \geq 0$ and all octahedral combinations of n elements of M.

Second construction: we can characterize $\mathbb{R}$-vector spaces as those $\mathbb{Z}_\infty$-modules, on which all elements of $S := \mathbb{Z}_\infty - \{0\}$ act bijectively; then the required universal property of $M_{(\mathbb{R})}$ is that of $S^{-1}M$, so we have to compute $S^{-1}M$, and this can be done by means of filtered inductive limits in the same way as in **2.13.4**. In particular, this means that all elements of $M_{(\mathbb{R})}$ can be written in form x/s for some $x \in M$ and $s \in \mathbb{Z}_\infty$, $s \neq 0$, and $x/s = y/t$ iff there is some $u \neq 0$, such that $utx = usy$.

Finally, we can develop first the theory of tensor products in $\mathbb{Z}_\infty$-*Mod*, and construct $M_{(\mathbb{R})}$ as $\mathbb{R} \otimes_{\mathbb{Z}_\infty} M$.

Definition 2.14.17 *We say that a $\mathbb{Z}_\infty$-module M is* a torsion-free $\mathbb{Z}_\infty$-module *if the canonical map $i_M : M \to M_{(\mathbb{R})}$ is injective, i.e. a monomorphism in $\mathbb{Z}_\infty$-Mod. We say that M is* a (pure) torsion $\mathbb{Z}_\infty$-module *if $M_{(\mathbb{R})} = 0$. Finally, we denote by M_{tf} the image of M in $M_{(\mathbb{R})}$; it is the largest torsion-free quotient of M.*

Note that by the universal property of $M_{(\mathbb{R})}$, we have that M is torsion-free iff there exists an embedding (=monomorphism) of M into an $\mathbb{R}$-vector space. This shows that $M_{tf} \subset M_{(\mathbb{R})}$ is indeed torsion-free. Any map $f : M \to N$ from M to a torsion-free N induces an $\mathbb{R}$-linear map $f_{(\mathbb{R})} : M_{(\mathbb{R})} \to N_{(\mathbb{R})}$; since $i_N : N \to N_{(\mathbb{R})}$ is injective, and $f_{(\mathbb{R})} \circ i_M = i_N \circ f$, this implies immediately that f factorizes through $i_M(M) = M_{tf}$, as stated above. In other words, $M \mapsto M_{tf}$ is a left adjoint to the embedding of the full subcategory of torsion-free $\mathbb{Z}_\infty$-modules into $\mathbb{Z}_\infty$-*Mod*.

2.14.18. Let us show that the torsion-free $\mathbb{Z}_\infty$-modules are exactly those which lie in the essential image of $I : \mathbb{Z}_\infty$-*Fl.Mod* $\to \mathbb{Z}_\infty$-*Mod*, i.e. which are isomorphic to some $I(A)$. This will characterize internally $\mathbb{Z}_\infty$-*Fl.Mod* as a subcategory of $\mathbb{Z}_\infty$-*Mod*, and it will show that $M \mapsto M_{tf}$ is a left adjoint to I.

Clearly, any $\mathbb{Z}_\infty$-module of form $I(A) = A_{\mathbb{Z}_\infty}$ for some object $A = (A_{\mathbb{Z}_\infty}, A_{\mathbb{R}})$ embeds into some $\mathbb{R}$-vector space, namely, $A_{\mathbb{R}}$, so it must be torsion-free. Conversely, if M is torsion-free, we put $A_{\mathbb{R}} := M_{(\mathbb{R})}$ and $A_{\mathbb{Z}_\infty} := M_{tf} = i_M(M) \cong M$. Then $A_{\mathbb{Z}_\infty} \subset A_{\mathbb{R}}$ is stable under all octahedral combinations of its elements, hence it is symmetric and convex; it also generates the whole of $A_{\mathbb{R}} = M_{(\mathbb{R})}$, as it easily follows either from the universal property or from any of the three constructions of $M_{(\mathbb{R})}$ given in **2.14.16**, hence $A \in \operatorname{Ob} \mathbb{Z}_\infty$-*Fl.Mod* and $M \cong I(A)$.

2.14.19. (Torsion $\mathbb{Z}_\infty$-modules.) By definition, M is a torsion $\mathbb{Z}_\infty$-module iff $M_{(\mathbb{R})} = S^{-1}M = 0$, where $S = \mathbb{Z}_\infty - \{0\}$. Our description of localizations shows immediately that this condition is equivalent to the following: For any $x \in M$ there is some $\lambda \in S$ (i.e. a non-zero $\lambda \in \mathbb{Z}_\infty$), such that $\lambda x = 0$ in M. For example, the $\mathbb{Z}_\infty$-module $\mathbb{F}_\infty$ considered in **2.14.13** is torsion, since $(1/2)x = 0$ for any $x \in \mathbb{F}_\infty$.

2.14.20. (Limits in $\mathbb{Z}_\infty$-*Fl.Mod* and $\mathbb{Z}_\infty$-*Mod*.) Since $I : \mathbb{Z}_\infty$-*Fl.Mod* $\to$ $\mathbb{Z}_\infty$-*Mod* has a left adjoint $M \mapsto M_{tf}$, it commutes with arbitrary projective limits, i.e. arbitrary projective limits of torsion-free $\mathbb{Z}_\infty$-modules can be still computed as in **2.8**. This is in general not true for arbitrary inductive limits: we have only the formula $I(\varinjlim_\alpha M_\alpha) = (\varinjlim_\alpha I(M_\alpha))_{tf}$, which follows immediately from the fact that I is fully faithful and admits a left adjoint $M \mapsto M_{tf}$.

However, this is still true for filtered inductive limits, since in both categories they can be essentially computed in *Sets*, and for arbitrary direct sums as well. In the latter case we are to check that arbitrary direct sums of torsion-free modules are still torsion-free. For this we first write a direct sum over an arbitrary index set I as a filtered inductive limit of direct sums over all finite subsets $J \subset I$, thus reducing to the case of a finite direct sum, and then by induction it suffices to check the statement for a direct sum $M \oplus N$.

So let M and N be torsion-free $\mathbb{Z}_\infty$-modules, $P := M \oplus N$ be their direct sum in $\mathbb{Z}_\infty$-*Mod*. Notice that $i_1 : M \to M \oplus N$ is injective, since it admits a left inverse $\pi_1 : M \oplus N \to M \oplus 0 \cong M$, and similarly for $i_2 : N \to M \oplus N$, so we can identify M and N with subsets of $M \oplus N$. Let us check that P is torsion-free, i.e. that $P \to S^{-1}P$ is injective. Let z, $z' \in P$ be two elements of P, such that $z/1 = z'/1$ in $S^{-1}P$, i.e. $\nu z = \nu z'$ for some $0 < \nu \leq 1$. We'll check in **4.6.15** that elements z, $z' \in P = M \oplus N$ can be always represented as octahedral combinations $z = \lambda x + \mu y$, $z' = \lambda' x' + \mu' y'$ with x, $x' \in M$, y, $y' \in N$. Applying $\pi_1 : P \to M$ to $\nu z = \nu z'$, we obtain $\nu\lambda x = \nu\lambda' x'$, hence $\lambda x = \lambda' x'$, M being torsion-free. Furthermore, if $|\lambda| \geq |\lambda'|$, we get $\lambda \cdot (\lambda'/\lambda) x' = \lambda x$, whence $x = (\lambda'/\lambda)x'$, unless both $\lambda = \lambda' = 0$. Substituting this into $z = \lambda x + \mu y$, we get $z = \lambda' x' + \mu y$, i.e. we can assume $\lambda = \lambda'$, $x = x'$. Case $|\lambda'| \geq |\lambda|$ is dealt with similarly, and if both $\lambda = \lambda' = 0$, then z and $z' \in N$, hence $z = z'$, N being torsion-free, so this case can be excluded. Next, we can consider cases $|\mu| \geq |\mu'|$, $|\mu'| \geq |\mu|$ and $\mu = \mu' = 0$, and deduce from N being torsion-free that we can assume $\mu = \mu'$, $y = y'$ as well. We see that $z = \lambda x + \mu y = z'$, so $P \to S^{-1}P$ is indeed injective and $P = M \oplus N$ is torsion-free, q.e.d.

2.14.21. (Linear algebra, $\otimes$-structure etc.) Given any three $\mathbb{Z}_\infty$-modules M, N and P, we say that a map $\Phi : M \times N \to P$ is $\mathbb{Z}_\infty$-*bilinear* if for any $x \in M$ the map $s_\Phi(x) : y \mapsto \Phi(x, y)$ is a $\mathbb{Z}_\infty$-linear map $N \to P$, and, similarly, for any $y \in N$ the map $d_\Phi(y) : x \mapsto \Phi(x, y)$ is $\mathbb{Z}_\infty$-linear.

Proceeding as in **2.10**, we construct tensor products and inner Homs in $\mathbb{Z}_\infty$-*Mod*, obtaining an ACU $\otimes$-structure on this category. This leads to a natural definition of a $\mathbb{Z}_\infty$-algebra and of a module over such an algebra; we can construct their localization theory and their spectra exactly in the same way we did it in **2.13**.

Again, we shall return to all these questions later in a more general context. We would like to finish this quite long chapter by remarking that $I : \mathbb{Z}_\infty$-*Fl.Mod* $\to \mathbb{Z}_\infty$-*Mod* commutes with tensor products and inner Homs (the reader is invited to check this easy statement once tensor products and inner Homs over arbitrary generalized rings are defined in **5.3**), so almost all our previous considerations and computations inside $\mathbb{Z}_\infty$-*Fl.Mod* will retain their validity in $\mathbb{Z}_\infty$-*Mod*.

3 Generalities on monads

In this chapter we want to collect some general facts on monads, which will be of use to us in the remaining part of this work. Almost all of these facts are very well known and can be found, for example, in [MacLane]. We recall also some proofs, especially in the cases where we need to generalize them later to the case of inner monads.

After that we study the concept of inner endofunctors and inner monads on a Cartesian closed category (e.g. a topos). This theory doesn't seem to be widely known; however, most definitions and proofs appear just as natural generalizations of their counterparts in the theory of monads.

Later we shall usually consider monads on the category of sets, and inner monads on topoi; we shall see that the latter intuitively correspond to sheaves of monads over the category of sets, hence their significance for our further developments.

3.1. (AU $\otimes$-categories.) Let $\mathcal{A}$ be an AU (associative with unity) $\otimes$-category. This means that we are given a bifunctor $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, an associativity constraint α, i.e. a family of functorial isomorphisms $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$ satisfying the pentagon axiom for quadruple tensor products, and that we are given an unit object $\mathbf{1}_{\mathcal{A}}$ together with functorial isomorphisms $\mathbf{1}_{\mathcal{A}} \otimes X \cong X \cong X \otimes \mathbf{1}_{\mathcal{A}}$, compatible with the associativity constraint in a natural way which need not be explained explicitly here. By a well-known theorem of MacLane, this data allows us to define multiple tensor products of any (possibly empty) finite sequence of objects of $\mathcal{A}$, and to establish canonical isomorphisms between tensor products of such multiple products. In short, we get a notion of tensor product on $\mathcal{A}$, enjoying all usual properties of tensor product of modules over a commutative ring, *apart from commutativity*, since we don't impose any commutativity constraint. We'll profit by this remark by working with this tensor product in the customary way, without bothering to write down explicitly all arising canonical isomorphisms.

3.1.1. (External $\otimes$-action.) Once the AU $\otimes$-category $\mathcal{A}$ is fixed, we can consider its *external (left) $\otimes$-action,* or *$\oslash$-action,* on some category $\mathcal{B}$. This means that we are given some "external tensor product" $\oslash : \mathcal{A} \times \mathcal{B} \to \mathcal{B}$ (which will be also denoted by the same symbol $\otimes$ as the tensor product of $\mathcal{A}$, when no confusion can arise), together with some external associativity constraint, i.e. a family of functorial isomorphisms $(X \otimes Y) \oslash M \xrightarrow{\sim} X \oslash (Y \oslash M)$, satisfying a variant of the pentagon axiom with respect to quadruple products of sort $X \otimes Y \otimes Z \oslash M$, where X, Y, Z are in $\mathcal{A}$ and M is in $\mathcal{B}$. We also require some functorial isomorphisms $\mathbf{1}_{\mathcal{A}} \oslash M \xrightarrow{\sim} M$ to be given and to be compatible with given external associativity constraint.

As before, we don't use these canonical isomorphisms explicitly, but simply write down multiple tensor products $X_1 \otimes \cdots \otimes X_n \obslash M$, for any X_i in $\mathcal{A}$ and M in $\mathcal{B}$, and use different canonical isomorphisms between them without explaining or even naming them.

We have also the notion of a *right* external $\otimes$-action (also called $\oslash$-action) of $\mathcal{A}$ on $\mathcal{B}$, given by some bifunctor $\oslash : \mathcal{B} \times \mathcal{A} \to \mathcal{B}$ together with some external associativity and unit constraints. This case can be reduced to one considered before by permuting the order of arguments to $\otimes$ and $\oslash$.

3.1.2. Of course, we have both a left and a right external $\otimes$-action of $\mathcal{A}$ on itself. Another example, even more trivial, is given by taking for $\mathcal{B}$ the empty or the point category.

3.1.3. We have the notion of a *$\otimes$-functor* $F : \mathcal{A} \to \mathcal{A}'$ between two AU $\otimes$-categories $\mathcal{A}$ and $\mathcal{A}'$. By definition, this is a functor $F : \mathcal{A} \to \mathcal{A}'$, together with a family of functorial isomorphisms $F(X \otimes_{\mathcal{A}} Y) \xrightarrow{\sim} F(X) \otimes_{\mathcal{A}'} F(Y)$ and an isomorphism $F(\mathbf{1}_{\mathcal{A}}) \xrightarrow{\sim} \mathbf{1}_{\mathcal{A}'}$, compatible with associativity and unit constraints on $\mathcal{A}$ and $\mathcal{A}'$.

If we are given some external $\otimes$-action of $\mathcal{A}$ on some $\mathcal{B}$, and of $\mathcal{A}'$ on $\mathcal{B}'$, we can speak about *external $\otimes$-functors (or $\obslash$-functors) $G : \mathcal{B} \to \mathcal{B}'$, compatible with F*. By definition, this means that we are given some functor $G : \mathcal{B} \to \mathcal{B}'$, together with functorial isomorphisms $G(X \obslash_{\mathcal{B}} M) \xrightarrow{\sim} F(X) \obslash_{\mathcal{B}'} G(M)$, compatible with external associativity and unit constraints on $\mathcal{B}$ and $\mathcal{B}'$ in a natural way.

In particular, we can apply this definition in case $\mathcal{A}' = \mathcal{A}$, $F = \mathrm{Id}_{\mathcal{A}}$. Then we are given external $\otimes$-actions of $\mathcal{A}$ on two categories $\mathcal{B}$ and $\mathcal{B}'$, and study $\obslash$-functors $G : \mathcal{B} \to \mathcal{B}'$ compatible with these $\otimes$-actions.

Another special case is given by $\mathcal{B} = \mathcal{A}$, $\mathcal{B}' = \mathcal{A}'$. Then we see that F is compatible with itself.

3.1.4. We have also a notion of natural transformation $\zeta : G \to G'$ of two $\obslash$-functors $G, G' : \mathcal{B} \to \mathcal{B}'$, compatible with same $\otimes$-functor $F : \mathcal{A} \to \mathcal{A}'$. This is simply a natural transformation $\zeta : G \to G'$ of corresponding underlying functors, such that the following square is commutative for all $X \in \mathrm{Ob}\,\mathcal{A}$ and $Y \in \mathrm{Ob}\,\mathcal{B}$, where the horizontal arrows come from the $\obslash$-structure on G and G':

$$\begin{tikzcd}
G(X \obslash M) \arrow[r, "\sim"] \arrow[d, "\zeta_{X \obslash M}"] & F(X) \obslash G(M) \arrow[d, "1_{F(X)} \obslash \zeta_M"] \\
G'(X \obslash M) \arrow[r, "\sim"] & F(X) \obslash G'(M)
\end{tikzcd} \tag{3.1.4.1}$$

Natural transformations of $\oslash$-functors are defined similarly.

3.1.5. Let's mention some important examples of ⊗-categories and ⊗-actions.

a) Take $\mathcal{A} := \mathit{Sets}$, and $\otimes := \times$, i.e. we consider the the cartesian products on the category of sets. Of course, this is an AU (even ACU) ⊗-category, and it acts on itself.

b) Take $\mathcal{A} := K\text{-}\mathit{Mod}$ for any commutative ring K, and let $\otimes := \otimes_K$ be the usual tensor product of K-modules. Again, this is clearly an AU (even ACU) ⊗-category, acting on itself.

c) Let K' be any (associative) K-algebra. Let $\mathcal{A} := K\text{-}\mathit{Mod}$ as before, and $\mathcal{B} := K'\text{-}\mathit{Mod}$. Then any tensor product $X \otimes_K M$, where X is a K-module and M is a (left) K'-module, has a canonical K'-module structure, so we get an external tensor product $\oslash : \mathcal{A} \times \mathcal{B} \to \mathcal{B}$. It satisfies indeed all requirements for an external ⊗-action, so we get an external ⊗-action of $K\text{-}\mathit{Mod}$ on $K'\text{-}\mathit{Mod}$.

d) If $F : \mathcal{A} \to \mathcal{A}'$ is an ⊗-functor, then it induces both a left and a right ⊗-action of $\mathcal{A}$ on $\mathcal{A}'$, given by $X \oslash M := F(X) \otimes_{\mathcal{A}'} M$ (resp. $M \oslash X := M \otimes_{\mathcal{A}'} F(X)$). If K' was commutative in the previous example, we see that that example can be recovered in this manner starting from the ⊗-functor $F : K\text{-}\mathit{Mod} \to K'\text{-}\mathit{Mod}$ given by $X \mapsto X_{(K')} = K' \otimes_K M$.

3.1.6. (Algebras.) By definition, an *algebra* in an AU ⊗-category $\mathcal{A}$ is a triple $A = (A, \mu, \varepsilon)$, consisting of an object A of $\mathcal{A}$, a "multiplication" morphism $\mu : A \otimes A \to A$ and a "unit" morphism $\varepsilon : \mathbf{1} \to A$, subject to usual associativity and unit axioms: $\mu \circ (1_A \otimes \mu) = \mu \circ (\mu \otimes 1_A) : A \otimes A \otimes A \to A$ and $\mu \circ (1_A \otimes \varepsilon) = 1_A = \mu \circ (\varepsilon \otimes 1_A)$. Note that in the first of these axioms we have to identify $A \otimes (A \otimes A)$ with $(A \otimes A) \otimes A$ by means of the associativity constraint, and, similarly, we have implicitly used the unit constraint in the second axiom. So AU ⊗-categories are actually the minimal possible context sufficient to consider algebras. Note that we cannot impose any commutativity conditions on A without a commutativity constraint on $\mathcal{A}$.

An *algebra homomorphism* $f : (A, \mu_A, \varepsilon_A) \to (B, \mu_B, \varepsilon_B)$ *in* $\mathcal{A}$ is by definition a morphism $f : A \to B$, such that $\mu_B \circ (f \otimes f) = f \circ \mu_A$ and $f \circ \varepsilon_A = \varepsilon_B$. In this way we construct the *category of algebras in* $\mathcal{A}$, denoted by $\mathit{Alg}(\mathcal{A})$.

This category has an initial object, called *unit* or *initial algebra*, given by the natural algebra structure on $\mathbf{1}_{\mathcal{A}}$ coming from the canonical isomorphism $\mathbf{1}_{\mathcal{A}} \otimes \mathbf{1}_{\mathcal{A}} \cong \mathbf{1}_{\mathcal{A}}$. Indeed, the only algebra homomorphism from $\mathbf{1}_{\mathcal{A}}$ to an arbitrary algebra $A = (A, \mu, \varepsilon)$ is defined by $\varepsilon : \mathbf{1}_{\mathcal{A}} \to A$. If $\mathcal{A}$ has a final object $e_{\mathcal{A}}$, it admits a unique algebra structure, thus becoming the final object in $\mathit{Alg}(\mathcal{A})$.

3.1.7. Of course, this definition is motivated by the situation of **3.1.5**b), where we recover the notion of an associative K-algebra with unity. In other situations we obtain other interesting notions, e.g. in **3.1.5**a) we recover the category of monoids.

3.1.8. (Projective limits of algebras.) Note that all the projective limits which exist in $\mathcal{A}$, exist also in $Alg(\mathcal{A})$, and they can be computed in $\mathcal{A}$. Indeed, if we have to compute $\varprojlim(A_\alpha, \mu_\alpha, \varepsilon_\alpha)$, we compute first $A := \varprojlim A_\alpha$ in $\mathcal{A}$, denote the canonical projections by $\pi_\alpha : A \to A_\alpha$, and define $\mu : A \otimes A \to A$ and $\varepsilon : \mathbf{1}_{\mathcal{A}} \to A$ starting from compatible families of morphisms $\mu_\alpha(\pi_\alpha \otimes \pi_\alpha) : A \otimes A \to A_\alpha$ and $\varepsilon_\alpha : \mathbf{1}_{\mathcal{A}} \to A$. We see that the forgetful functor $Alg(\mathcal{A}) \to \mathcal{A}$ commutes with arbitrary projective limits, and in particular is left exact, if finite projective limits exist in $\mathcal{A}$. In this situation it preserves monomorphisms, and since in addition it is faithful, we see that $\varphi : A' \to A$ is a monomorphism in $Alg(\mathcal{A})$ iff φ is a monomorphism in $\mathcal{A}$.

3.1.9. (Subalgebras.) Given an algebra $A = (A, \mu, \varepsilon)$ in $Alg(\mathcal{A})$ and a monomorphism $i : A' \to A$, we see that there is at most one algebra structure (μ', ε') on A' compatible with i, since we must have $i\mu' = \mu(i \otimes i)$ and $i\varepsilon' = \varepsilon$. Note that whenever μ' and ε' satisfying these relations exist, they satisfy the associativity and unit axioms automatically, thus giving required algebra structure on A'. In particular, we can apply this to embeddings $A' \to A$ of subobjects $A' \subset A$. If a compatible algebra structure on A' exists, we call it the *induced structure*, and say that A' is a *subalgebra* of A. If finite projective limits exist in $\mathcal{A}$, then the subalgebras of a fixed algebra A are the same thing as the subobjects of A in $Alg(\mathcal{A})$.

3.1.10. (Modules over an algebra.) Suppose we are given a (left) external $\otimes$-action of $\mathcal{A}$ on $\mathcal{B}$, and an algebra $A = (A, \mu, \varepsilon)$ in $\mathcal{A}$. Then a *(left) A-module in $\mathcal{B}$* is by definition a pair $M = (M, \alpha)$, consisting of an object M of $\mathcal{B}$, and a left action α of A on M, i.e. a morphism $\alpha : A \oslash M \to M$, such that $\alpha \circ (\mu \oslash 1_M) = \alpha \circ (1_A \oslash \alpha)$ and $\alpha \circ (\varepsilon \oslash 1_M) = 1_M$. Note that we have again implicitly used here the external associativity and unity constraint for the $\otimes$-action of $\mathcal{A}$ on $\mathcal{B}$, so our context seems to be the most general context in which the notion of an A-module makes sense.

Of course, a *morphism of A-modules $f : (M, \alpha_M) \to (N, \alpha_N)$ in $\mathcal{B}$* is just a morphism $f : M \to N$ in $\mathcal{B}$, compatible with given A-actions, i.e. $f \circ \alpha_M = \alpha_N \circ (1_A \oslash f)$. In this way we obtain the category of A-modules in $\mathcal{B}$, denoted by $\mathcal{B}^A$.

If we have a *right* $\otimes$-action $\oslash : \mathcal{B} \times \mathcal{A} \to \mathcal{B}$ of $\mathcal{A}$ on $\mathcal{B}$, we can define *right* A-modules $M = (M, \alpha)$ in $\mathcal{B}$ in a similar way. In this case $\alpha : M \oslash A \to M$ is required to satisfy $\alpha \circ (1_M \oslash \mu) = \alpha \circ (\alpha \oslash 1_A)$ and $\alpha \circ (1_M \oslash \varepsilon) = 1_M$. This yields the category $\mathcal{B}_A$ of right A-modules in $\mathcal{B}$. Usually we will state and prove the statements only for left modules, leaving their counterparts for right modules to the reader.

In particular, we can always take the canonical left and right $\otimes$-action of $\mathcal{A}$ on itself. This leads to a definition of left and right A-modules in $\mathcal{A}$, which are usually called simply left and right A-modules. For example, the multiplication μ gives both a left and a right A-module structure on A itself.

3.1.11. Of course, this definition is motivated by the situation of **3.1.5**b) again, where we recover the usual notions of a left or right A-module over an associative K-algebra A. In the situation of **3.1.5**a) we recover the notion of a (left or right) action of a monoid on a set, and in **3.1.5**c) we obtain the notion of an $A \otimes_K K'$-module.

3.1.12. (Submodules and projective limits.) Similarly to what we had in **3.1.8**, all projective limits existing in $\mathcal{B}$ exist also in $\mathcal{B}^A$ and can be essentially computed in $\mathcal{B}$; if finite projective limits exist in $\mathcal{B}$, then $\varphi : N \to M$ is a monomorphism in $\mathcal{B}^A$ iff it is a monomorphism in $\mathcal{B}$. In this situation we have a notion of *submodules* M' of a module $M = (M, \alpha)$, i.e. subobjects $M' \subset M$ in $\mathcal{B}$, such that $\alpha(1_A \oslash i)$ factorizes into $i\alpha'$, thus yielding an A-module structure on M'; here $i : M' \to M$ is the canonical embedding. Again, when finite projective limits exist in $\mathcal{B}$, the set of submodules of M is actually the set of subobjects of M in $\mathcal{B}^A$.

3.1.13. (Scalar restriction.) We have a very natural notion of scalar restriction in this setup. Namely, if $f : A \to A'$ is an algebra homomorphism in $\mathcal{A}$, and we are given a (left) A'-module $N = (N, \alpha')$ in some category $\mathcal{B}$ on which $\mathcal{A}$ $\otimes$-acts, then we can define its *scalar restriction with respect to* f, denoted by f^*N, to be the same object N of $\mathcal{B}$ equipped with (left) A-action $\alpha := \alpha' \circ (f \oslash 1_N) : A \oslash N \to N$. As usual, this gives us a functor $f^* : \mathcal{B}^{A'} \to \mathcal{B}^A$.

3.1.14. (Scalar extension.) In some cases the scalar restriction functor has a left adjoint, called the *scalar extension* functor. In general it can be not so easy to construct it, but we'll consider now a very simple special case. Fix an algebra $A = (A, \mu, \varepsilon)$ and consider the scalar restriction $\varepsilon^* : \mathcal{B}^A \to \mathcal{B}^{\mathbf{1}_\mathcal{A}}$ with respect to the only homomorphism $\varepsilon : \mathbf{1}_\mathcal{A} \to A$ from the initial algebra $\mathbf{1}_\mathcal{A}$ into A. Clearly, $\mathcal{B}^{\mathbf{1}_\mathcal{A}}$ is canonically equivalent (even isomorphic) to $\mathcal{B}$, since any object of $\mathcal{B}$ admits exactly one $\mathbf{1}_\mathcal{A}$-module structure, and ε^* can be identified with the forgetful functor $\Gamma = \Gamma_A : \mathcal{B}^A \to \mathcal{B}$, $(M, \alpha) \mapsto M$.

Proposition. *For any algebra $A = (A, \mu, \varepsilon)$ in $\mathcal{A}$ and any $\oslash$-action of $\mathcal{A}$ on $\mathcal{B}$ the forgetful functor $\Gamma = \Gamma_A : \mathcal{B}^A \to \mathcal{B}$, $(M, \alpha) \mapsto M$, has a left adjoint $L = L_A : \mathcal{B} \to \mathcal{B}^A$, given by $X \mapsto (A \oslash X, \mu \oslash 1_X)$. The functorial isomorphism $\mathrm{Hom}_{\mathcal{B}^A}(L(X), M) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{B}}(X, \Gamma(M))$ maps an A-module homomorphism $\varphi : A \oslash X \to M$ into $\varphi^\flat : X \to M$ given by $\varphi^\flat := \varphi \circ (\varepsilon \oslash 1_X)$, and* *its inverse maps a morphism $\psi : X \to M$ into $\psi^\sharp := \alpha \circ (1_A \oslash \psi)$, where $\alpha : A \oslash M \to M$ is the multiplication of M. Moreover, the adjointness natural transformations $\xi : \mathrm{Id}_\mathcal{B} \to \Gamma L$ and $\eta : L\Gamma \to \mathrm{Id}_{\mathcal{B}^A}$ can be described explicitly as follows: $\xi_X = \varepsilon \oslash 1_X : X \to A \oslash X$ for any $X \in \mathrm{Ob}\,\mathcal{B}$, and $\eta_{(M,\alpha)} = \alpha : (A \oslash M, \mu \oslash 1_M) \to (M, \alpha)$.*

Proof. We have to check that $\psi^\sharp = \alpha \circ (1_A \oslash \psi)$ is indeed an A-module homomorphism $(A \oslash X, \mu \oslash 1_X) \to (M, \alpha)$ for any $\psi : X \to M$. This means $\psi^\sharp \circ (\mu \oslash 1_X) = \alpha \circ (1_A \oslash \psi^\sharp)$, i.e. $\alpha \circ (1_A \oslash \psi) \circ (\mu \oslash 1_X) = \alpha \circ (1_A \oslash \alpha) \circ (1_{A\otimes A} \oslash \psi)$. This is clear since $\alpha \circ (1_A \oslash \alpha) = \alpha \circ (\mu \oslash 1_M)$ and $(1_A \oslash \psi) \circ (\mu \oslash 1_X) = \mu \oslash \psi = (\mu \oslash 1_M) \circ (1_{A\otimes A} \oslash \psi)$. Next, we have to check that the maps $\varphi \mapsto \varphi^\flat$ and $\psi \mapsto \psi^\sharp$ are inverse to each other. This is also quite straightforward: $(\varphi^\flat)^\sharp = \alpha \circ (1_A \oslash \varphi^\flat) = \alpha \circ (1_A \oslash \varphi) \circ (1_A \otimes \varepsilon \oslash 1_X) = \varphi \circ (\mu \oslash 1_X) \circ (1_A \otimes \varepsilon \oslash 1_X) = \varphi \circ \bigl((\mu \circ (1_A \otimes \varepsilon)) \oslash 1_X\bigr) = \varphi$ and $(\psi^\sharp)^\flat = \psi^\sharp \circ (\varepsilon \oslash 1_X) = \alpha \circ (1_A \oslash \psi) \circ (\varepsilon \oslash 1_X) = \alpha \circ (\varepsilon \oslash \psi) = \alpha \circ (\varepsilon \oslash 1_M) \circ \psi = \psi$. This completes the proof of adjointness of L and Γ. The formulas for ξ and η follow immediately from $\xi_X = (1_{L(X)})^\flat$ and $\eta_{(M,\alpha)} = (1_M)^\sharp$.

3.1.15. (Application of functors.) Given an $\otimes$-functor $F : \mathcal{A} \to \mathcal{A}'$, we can apply it to any algebra $A = (A, \mu, \varepsilon)$ in $\mathcal{A}$, obtaining an algebra $F(A) = \bigl(F(A), F(\mu), F(\varepsilon)\bigr)$, where we identify for simplicity $F(A \otimes A)$ with $F(A) \otimes F(A)$, and $F(\mathbf{1}_\mathcal{A})$ with $\mathbf{1}_{\mathcal{A}'}$. Clearly, F transforms algebra homomorphisms into algebra homomorphisms, hence it induces a functor $\mathit{Alg}(F)$ from the category $\mathit{Alg}(\mathcal{A})$ of algebras in $\mathcal{A}$ into the category $\mathit{Alg}(\mathcal{A}')$ of algebras in $\mathcal{A}'$; this functor will be usually denoted by the same letter F.

Similarly, if we have an $\otimes$-action of $\mathcal{A}$ on $\mathcal{B}$ and of $\mathcal{A}'$ on $\mathcal{B}'$, and an external $\otimes$-functor $G : \mathcal{B} \to \mathcal{B}'$ compatible with F, then we can apply G to any A-module $M = (M, \alpha)$ in $\mathcal{B}$, thus obtaining an $F(A)$-module $G(M) = \bigl(G(M), G(\alpha)\bigr)$ in $\mathcal{B}'$. It is clear again that this gives us actually a functor $\tilde{G}$ (or simply G) from the category $\mathcal{B}^A$ of A-modules in $\mathcal{B}$ into the category $\mathcal{B}'^{F(A)}$ of $F(A)$-modules in $\mathcal{B}'$.

Furthermore, any natural transformation of $\oslash$-functors $\zeta : G \to G'$ induces an $F(A)$-module homomorphism $\zeta_M : G(M) \to G'(M)$ in $\mathcal{B}'$ for any A-module M in $\mathcal{B}$, thus effectively defining a natural transformation $\tilde{\zeta} : \tilde{G} \to \tilde{G}'$ of functors $\mathcal{B}^A \to \mathcal{B}'^{F(A)}$.

3.2. (Categories of functors.) Recall that the category of all categories Cat is actually a (strictly associative) 2-category. In other words, the set of functors $\mathit{Funct}(\mathcal{C}, \mathcal{D}) = \mathrm{Hom}_{\mathit{Cat}}(\mathcal{C}, \mathcal{D})$ between two arbitrary categories $\mathcal{C}$ and $\mathcal{D}$ is not just a set, but a category, since we have the notion of a natural transformation between two functors from $\mathcal{C}$ to $\mathcal{D}$, and the composition map

$\circ : \mathit{Funct}(\mathcal{D},\mathcal{E}) \times \mathit{Funct}(\mathcal{C},\mathcal{D}) \to \mathit{Funct}(\mathcal{C},\mathcal{E})$ is not just a map of sets, but a functor.

This means that, apart from the usual composition of natural transformations $\xi' \circ \xi = \xi'\xi : F \to F''$, defined for any natural transformations $\xi' : F' \to F''$, $\xi : F \to F'$ between functors $F, F', F'' : \mathcal{C} \to \mathcal{D}$, we have also the notion of $\star$-*composition* $\eta \star \xi : GF \to G'F'$ for any natural transformations $\xi : F \to F' : \mathcal{C} \to \mathcal{D}$ and $\eta : G \to G' : \mathcal{D} \to \mathcal{E}$. The requirement for $\circ$ to be a functor means that $\mathrm{id}_G \star \mathrm{id}_F = \mathrm{id}_{GF}$ and $(\eta' \star \xi')(\eta \star \xi) = (\eta'\eta) \star (\xi'\xi)$.

We adopt the usual convention and denote $\eta \star \mathrm{id}_F$ simply by $\eta \star F$, and $\mathrm{id}_G \star \xi$ by $G \star \xi$. Hence we have $\eta \star \xi = (\eta \star F')(G \star \xi) = (G' \star \xi)(\eta \star F)$.

3.2.1. Let's make these operations with natural transformations of functors more explicit. First of all, a natural transformation $\xi : F \to G$ between two functors from $\mathcal{C}$ to $\mathcal{D}$ is by definition a collection of morphisms in $\mathcal{D}$ $(\xi_X)_{X \in \mathrm{Ob}\,\mathcal{C}}$, indexed by objects of $\mathcal{C}$, where $\xi_X : F(X) \to G(X)$ are required to satisfy $\xi_Y \circ F(\varphi) = G(\varphi) \circ \xi_X$ for any morphism $\varphi : X \to Y$ in $\mathcal{C}$. Therefore, we would like to have explicit descriptions of our operations with natural transformations in terms of such families.

a) First of all, the usual composition $\xi'\xi = \xi' \circ \xi : F \to F''$ of two natural transformations $\xi : F \to F'$ and $\xi' : F' \to F''$ of functors from $\mathcal{C}$ to $\mathcal{D}$ is computed in the obvious way: $(\xi'\xi)_X := \xi'_X \circ \xi_X : F(X) \to F''(X)$.

b) Now, given a functor $G : \mathcal{D} \to \mathcal{E}$, we would like to obtain a natural transformation $G \star \xi = \mathrm{id}_G \star \xi : GF \to GF'$. This is done in a very natural way: $(G \star \xi)_X := G(\xi_X) : GF(X) \to GF'(X)$.

c) Similarly, given a natural transformation $\eta : G \to G'$, we have to define $\eta \star F = \eta \star \mathrm{id}_F : GF \to G'F$. Again, the most natural way of doing this is correct: $(\eta \star F)_X := \eta_{F(X)} : GF(X) \to G'F(X)$.

d) Now we combine these two particular cases into a general formula for $\eta \star \xi : GF \to G'F'$ for any two $\xi : F \to F'$ and $\eta : G \to G'$. We must have $\eta \star \xi = (\eta \star F')(G \star \xi) = (G' \star \xi)(\eta \star F)$, hence $(\eta \star \xi)_X = \eta_{F'(X)} \circ G(\xi_X) = G'(\xi_X) \circ \eta_{F(X)}$:

$$\begin{tikzcd} GF(X) \arrow[r, "G(\xi_X)"] \arrow[d, "\eta_{F(X)}"] \arrow[dr, "(\eta\star\xi)_X"] & GF'(X) \arrow[d, "\eta_{F'(X)}"] \\ G'F(X) \arrow[r, "G'(\xi_X)"] & G'F'(X) \end{tikzcd} \tag{3.2.1.1}$$

These two descriptions of $(\eta\star\xi)_X$ coincide, i.e. the above square is commutative, simply because of compatibility of $\eta_{F(X)}$ and $\eta_{F'(X)}$ with the morphism $\xi_X : F(X) \to F'(X)$ in $\mathcal{D}$.

e) Once the equivalence of these two descriptions of $\eta \star \xi$ is checked, we deduce immediately the formula $(\eta' \star \xi')(\eta \star \xi) = \eta'\eta \star \xi'\xi$, which actually means that *Cat* is a (strictly associative) 2-category.

3.2.2. Now we can exploit this 2-category structure on *Cat* to obtain some important examples of AU $\otimes$-categories and their external $\otimes$-actions. First of all, consider the category of endofunctors $\mathit{Endof}(\mathcal{C}) := \mathit{Funct}(\mathcal{C},\mathcal{C})$ on any category $\mathcal{C}$. Let's take the composition functor $\circ : \mathit{Endof}(\mathcal{C}) \times \mathit{Endof}(\mathcal{C}) \to \mathit{Endof}(\mathcal{C})$ as the tensor product on $\mathit{Endof}(\mathcal{C})$. In other words, we put $G \otimes F := GF$ for any two endofunctors G and F on $\mathcal{C}$, and for any two natural transformations $\eta : G \to G'$ and $\xi : F \to F'$ of such endofunctors we put $\eta \otimes \xi := \eta \star \xi : GF \to G'F'$. Of course, the identity functor $\mathrm{Id}_{\mathcal{C}}$ of $\mathcal{C}$ is the unity for this $\otimes$-structure.

In this way we obtain an AU $\otimes$-structure on $\mathit{Endof}(\mathcal{C})$. Clearly, it is even strictly associative (i.e. we have $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$, and the associativity constraint is identity), since *Cat* is strictly associative (i.e. $(XY)Z = X(YZ)$).

3.2.3. Moreover, for any other category $\mathcal{D}$ we have a left $\otimes$-action of $\mathcal{A} := \mathit{Endof}(\mathcal{C})$ on $\mathit{Funct}(\mathcal{D},\mathcal{C})$, and a right $\otimes$-action of $\mathcal{A}$ on $\mathit{Funct}(\mathcal{C},\mathcal{D})$, given in both cases by the composition functor $\circ : \mathit{Endof}(\mathcal{C}) \times \mathit{Funct}(\mathcal{D},\mathcal{C}) \to \mathit{Funct}(\mathcal{D},\mathcal{C})$ (resp. $\circ : \mathit{Funct}(\mathcal{C},\mathcal{D}) \times \mathit{Endof}(\mathcal{C}) \to \mathit{Funct}(\mathcal{C},\mathcal{D})$). Therefore, in both cases we have $F \oslash H = FH$ (resp. $H \oslash F = HF$) for any $F \in \operatorname{Ob} \mathit{Endof}(\mathcal{C})$ and $H : \mathcal{D} \to \mathcal{C}$ (resp. $H : \mathcal{C} \to \mathcal{D}$), and the action on natural transformations is given by the $\star$-product.

3.2.4. Any functor $H : \mathcal{D} \to \mathcal{D}'$ defines functors $H_* : \mathit{Funct}(\mathcal{C},\mathcal{D}) \to \mathit{Funct}(\mathcal{C},\mathcal{D}')$ and $H^* : \mathit{Funct}(\mathcal{D}',\mathcal{C}) \to \mathit{Funct}(\mathcal{D},\mathcal{C})$, given again by the composition functors in *Cat*, i.e. $H_* : G \mapsto HG, \eta \mapsto H \star \eta$ and $H^* : G \mapsto GH, \eta \mapsto \eta \star H$. It is immediate that H_* (resp. H^*) is compatible with right (resp. left) external $\otimes$-action of $\mathcal{A} = \mathit{Endof}(\mathcal{C})$ on corresponding categories, i.e. H_* is an $\oslash$-functor, and H^* is an $\oslash$-functor.

Moreover, any natural transformation $\zeta : H \to H'$ induces a natural transformation of $\oslash$-functors $\zeta^* : H^* \to H'^*$, given by $(\zeta^*)_G := G \star \zeta : GH \to GH'$, and a natural transformation of $\oslash$-functors $\zeta_* : H_* \to H'_*$, given by $(\zeta_*)_G := \zeta \star G : HG \to H'G$.

3.2.5. So far this reasoning was valid in any 2-category. Let's do something more specific now. Put $\mathcal{D} := \langle * \rangle$ — the point (or final) category, consisting of exactly one object $*$ and only one morphism — the identity of $*$. Then we get a right external $\otimes$-action of $\mathcal{A} = \mathit{Endof}(\mathcal{C})$ on $\mathit{Funct}(\mathcal{C}, \langle * \rangle)$, which is not really interesting since this category is equivalent (even isomorphic) to $\langle * \rangle$, and a left $\otimes$-action of $\mathcal{A}$ on $\mathit{Funct}(\langle * \rangle, \mathcal{C})$. This is more interesting, since $\mathit{Funct}(\langle * \rangle, \mathcal{C})$ is canonically isomorphic to $\mathcal{C}$ itself.

Therefore, we get an external $\otimes$-action of $\mathcal{A} = \mathit{Endof}(\mathcal{C})$ on $\mathcal{C}$, $\oslash$:

$Endof(\mathcal{C}) \times \mathcal{C} \to \mathcal{C}$, which is easily seen to be the evaluation map: $F \oslash X = F(X)$ for any endofunctor F and any object X of $\mathcal{C}$, and $\xi \oslash f = \xi_Y \circ F(f) = F'(f) \circ \xi_X : F(X) \to F'(Y)$ for any natural transformation $\xi : F \to F'$ and any morphism $f : X \to Y$ in $\mathcal{C}$. In particular, $\xi \oslash X = \xi_X$ and $F \oslash f = F(f)$.

The situation of **3.2.4** also admits a similar special case, if we put $\mathcal{D} := \langle * \rangle$, and rename $\mathcal{D}'$ to $\mathcal{D}$. We see that for any object $Z \in \operatorname{Ob} \mathcal{D}$ the corresponding "evaluation functor" $Z^* : \mathit{Funct}(\mathcal{D}, \mathcal{C}) \to \mathcal{C}$, $Z^* : G \mapsto G(Z), \eta \mapsto \eta_Z$ is an $\oslash$-functor with respect to the external $\otimes$-action of $\mathcal{A}$ on $\mathit{Funct}(\mathcal{D}, \mathcal{C})$ and on $\mathcal{C}$. In this case any morphism $\zeta : Z \to Z'$ in $\mathcal{D}$ gives rise to a natural $\oslash$-transformation $\zeta^* : Z^* \to Z'^*$ of corresponding evaluation functors, given by $\zeta^*_G := G(\zeta)$.

3.2.6. (Set-theoretical issues.) Strictly speaking, the category of all categories is no more legal mathematical object than the set of all sets. Therefore, we encounter some set-theoretical complications, which really do not affect anything important for us. However, we would like to explain here how we are going to avoid them, without paying too much attention to similar issues in future.

To do this we work in Tarski–Grothendieck set theory, i.e. we accept the usual Zermelo–Frenkel axiomatics for set theory and mathematics (recall that all mathematical objects are sets in this picture), and we accept Grothendieck's universe axiom as well, which tells us that any set can be embedded into an *universe*, i.e. a set $\mathcal{U}$ closed under all usual set-theoretical operations with its elements (e.g. the union of a family of sets $\in \mathcal{U}$ indexed by a set $\in \mathcal{U}$ also belongs to $\mathcal{U}$). Usually we fix such an universe from the very beginning, choosing it as to contain all sets of interest to us (e.g. the set of integers $\mathbb{Z}$). Then all usual constructions produce sets $\in \mathcal{U}$, and we restrict ourselves to working with such sets (cf. appendix to SGA 4 I).

A set is said to be $\mathcal{U}$*-small* (or just *small*, when $\mathcal{U}$ is fixed) if it belongs to $\mathcal{U}$, or in some cases if we can establish a bijection between this set and an element of $\mathcal{U}$. A category is said to be $\mathcal{U}$-small, or just small when $\mathcal{U}$ is fixed, if it belongs to $\mathcal{U}$, i.e. if both its set of objects and its set of morphisms are $\mathcal{U}$-small. There is also a notion of a $\mathcal{U}$-category $\mathcal{C}$, where the only requirement is for all individual Hom-sets $\operatorname{Hom}_{\mathcal{C}}(x, y)$ to be $\mathcal{U}$-small for the less restrictive understanding of $\mathcal{U}$-smallness (cf. SGA 4 I for more details).

When we consider the categories *Sets*, *Ab*, ... of, say, sets, abelian groups and so on, we actually consider the categories $\mathcal{U}$-*Sets*, $\mathcal{U}$-*Ab* ... of $\mathcal{U}$-small sets, $\mathcal{U}$-small abelian groups and so on (i.e. we require these sets, abelian groups, ... to belong to $\mathcal{U}$). In this way we obtain $\mathcal{U}$-categories, which, however, are not $\mathcal{U}$-small.

Now, if we consider the category $\mathcal{U}$-*Cat* of $\mathcal{U}$-small categories, this perfectly makes sense, and all previous constructions can be applied. For example, for any two $\mathcal{U}$-small categories $\mathcal{C}$ and $\mathcal{D}$ the category of functors *Funct*$(\mathcal{C}, \mathcal{D})$ is also $\mathcal{U}$-small.

However, this approach is too restrictive for our purposes, since we cannot even consider the category of endofunctors on the category of sets. Therefore, we choose a larger universe $\mathcal{V} \ni \mathcal{U}$, and consider the category *Cat* $:= \mathcal{V}$-$\mathcal{U}$-*Cat* of $\mathcal{V}$-small $\mathcal{U}$-categories. Then all previous constructions begin to make sense; however, observe that even in this case *Funct*$(\mathcal{C}, \mathcal{D})$ usually will not be an $\mathcal{U}$-category, but only a $\mathcal{V}$-category. For example, *Endof*(*Sets*) is not an $\mathcal{U}$-category.

We'll deal with this minor problem later by restricting our attention to certain full subcategories of *Funct*$(\mathcal{C}, \mathcal{D})$, which will be actually $\mathcal{U}$-categories. Then the output of our constructions won't involve the larger universe $\mathcal{V}$, and won't depend on its choice, even if it is required at some intermediate stages. One example of such full $\mathcal{U}$-subcategory of a category of functors is given by the subcategory of *algebraic* endofunctors in *Endof*(*Sets*), which will be studied in the next chapter.

3.3. (Monads.) Now we combine the AU $\otimes$-structure on the categories of endofunctors *Endof*$(\mathcal{C})$ studied in **3.2** with the definition **3.1.6** of algebras in $\otimes$-categories and of modules over such algebras. Of course, we thus obtain just the definition of monads. Our choice of this approach is motivated by the theory of inner endofunctors and inner monads to be developed later. The statements for this second case will be almost the same, but direct proofs tend to be more lengthy and slightly more complicated.

Definition 3.3.1 *A* monad Σ on *(or* over*)* some category $\mathcal{C}$*) is an algebra (cf.* **3.1.6***) in the category Endof*$(\mathcal{C})$ *with respect to its AU $\otimes$-structure defined in* **3.2.2**. *In other words, a monad Σ over $\mathcal{C}$ is a triple $\Sigma = (\Sigma, \mu, \varepsilon)$ consisting of an endofunctor $\Sigma : \mathcal{C} \to \mathcal{C}$ and two natural transformations: the* multiplication $\mu : \Sigma^2 = \Sigma\Sigma \to \Sigma$ *and the* unit $\varepsilon : \mathrm{Id}_{\mathcal{C}} \to \Sigma$, *required to satisfy the associativity axiom $\mu \circ (\Sigma \star \mu) = \mu \circ (\mu \star \Sigma) : \Sigma^3 \to \Sigma$ and the unit axiom $\mu \circ (\Sigma \star \varepsilon) = \mathrm{id}_\Sigma = \mu \circ (\varepsilon \star \Sigma)$.*

A morphism of monads $\varphi : \Sigma \to \Xi$ *is simply a morphism of algebras in Endof*$(\mathcal{C})$*, i.e. a natural transformation of underlying endofunctors $\varphi : \Sigma \to \Xi$, such that $\varphi \circ \varepsilon_\Sigma = \varepsilon_\Xi$ and $\varphi \circ \mu_\Sigma = \mu_\Xi \circ (\varphi \star \varphi)$. The category of all monads over $\mathcal{C}$ will be denoted by Monads*$(\mathcal{C})$*. Hence Monads*$(\mathcal{C}) =$ *Alg*(*Endof*$(\mathcal{C})$).

3.3.2. The definition of a monad $\Sigma = (\Sigma, \mu, \varepsilon)$ over some category $\mathcal{C}$ can be made even more explicit in terms of individual components of natural transformations $\mu : \Sigma^2 \to \Sigma$ and $\varepsilon : \mathrm{Id}_{\mathcal{C}} \to \Sigma$, i.e. morphisms $\mu_X : \Sigma^2(X) \to \Sigma(X)$ and $\varepsilon_X : X \to \Sigma(X)$ parametrized by $X \in \mathrm{Ob}\,\mathcal{C}$. The requirement for

μ and ε to be natural transformations translates into $\Sigma(f)\circ\mu_X = \mu_Y\circ\Sigma^2(f)$ and $\Sigma(f)\circ\varepsilon_X = \varepsilon_Y\circ f$ for any morphism $f : X \to Y$ in $\mathcal{C}$, and our axioms translate into $\mu_X\circ\Sigma(\mu_X) = \mu_X\circ\mu_{\Sigma(X)} : \Sigma^3(X) \to \Sigma(X)$ and $\mu_X\circ\Sigma(\varepsilon_X) = \mathrm{id}_{\Sigma(X)} = \mu_X\circ\varepsilon_{\Sigma(X)}$ for all $X \in \operatorname{Ob}\mathcal{C}$.

Similarly, a morphism of monads $\varphi : \Sigma \to \Xi$ is simply a collection of morphisms $\varphi_X : \Sigma(X) \to \Xi(X)$, parametrized by $X \in \operatorname{Ob}\mathcal{C}$, such that $\varphi_Y\circ\Sigma(f) = \Xi(f)\circ\varphi_X$ for any morphism $f : X \to Y$ in $\mathcal{C}$, and $\varphi_X\circ\varepsilon_{\Sigma,X} = \varepsilon_{\Xi,X}$ and $\varphi_X\circ\mu_{\Sigma,X} = \mu_{\Xi,X}\circ\Xi(\varphi_X)\circ\varphi_{\Sigma(X)} = \mu_{\Xi,X}\circ\varphi_{\Xi(X)}\circ\Sigma(\varphi_X)$ for all $X \in \operatorname{Ob}\mathcal{C}$. Here, of course, $\Xi(\varphi_X)\circ\varphi_{\Sigma(X)} = (\varphi\star\varphi)_X = \varphi_{\Xi(X)}\circ\Sigma(\varphi_X)$, so only one equality has to be checked in the last axiom.

3.3.3. (Submonads and projective limits of monads.) Note that all projective limits that exist in $\mathcal{C}$, exist also in $\mathcal{A} = \mathit{Endof}(\mathcal{C})$, since they can be computed componentwise: $(\varprojlim F_\alpha)(X) := \varprojlim F_\alpha(X)$. We say that F' is a *subfunctor* of F, if $F'(X) \subset F(X)$ for all $X \in \operatorname{Ob}\mathcal{C}$. When $\mathcal{C}$ has finite projective limits, the subfunctors of some endofunctor F are actually its subobjects in $\mathcal{A}$, and we are in position to apply **3.1.8** and **3.1.9** to $\mathit{Monads}(\mathcal{C}) = \mathit{Alg}(\mathcal{A})$.

We see that all projective limits existing in $\mathcal{C}$ exist also in $\mathit{Monads}(\mathcal{C})$, and they can be computed componentwise. Moreover, we obtain the notion of a *submonad* Σ' of a monad Σ: by definition it is a subfunctor $\Sigma' \subset \Sigma$ stable under multiplication and unit of Σ, or equivalently a monad Σ', such that $\Sigma'(X) \subset \Sigma(X)$ for all $X \in \operatorname{Ob}\mathcal{C}$, and $\Sigma' \to \Sigma$ is a morphism of monads.

3.3.4. Once we have the definition of a monad Σ over $\mathcal{C}$ as an algebra in AU $\otimes$-category $\mathcal{A} := \mathit{Endof}(\mathcal{C})$, we can use the left (resp. right) $\otimes$-action of this category on $\mathit{Funct}(\mathcal{D},\mathcal{C})$ and $\mathcal{C}$ (resp. on $\mathit{Funct}(\mathcal{C},\mathcal{D})$) explained in **3.2.3** and **3.2.5** to define the categories of left (resp. right) Σ-modules in $\mathit{Funct}(\mathcal{D},\mathcal{C})$ and $\mathcal{C}$ (resp. in $\mathit{Funct}(\mathcal{C},\mathcal{D})$) according to **3.1.10**; the corresponding categories of modules are denoted by $\mathit{Funct}(\mathcal{D},\mathcal{C})^\Sigma$ and $\mathcal{C}^\Sigma$ (resp. $\mathit{Funct}(\mathcal{C},\mathcal{D})_\Sigma$), according to conventions of *loc.cit.* In all of these situations we have a faithful forgetful functor Γ, acting from corresponding category of modules into the underlying category.

3.3.5. (Category of Σ-modules.) Let's make these definitions more explicit, starting from the category $\mathcal{C}^\Sigma$ of (left) Σ-modules in $\mathcal{C}$. Its objects are pairs $M = (M,\alpha)$, consisting of an object M of $\mathcal{C}$, and a Σ-*structure* or Σ-*action* α on M, i.e. a morphism $\alpha : \Sigma \oslash M = \Sigma(M) \to M$, such that $\alpha\circ\mu_M = \alpha\circ\Sigma(\alpha) : \Sigma^2(M) \to M$ and $\alpha\circ\varepsilon_M = \mathrm{id}_M$. Objects of $\mathcal{C}^\Sigma$, i.e. these pairs $M = (M,\alpha) = (M,\alpha_M)$ are called Σ-*modules* or Σ-*objects*; if $\mathcal{C} = \mathit{Sets}$, we also call them Σ-*sets*. Another classical terminology is "Σ-algebras"; we avoid it since in our setup these are modules over algebra Σ, not algebras.

Morphisms $f : (M,\alpha_M) \to (N,\alpha_N)$ in $\mathcal{C}^\Sigma$ are simply those morphisms $f : M \to N$ in $\mathcal{C}$, which agree with given Σ-structures, i.e. such that $f\circ$

$\alpha_M = \alpha_N \circ \Sigma(f)$. When no confusion can arise, the subset $\mathrm{Hom}_{\mathcal{C}^\Sigma}(M,N) \subset \mathrm{Hom}_{\mathcal{C}}(M,N)$ is denoted simply by $\mathrm{Hom}_\Sigma(M,N)$.

3.3.6. Let's consider the category $\mathit{Funct}(\mathcal{D},\mathcal{C})^\Sigma$ now. First of all, any functor $H : \mathcal{D}' \to \mathcal{D}$ induces an $\oslash$-functor $H^* : \mathit{Funct}(\mathcal{D},\mathcal{C}) \to \mathit{Funct}(\mathcal{D}',\mathcal{C})$, hence a functor $H^* : \mathit{Funct}(\mathcal{D},\mathcal{C})^\Sigma \to \mathit{Funct}(\mathcal{D}',\mathcal{C})^\Sigma$, and natural transformations $\zeta : H \to H'$ induce natural transformations $\zeta^* : H^* \to H'^*$ (cf. **3.2.4** and **3.1.15**).

Similarly, the evaluation functor $Z^* : \mathit{Funct}(\mathcal{D},\mathcal{C}) \to \mathcal{C}$, $M \mapsto M(Z)$, defined by any $Z \in \mathrm{Ob}\,\mathcal{D}$, is an $\oslash$-functor (cf. **3.2.5**), hence it induces a functor $Z^* : \mathit{Funct}(\mathcal{D},\mathcal{C})^\Sigma \to \mathcal{C}^\Sigma$, and morphisms $\zeta : Z \to Z'$ induce natural transformations $\zeta^* : Z^* \to Z'^*$ of these functors.

In other words, for any $M = (M,\alpha) \in \mathrm{Ob}\,\mathit{Funct}(\mathcal{D},\mathcal{C})^\Sigma$, where $M : \mathcal{D} \to \mathcal{C}$ is a functor and $\alpha : M \to \Sigma M$ is a natural transformation, such that $\alpha \circ (\mu \star M) = \alpha \circ (\Sigma \star \alpha) : \Sigma^2 M \to M$, we get an object Z^*M of $\mathcal{C}^\Sigma$, easily seen to be $M(Z) = \big(M(Z), \alpha_Z\big)$, and any $\zeta : Z \to Z'$ induces a Σ-morphism $\zeta^* M = M(\zeta) : M(Z) \to M(Z')$.

In this way we obtain a functor $\tilde{M} : \mathcal{D} \to \mathcal{C}^\Sigma$, which is usually denoted with the same letter M, since it coincides with M on the level of the underlying objects; applying this for different Σ-modules M and their homomorphisms, we construct a functor $\mathit{Funct}(\mathcal{D},\mathcal{C})^\Sigma \to \mathit{Funct}(\mathcal{D},\mathcal{C}^\Sigma)$, which is now easily seen to be an equivalence, and even an isomorphism of categories. One can say that $\mathcal{C}^\Sigma$ "represents" the functor $\mathit{Funct}(-,\mathcal{C})^\Sigma$ on the category of all categories. Let's state this result separately:

Proposition 3.3.7 *For any monad Σ over a category $\mathcal{C}$ and any category $\mathcal{D}$ we have a canonical* isomorphism *of categories $\mathit{Funct}(\mathcal{D},\mathcal{C})^\Sigma \to \mathit{Funct}(\mathcal{D},\mathcal{C}^\Sigma)$ which maps $(M, \alpha : M \to \Sigma M)$ into the functor $Z \mapsto (M(Z), \alpha_Z)$. This isomorphism of categories is compatible with corresponding forgetful functors, i.e. it identifies $\mathit{Funct}(\mathcal{D},\mathcal{C})^\Sigma \to \mathit{Funct}(\mathcal{D},\mathcal{C})$ with $(\Gamma_\Sigma)_* : \mathit{Funct}(\mathcal{D},\mathcal{C}^\Sigma) \to \mathit{Funct}(\mathcal{D},\mathcal{C})$, where $\Gamma_\Sigma : \mathcal{C}^\Sigma \to \mathcal{C}$ is the forgetful functor for $\mathcal{C}^\Sigma$. Moreover, the functor $\mathit{Funct}(\mathcal{D},\mathcal{C})^\Sigma \to \mathit{Funct}(\mathcal{D}',\mathcal{C})^\Sigma$ induced by any functor $H : \mathcal{D}' \to \mathcal{D}$ is identified with $H_* : \mathit{Funct}(\mathcal{D},\mathcal{C}^\Sigma) \to \mathit{Funct}(\mathcal{D}',\mathcal{C}^\Sigma)$, and this statement extends to natural transformations arising from any $\zeta : H \to H'$. A similar statement is also true for evaluation functors $Z^* : \mathit{Funct}(\mathcal{D},\mathcal{C})^\Sigma \to \mathcal{C}^\Sigma$ arising from objects Z of $\mathcal{D}$.*

Proof. The functor mentioned in the statement has been constructed before. It remains to check that it is an isomorphism of categories, but this is quite straightforward, once we understand that this statement actually means that to give a left Σ-module structure on a functor $F : \mathcal{D} \to \mathcal{C}$ is the same thing as to give a left Σ-structure on each individual $F(X)$, depending functorially on $X \in \mathrm{Ob}\,\mathcal{D}$. Another possibility — construct the inverse functor directly, by first introducing a left Σ-functor structure on $\Gamma : \mathcal{C}^\Sigma \to \mathcal{C}$, and then applying $\tilde{F}^*$ to it for any $\tilde{F} : \mathcal{D} \to \mathcal{C}^\Sigma$, thus obtaining a left Σ-functor structure on $\Gamma\tilde{F} : \mathcal{D} \to \mathcal{C}$.

3.3.8. In particular, we see that the category $\mathcal{A}^\Sigma$ of left Σ-modules in $\mathcal{A} := \mathit{Endof}(\mathcal{C})$ is isomorphic to $\mathit{Funct}(\mathcal{C}, \mathcal{C}^\Sigma)$. Since the multiplication μ defines a left Σ-action on Σ itself, we obtain a functor $L = L_\Sigma : \mathcal{C} \to \mathcal{C}^\Sigma$, such that $L(X) = \big(\Sigma(X), \mu_X\big)$ for any object X in $\mathcal{C}$, and $L(f)$ is given by $\Sigma(f)$ for any morphism $f : X \to Y$ in $\mathcal{C}$. Moreover, $L(X)$ can be described as $(\Sigma \oslash X, \mu \oslash \mathrm{id}_X)$ in terms of the $\oslash$-action of $\mathcal{A}$ on $\mathcal{C}$, so we are in position to apply **3.1.14**:

Proposition. *The forgetful functor $\Gamma = \Gamma_\Sigma : \mathcal{C}^\Sigma \to \mathcal{C}$, $(M, \alpha) \mapsto M$, has a left adjoint $L = L_\Sigma : \mathcal{C} \to \mathcal{C}^\Sigma$, given by $X \mapsto \big(\Sigma(X), \mu_X\big)$. The adjointness isomorphism $\mathrm{Hom}_{\mathcal{C}^\Sigma}\big(L(X), (M, \alpha)\big) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, M)$ transforms a Σ-morphism $\varphi : L(X) = \Sigma(X) \to M$ into $\varphi^\flat := \varphi \circ \varepsilon_X$, and its inverse transforms $\psi : X \to \Gamma(M) = M$ into $\psi^\sharp := \alpha \circ \Sigma(\psi)$.*

Moreover, $\Gamma L = \Sigma$, and the natural transformations $\xi : \mathrm{Id}_{\mathcal{C}} \to \Gamma L = \Sigma$ and $\eta : L\Gamma \to \mathrm{Id}_{\mathcal{C}^\Sigma}$ defined by adjointness of L and Γ are given by $\xi = \varepsilon$ and $\eta_{(M,\alpha)} = \alpha$.

Finally, the pair of adjoint functors L and Γ completely determines the monad $\Sigma = (\Sigma, \mu, \varepsilon)$ since $\Sigma = \Gamma L$, $\varepsilon = \xi$ and $\mu = \Gamma \star \eta \star L : \Gamma L \Gamma L = \Sigma^2 \to \Gamma L = \Sigma$.

Proof. All statements but the last one follow immediately from **3.1.14**. Let's check the formula for μ: $(\Gamma \star \eta \star L)_X = \Gamma(\eta_{L(X)}) = \Gamma(\eta_{(\Sigma(X), \mu_X)}) = \Gamma(\mu_X) = \mu_X$.

3.3.9. Since L_Σ is a left adjoint to the forgetful functor Γ_Σ, the Σ-object $L_\Sigma(X) = (\Sigma(X), \mu_X)$ will be called *the free Σ-object (or Σ-module) generated by X*, and Σ-objects isomorphic to some $L_\Sigma(X)$ will be called *free.* Since the underlying object of $L_\Sigma(X)$ is equal to $\Sigma(X)$, this Σ-object will be often denoted simply by $\Sigma(X)$, when no confusion can arise. In other words, when we consider $\Sigma(X)$ as a Σ-object, we endow it with the Σ-structure given by μ_X, unless otherwise specified.

3.3.10. We have seen that the monad Σ over $\mathcal{C}$ is completely determined by the pair of adjoint functors $L : \mathcal{C} \to \mathcal{C}^\Sigma$ and $\Gamma : \mathcal{C}^\Sigma \to \mathcal{C}$.

This construction can be generalized as follows. Suppose we are given two adjoint functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ with adjointness transformations $\xi : \mathrm{Id}_{\mathcal{C}} \to GF$ and $\eta : FG \to \mathrm{Id}_{\mathcal{D}}$, i.e. we have $(\xi \star G)(G \star \eta) = \mathrm{id}_G$ and

$(\eta \star F)(F \star \xi) = \mathrm{id}_F$. Put $\Sigma := GF \in \operatorname{Ob} \mathit{Endof}(\mathcal{C})$, $\mu := G \star \eta \star F : \Sigma^2 = GFGF \to \Sigma = GF$ and $\varepsilon := \xi : \mathrm{Id}_{\mathcal{C}} \to \Sigma$. Then it is easy to see that $\Sigma := (\Sigma, \mu, \varepsilon)$ is a monad over $\mathcal{C}$. Indeed, $\mu(\Sigma \star \mu) = (G \star \eta \star F)(GFG \star \eta \star F) = G \star \eta \star \eta \star F = \mu(\mu \star \Sigma)$, $\mu(\Sigma \star \varepsilon) = (G \star \eta \star F)(GF \star \xi) = G \star \mathrm{id}_F = \mathrm{id}_\Sigma$, and $\mu(\varepsilon \star \Sigma) = \mathrm{id}_\Sigma$ is checked similarly.

Moreover, $\alpha := G \star \eta : \Sigma G \to G$ determines a left Σ-module structure on $G : \mathcal{D} \to \mathcal{C}$: indeed, $\alpha(\Sigma \star \alpha) = (G \star \eta)(GFG \star \eta) = G \star \eta \star \eta = (G \star \eta)(G \star \eta \star FG) = \alpha(\mu \star G)$ and $\alpha(\varepsilon \star G) = (G \star \eta)(\xi \star G) = \mathrm{id}_G$. Therefore, according to **3.3.7** our left Σ-functor $(G, G \star \eta) \in \operatorname{Ob} \mathit{Funct}(\mathcal{D}, \mathcal{C})^\Sigma$ determines in a unique way a functor $\tilde{G} : \mathcal{D} \to \mathcal{C}^\Sigma$, such that $\tilde{G}(Z) = \big(G(Z), G(\eta_Z)\big)$ for any $Z \in \operatorname{Ob} \mathcal{D}$. Clearly, $G = \Gamma_\Sigma \tilde{G}$, i.e. we have shown that $G : \mathcal{D} \to \mathcal{C}$ factorizes through $\mathcal{C}^\Sigma$ in a natural way. It is also easy to see that $\tilde{G}F = L_\Sigma$.

Definition 3.3.11 *A functor $G : \mathcal{D} \to \mathcal{C}$ is called* monadic, *if it admits a left adjoint $F : \mathcal{C} \to \mathcal{D}$, and if the induced functor $\tilde{G} : \mathcal{D} \to \mathcal{C}^\Sigma$ is an equivalence of categories, where Σ is the monad over $\mathcal{C}$ defined by F and G as described above.*

Clearly, in this case we can replace $\mathcal{D}$ with $\mathcal{C}^\Sigma$, G with Γ_Σ, and F with the left adjoint L_Σ of Γ_Σ.

Note that the monadicity of G doesn't depend on the choice of its left adjoint F, since such an adjoint is unique up to a unique isomorphism.

3.3.12. The concept of monadicity is very important since it allows us to replace some categories $\mathcal{D}$ with the categories of Σ-objects inside a simpler category $\mathcal{C}$ with respect to a certain monad Σ over $\mathcal{C}$, thus providing a uniform description of these categories $\mathcal{D}$. There are several interesting criteria of monadicity, e.g. the *Beck's monadicity theorem* (cf. [MacLane]). However, what is really important for us is that if $\mathcal{C}$ is the category of sets and $\mathcal{D}$ is a category defined by some algebraic structure (e.g. the category of groups, monoids, rings, left modules over a fixed ring, ...), then the forgetful functor $F : \mathcal{D} \to \mathit{Sets}$ is monadic, i.e. any of these "algebraic" categories is equivalent (usually even isomorphic) to Sets^Σ for a suitable monad Σ over Sets. In fact, such categories are defined by a certain special class of monads, which we call *algebraic*; they will be studied in some detail in the next chapter.

3.3.13. We have seen in **3.3.7** that $\mathcal{C}^\Sigma$ in a certain sense represents the category of left Σ-functors $\mathit{Funct}(\mathcal{D}, \mathcal{C})^\Sigma$ for variable category $\mathcal{D}$. Is a similar result true for the categories of right Σ-functors $\mathit{Funct}(\mathcal{C}, \mathcal{D})_\Sigma$? The answer turns out to be positive:

Proposition. *Given a monad $\Sigma = (\Sigma, \mu, \varepsilon)$ over a category $\mathcal{C}$, denote by $\mathcal{C}_\Sigma$ the category with the same objects as $\mathcal{C}$, and with morphisms given* *by* $\mathrm{Hom}_{\mathcal{C}_\Sigma}(X,Y) := \mathrm{Hom}_{\mathcal{C}}(X,\Sigma(Y)) \cong \mathrm{Hom}_{\mathcal{C}^\Sigma}(L_\Sigma(X),L_\Sigma(Y))$. *The composition of morphisms is given by the second description of* $\mathrm{Hom}_{\mathcal{C}_\Sigma}$*; in terms of the first description it is given by* $(\psi,\varphi)\mapsto \mu_Z\circ\Sigma(\psi)\circ\varphi$*, where* $\varphi\in\mathrm{Hom}_{\mathcal{C}_\Sigma}(X,Y)$ *and* $\psi\in\mathrm{Hom}_{\mathcal{C}_\Sigma}(Y,Z)$*. In other words,* $\mathcal{C}_\Sigma$ *is equivalent to the full subcategory of free objects in* $\mathcal{C}^\Sigma$*.*

Consider the functor $I = I_\Sigma : \mathcal{C}\to\mathcal{C}_\Sigma$*, which is identical on objects and maps* $\varphi : X\to Y$ *into* $\varepsilon_Y\circ\varphi\in\mathrm{Hom}_{\mathcal{C}_\Sigma}(X,Y)$*. Then the category Funct*$(\mathcal{C},\mathcal{D})_\Sigma$ *of right* Σ*-functors* $\mathcal{C}\to\mathcal{D}$ *is canonically isomorphic to the category Funct*$(\mathcal{C}_\Sigma,\mathcal{D})$*, and under this isomorphism the forgetful functor is identified with* $I^* :$ *Funct*$(\mathcal{C}_\Sigma,\mathcal{D})\to$ *Funct*$(\mathcal{C},\mathcal{D})$*.*

More explicitly, $F = (F,\alpha)\in\mathrm{Ob}\,$*Funct*$(\mathcal{C},\mathcal{D})_\Sigma$*, where* $F:\mathcal{C}\to\mathcal{D}$ *and* $\alpha : F\Sigma\to F$*, corresponds under this isomorphism to the functor* $\tilde{F}:\mathcal{C}_\Sigma\to\mathcal{D}$ *which coincides with* F *on objects, and transforms* $\varphi : X\to\Sigma(Y)$ *into* $\tilde{F}(\varphi) := \alpha_Y\circ F(\varphi) : F(X)\to F(Y)$*.*

Proof. Let's check first the equivalence of the two descriptions of composition in $\mathcal{C}_\Sigma$. In the notations of **3.3.8** this amounts to check $\psi^\sharp\circ\varphi^\sharp = (\mu_Z\circ\Sigma(\psi)\circ\varphi)^\sharp$, i.e. $\mu_Z\circ\Sigma(\psi)\circ\mu_Y\circ\Sigma(\varphi) = \mu_Z\circ\Sigma(\mu_Z)\circ\Sigma^2(\psi)\circ\Sigma(\varphi)$; this equality follows immediately from $\Sigma(\psi)\circ\mu_Y = \mu_{\Sigma(Z)}\circ\Sigma^2(\psi)$ and from the associativity condition $\mu_Z\circ\Sigma(\mu_Z) = \mu_Z\circ\mu_{\Sigma(Z)}$.

Now we want to construct a functor *Funct*$(\mathcal{C}_\Sigma,\mathcal{D})\to$ *Funct*$(\mathcal{C},\mathcal{D})_\Sigma$, inverse to one constructed above. To do this we construct first a right Σ-structure $\beta : I\Sigma\to I$ on $I:\mathcal{C}\to\mathcal{C}_\Sigma$, and then apply to it the $\oslash$-functor $\tilde{F}_*$ for any $\tilde{F}:\mathcal{C}_\Sigma\to\mathcal{D}$, thus obtaining a right Σ-functor $\tilde{F}_*(I,\beta) = (\tilde{F}I,\tilde{F}\star\beta)$; this construction, being functorial in $\tilde{F}$, yields the required functor *Funct*$(\mathcal{C}_\Sigma,\mathcal{D})\to$ *Funct*$(\mathcal{C},\mathcal{D})_\Sigma$.

So we have to construct $\beta : I\Sigma\to I$. For any $X\in\mathrm{Ob}\,\mathcal{C}$ we must produce some $\beta_X\in\mathrm{Hom}_{\mathcal{C}_\Sigma}(\Sigma(X),X) = \mathrm{Hom}_{\mathcal{C}}(\Sigma(X),\Sigma(X))$; of course, we take $\mathrm{id}_{\Sigma(X)}$. It remains to check that β is indeed a right Σ-action on I, and that our two functors between *Funct*$(\mathcal{C},\mathcal{D})_\Sigma$ and *Funct*$(\mathcal{C}_\Sigma,\mathcal{D})$ are inverse to each other; we leave these verifications to the reader.

3.3.14. Note that $I = I_\Sigma:\mathcal{C}\to\mathcal{C}_\Sigma$ admits a right adjoint $K = K_\Sigma:\mathcal{C}_\Sigma\to\mathcal{C}$, given by $K(X) := \Sigma(X)$, $K(\varphi) := \mu_Y\circ\Sigma(\varphi)$ for any $\varphi : X\to\Sigma(Y)$. Clearly, $KI = \Sigma$, and $IK : \mathcal{C}_\Sigma\to\mathcal{C}_\Sigma$ coincides with Σ on objects and transforms $\varphi : X\to\Sigma(Y)$ into $\varepsilon_{\Sigma(Y)}\circ\mu_Y\circ\Sigma(\varphi)$. The adjointness natural transformations $\xi : \mathrm{Id}_{\mathcal{C}}\to KI = \Sigma$ and $\eta : IK\to\mathrm{Id}_{\mathcal{C}_\Sigma}$ are computed as follows: $\xi = \varepsilon$ and $\eta_X\in\mathrm{Hom}_{\mathcal{C}_\Sigma}(\Sigma(X),X) = \mathrm{Hom}_{\mathcal{C}}(\Sigma(X),\Sigma(X))$ is given again by $\mathrm{id}_{\Sigma(X)}$.

It is easy to see that this pair of adjoint functors $I:\mathcal{C}\to\mathcal{C}_\Sigma$ and $K:\mathcal{C}_\Sigma\to\mathcal{C}$ determines by the recipe of **3.3.10** the original monad $\Sigma = (\Sigma,\mu,\varepsilon)$: indeed, we have already seen that $KI = \Sigma$ and $\xi = \varepsilon$, and $K\star\eta\star I = \mu$ is quite easy to check: $(K \star \eta \star I)_X = K(\eta_{I(X)}) = K(\mathrm{id}_{\Sigma(X)}) = \mu_X$.

3.3.15. Moreover, if we are given two adjoint functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ with adjointness natural transformations $\xi : \mathrm{Id}_{\mathcal{C}} \to GF$ and $\eta : FG \to \mathrm{Id}_{\mathcal{D}}$, defining a monad $\Sigma = (GF, G{\star}\eta{\star}F, \xi)$ as described in **3.3.10**, we obtain a right Σ-action $\eta \star F : F\Sigma \to F$ on F, hence by **3.3.13** there is a unique functor $\tilde{F} : \mathcal{C}_\Sigma \to \mathcal{D}$, such that $\tilde{F}I = F$ and $\tilde{F} \star \beta = \eta \star F$.

In this respect the situation is quite similar (or rather dual) to that of **3.3.10**, where we obtained a functor $\tilde{G} : \mathcal{D} \to \mathcal{C}^\Sigma$. We see that among all pairs of adjoint functors (F, G), defining the same monad Σ on $\mathcal{C}$, we have an initial object (I_Σ, K_Σ) and a final object $(L_\Sigma, \Gamma_\Sigma)$.

3.3.16. Proposition **3.3.13** is important for us since it allows us to identify the category $\mathcal{A}_\Sigma$ of right Σ-modules in $\mathcal{A} := \mathit{Endof}(\mathcal{C})$ with the category $\mathit{Funct}(\mathcal{C}_\Sigma, \mathcal{C})$ of functors from $\mathcal{C}_\Sigma$ to $\mathcal{C}$.

Combining this with **3.3.7**, we see that, given a monad Σ over $\mathcal{C}$ and a monad Ξ over $\mathcal{D}$, the category $\mathit{Funct}(\mathcal{D}, \mathcal{C})^\Sigma_\Xi$ of Σ-Ξ-bimodules in $\mathit{Funct}(\mathcal{D}, \mathcal{C})$ (i.e. we consider the category of triples (M, α, β), where $M : \mathcal{D} \to \mathcal{C}$, $\alpha : \Sigma M \to M$ is a left Σ-action, $\beta : M\Xi \to M$ is a right Ξ-action, and these actions commute: $\alpha(\Sigma \star \beta) = \beta(\alpha \star \Xi)$) is isomorphic to the category $\mathit{Funct}(\mathcal{D}_\Xi, \mathcal{C}^\Sigma)$.

In particular, the category $\mathcal{A}^\Sigma_\Sigma$ of Σ-bimodules in $\mathcal{A}$ is isomorphic to $\mathit{Funct}(\mathcal{C}_\Sigma, \mathcal{C}^\Sigma)$. In other words, a Σ-bimodule in $\mathcal{A}$ is essentially the same thing as a functor from the subcategory of free objects in $\mathcal{C}^\Sigma$ into the whole of $\mathcal{C}^\Sigma$. Since Σ is canonically a bimodule over itself, we obtain a functor $Q = Q_\Sigma : \mathcal{C}_\Sigma \to \mathcal{C}^\Sigma$, such that $Q_\Sigma I_\Sigma = L_\Sigma$ and $\Gamma_\Sigma Q_\Sigma = K_\Sigma$; explicitly, $Q(X) = L(X) = (\Sigma(X), \mu_X)$ on objects, and $Q(\varphi) = \mu_Y \circ \Sigma(\varphi)$ for any $\varphi \in \mathrm{Hom}_{\mathcal{C}_\Sigma}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, \Sigma(Y)) \cong \mathrm{Hom}_{\mathcal{C}^\Sigma}(L(X), L(Y))$. We see again that this functor $Q : \mathcal{C}_\Sigma \to \mathcal{C}^\Sigma$ is fully faithful (cf. **3.3.13**). Different functors between categories $\mathcal{C}$, $\mathcal{C}^\Sigma$ and $\mathcal{C}_\Sigma$ fit into the following commutative diagram:

$$\begin{tikzcd}
 & \mathcal{C}_\Sigma \arrow[r, "Q_\Sigma"] \arrow[drr, "K_\Sigma"] & \mathcal{C}^\Sigma \arrow[dr, "\Gamma_\Sigma"] & \\
\mathcal{C} \arrow[ur, "I_\Sigma"] \arrow[urr, "L_\Sigma"] \arrow[rrr, "\Sigma"] & & & \mathcal{C}
\end{tikzcd} \tag{3.3.16.1}$$

3.3.17. (Scalar restriction.) Given a morphism $\rho : \Sigma \to \Xi$ of monads over $\mathcal{C}$, we obtain the *scalar restriction functors* $\rho^* : \mathcal{C}^\Xi \to \mathcal{C}^\Sigma$, $\rho^* : \mathit{Funct}(\mathcal{D}, \mathcal{C})^\Xi \to \mathit{Funct}(\mathcal{D}, \mathcal{C})^\Sigma$ and $\rho^* : \mathit{Funct}(\mathcal{C}, \mathcal{D})_\Xi \to \mathit{Funct}(\mathcal{C}, \mathcal{D})_\Sigma$ according to the general recipe of **3.1.13**. It is easy to see that the second of these functors can be identified with $(\rho^*)_* : \mathit{Funct}(\mathcal{D}, \mathcal{C}^\Xi) \to \mathit{Funct}(\mathcal{D}, \mathcal{C}^\Sigma)$, $H \mapsto \rho^* \circ H$ (cf. **3.3.7**). Similarly, the third of these functors can be identified according to **3.3.13**

with a functor $Funct(\mathcal{C}_\Xi, \mathcal{D}) \to Funct(\mathcal{C}_\Sigma, \mathcal{D})$, and this construction is functorial in $\mathcal{D}$, hence "by Yoneda" this functor has to be induced by some functor $\rho_* : \mathcal{C}_\Sigma \to \mathcal{C}_\Xi$, acting in the opposite direction.

Let's describe these two functors $\rho^* : \mathcal{C}^\Xi \to \mathcal{C}^\Sigma$ and $\rho_* : \mathcal{C}_\Sigma \to \mathcal{C}_\Xi$ more explicitly. Clearly, $\rho^* : \mathcal{C}^\Xi \to \mathcal{C}^\Sigma$ transforms a Ξ-module $(N, \alpha : \Xi(N) \to N)$ into Σ-module $\rho^*(N) := (N, \alpha \circ \rho_N)$, and acts identically on morphisms. It is also easy to check that $\rho_* : \mathcal{C}_\Sigma \to \mathcal{C}_\Xi$ acts identically on $\mathrm{Ob}\,\mathcal{C}_\Sigma = \mathrm{Ob}\,\mathcal{C} = \mathrm{Ob}\,\mathcal{C}_\Xi$, and transforms a morphism $f \in \mathrm{Hom}_{\mathcal{C}_\Sigma}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, \Sigma(Y))$ into $\rho_Y \circ f \in \mathrm{Hom}_{\mathcal{C}_\Xi}(X, Y)$. Note that $\Gamma_\Sigma \rho^* = \Gamma_\Xi$ and $\rho_* I_\Sigma = I_\Xi$.

3.3.18. (Base change.) Since Q_Σ establishes an equivalence between $\mathcal{C}_\Sigma$ and the full subcategory of free Σ-modules in $\mathcal{C}^\Sigma$, and similarly for Q_Ξ, we see that $\rho_* : \mathcal{C}_\Sigma \to \mathcal{C}_\Xi$ can be considered as a partially defined functor $\tilde{\rho}_* : \mathcal{C}^\Sigma \dashrightarrow \mathcal{C}^\Xi$, defined only for free Σ-modules. Clearly, $\tilde{\rho}_*$ transforms $L_\Sigma(X) = \Sigma(X)$ into $\Xi(X)$, and $f : \Sigma(X) \to \Sigma(Y)$ into $(\rho_Y \circ f^\flat)^\sharp = \mu_{\Xi,Y} \circ \Xi(\rho_Y \circ f \circ \varepsilon_{\Sigma,X})$ (cf. **3.3.8**). We claim that *$\tilde{\rho}_* : \mathcal{C}^\Sigma \dashrightarrow \mathcal{C}^\Xi$ is a partially defined left adjoint to $\rho^* : \mathcal{C}^\Xi \to \mathcal{C}^\Sigma$*. Indeed, $\mathrm{Hom}_{\mathcal{C}^\Xi}(\tilde{\rho}_* L_\Sigma(X), N) = \mathrm{Hom}_{\mathcal{C}^\Xi}(L_\Xi(X), N) \cong \mathrm{Hom}_{\mathcal{C}}(X, \Gamma_\Xi(N)) = \mathrm{Hom}_{\mathcal{C}}(X, \Gamma_\Sigma \rho^*(N)) \cong \mathrm{Hom}_{\mathcal{C}^\Sigma}(L_\Sigma(X), \rho^* N)$. In some situations $\tilde{\rho}_*$ can be extended to a well-defined *base change* (or *scalar extension) functor* $\rho_* : \mathcal{C}^\Sigma \to \mathcal{C}^\Xi$, left adjoint to $\rho^* : \mathcal{C}^\Xi \to \mathcal{C}^\Sigma$.

Proposition 3.3.19 *If cokernels of pairs of morphisms exist in $\mathcal{C}^\Xi$, then $\tilde{\rho}_*$ extends to a well-defined* scalar extension *(or* base change*) functor $\rho_* : \mathcal{C}^\Sigma \to \mathcal{C}^\Xi$, left adjoint to $\rho^* : \mathcal{C}^\Xi \to \mathcal{C}^\Sigma$. Moreover, in this situation ρ^* is monadic, i.e. it induces an equivalence and even an isomorphism of categories $\mathcal{C}^\Xi \to (\mathcal{C}^\Sigma)^{\Xi/\Sigma}$, where $\Xi/\Sigma := \rho^*\rho_*$ is the monad over $\mathcal{C}^\Sigma$ defined by ρ_* and ρ^* (cf.* **3.3.10***).*

Proof. According to Lemma **3.3.20** below, any Σ-module $M = (M, \alpha)$ in $\mathcal{C}$ can be represented (even in a functorial way) as a cokernel of two Σ-morphisms between two free Σ-modules: $M = \mathrm{Coker}(p, q : L_\Sigma(R) \rightrightarrows L_\Sigma(X))$. If ρ_* exists, it must preserve arbitrary inductive limits and transform any free Σ-object $L_\Sigma(Z)$ into free Ξ-object $L_\Xi(Z)$, hence $\rho_*(M)$ must be isomorphic to $\mathrm{Coker}(\tilde{\rho}_*(p), \tilde{\rho}_*(q))$. We put $\rho_*(M) := \mathrm{Coker}(\tilde{\rho}_*(p), \tilde{\rho}_*(q) : L_\Xi(R) \rightrightarrows L_\Xi(X))$. It remains to check $\mathrm{Hom}_{\mathcal{C}^\Xi}(\rho_* M, N) \cong \mathrm{Hom}_{\mathcal{C}^\Sigma}(M, \rho^* N)$, but this is quite clear: $\mathrm{Hom}_{\mathcal{C}^\Xi}(\rho_* M, N) = \mathrm{Hom}_{\mathcal{C}^\Xi}\bigl(\mathrm{Coker}(\tilde{\rho}_*(p), \tilde{\rho}_*(q)), N\bigr) \cong \mathrm{Ker}\bigl(\mathrm{Hom}_{\mathcal{C}^\Xi}(L_\Xi(X), N) \rightrightarrows \mathrm{Hom}_{\mathcal{C}^\Xi}(L_\Xi(R), N)\bigr)$, and since $\tilde{\rho}_*$ is a partial left adjoint to ρ^* and $L_\Xi(X) = \tilde{\rho}_* L_\Sigma(X)$, this kernel turns out to be isomorphic to $\mathrm{Ker}\bigl(\mathrm{Hom}_{\mathcal{C}^\Sigma}(L_\Sigma(X), \rho^* N) \rightrightarrows \mathrm{Hom}_{\mathcal{C}^\Sigma}(L_\Sigma(R), \rho^* N)\bigr) \cong \mathrm{Hom}_{\mathcal{C}^\Sigma}(M, \rho^* N)$, hence $\rho^* M$ represents indeed whatever it has to represent. The monadicity of ρ^* will not be used in the sequel, so we leave it as an exercise for the reader; an important point here is to construct natural transformations $\zeta =$

$(\rho_*) \star \eta_\Sigma : L_\Xi\Gamma_\Sigma \cong \rho_* L_\Sigma\Gamma_\Sigma \to \rho_*$ and $\Gamma_\Xi \star \zeta : \Xi\Gamma_\Sigma \to \Gamma_\Sigma(\Xi/\Sigma) : \mathcal{C}^\Sigma \to \mathcal{C}$, and to use them for constructing the inverse functor $(\mathcal{C}^\Sigma)^{\Xi/\Sigma} \to \mathcal{C}^\Xi$. Another possibility is to use Beck's monadicity theorem to check that the monadicity of ρ^* follows from the existence of a left adjoint to ρ^* and from the monadicity of Γ_Σ and that of $\Gamma_\Sigma\rho^* = \Gamma_\Xi$.

Lemma 3.3.20 *Any Σ-module $M = (M, \alpha) \in \operatorname{Ob}\mathcal{C}^\Sigma$ can be represented (even in a functorial way) as a cokernel of a pair of morphisms between two free Σ-objects: $M \cong \operatorname{Coker}(p, q : L_\Sigma(R) \rightrightarrows L_\Sigma(X))$. Moreover, one can take $X := M$, $R := \Sigma(M)$, $p := \mu_M$, $q := \Sigma(\alpha) : \Sigma^2(M) \to \Sigma(M)$, and α for the strict epimorphism $L_\Sigma(X) = \Sigma(M) \to M$.*

Proof. We want to check that $\Sigma^2(M) \overset{p,q}{\rightrightarrows} \Sigma(M) \xrightarrow{\alpha} M$ is right exact in $\mathcal{C}^\Sigma$. First of all, notice that $\sigma := \varepsilon_M : M \to \Sigma(M)$ and $\tau := \varepsilon_{\Sigma(M)} : \Sigma(M) \to \Sigma^2(M)$ provide a splitting of this diagram in $\mathcal{C}$, i.e. we have $\alpha \circ \sigma = \mathrm{id}_M$, $p \circ \tau = \mathrm{id}_{\Sigma(M)}$ and $q \circ \tau = \sigma \circ \alpha$. This together with $\alpha \circ p = \alpha \circ q$ implies the right exactness of this diagram in $\mathcal{C}$, and, since this splitting is preserved by any functor, the right exactness of this diagram after an application of Σ.

Now, given any Σ-morphism $\varphi : (\Sigma(M), \mu_M) \to (N, \beta)$, such that $\varphi \circ p = \varphi \circ q$, we want to show the existence and uniqueness of a Σ-morphism $\psi : M \to N$, such that $\varphi = \psi \circ \alpha$. Right exactness in $\mathcal{C}$ implies existence and uniqueness of such a morphism ψ in $\mathcal{C}$; it remains to check that this is a Σ-morphism, i.e. that $\psi \circ \alpha = \beta \circ \Sigma(\psi)$. Since $\Sigma(\alpha)$ is a (split) epimorphism in $\mathcal{C}$, this follows from $\psi \circ \alpha \circ \Sigma(\alpha) = \psi \circ \alpha \circ \mu_M = \varphi \circ \mu_M = \beta \circ \Sigma(\varphi) = \beta \circ \Sigma(\psi) \circ \Sigma(\alpha)$.

3.3.21. We have seen that any monad homomorphism $\rho : \Sigma \to \Xi$ induces a scalar restriction functor $\rho^* : \mathcal{C}^\Xi \to \mathcal{C}^\Sigma$, such that $\Gamma_\Sigma\rho^* = \Gamma_\Xi$. We claim that *any functor $H : \mathcal{C}^\Xi \to \mathcal{C}^\Sigma$, such that $\Gamma_\Sigma H = \Gamma_\Xi$, is equal to ρ^* for a uniquely determined monad homomorphism $\rho : \Sigma \to \Xi$.* To show this we first observe that by **3.3.7** such functors $H : \mathcal{C}^\Xi \to \mathcal{C}^\Sigma$ are in one-to-one correspondence with left Σ-structures $\beta : \Sigma\Gamma_\Xi \to \Gamma_\Xi$ on Γ_Ξ. For example, $H = \rho^*$ corresponds to $\beta = (\Gamma_\Xi \star \eta_\Xi) \circ (\rho \star \Gamma_\Xi)$, where $\eta_\Xi : L_\Xi\Gamma_\Xi \to \mathrm{Id}_{\mathcal{C}^\Xi}$ and $\xi_\Xi = \varepsilon_\Xi : \mathrm{Id}_\mathcal{C} \to \Xi = \Gamma_\Xi L_\Xi$ are the usual adjointness natural transformations. Notice that ρ is uniquely determined by β since $\rho = \rho \circ \mu_\Sigma \circ (\Sigma \star \varepsilon_\Sigma) = \mu_\Xi \circ (\rho \star \rho\varepsilon_\Sigma) = \mu_\Xi \circ (\rho \star \varepsilon_\Xi) = \mu_\Xi \circ (\rho \star \Xi) \circ (\Sigma \star \varepsilon_\Xi) = (\beta \star L_\Xi) \circ (\Sigma \star \varepsilon_\Xi)$. In general we put $\rho := (\beta \star L_\Xi) \circ (\Sigma \star \varepsilon_\Xi)$ for an arbitrary left Σ-action β on Γ_Ξ. We check that $\rho : \Sigma \to \Xi$ is a monad homomorphism using the fact that β is a left Σ-action. For example, we use $\beta \circ (\varepsilon_\Sigma \star \Gamma_\Xi) = \mathrm{id}_{\Gamma_\Xi}$ to prove $\rho \circ \varepsilon_\Sigma = (\beta \star L_\Xi) \circ (\Sigma \star \varepsilon_\Xi) \circ \varepsilon_\Sigma = (\beta \star L_\Xi) \circ (\varepsilon_\Sigma \star \varepsilon_\Xi) = (\beta \star L_\Xi) \circ (\varepsilon_\Sigma \star \Gamma_\Xi L_\Xi) \circ \varepsilon_\Xi = \varepsilon_\Xi$. The compatibility with multiplication $\mu_\Xi \circ (\rho \star \rho) = \rho \circ \mu_\Sigma$ is checked similarly but longer, using $\mu_\Xi = \Gamma_\Xi \star \eta_\Xi \star L_\Xi$, $(\Gamma_\Xi \star \eta_\Xi) \circ (\varepsilon_\Xi \star \Gamma_\Xi) = \mathrm{id}_{\Gamma_\Xi}$ and

$\beta \circ (\Sigma \star \beta) = \beta \circ (\mu_\Sigma \star \Gamma_\Xi)$. Finally, we have to check that $(\Gamma_\Xi \star \eta_\Xi) \circ (\rho \star \Gamma_\Xi)$ will give back the original β. Indeed, this expression equals $(\Gamma_\Xi \star \eta_\Xi) \circ (\beta \star L_\Xi \Gamma_\Xi) \circ (\Sigma \star \varepsilon_\Xi \star \Gamma_\Xi) = (\beta \star \eta_\Xi) \circ (\Sigma \star \varepsilon_\Xi \star \Gamma_\Xi) = \beta \circ (\Sigma \Gamma_\Xi \star \eta_\Xi) \circ (\Sigma \star \varepsilon_\Xi \star \Gamma_\Xi) = \beta$.

3.4. (Examples of monads.) Now we would like to give some examples of monads which will be important for us later. The most important of them are those defined by pairs of adjoint functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ (cf. **3.3.10**), where usually $\mathcal{C} := \mathit{Sets}$, $\mathcal{D}$ is some "algebraic category" (category of sets with some algebraic structure, e.g. the category of groups, rings and so on), $G = \Gamma : \mathcal{D} \to \mathit{Sets}$ is the forgetful functor, and F is a left adjoint of G. This gives us a monad Σ on the endofunctor GF over the category of sets, and the forgetful functor G usually turns out to be monadic (cf. **3.3.11**), i.e. the induced functor $\tilde{G} : \mathcal{D} \to \mathit{Sets}^\Sigma$ turns out to be an equivalence and even an isomorphism of categories, thus identifying $\mathcal{D}$ with Sets^Σ. Another important source of monads is obtained by considering submonads of previously constructed monads (cf. **3.3.3**).

3.4.1. (The monad of words.) Let's take $\mathcal{C} := \mathit{Sets}$, $\mathcal{D} := \mathit{Mon}$ be the category of monoids, $G : \mathit{Mon} \to \mathit{Sets}$ be the forgetful functor, and $F : \mathit{Sets} \to \mathit{Mon}$ be its left adjoint, which maps any set X into the free monoid $F(X)$ generated by X. Recall that $F(X)$ consists of all finite words $x_1 \ldots x_n$ in alphabet X, i.e. all $x_i \in X$, $n \geq 0$, the multiplication on $F(X)$ is given by the concatenation of words, and the unit of $F(X)$ is the empty word $\emptyset$, i.e. the only word of length zero. According to **3.3.10**, this defines a monad $W = (W, \mu, \varepsilon)$ over the category of sets, which will be called *the monad of words.*

We see that $W(X) = \bigsqcup_{n \geq 0} X^n$ for any set X. Sequences $(x_1, x_2, \ldots, x_n)$ lying in $X^n \subset W(X)$ are usually called *words of length n over X*, and are written simply as $x_1 x_2 \ldots x_n$; the individual components x_i of a word are usually called *letters.* When some of x_i are replaced by more complicated expressions, or when there is some other source of potential confusion, we enclose some or all of them into braces, thus writing $\{x_1\}\{x_2\} \ldots \{x_n\}$ instead of $x_1 x_2 \ldots x_n$ or $(x_1, x_2, \ldots, x_n)$. For example, $\{x\}$ is the one-letter word corresponding to an element $x \in X$. The *empty word* will be usually denoted by $\emptyset$, or on some occasions by $\{\}$.

Now the unit $\varepsilon : \mathrm{Id}_{\mathit{Sets}} \to W$ and the multiplication $\mu : W^2(X) \to W(X)$ can be described explicitly as follows. $\varepsilon_X : X \to W(X)$ maps any $x \in X$ into the corresponding one-letter word $\{x\} \in W(X)$. As to the multiplication $\mu_X : W^2(X) \to W(X)$, it "removes one layer of braces", i.e. maps $\{x_1 x_2 \ldots x_n\}\{y_1 y_2 \ldots\} \ldots \{z_1 \ldots z_m\}$ into $x_1 x_2 \ldots x_n y_1 y_2 \ldots z_1 \ldots z_m$. The axioms of a monad can be checked now directly. For example, the associativity condition $\mu_X \circ \mu_{W(X)} = \mu_X \circ W(\mu_X) : W^3(X) \to W(X)$ means simply that if we have an expression with two layers of braces, and remove first the inner layer and then the outer layer, or conversely, then the resulting word in $W(X)$ is the same in both cases. So we might have defined the monad $W = (W, \mu, \varepsilon)$ directly, without any reference to the category of monoids.

3.4.2. Of course, the forgetful functor $G : Mon \to Sets$ is monadic, i.e. the induced functor $\tilde{G} : Mon \to Sets^W$ is an equivalence, and even an isomorphism of categories. In other words, the structure of a monoid on some set X is essentially the same thing as a W-structure $\alpha : W(X) \to X$, and W-morphisms are exactly monoid morphisms.

Let's show how this works in this situation. If we have a monoid structure $(X, *, e)$ on X, where $* : X \times X \to X$ is the multiplication and $e \in X$ is the unit, then the corresponding W-structure $\alpha : W(X) \to X$ simply maps a word $x_1 x_2 \ldots x_n \in W(X)$ into the corresponding product $x_1 * x_2 * \cdots * x_n$ (this actually follows from the description of $\tilde{G}$ given in **3.3.10**). Conversely, if we are given a W-structure $\alpha : W(X) \to X$ on a set X, we can recover a monoid structure on X by setting $e := \alpha(\{\})$ and $x * y := \alpha(\{x\}\{y\}) = \alpha(xy)$. The associativity and unit axioms for this choice of operations on X follow from the axioms $\alpha \circ W(\alpha) = \alpha \circ \mu_X$ and $\alpha \circ \varepsilon_X = \mathrm{id}_X$ for α. For example, $(x * y) * z = \alpha(\{\alpha(xy)\}\{z\}) = \alpha(\{\alpha(xy)\}\{\alpha(z)\}) = (\alpha \circ W(\alpha))(\{xy\}\{z\}) = \alpha(\mu_X(\{xy\}\{z\})) = \alpha(xyz)$, and $x * (y * z) = \alpha(xyz)$ is shown similarly, thus proving the associativity of $*$.

The fact that W-morphisms $f : (X, \alpha) \to (Y, \beta)$ are exactly monoid homomorphisms is also checked similarly. For example, if f is a W-morphism, i.e. $\beta \circ W(f) = f \circ \alpha$, then $f(x *_X y) = f(\alpha(xy)) = (\beta \circ W(f))(\{x\}\{y\}) = \beta(\{f(x)\}\{f(y)\}) = f(x) *_Y f(y)$.

3.4.3. This construction generalizes to any (Grothendieck) topos $\mathcal{E}$, where we can also put $W_{\mathcal{E}}(X) := \bigsqcup_{n \geq 0} X^n$ for any $X \in \operatorname{Ob} \mathcal{E}$. Then $\mathcal{E}^{W_{\mathcal{E}}}$ is of course just the category $Mon(\mathcal{E})$ of monoids in $\mathcal{E}$. If $\mathcal{E}$ is the category $\tilde{\mathcal{S}}$ of sheaves (of sets) on some site $\mathcal{S}$, then $\mathcal{E}^{W_{\mathcal{E}}}$ is the category of sheaves of monoids on $\mathcal{S}$. We'll see later that both W and $W_{\mathcal{E}}$ are examples of algebraic monads, and that $W_{\mathcal{E}}$ is the pullback of W with respect to the canonical morphism of topoi $\mathcal{E} \to Sets$.

3.4.4. Given a monad $\Sigma = (\Sigma, \mu, \varepsilon)$ over a category $\mathcal{C}$ with finite coproducts, and an object $U \in \operatorname{Ob} \mathcal{C}$, we can construct a new monad $\Sigma_U = (\Sigma_U, \mu', \varepsilon')$ with the underlying functor given by $\Sigma_U(X) := \Sigma(U \sqcup X)$ and $\Sigma_U(f) := \Sigma(\mathrm{id}_U \sqcup f)$. Informally, we want to add some "constants" from U to the "variables" from X. Let's define the multiplication and the unit of this monad. Denote for this by $i_X : U \to U \sqcup X$ and $j_X : X \to U \sqcup X$ the natural embeddings, and by $k_X : U \sqcup \Sigma(U \sqcup X) \to \Sigma(U \sqcup X)$ the morphism

$k_X := \langle \Sigma(i_X) \circ \varepsilon_U, \mathrm{id}\rangle$ with components $\Sigma(i_X) \circ \varepsilon_U = \varepsilon_{U \sqcup X} \circ i_X : U \to \Sigma(U \sqcup X)$ and $\mathrm{id}_{\Sigma(U\sqcup X)}$, i.e. $k_X \circ i_{\Sigma_U(X)} = \Sigma(i_X) \circ \varepsilon_U = \varepsilon_{U\sqcup X} \circ i_X$ and $k_X \circ j_{\Sigma_U(X)} = \mathrm{id}$. Now put $\varepsilon'_X := \Sigma(j_X) \circ \varepsilon_X = \varepsilon_{U\sqcup X} \circ j_X : X \to \Sigma_U(X)$, and define $\mu'_X : \Sigma(U \sqcup \Sigma(U \sqcup X)) \to \Sigma(U \sqcup X)$ by $\mu'_X := \mu_{U\sqcup X} \circ \Sigma(k_X)$.

Let's check first that ε' is a unit for μ'. Indeed, $\mu'_X \circ \Sigma_U(\varepsilon'_X) = \mu_{U\sqcup X} \circ \Sigma(k_X) \circ \Sigma(\mathrm{id}_U \sqcup \varepsilon'_X) = \mu_{U\sqcup X} \circ \Sigma\bigl(\langle \varepsilon_{U\sqcup X} \circ i_X, \varepsilon_{U\sqcup X} \circ j_X\rangle\bigr) = \mu_{U\sqcup X} \circ \Sigma(\varepsilon_{U\sqcup X}) = \mathrm{id}_{\Sigma(U\sqcup X)}$ and $\mu'_X \circ \varepsilon'_{\Sigma_U(X)} = \mu_{U\sqcup X} \circ \Sigma(k_X) \circ \Sigma(j_{\Sigma_U(X)}) \circ \varepsilon_{\Sigma_U(X)} = \mu_{U\sqcup X} \circ \Sigma(\mathrm{id}) \circ \varepsilon_{\Sigma(U\sqcup X)} = \mathrm{id}_{\Sigma(U\sqcup X)}$.

Next we have to check the associativity axiom for μ'. Before doing this let's introduce $\lambda_X : U \sqcup \Sigma^2(U \sqcup X) \to \Sigma^2(U \sqcup X)$ by $\lambda_X := \langle \varepsilon_{\Sigma(U\sqcup X)} \circ \varepsilon_{U\sqcup X} \circ i_X, \mathrm{id}\rangle$. Now the associativity condition $\mu'_X \circ \Sigma_U(\mu'_X) = \mu'_X \circ \mu'_{\Sigma_U(X)}$ will be a consequence of the commutativity of the outer circuit of the following diagram:

$$\begin{tikzcd}
\Sigma(U \sqcup \Sigma_U^2(X)) \arrow[r, "\Sigma(k_{\Sigma_U(X)})"] \arrow[d, "\Sigma(\mathrm{id}_U \sqcup \Sigma(k_X))"] & \Sigma^2(U \sqcup \Sigma_U(X)) \arrow[r, "\mu_{U\sqcup\Sigma_U(X)}"] \arrow[d, "\Sigma^2(k_X)"] & \Sigma(U \sqcup \Sigma_U(X)) \arrow[d, "\Sigma(k_X)"'] \\
\Sigma(U \sqcup \Sigma^2(U \sqcup X)) \arrow[r, "\Sigma(\lambda_X)"] \arrow[d, "\Sigma(\mathrm{id}_U \sqcup \mu_{U\sqcup X})"] & \Sigma^3(U \sqcup X) \arrow[r, "\mu_{\Sigma(U\sqcup X)}"] \arrow[d, "\Sigma(\mu_{U\sqcup X})"] & \Sigma^2(U \sqcup X) \arrow[d, "\mu_{U\sqcup X}"'] \\
\Sigma(U \sqcup \Sigma(U \sqcup X)) \arrow[r, "\Sigma(k_X)"] & \Sigma^2(U \sqcup X) \arrow[r, "\mu_{U\sqcup X}"] & \Sigma(U \sqcup X)
\end{tikzcd}$$

The upper right square is commutative simply because the natural transformation μ is compatible with k_X, and the commutativity of the lower right square follows from the associativity of μ. The two remaining squares are obtained by applying Σ to some simpler diagrams, and it is sufficient to check the commutativity of these simpler diagrams. Both of them assert equality of some morphisms from certain coproducts $U \sqcup \ldots$, hence it is sufficient to check the equality of the restrictions of these morphisms to individual components of the coproduct.

Now restricting to U in the (counterpart of) the upper left square yields $\lambda_X \circ i_{\Sigma^2(U\sqcup X)} = \varepsilon_{\Sigma(U\sqcup X)} \circ \varepsilon_{U\sqcup X} \circ i_X = \Sigma(\varepsilon_{U\sqcup X} \circ i_X) \circ \varepsilon_U = \Sigma(k_X \circ i_{\Sigma_U(X)}) \circ \varepsilon_U = \Sigma(k_X) \circ k_{\Sigma_U(X)} \circ i_{\Sigma_U^2(X)}$, and restricting to $\Sigma_U^2(X)$ yields $\Sigma(k_X)$ for both paths; this proves the commutativity of this square.

Restricting to U in the lower left square we get $\mu_{U\sqcup X} \circ \lambda_X \circ i_{\Sigma^2(U\sqcup X)} = \mu_{U\sqcup X} \circ \varepsilon_{\Sigma(U\sqcup X)} \circ \varepsilon_{U\sqcup X} \circ i_X = \varepsilon_{U\sqcup X} \circ i_X = k_X \circ i_{\Sigma(U\sqcup X)}$, and restricting to $\Sigma^2(U \sqcup X)$ we get $\mu_{U\sqcup X}$ for both paths. This finishes the proof of the commutativity of the whole diagram, hence also that of the associativity of μ'.

3.4.5. Note that the above construction is functorial in U, so we get canonical monad homomorphisms $\zeta_f : \Sigma_U \to \Sigma_V$ for any $f : U \to V$, given by $\zeta_{f,X} := \Sigma(f \sqcup \mathrm{id}_X) : \Sigma(U \sqcup X) \to \Sigma(V \sqcup X)$, and homomorphisms $\zeta_U : \Sigma \to \Sigma_U$, $\zeta_{U,X} := \Sigma(j_X) : \Sigma(X) \to \Sigma(U \sqcup X)$ as well.

3.4.6. In particular, we can apply this to the monad of words W on the category of sets, or to its topos counterpart $W_{\mathcal{E}}$, thus obtaining a family of new monads $W\langle U\rangle = W_U$ parametrized by sets (resp. objects of $\mathcal{E}$) U. The alternative notation $W\langle U\rangle$ reflects the fact that this is something like "the algebra of polynomials over W in non-commuting variables from U". More precisely, Sets^{W_U} is the category of pairs (M, φ), consisting of a monoid M and some "choice of constants", i.e. an arbitrary map of sets $\varphi : U \to M$.

Clearly, the elements of $W_U(X) = W(U \sqcup X)$ are simply words in the alphabet $U \sqcup X$. Usually we don't enclose the letters from U in braces, even if we do it with some other letters of the same word, thus obtaining expressions like $u_1\{x_1\}u_2\{x_2\}\{x_3\}$. The unit morphism $\varepsilon'_X : X \to W_U(X)$ still maps any $x \in X$ into the corresponding "variable" (one-letter word) $\{x\}$, and the multiplication $\mu'_X : W_U^2(X) \to W_U(X)$ is given by the concatenation of words and subsequent identification of "constants", i.e. letters from U, coming from different layers. For example, $\mu'_X(u_1\{u_2\{x_1\}\{x_2\}\}\{u_3\}) = u_1u_2\{x_1\}\{x_2\}u_3$. This property is actually the reason which allows us to omit braces around letters from U in almost all situations.

3.4.7. The importance of these monads W_U obtained from the monad of words W is due to the fact that *any algebraic monad appears as a subquotient of a monad of this form.* We'll return to this issue in the next chapter.

3.4.8. (Monads defined by rings.) Now fix any ring R (required to be associative, with unity, but not necessarily commutative) and consider the forgetful functor $\Gamma_R : R\text{-}\mathit{Mod} \to \mathit{Sets}$ and its left adjoint $L_R : \mathit{Sets} \to R\text{-}\mathit{Mod}$, which transforms any set X into the corresponding free module $R^{(X)}$. As usual, this defines a monad $\Sigma_R = (\Sigma_R, \mu, \varepsilon)$ over *Sets* with the underlying endofunctor equal to $\Gamma_R L_R$. In other words, $\Sigma_R(X)$ is simply the underlying set of $R^{(X)}$, i.e. the set of all formal R-linear combinations $\lambda_1\{x_1\} + \cdots + \lambda_n\{x_n\}$, where $n \geq 0$, all $\lambda_i \in R$, $x_i \in X$, and where we have denoted by $\{x\}$ the basis element corresponding to $x \in X$. Another description: $\Sigma_R(X) = \{\lambda : X \to R \mid \lambda(x) = 0 \text{ for almost all } x \in X\}$.

Clearly, the unit $\varepsilon_X : X \to \Sigma_R(X)$ maps any $x \in X$ into the corresponding basis element $\{x\}$, and $\mu_X : \Sigma_R^2(X) \to \Sigma_R(X)$ computes linear combinations of formal linear combinations (cf. **2.14.8**): $\mu_X\bigl(\sum_i \lambda_i\{\sum_j \mu_{ij}\{x_j\}\}\bigr)$ equals $\sum_{i,j} \lambda_i\mu_{ij}\{x_j\}$.

Now a Σ_R-structure $\alpha : \Sigma_R(X) \to X$ on some set X can be considered as a way of evaluating formal linear combinations of elements of X. In particular, we can define an addition and an action of R on X by setting $x + y := \alpha(\{x\}+\{y\})$ and $\lambda\cdot x := \alpha(\lambda\{x\})$. Similarly to what we had in **3.4.2**, one can check directly that this addition and R-action satisfy the axioms for a left R-module, thus obtaining a one-to-one correspondence between Σ_R-structures and left R-module structures. This shows that the categories R-*Mod* and Sets^{Σ_R} are equivalent and actually isomorphic, hence Γ_R is monadic.

3.4.9. (The underlying set of a monad.) Can we recover the ring R, or at least its underlying set, starting from the corresponding monad Σ_R? The answer is *positive*, since $R = \Sigma_R(\mathbf{1})$, where $\mathbf{1} = \{1\}$ is the standard one-element set (i.e. a final object of *Sets*). Moreover, we can recover the multiplication on R, as well as the action of R on the underlying set of any Σ_R-module X.

To do this observe that $R_s := L_{\Sigma_R}(\mathbf{1})$ is actually the ring R, considered as a left module over itself. Now $R \cong \mathrm{End}_R(R_s)^{op} = \mathrm{End}_{\Sigma_R}(L_{\Sigma_R}(\mathbf{1}))^{op}$, and this is an isomorphism of monoids, if we consider the multiplication on End's defined by composition of endomorphisms. Moreover, the action of R on any R-module (i.e. Σ_R-module) M can be recovered from the canonical isomorphism $\mathrm{Hom}_{\mathit{Sets}}(\mathbf{1}, M) \cong \mathrm{Hom}_{\Sigma_R}(L_{\Sigma_R}(\mathbf{1}), M) = \mathrm{Hom}_R(R_s, M)$, and from the right action of $\mathrm{End}_R(R_s)$ on this set, given again by the composition of homomorphisms.

This construction can be generalized to any monad Σ over *Sets*. For any such monad we define its *underlying set* $|\Sigma|$ (sometimes denoted simply by Σ) by $|\Sigma| := \Sigma(\mathbf{1})$. Clearly, $|\Sigma| \cong \mathrm{Hom}_{\mathit{Sets}}(\mathbf{1}, \Sigma(\mathbf{1})) \cong \mathrm{End}_\Sigma(L_\Sigma(\mathbf{1})) \cong \mathrm{End}_{\mathit{Sets}_\Sigma}(\mathbf{1})$, and we introduce a monoid structure on $|\Sigma|$ by transporting the natural monoid structure of $\mathrm{End}_\Sigma(L_\Sigma(\mathbf{1}))$ given by composition of endomorphisms and taking the opposite. This defines the *underlying monoid* of Σ, which will be also denoted by $|\Sigma|$. Clearly, $\{1\} := \varepsilon_{\mathbf{1}}(1) \in |\Sigma|$ is the unit for this multiplication.

Moreover, for any Σ-module M we get a canonical right action of monoid $\mathrm{End}_\Sigma(L_\Sigma(\mathbf{1}))$ on $\mathrm{Hom}_\Sigma(L_\Sigma(\mathbf{1}), M) \cong \mathrm{Hom}_{\mathit{Sets}}(\mathbf{1}, M) \cong M$, hence a left action of the opposite monoid $|\Sigma| \cong \mathrm{End}_\Sigma(L_\Sigma(\mathbf{1}))^{op}$. All these constructions are functorial: for example, any Σ-homomorphism $f : M \to N$ is at the same time a map of $|\Sigma|$-sets, any homomorphism of monads $\rho : \Sigma \to \Xi$ induces a monoid homomorphism $|\rho| := \rho_{\mathbf{1}} : |\Sigma| \to| \Xi|$, and the scalar restriction functors $\rho^* : \mathit{Sets}^\Xi \to \mathit{Sets}^\Sigma$ are compatible with these constructions as well, i.e. the $|\Sigma|$-action on $\rho^* N$ coincides with the scalar restriction of the $|\Xi|$-action on N with respect to $|\rho|$.

Of course, in any particular case there is some extra information which cannot be recovered by this general construction. For example, the addition of R or of an R-module is not recovered in this way from Σ_R.

3.4.10. (Functoriality of Σ_R.) Any ring homomorphism $\rho : R \to S$ induces a homomorphism $\Sigma_\rho : \Sigma_R \to \Sigma_S$ of corresponding monads, explicitly given by the collection of maps $\Sigma_{\rho,X} : \Sigma_R(X) \to \Sigma_S(X)$, $\sum_{i=1}^n \lambda_i\{x_i\} \mapsto \sum_{i=1}^n \rho(\lambda_i)\{x_i\}$. It is easy to see that this natural transformation is indeed compatible with the multiplication and the unit of these monads, and that $\Sigma_\rho^* : \mathit{Sets}^{\Sigma_S} = S\text{-}\mathit{Mod} \to \mathit{Sets}^{\Sigma_R} = R\text{-}\mathit{Mod}$ is just the usual scalar restriction functor $\rho^* : S\text{-}\mathit{Mod} \to R\text{-}\mathit{Mod}$; by uniqueness of adjoints we see that the scalar extension functor $(\Sigma_\rho)_*$ constructed in **3.3.19** is isomorphic to the usual scalar extension functor $\rho_* : R\text{-}\mathit{Mod} \to S\text{-}\mathit{Mod}$, $M \mapsto S \otimes_R M$.

Note that $|\Sigma_\rho| = \rho$, hence ρ can be recovered from $|\Sigma_\rho|$. Hence the functor $R \mapsto \Sigma_R$ from the category of rings into the category of monads over *Sets* is at least faithful. We claim that *the functor $R \mapsto \Sigma_R$ is fully faithful, hence it identifies the category of rings with a full subcategory of the category of monads.* In other words, any monad homomorphism $\zeta : \Sigma_R \to \Sigma_S$ is of form Σ_ρ for a (uniquely determined) ring homomorphism $\rho : R \to S$. Clearly, we must have $\rho = |\zeta| : R \to S$. We postpone some details of the proof until **4.8**, where we develop a suitable machinery for such tasks. Up to these details the proof goes as follows.

We know already that $\rho := |\zeta|$ is a monoid homomorphism, and we'll check later that it respects addition as well. Thus we are reduced to check $\Sigma_{\rho,X} = \zeta_X : \Sigma_R(X) \to \Sigma_S(X)$ for any set X. Since both Σ_R and Σ_S commute with filtered inductive limits and any set is a filtered inductive limit of its finite subsets, we can assume X to be finite, e.g. $X = \mathbf{n} = \{1, 2, \ldots, n\}$. Then we construct canonical projections $\mathrm{pr}_i : \Sigma_R(\mathbf{n}) \to \Sigma_R(\mathbf{1}) = |\Sigma_R| = R$, using only the monad structure of Σ_R, and combine them to obtain a canonical isomorphism $\Sigma_R(\mathbf{n}) \xrightarrow{\sim} R^n$. Since this construction uses only the monad structure of Σ_R, it has to be compatible with ζ, i.e. it fits into a commutative diagram relating $\zeta_\mathbf{n}$ with $(\zeta_\mathbf{1})^n = \rho^n$. This proves $\zeta_\mathbf{n} = \Sigma_{\rho,\mathbf{n}}$. This also proves that ρ respects addition as well, since the only map $\mathbf{2} = \{1, 2\} \to \mathbf{1}$ induces a map $R^2 \cong \Sigma_R(\mathbf{2}) \to R = \Sigma_R(\mathbf{1})$, which is nothing else than the addition of R. Now the functoriality of this construction with respect to $\zeta : \Sigma_R \to \Sigma_S$ and the identification of $\zeta_\mathbf{2}$ with ρ^2 proves the compatibility of ρ with the addition on R and S.

3.4.11. (Monads as generalized rings.) We have embedded the category of rings into the category of monads over *Sets* as a full subcategory. We will often identify a ring R with the corresponding monad Σ_R, thus writing L_R instead of L_{Σ_R}, and on some occasion even $R(X)$ instead of $\Sigma_R(X)$. Moreover, we can treat arbitrary monads over *Sets* as some sort of non-commutative generalized rings, and transport some notations usual for rings into this more general context. For example, we write $\Sigma\text{-}\mathit{Mod}$ instead of Sets^Σ, Hom_Σ instead of $\mathrm{Hom}_{\mathit{Sets}^\Sigma}$, and even $\Sigma^{(X)}$ instead of $\Sigma(X)$ or $L_\Sigma(X)$.

However, one should be careful with these identifications, since to obtain the correct category of non-commutative generalized rings we have to impose the algebraicity condition, satisfied by all monads we considered so far. For example, infinite projective limits computed in the category of all monads (cf. **3.3.3**) and in the category of algebraic monads are different.

Notice that $R \mapsto \Sigma_R$ *is left exact, i.e. it commutes with finite projective limits.* Indeed, given such a limit $R = \varprojlim R_\alpha$, we first observe that $\Sigma_R(X) \cong \varprojlim \Sigma_{R_\alpha}(X)$ is true for a finite X, since in this case $\Sigma_R(\mathbf{n}) \cong R^n$ and similarly for Σ_{R_α}, and then extend this result to the case of an arbitrary X by representing X as a filtered inductive limit of finite sets and observing that filtered inductive limits commute with finite projective limits.

Nevertheless, if we compute for example $\hat{\mathbb{Z}}_p := \varprojlim \mathbb{Z}/p^n\mathbb{Z}$ in the category of monads, we obtain a monad $\hat{\mathbb{Z}}_p$ which is not algebraic and in particular is different from $\mathbb{Z}_p$, regardless of the fact that $|\hat{\mathbb{Z}}_p| = \mathbb{Z}_p$. Actually the category of $\hat{\mathbb{Z}}_p$-modules turns out to be quite similar to that of *topological* $\mathbb{Z}_p$-modules. In particular, it doesn't have the usual properties of a category of modules over a ring.

3.4.12. (Subrings and submonads.) Clearly, if R' is a subring of R, then $\Sigma_{R'}(X) \subset \Sigma_R(X)$ for any set X, i.e. $\Sigma_{R'}$ is a *submonad* of Σ_R (cf. **3.3.3**). However, not all submonads of Σ_R come from subrings of R. Let's give some examples of such submonads.

a) Consider the submonad $\mathbb{Z}_{\geq 0}$ of $\mathbb{Z}$, defined by $\mathbb{Z}_{\geq 0}^{(X)} := \{$formal linear combinations of elements of X with *non-negative* integer coefficients$\}$. Clearly, $\mathbb{Z}_{\geq 0}$ is a subfunctor of $\mathbb{Z} = \Sigma_{\mathbb{Z}}$, and it is compatible with the multiplication and the unit of $\Sigma_{\mathbb{Z}}$, since all basis vectors $\{x\}$ belong to $\mathbb{Z}_{\geq 0}^{(X)}$, and a linear combination with non-negative integer coefficients of several such formal linear combinations is again a formal linear combination with such coefficients. Hence $\mathbb{Z}_{\geq 0}$ is indeed a submonad of $\mathbb{Z}$. Notice that $|\mathbb{Z}_{\geq 0}|$ is indeed the multiplicative monoid of all non-negative integers, as suggested by our notation. It is easy to see that $\mathbb{Z}_{\geq 0}$-*Mod* is actually the category of commutative monoids, and that the scalar restriction functor for the inclusion $\mathbb{Z}_{\geq 0} \to \mathbb{Z}$ is simply the forgetful functor from the category of abelian groups to the category of commutative monoids. We have also a unique (surjective) homomorphism of monads $W \to \mathbb{Z}_{\geq 0}$, and the corresponding scalar restriction functor is simply the embedding of the category of commutative monoids into the category of all monoids.

b) Now consider the submonad $\mathbb{Z}_\infty \subset \mathbb{R}$, defined by the following property: $\mathbb{Z}_\infty^{(X)} \subset \mathbb{R}^{(X)}$ consists of all *octahedral* formal combinations $\sum_i \lambda_i\{x_i\}$ of elements of X, i.e. we require $\sum_i |\lambda_i| \leq 1$, $\lambda_i \in \mathbb{R}$. Clearly, any octahedral combination of formal octahedral combinations is an octahedral combination again, and all basis vectors lie in $\mathbb{Z}_\infty^{(X)}$, hence $\mathbb{Z}_\infty$ is indeed a submonad of $\mathbb{R}$. Of course, $\mathbb{Z}_\infty$-*Mod* coincides with the category denoted in the same way in **2.14.4**, where it has been defined as the category of Σ_∞-modules, for a monad Σ_∞ isomorphic to our $\mathbb{Z}_\infty$.

Clearly, $|\mathbb{Z}_\infty|$ is the segment $[-1,1] \subset \mathbb{R}$ with the multiplication induced by that of $\mathbb{R}$.

c) Similarly, we can consider $\bar{\mathbb{Z}}_\infty \subset \mathbb{C}$, such that $\bar{\mathbb{Z}}_\infty^{(X)} \subset \mathbb{C}^{(X)}$ consists of all formal $\mathbb{C}$-linear combinations $\sum_i \lambda_i\{x_i\}$, subject to condition $\sum_i |\lambda_i| \leq 1$, $\lambda_i \in \mathbb{C}$. Again, $|\bar{\mathbb{Z}}_\infty|$ is just the unit disk in $\mathbb{C}$ with the multiplication induced by that of $\mathbb{C}$.

d) Given two submonads Σ', Σ'' of the same monad Σ, we can construct a new submonad $\Sigma' \cap \Sigma'' := \Sigma' \times_\Sigma \Sigma'' \subset \Sigma$. Since projective limits of monads are computed componentwise (cf. **3.3.3**), $(\Sigma' \cap \Sigma'')(X) = \Sigma'(X) \cap \Sigma''(X) \subset \Sigma(X)$.

In particular, we can define the "non-completed localization of $\mathbb{Z}$ at infinity" $\mathbb{Z}_{(\infty)} := \mathbb{Z}_\infty \cap \mathbb{Q} \subset \mathbb{R}$, similarly to the p-adic case $\mathbb{Z}_{(p)} = \mathbb{Z}_p \cap \mathbb{Q} \subset \mathbb{Q}_p$. Clearly, $\mathbb{Z}_{(\infty)}^{(X)}$ consists of all formal octahedral combinations of elements of X with rational coefficients. This generalized ring will be important in the construction of $\widehat{\operatorname{Spec}\mathbb{Z}}$, where it will appear as the local ring at infinity.

We can intersect further and consider $\mathbb{Z}[1/p] \cap \mathbb{Z}_{(\infty)}$ for any prime p. This corresponds to considering formal octahedral combinations with p-rational numbers (i.e. rational numbers of form x/p^n) as coefficients. Later we'll construct $\widehat{\operatorname{Spec}\mathbb{Z}}$ by gluing together $\operatorname{Spec}\mathbb{Z}$ and $\operatorname{Spec}(\mathbb{Z}[1/p] \cap \mathbb{Z}_{(\infty)})$ along their principal open subsets isomorphic to $\operatorname{Spec}\mathbb{Z}[1/p]$.

e) Put $\mathbb{F}_{\pm 1} := \mathbb{Z} \cap \mathbb{Z}_{(\infty)} \subset \mathbb{Q}$, or equivalently $\mathbb{F}_{\pm 1} := \mathbb{Z} \cap \mathbb{Z}_\infty \subset \mathbb{R}$. Clearly, if we construct $\widehat{\operatorname{Spec}\mathbb{Z}}$ as described above, $\mathbb{F}_{\pm 1}$ must be the (generalized) ring of global sections of the structural sheaf of $\widehat{\operatorname{Spec}\mathbb{Z}}$. We see that $\mathbb{F}_{\pm 1}^{(X)}$ consists of all formal octahedral combinations of elements of X with *integer* coefficients. This means that there can be at most one non-zero coefficient, and it must be equal to ± 1. In other words, $\mathbb{F}_{\pm 1}^{(X)}$ consists of 0, basis elements $\{x\}$, and their opposites $-\{x\}$, i.e. $\mathbb{F}_{\pm 1}^{(X)} = 0 \sqcup X \sqcup -X$. Hence an $\mathbb{F}_{\pm 1}$-structure α on a set X is a map $0 \sqcup X \sqcup -X \to X$, identical on X. Its restriction to 0 yields a marked (or zero) element $\alpha(0)$ of X, also denoted by 0 or 0_X. The restriction of α to $-X$ gives a map $-_X : X \to X$, which can be shown to be an involution preserving 0_X. Therefore, $\mathbb{F}_{\pm 1}$-*Mod* is the category of sets with a marked point and an involution preserving this marked point.

We introduce an alternative notation $\mathbb{F}_{1^2}$ for $\mathbb{F}_{\pm 1}$. This is due to the fact that we'll construct later generalized rings $\mathbb{F}_{1^n}$, such that $\mathbb{F}_{1^n}$-*Mod* is the category of sets with a marked point and a permutation of order n which fixes the marked point.

f) We can also consider the "field with one element" $\mathbb{F}_1 := \mathbb{Z}_{\geq 0} \cap \mathbb{F}_{\pm 1} =$

$\mathbb{Z}_{\geq 0} \cap \mathbb{Z}_\infty$. The positivity condition implies that $\mathbb{F}_1^{(X)} = 0 \sqcup X$, and $\mathbb{F}_1$-*Mod* is simply the category of sets with a marked point.

g) Here are some more examples of submonads: $\boldsymbol{\Delta} \subset \mathbb{R}$ corresponds to baricentric combinations $\sum_i \lambda_i \{x_i\}$, $\lambda_i \in \mathbb{R}_{\geq 0}$, $\sum_i \lambda_i = 1$; then $\boldsymbol{\Delta}$-*Mod* is the category of abstract convex sets. We also define $\mathbb{R}_{\geq 0} \subset \mathbb{R}$ by considering linear combinations with non-negative real coefficients. For any ring R we define $\mathrm{Aff}_R \subset R$, obtained by considering formal R-linear combinations with the sum of coefficients equal to one; then Aff_R-*Mod* is the category of (abstract) affine spaces over R.

h) Finally, the identity functor Id_{Sets} admits a natural monad structure, thus becoming the initial object in the category of monads over *Sets* (cf. **3.1.6**). This monad will be denoted by $\mathbb{F}_\varnothing$, and sometimes called *the field without elements*. Note that the category of monads has a final object $\mathbf{1}$ as well, such that $\mathbf{1}^{(X)} = \{0\}$ for any set X. Clearly, this is just the monad corresponding to the classical trivial ring 0.

3.4.13. All monads of the above examples a)–h) turn out to be algebraic, i.e. they commute with filtered inductive limits of sets. Let's construct a monad $\hat{\mathbb{Z}}_\infty \supset \mathbb{Z}_\infty$ which is *not* algebraic. For this we take $\hat{\mathbb{Z}}_\infty^{(X)}$ to be the set of all expressions $\sum_x \lambda_x \{x\}$, where all λ_x are real, and $\sum_x |\lambda_x| \leq 1$. (In particular, only countably many of λ_x are $\neq 0$, otherwise this sum would be $+\infty$.) In several respects the relationship between $\mathbb{Z}_\infty$ and $\hat{\mathbb{Z}}_\infty$ is similar to the relationship between $\mathbb{Z}_p$ and $\hat{\mathbb{Z}}_p$ (cf. **3.4.11**).

3.4.14. All of the above monads, maybe with the only exception of W and W_U, behave themselves very much like ordinary rings. Let's consider an example where this is definitely not so. For this consider the monad Σ defined by the forgetful functor from the category of commutative rings to the category of sets. This functor turns out to be monadic, so Σ-*Mod* is actually the category of commutative rings. Furthermore, $\Sigma(X) = \mathbb{Z}[X]$ is the set of polynomials in variables from X with integer coefficients. In particular, $|\Sigma|$ is the set $\Sigma[T]$ of polynomials in one variable T, and the monoid structure on $|\Sigma|$ is given by the application of polynomials: $f * g = f(g)$. This is definitely not what somebody expects from the multiplicative monoid of a ring.

3.5. (Inner functors.) Given a monad Σ over a topos $\mathcal{E}$, we would like to be able to localize it, i.e. to obtain a monad Σ_S over $\mathcal{E}/S$ for any $S \in \mathrm{Ob}\,\mathcal{E}$. However, this requires some additional structure on functor Σ. We suggest one possible choice of such a structure, sufficient for our purposes.

Definition 3.5.1 *Let $P : \mathcal{C} \to \mathcal{D}$ be a functor between two cartesian categories (i.e. categories with finite products). A* diagonal structure (ρ, θ) *on P consists of a morphism $\theta : e_\mathcal{D} \to P(e_\mathcal{C})$, where $e_\mathcal{C}$ and $e_\mathcal{D}$ are the final objects* *of $\mathcal{C}$ and $\mathcal{D}$, and a family of morphisms $\{\rho_{X,Y} : P(X) \times P(Y) \to P(X \times Y)\}$ in $\mathcal{D}$, parametrized by couples of objects X, Y of $\mathcal{C}$, required to satisfy the following axioms:*

$\rho 0)$ *$\rho_{X,Y} : P(X) \times P(Y) \to P(X \times Y)$ is functorial in X and Y, i.e. for any two morphisms $f : X \to X'$ and $g : Y \to Y'$ we have $P(f \times g) \circ \rho_{X,Y} = \rho_{X',Y'} \circ (P(f) \times P(g))$;*

$\rho 1)$ *$P(X) \xrightarrow{\theta \times \mathrm{id}_{P(X)}} P(e_{\mathcal{C}}) \times P(X) \xrightarrow{\rho_{e_{\mathcal{C}},X}} P(e_{\mathcal{C}} \times X) \cong P(X)$ is the identity of $P(X)$ for any $X \in \mathrm{Ob}\,\mathcal{C}$;*

$\rho 2)$ *For any X, Y, $Z \in \mathrm{Ob}\,\mathcal{C}$ we have $\rho_{X\times Y,Z} \circ (\rho_{X,Y} \times \mathrm{id}_{P(Z)}) = \rho_{X,Y\times Z} \circ (\mathrm{id}_{P(X)} \times \rho_{Y,Z})$:*

$$\begin{tikzcd}
P(X) \times P(Y) \times P(Z) \arrow[r, "\rho_{X,Y} \times \mathrm{id}_{P(Z)}"] \arrow[d, "\mathrm{id}_{P(X)} \times \rho_{Y,Z}"] & P(X \times Y) \times P(Z) \arrow[d, "\rho_{X\times Y,Z}"] \\
P(X) \times P(Y \times Z) \arrow[r, "\rho_{X,Y\times Z}"] & P(X \times Y \times Z)
\end{tikzcd}$$

A diagonal (inner) functor *$P = (P, \rho, \theta) : \mathcal{C} \to \mathcal{D}$ is a functor $P : \mathcal{C} \to \mathcal{D}$ endowed with a diagonal structure. A* natural transformation *$\xi : (P, \rho, \theta) \to (P', \rho', \theta')$ of diagonal functors is simply a natural transformation $\xi : P \to P'$, compatible with diagonal structures: $\xi_{e_{\mathcal{C}}} \circ \theta = \theta'$ and $\xi_{X\times Y} \circ \rho_{X,Y} = \rho'_{X,Y} \circ (\xi_X \times \xi_Y)$. The category of all diagonal functors between two cartesian categories will be denoted by DiagFunct($\mathcal{C}, \mathcal{D}$); if $\mathcal{D} = \mathcal{C}$, we also write DiagEndof($\mathcal{C}$).*

3.5.2. Any left exact functor $P : \mathcal{C} \to \mathcal{D}$ admits a unique diagonal structure, given by canonical isomorphisms $P(X \times Y) \cong P(X) \times P(Y)$ and $P(e_{\mathcal{C}}) \cong e_{\mathcal{D}}$. In particular, $\mathrm{Id}_{\mathcal{C}}$ can be considered as a diagonal endofunctor on $\mathcal{C}$.

3.5.3. If $(P, \rho, \theta) : \mathcal{C} \to \mathcal{D}$ and $(Q, \rho', \theta') : \mathcal{D} \to \mathcal{E}$ are two diagonal functors, we obtain a diagonal structure (ρ'', θ'') on $QP : \mathcal{C} \to \mathcal{E}$ as follows. We put $\theta'' := Q(\theta) \circ \theta' : e_{\mathcal{E}} \to QP(e_{\mathcal{C}})$, and $\rho''_{X,Y} := Q(\rho_{X,Y}) \circ \rho'_{P(X),P(Y)}$. This actually defines a *functor* $\circ$: *DiagFunct*$(\mathcal{D}, \mathcal{E}) \times$ *DiagFunct*$(\mathcal{C}, \mathcal{D}) \to$ *DiagFunct*$(\mathcal{C}, \mathcal{E})$. Combining together these categories of diagonal functors, we obtain a strictly associative 2-category with cartesian categories for objects and diagonal functors for morphisms. We can replace the 2-category of all categories with this new 2-category in our previous considerations, thus obtaining an AU $\otimes$-structure on *DiagEndof*$(\mathcal{C})$, its $\oslash$-action on *DiagFunct*$(\mathcal{D}, \mathcal{C})$ and $\mathcal{C}$, and its $\oslash$-action on *DiagFunct*$(\mathcal{C}, \mathcal{D})$.

3.5.4. Of course, a *diagonal (inner) monad* Σ over a cartesian category $\mathcal{C}$ is simply an algebra $\Sigma = (\Sigma, \mu, \varepsilon)$ in *DiagEndof*$(\mathcal{C})$. Natural transformations $\mu : \Sigma^2 \to \Sigma$ and $\varepsilon : \mathrm{Id}_{\mathcal{C}} \to \Sigma$ define a monad structure on the underlying endofunctor Σ, and the category $\mathcal{C}^{\Sigma}$ is the same for the diagonal monad Σ and for its underlying usual monad, also denoted by Σ.

We see that a diagonal monad Σ over $\mathcal{C}$ is just a usual monad $\Sigma = (\Sigma, \mu, \varepsilon)$ over $\mathcal{C}$, together with a diagonal structure (ρ, θ) on Σ, compatible in a certain sense.

For example, the compatibility of ε with θ actually means $\theta = \varepsilon_{e_{\mathcal{C}}} : e_{\mathcal{C}} \to \Sigma(e_{\mathcal{C}})$, so we don't have to specify θ separately. Compatibility of ε with ρ means $\rho_{X,Y} \circ (\varepsilon_X \times \varepsilon_Y) = \varepsilon_{X \times Y}$. Next, the compatibility of μ with $\theta = \varepsilon_{e_{\mathcal{C}}}$ means $\rho_{e_{\mathcal{C}},X} \circ (\varepsilon_{e_{\mathcal{C}}} \times \mathrm{id}_{\Sigma(X)}) = \mathrm{id}_{\Sigma(X)}$, and the remaining compatibility of μ with ρ means $\mu_{X\times Y} \circ \Sigma(\rho_{X,Y}) \circ \rho_{\Sigma(X),\Sigma(Y)} = \rho_{X,Y} \circ (\mu_X \times \mu_Y)$:

$$\begin{tikzcd}
\Sigma^2(X) \times \Sigma^2(Y) \arrow[r, "\rho_{\Sigma(X),\Sigma(Y)}"] \arrow[d, "\mu_X \times \mu_Y"] & \Sigma(\Sigma(X) \times \Sigma(Y)) \arrow[r, "\Sigma(\rho_{X,Y})"] & \Sigma^2(X \times Y) \arrow[d, "\mu_{X\times Y}"] \\
\Sigma(X) \times \Sigma(Y) \arrow[rr, "\rho_{X,Y}"] & & \Sigma(X \times Y)
\end{tikzcd} \tag{3.5.4.1}$$

Definition 3.5.5 *Given a diagonal functor* $(P, \rho, \theta) : \mathcal{C} \to \mathcal{D}$*, we define a* P-inner functor $F = (F, \alpha) : \mathcal{C} \to \mathcal{D}$ *as follows:* F *is a functor* $\mathcal{C} \to \mathcal{D}$*, and* α *is a* P-inner structure *on* F*, i.e. a family of morphisms* $\{\alpha_{X,Y} : P(X) \times F(Y) \to F(X \times Y)\}_{X,Y \in \mathrm{Ob}\,\mathcal{C}}$*, subject to the following conditions, similar to* $\rho0)$–$\rho2)$ *of* **3.5.1***:*

$\alpha0)$ $\alpha_{X,Y} : P(X) \times F(Y) \to F(X \times Y)$ *is functorial in* X *and* Y*, i.e. for any two morphisms* $f : X \to X'$ *and* $g : Y \to Y'$ *we have* $F(f \times g) \circ \alpha_{X,Y} = \alpha_{X',Y'} \circ (P(f) \times F(g))$*;*

$\alpha1)$ $F(X) \xrightarrow{\theta \times \mathrm{id}_{F(X)}} P(e_{\mathcal{C}}) \times F(X) \xrightarrow{\alpha_{e_{\mathcal{C}},X}} F(e_{\mathcal{C}} \times X) \cong F(X)$ *is the identity of* $F(X)$ *for any* $X \in \mathrm{Ob}\,\mathcal{C}$*;*

$\alpha2)$ *For any* X, Y, $Z \in \mathrm{Ob}\,\mathcal{C}$ *we have* $\alpha_{X\times Y,Z} \circ (\rho_{X,Y} \times \mathrm{id}_{F(Z)}) = \alpha_{X,Y\times Z} \circ (\mathrm{id}_{P(X)} \times \alpha_{Y,Z})$*:*

$$\begin{tikzcd}
P(X) \times P(Y) \times F(Z) \arrow[r, "\rho_{X,Y} \times \mathrm{id}_{F(Z)}"] \arrow[d, "\mathrm{id}_{P(X)} \times \alpha_{Y,Z}"] & P(X \times Y) \times F(Z) \arrow[d, "\alpha_{X\times Y,Z}"] \\
P(X) \times F(Y \times Z) \arrow[r, "\alpha_{X,Y\times Z}"] & F(X \times Y \times Z)
\end{tikzcd}$$

A natural transformation *of P-inner functors* $\zeta : (F, \alpha) \to (F', \alpha')$ *is a natural transformation* $\zeta : F \to F'$*, such that* $\zeta_{X\times Y} \circ \alpha_{X,Y} = \alpha'_{X,Y} \circ (\mathrm{id}_{P(X)} \times \zeta_Y) : P(X)\times F(Y) \to F'(X\times Y)$*. The category of P-inner functors will be denoted by* $\mathit{InnFunct}_{/P}(\mathcal{C}, \mathcal{D})$*.*

3.5.6. Given a P-inner functor $(F, \alpha) : \mathcal{C} \to \mathcal{D}$ and a Q-inner functor $(G, \alpha') : \mathcal{D} \to \mathcal{E}$, we have a canonical QP-inner structure α'' on $GF : \mathcal{C} \to \mathcal{E}$, given by $\alpha''_{X,Y} = G(\alpha_{X,Y}) \circ \alpha'_{P(X),F(Y)} : QP(X) \times GF(Y) \to GF(X \times Y)$. In this way we construct the composition functors $\circ : \mathit{InnFunct}_{/Q}(\mathcal{D}, \mathcal{E}) \times \mathit{InnFunct}_{/P}(\mathcal{C}, \mathcal{D}) \to \mathit{InnFunct}_{/QP}(\mathcal{C}, \mathcal{E})$.

3.5.7. Notice that $P = (P, \rho, \theta)$ is itself a P-inner functor, since the axioms α0)–α2) specialize to ρ0)–ρ2) when we put $F := P$, $\alpha := \rho$. Hence the statements proved for inner functors over a diagonal functor can be transferred to diagonal inner functors themselves.

3.5.8. If $\mathcal{C} = \mathcal{D}$, we can take $P = \mathrm{Id}_{\mathcal{C}}$, thus obtaining the *category of plain inner endofunctors* $\mathit{InnEndof}(\mathcal{C})$, which consists of endofunctors $F : \mathcal{C} \to \mathcal{C}$, endowed with a plain inner structure α, $\alpha_{X,Y} : X \times F(Y) \to F(X \times Y)$. We have an AU $\otimes$-structure on $\mathit{InnEndof}(\mathcal{C})$; the algebras with respect to this $\otimes$-structure will be called *(plain) inner monads.* Clearly, a plain inner monad $\Sigma = (\Sigma, \mu, \varepsilon, \alpha)$ is simply a usual monad $(\Sigma, \mu, \varepsilon)$ over $\mathcal{C}$, together with some plain inner (i.e. $\mathrm{Id}_{\mathcal{C}}$-inner) structure α on Σ, subject to some compatibility conditions similar to those considered before. Namely, we must have

$$\varepsilon_{X\times Y} = \alpha_{X,Y} \circ (\mathrm{id}_X \times \varepsilon_Y) \tag{3.5.8.1}$$

$$\mu_{X\times Y} \circ \Sigma(\alpha_{X,Y}) \circ \alpha_{X,\Sigma(Y)} = \alpha_{X,Y} \circ (\mathrm{id}_X \times \mu_Y) : X \times \Sigma^2(Y) \to \Sigma(X \times Y) \tag{3.5.8.2}$$

Proposition 3.5.9 *Any endofunctor F over Sets admits a unique plain inner structure, i.e. the $\otimes$-categories $\mathit{InnEndof}(\mathit{Sets})$ and $\mathit{Endof}(\mathit{Sets})$ are isomorphic. In particular, any monad over Sets uniquely extends to a plain inner monad.*

Proof. For any set X and any element $x \in X$ denote by $\tilde{x} : e_{\mathit{Sets}} = \mathbf{1} = \{1\} \to X$ the only map with image x. Then $\tilde{x} \times \mathrm{id}_Y : Y \to X \times Y$ is the map $y \mapsto (x, y)$ for any set Y. A plain inner structure $\{\alpha_{X,Y} : X \times F(Y) \to F(X \times Y)\}$ is functorial with respect to X and Y; applying this to $\tilde{x}$ and taking α1) into account we see that $\alpha_{X,Y} \circ (\tilde{x} \times \mathrm{id}_{F(Y)}) = F(\tilde{x} \times \mathrm{id}_Y) : F(Y) \to F(X \times Y)$, i.e. $\alpha_{X,Y}(x, w) = (F(\tilde{x} \times \mathrm{id}_Y))(w)$ for any $x \in X$, $w \in F(Y)$. This shows the uniqueness of α, and also gives a way of defining it. It is immediate that the family $\{\alpha_{X,Y}\}$ defined by this formula indeed satisfies α0)–α2).

3.5.10. For example, the plain inner structure on the monad of words W is given by $\alpha_{X,Y}(x, y_1y_2\ldots y_n) = \{x, y_1\}\{x, y_2\}\ldots\{x, y_n\}$, where we write $\{x, y\}$ instead of $\{(x, y)\}$. For the monad W_U of words with constants from U the situation is similar, but the constants are not affected, e.g. $\alpha_{X,Y}(x, y_1u_1y_2y_3u_2) = \{x, y_1\}u_1\{x, y_2\}\{x, y_3\}u_2$.

3.5.11. (Alternative description of inner structures.) A P-inner structure $\{\alpha_{X,Y} : P(X)\times F(Y) \to F(X\times Y)\}$ on a functor $F : \mathcal{C} \to \mathcal{D}$ admits an alternative description in terms of a natural transformation γ of functors $\mathcal{C}^0\times\mathcal{C}^0\times\mathcal{C} \to \mathit{Sets}$, $\gamma_{X,Y,Z} : \mathrm{Hom}_{\mathcal{C}}(X\times Y, Z) \to \mathrm{Hom}_{\mathcal{D}}(P(X)\times F(Y), F(Z))$. Namely, for any $\varphi : X \times Y \to Z$ we put $\gamma_{X,Y,Z}(\varphi) := F(\varphi)\circ\alpha_{X,Y}$, and conversely, any such γ defines (by Yoneda) a family $\alpha_{X,Y}$, functorial in X and Y; actually, $\alpha_{X,Y} = \gamma_{X,Y,X\times Y}(\mathrm{id}_{X\times Y})$. Moreover, the axioms $\alpha0)$–$\alpha2)$ can be rewritten directly in terms of γ:

$\gamma0)$ $\gamma_{X,Y,Z} : \mathrm{Hom}_{\mathcal{C}}(X \times Y, Z) \to \mathrm{Hom}_{\mathcal{D}}(P(X) \times F(Y), F(Z))$ is functorial in X, Y and $Z \in \mathrm{Ob}\,\mathcal{C}$;

$\gamma1)$ The following diagram is commutative for any X and Y in $\mathcal{C}$:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(e_{\mathcal{C}}\times X, Y) \arrow[r, "\gamma_{e_{\mathcal{C}},X,Y}"] \arrow[d, "\sim"] & \mathrm{Hom}_{\mathcal{D}}(P(e_{\mathcal{C}})\times F(X), F(Y)) \arrow[d, "(\theta\times\mathrm{id}_{F(X)})^*"] \\
\mathrm{Hom}_{\mathcal{C}}(X, Y) \arrow[r, "F(\cdot)"] & \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))
\end{tikzcd} \tag{3.5.11.1}$$

$\gamma2)$ For any morphisms $\varphi : U \times X \to Y$ and $\psi : V \times Y \to Z$ in $\mathcal{C}$ define $\chi : V\times U\times X \to Z$ by $\chi := \psi\circ(\mathrm{id}_V\times\varphi)$. Then $\gamma_{V\times U,X,Z}(\chi)\circ(\rho_{V,U}\times \mathrm{id}_{F(X)}) = \gamma_{V,Y,Z}(\psi)\circ(\mathrm{id}_{P(V)}\times\gamma_{U,X,Y}(\varphi))$:

$$\begin{tikzcd}
P(V)\times P(U)\times F(X) \arrow[r, "\mathrm{id}_{P(V)}\times\gamma_{U,X,Y}(\varphi)"] \arrow[d, "\rho_{V,U}\times\mathrm{id}_{F(X)}"] & P(V)\times F(Y) \arrow[d, "\gamma_{V,Y,Z}(\psi)"] \\
P(V\times U)\times F(X) \arrow[r, "\gamma_{V\times U,X,Z}(\chi)"] & F(Z)
\end{tikzcd} \tag{3.5.11.2}$$

The verification of the equivalence of $\alpha0)$–$\alpha2)$ and $\gamma0)$–$\gamma2)$ doesn't present any difficulties. For example, $\alpha2)$ is obtained from $\gamma2)$ by setting $\varphi = \mathrm{id}$, $\psi = \mathrm{id}$, and the converse follows from the explicit formula for γ and the functoriality of α.

3.5.12. (Inner structures over cartesian closed categories.) Now suppose that both $\mathcal{C}$ and $\mathcal{D}$ are *cartesian closed,* i.e. they are cartesian, and the functor $\mathrm{Hom}(-\times Y, Z)$ is representable by some object $\mathbf{Hom}(Y, Z)$ (*local* or *inner* Hom), for any couple of objects Y and Z from $\mathcal{C}$ or from $\mathcal{D}$. The canonical bijection $\mathrm{Hom}(X \times Y, Z) \xrightarrow{\sim} \mathrm{Hom}(X, \mathbf{Hom}(Y, Z))$ will be denoted $\varphi \mapsto \varphi^\flat$, and its inverse $\psi \mapsto \psi^\sharp$. We also have the *evaluation morphisms* $\mathrm{ev}_{Y,Z} := (\mathrm{id}_{\mathbf{Hom}(Y,Z)})^\sharp : \mathbf{Hom}(Y, Z) \times Y \to Z$, and the *functor of global sections* $\Gamma_{\mathcal{C}}(-) := \mathrm{Hom}_{\mathcal{C}}(e_{\mathcal{C}}, -) : \mathcal{C} \to \mathit{Sets}$, and similarly for $\mathcal{D}$. Clearly, $\Gamma_{\mathcal{C}}(\mathbf{Hom}_{\mathcal{C}}(Y, Z)) \cong \mathrm{Hom}_{\mathcal{C}}(Y, Z)$.

3.5.13. Consider now a P-inner structure α on some functor $F : \mathcal{C} \to \mathcal{D}$. We know already that it admits a description in terms of morphisms $\gamma_{X,Y,Z} : \mathrm{Hom}_{\mathcal{C}}(X \times Y, Z) \to \mathrm{Hom}_{\mathcal{D}}(P(X) \times F(Y), F(Z))$, functorial in X, Y and Z (cf. **3.5.11**). Since we have assumed $\mathcal{C}$ and $\mathcal{D}$ to be cartesian closed, we can interpet $\gamma_{X,Y,Z}$ as a morphism $\mathrm{Hom}_{\mathcal{C}}(X, \mathbf{Hom}_{\mathcal{C}}(Y, Z)) \to \mathrm{Hom}_{\mathcal{D}}(P(X), \mathbf{Hom}_{\mathcal{D}}(F(Y), F(Z))$. By Yoneda such a functorial family of morphisms is determined by its value on $\mathrm{id}_{\mathbf{Hom}_{\mathcal{C}}(Y,Z)}$, so we put $\beta_{Y,Z} := \gamma_{\mathbf{Hom}(Y,Z),Y,Z}(\mathrm{ev}_{Y,Z})^\flat : P(\mathbf{Hom}_{\mathcal{C}}(Y, Z)) \to \mathbf{Hom}_{\mathcal{D}}(F(Y), F(Z))$. This gives another equivalent description of a P-inner structure on a functor between two cartesian closed categories, and in fact it is possible to restate the axioms in terms of these morphisms β. For example, $\beta 0)$ states that $\beta_{Y,Z} : P(\mathbf{Hom}_{\mathcal{C}}(Y, Z)) \to \mathbf{Hom}_{\mathcal{D}}(F(Y), F(Z))$ is functorial in Y and Z, axiom $\beta 1)$ relates β with the global section functors, and $\beta 2)$ relates β with the composition morphisms $o_{X,Y,Z} : \mathbf{Hom}(Y, Z) \times \mathbf{Hom}(X, Y) \to \mathbf{Hom}(X, Z)$. Actually, it states $\beta_{X,Z} \circ P(o_{X,Y,Z}) \circ \rho_{\mathbf{Hom}(Y,Z),\mathbf{Hom}(X,Y)} = o_{F(X),F(Y),F(Z)} \circ (\beta_{Y,Z} \times \beta_{X,Y})$. We omit the verification of equivalence of this set of axioms $\beta 0)$–$\beta 2)$ with our other sets of axioms. Let's just remark that this description shows that an inner structure can be thought of as an extension of $F(-) : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{D}}(F(Y), F(Z))$ to the local Homs. This also shows again that in case $\mathcal{C} = \mathcal{D} = \mathit{Sets}$ there is always a unique plain inner structure, since in this case local Homs coincide with the usual ones.

3.5.14. Here is one application of (plain) inner endofuctors: *Any plain inner endofunctor (F, α) over a category $\mathcal{C}$ with finite projective limits defines a family of endofunctors $F_{/S} : \mathcal{C}/S \to \mathcal{C}/S$, parametrized by objects S of $\mathcal{C}$.* For any object $p : X \to S$ of $\mathcal{C}/S$ we construct $F_{/S}(p) \in \mathrm{Ob}\,\mathcal{C}/S$ as follows. We consider the graph $\Gamma'_p = (p, \mathrm{id}_X) : X \to S \times X$, and define $F_{/S}(p)$ from the first line of the following diagram, involving a cartesian square:

$$\begin{tikzcd} F_{/S}(p) \arrow[r, dashed] \arrow[d, dashed] & S \times F(X) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\alpha_{S,X}"] & S \\ F(X) \arrow[r, "F(\Gamma'_p)"] & F(S \times X) & \end{tikzcd} \tag{3.5.14.1}$$

This construction is functorial in $F = (F, \alpha)$ and S, but in general it doesn't respect composition of functors, hence it cannot be used for the localization of monads. Notice that we don't even obtain a natural plain inner structure on functors $F_{/S}$. However, we'll see in the next chapter that this construction behaves very nicely when $\mathcal{C} = \mathcal{E}$ is a topos and F is an algebraic inner endofunctor or monad over $\mathcal{E}$; this is exactly the case we need.

3.5.15. Here is another useful application of plain inner monads. Given a monad Σ over *Sets*, any Σ-object $M = (M, \sigma)$ and any set I, the local Hom $\mathbf{Hom}(I, M)$ equals M^I, hence it has a canonical Σ-structure, namely, the product Σ-structure (cf. **3.1.12**). We want to generalize this to the case of a monad Σ over a cartesian closed category $\mathcal{C}$, an object $M = (M, \sigma) \in \operatorname{Ob} \mathcal{C}^\Sigma$, and any $I \in \operatorname{Ob} \mathcal{C}$. However, to obtain a Σ-structure on $H := \mathbf{Hom}_\mathcal{C}(I, M)$ we need Σ to be a plain inner monad, i.e. we need a compatible plain inner structure α on Σ. In this case we define a Σ-structure $\tau : \Sigma(H) \to H$ by the following diagram:

$$\begin{tikzcd} I \times \Sigma(H) \arrow[r, "\alpha_{I,H}"] & \Sigma(I \times H) \arrow[r, "\Sigma(\mathrm{ev}')"] & \Sigma(M) \arrow[d, "\sigma"] \\ \Sigma(H) \times I \arrow[u, "\sim"] \arrow[rr, "\tau^\sharp"] & & M \end{tikzcd} \tag{3.5.15.1}$$

Here $\mathrm{ev}' : I \times H = I \times \mathbf{Hom}(I, M) \to M$ is essentially the evaluation map (up to a permutation of arguments), and the left vertical arrow is the natural symmetry.

It is easy to check that in case $\mathcal{C} = \mathit{Sets}$, when any monad Σ admits a unique compatible plain inner structure (cf. **3.5.9**), we obtain again the product Σ-structure on $\mathbf{Hom}(I, M) = M^I$. In the general case we have to check that τ is indeed a Σ-structure on H, i.e. that $\tau \circ \varepsilon_H = \mathrm{id}_H$ and $\tau \circ \Sigma(\tau) = \tau \circ \mu_H$. Both these statements follow from the definition of τ and from the compatibility conditions for α recalled in **3.5.8**; we suggest the reader to skip their verification presented below.

a) First of all, denote by $\alpha'_X : \Sigma(X) \times I \to \Sigma(X \times I)$ the morphism deduced from $\alpha_{I,X} : I \times \Sigma(X) \to \Sigma(I \times X)$ by permuting the factors in direct products. Then $\tau^\sharp = \sigma \circ \Sigma(\mathrm{ev}) \circ \alpha'_H : \Sigma(H) \times I \to M$, where $\mathrm{ev} = \mathrm{ev}_{H,I} = \mathrm{id}_H^\sharp : H \times I \to M$ is the evaluation map. Now we compute $(\tau \circ \varepsilon_H)^\sharp = \tau^\sharp \circ (\varepsilon_H \times \mathrm{id}_I) = \sigma \circ \Sigma(\mathrm{ev}) \circ \alpha'_H \circ (\varepsilon_H \times \mathrm{id}_I)$; according to (3.5.8.1), this equals $\sigma \circ \Sigma(\mathrm{ev}) \circ \varepsilon_{H \times I} = \sigma \circ \varepsilon_M \circ \mathrm{ev} = \mathrm{ev} = (\mathrm{id}_H)^\sharp$. This proves $\tau \circ \varepsilon_H = \mathrm{id}_H$.

b) Notice that $\mathrm{ev} \circ (\tau \times \mathrm{id}_I) = \tau^\sharp$. Now compute $(\tau \circ \Sigma(\tau))^\sharp = \tau^\sharp \circ (\Sigma(\tau) \times \mathrm{id}_I) = \sigma \circ \Sigma(\mathrm{ev}) \circ \alpha'_H \circ (\Sigma(\tau) \times \mathrm{id}_I) = \sigma \circ \Sigma(\mathrm{ev}) \circ \Sigma(\tau \times \mathrm{id}_I) \circ \alpha'_{\Sigma(H)} = \sigma \circ \Sigma(\tau^\sharp) \circ \alpha'_{\Sigma(H)} = \sigma \circ \Sigma(\sigma) \circ \Sigma^2(\mathrm{ev}) \circ \Sigma(\alpha'_H) \circ \alpha'_{\Sigma(H)}$. Now we use (3.5.8.2) together with $\sigma \circ \Sigma(\sigma) = \sigma \circ \mu_M$, thus obtaining $\sigma \circ \mu_M \circ \Sigma^2(\mathrm{ev}) \circ \Sigma(\alpha'_H) \circ \alpha'_{\Sigma(H)} = \sigma \circ \Sigma(\mathrm{ev}) \circ \mu_{H \times I} \circ \Sigma(\alpha'_H) \circ \alpha'_{\Sigma(H)} = \sigma \circ \Sigma(\mathrm{ev}) \circ \alpha'_H \circ (\mu_H \times \mathrm{id}_I) = \tau^\sharp \circ (\mu_H \times \mathrm{id}_I) = (\tau \circ \mu_H)^\sharp$. This proves $\tau \circ \Sigma(\tau) = \tau \circ \mu_H$, hence $(H, \mu_H) \in \operatorname{Ob} \mathcal{C}^\Sigma$.

3.5.16. Given any diagonal inner structure ρ on a monad Σ over $\mathcal{C}$, we can retrieve a plain inner structure α on the same monad by putting $\alpha_{X,Y} := \rho_{X,Y} \circ (\varepsilon_X \times \mathrm{id}_{\Sigma(Y)})$. Conversely, we might try to extend a plain inner structure α to a diagonal inner structure ρ by defining $\rho_{X,Y} : \Sigma(X) \times \Sigma(Y) \to \Sigma(X \times Y)$ from the following commutative diagram:

$$\begin{tikzcd}
\Sigma(X) \times \Sigma(Y) \arrow[rrr, "\rho_{X,Y}"] \arrow[d, "\sim"] & & & \Sigma(X \times Y) \\
\Sigma(Y) \times \Sigma(X) \arrow[r, "\alpha_{\Sigma(Y),X}"] & \Sigma(\Sigma(Y) \times X) \arrow[r, "\sim"] & \Sigma(X \times \Sigma(Y)) \arrow[r, "\Sigma(\alpha_{X,Y})"] & \Sigma^2(X \times Y) \arrow[u, "\mu_{X \times Y}"']
\end{tikzcd} \tag{3.5.16.1}$$

Strictly speaking, we have to check that these formulas indeed define some plain inner structure α starting from a diagonal inner structure ρ, and study the exact conditions under which a plain inner monad extends to a diagonal inner monad in this manner; it is true in general that we always obtain a diagonal inner structure on the underlying endofunctor of Σ, which need not be compatible with the monad structure. The verifications are straightforward but lengthy; we omit them and consider an example instead.

Let $\mathcal{C} = \mathit{Sets}$, $\Sigma = W$ be the monad of words (cf. **3.4.1**), and consider the only compatible plain inner structure α on W (cf. **3.5.9**); $\alpha_{X,Y} : X \times W(Y) \to W(X \times Y)$ is given by $(x, y_1 y_2 \ldots y_n) \mapsto \{x, y_1\}\{x, y_2\} \ldots \{x, y_n\}$ (cf. **3.5.10**). Then the corresponding diagonal inner structure $\rho_{X,Y} : W(X) \times W(Y) \to W(X \times Y)$ maps a pair of words $(x_1 x_2 \ldots x_m, y_1 y_2 \ldots y_n)$ into the word $\{x_1, y_1\}\{x_1, y_2\} \ldots \{x_1, y_n\}\{x_2, y_1\} \ldots \{x_m, y_n\}$, i.e. we list all pairs of letters (x_i, y_j) of the two original words in the lexicographical order with respect to the pair of indices (i, j). Notice that this is indeed a diagonal inner structure on the underlying endofunctor W, but it is *not* compatible with the monad structure. For example, put $X := \{a, b\}$, $Y := \{x, y, z\}$ and consider the two maps $W^2(X) \times W^2(Y) \to W(X \times Y)$ which have to be equal by (3.5.4.1). The images of $(\{\{a\}\{b\}\}, \{\{x\}\{y\}\}\{\{z\}\}) \in W^2(X) \times W^2(Y)$ under these two maps are distinct: $\{a, x\}\{a, y\}\{a, z\}\{b, x\}\{b, y\}\{b, z\} \neq \{a, x\}\{a, y\}\{b, x\}\{b, y\}\{a, z\}\{b, z\}$.

3.5.17. Notice that the diagonal inner structure ρ on endofunctor Σ constructed above in general need not be *commutative*, i.e. the following diagram in general does *not* commute:

$$\begin{tikzcd}
\Sigma(X) \times \Sigma(Y) \arrow[r, "\rho_{X,Y}"] \arrow[d, "\sim"] & \Sigma(X \times Y) \arrow[d, "\sim"] \\
\Sigma(Y) \times \Sigma(X) \arrow[r, "\rho_{Y,X}"] & \Sigma(Y \times X)
\end{tikzcd} \tag{3.5.17.1}$$

It is easy to see that the family $\rho'_{X,Y} : \Sigma(X)\times\Sigma(Y) \to \Sigma(X\times Y)$, obtained from ρ by permuting factors in direct products, is another diagonal structure on Σ, also restricting to the same plain inner structure α.

We say that a diagonal (resp. plain) inner monad is *commutative* if $\rho' = \rho$ (resp. if this is true for diagonal inner structure constructed above), i.e. if this diagram commutes. Since any monad over *Sets* extends uniquely to a plain inner monad, the notion of commutativity makes sense in this context as well. For example, the monad of words is not commutative, while the monad Σ_R defined by an associative ring R is commutative iff R is commutative, since in this case $\rho_{X,Y}$ maps $\bigl(\sum_i \lambda_i\{x_i\}, \sum_j \mu_j\{y_j\}\bigr)$ into $\sum_{i,j} \lambda_i\mu_j\{x_i, y_j\}$.

An interesting statement is that *if the diagonal inner structure ρ defined by some plain inner structure α on a monad Σ is commutative, then it is compatible with the monad structure of* Σ. It seems to be known in similar contexts (cf. **3.5.19** below), so we omit the proof. We won't use it anyway.

3.5.18. Let $\Sigma = (\Sigma, \mu, \varepsilon, \alpha)$ be a plain inner monad over a cartesian closed category $\mathcal{C}$, and let $M = (M, \sigma)$ and $N = (N, \tau)$ be two objects of $\mathcal{C}^\Sigma$. Then we can construct "local inner Homs" $\mathbf{Hom}_{\mathcal{C},\Sigma}(M,N) = \mathbf{Hom}_\Sigma(M,N) \subset \mathbf{Hom}(M,N)$, such that $\Gamma_{\mathcal{C}}(\mathbf{Hom}_\Sigma(M,N)) = \mathrm{Hom}_\Sigma(M,N)$. To do this consider an arbitrary morphism $i : H' \to H := \mathbf{Hom}(M,N)$, which corresponds by adjointness to some $i^\sharp : H' \times M \to N$. We say that H' *acts by* Σ-*morphisms from* M *to* N if the following diagram commutes:

$$\begin{tikzcd} H' \times \Sigma(M) \arrow[rr, "\mathrm{id}_{H'}\times\sigma"] \arrow[d, "\alpha_{H',M}"] & & H' \times M \arrow[d, "i^\sharp"] \\ \Sigma(H' \times M) \arrow[r, "\Sigma(i^\sharp)"] & \Sigma(N) \arrow[r, "\tau"] & N \end{tikzcd} \tag{3.5.18.1}$$

In particular, we can take $H' = H$, $i = \mathrm{id}_H$ and obtain two morphisms $H \times \Sigma(M) \rightrightarrows N$, which correspond by adjointness to some $p, q : H \rightrightarrows \mathbf{Hom}(\Sigma(M), N)$. In general the above diagram commutes for some $i : H' \to H$ iff $p \circ i = q \circ i$, i.e. iff i factorizes through the kernel $H_0 = \mathbf{Hom}_\Sigma(M,N) := \mathrm{Ker}(p,q) \subset H$. In other words, H_0 is the largest subobject of $\mathbf{Hom}(M,N)$ which acts by Σ-homomorphisms, so it is a natural candidate for $\mathbf{Hom}_\Sigma(M,N)$. Notice that $\Gamma_{\mathcal{C}}(H_0)$ corresponds to morphisms $i : e_{\mathcal{C}} \to \mathbf{Hom}(M,N)$ which factorize through H_0, i.e. which make the above diagram commutative; using $\alpha 1)$ we see that this is equivalent to $i^\sharp : M \to N$ being a Σ-homomorphism, i.e. $\Gamma_{\mathcal{C}}(\mathbf{Hom}_\Sigma(M,N)) \cong \mathrm{Hom}_\Sigma(M,N)$.

Of course, one can check directly that the composition morphisms $\circ_{M,N,P} : \mathbf{Hom}(N,P) \times \mathbf{Hom}(M,N) \to \mathbf{Hom}(M,P)$ respect these subobjects $\mathbf{Hom}_\Sigma$ once we fix Σ-structures on M, N and P; we'll prove this later for algebraic monads over a topos in another way.

Notice that in general $\mathbf{Hom}_\Sigma(M,N)$ doesn't inherit a Σ-structure from $\mathbf{Hom}(M,N)$ (cf. **3.5.15**), i.e. it is a subobject of $\mathbf{Hom}(M,N)$, but not necessarily a Σ-stable subobject. Actually, we'll prove later that $\mathbf{Hom}_\Sigma(M,N)$ is a Σ-subobject of $\mathbf{Hom}(M,N)$ for all choices of M and N from $\mathcal{C}^\Sigma$ iff Σ is commutative.

3.5.19. A terminological remark: if we consider on a cartesian category $\mathcal{C}$ the ACU $\otimes$-structure defined by the direct product ($A \otimes B := A \times B$), then our notion of plain inner structure corresponds (up to some minor details) to what people call *tensorial strength*, and plain inner monads – to *strong monads*. Furthermore, our diagonal inner monads correspond to *monoidal* (or *tensorial*) *monads*, and our commutativity condition seems to be known either under the name of *commutativity* or *symmetry* condition in this setup.

4 Algebraic monads and algebraic systems

This chapter is dedicated to the study of a very important class of endofunctors and monads — the *algebraic* endofunctors and monads over the category of sets, and their topos counterparts — algebraic (plain inner) endofunctors and monads over a (Grothendieck) topos $\mathcal{E}$. These notions admit both category-theoretic descriptions (in terms of some functors and monads) and algebraic descriptions (as sets or sheaves with some operations on them, in the spirit of universal algebra). Both these approaches are important: for example, the first of them enables us to construct (arbitrary) projective limits of algebraic monads, while the second is equally indispensable for the construction of some inductive limits (e.g. tensor products). That's why we carefully develop some language and some notation to simplify transition between these two alternative descriptions.

Let us emphasize once more that the notion of an algebraic monad (resp. algebraic inner monad over a topos) is extremely important in the sequel since in our setup it corresponds to the notion of a "(non-commutative) generalized ring" (or a sheaf of such generalized rings); in this sense the content of this chapter should be viewed as some generalization of the basic theory of (sheaves of) associative rings and algebras.

Notations. All endofunctors and monads considered in this chapter, unless otherwise specified, will be defined over a base category $\mathcal{C}$, which will be supposed to be equal either to *Sets*, or to a topos $\mathcal{E}$. Sometimes $\mathcal{E}$ is the topos of sheaves (of sets) on some site $\mathcal{S}$; then we write $\mathcal{E} = \tilde{\mathcal{S}}$. We also denote by $\mathcal{A}$ the AU $\otimes$-category $\mathit{Endof}(\mathcal{C})$.

4.1. (Algebraic endofunctors on *Sets*.) The definition itself is very simple:

Definition 4.1.1 *We say that an endofunctor $\Sigma : \mathit{Sets} \to \mathit{Sets}$ is* algebraic *if it commutes with filtered inductive limits. The full $\otimes$-subcategory of $\mathcal{A} = \mathit{Endof}(\mathit{Sets})$, consisting of all algebraic endofunctors, will be denoted $\mathit{Endof}_{alg}(\mathit{Sets})$ or $\mathcal{A}_{alg}$. Thus an* algebraic monad *is simply an algebra in $\mathcal{A}_{alg}$, i.e. a monad over Sets, the underlying functor of which commutes with arbitrary inductive limits.*

Clearly, if F and G commute with filtered inductive limits, the same is true for $F \otimes G = F \circ G$, so $\mathcal{A}_{alg}$ is indeed a $\otimes$-subcategory of $\mathcal{A}$.

4.1.2. Notice that any set X is a filtered inductive limit of the ordered set of all its finite subsets $I \subset X$, hence an algebraic endofunctor Σ is completely determined by its values on finite sets: $\Sigma(X) = \varinjlim_{\text{finite } I \subset X} \Sigma(I)$. On the other hand, any finite set is isomorphic to some *standard finite set* $\mathbf{n} = \{1, 2, \ldots, n\}$, $n \geq 0$. Let us denote by $\underline{\mathbb{N}}$ the full subcategory of *Sets*, consisting of all standard finite sets. Sometimes we identify the objects of $\underline{\mathbb{N}}$ with the set of non-negative integers, and write n instead of $\mathbf{n}$. Let us denote by $J = J_{Sets} : \underline{\mathbb{N}} \to Sets$ the inclusion functor.

We see that J is fully faithful, and that its essential image is the category of all finite sets. Since any algebraic endofunctor Σ is completely determined by its restriction to the category of finite sets, which is equivalent to $\underline{\mathbb{N}}$, we see that the canonical functor $J^* : \mathcal{A} \to Funct(\underline{\mathbb{N}}, Sets) = Sets^{\underline{\mathbb{N}}}$, given by $F \mapsto F \circ J$, induces an equivalence between $\mathcal{A}_{alg}$ and a full subcategory of $Sets^{\underline{\mathbb{N}}}$. Let's make a more precise statement:

Proposition 4.1.3 *Consider the inclusion functor $J : \underline{\mathbb{N}} \to Sets$, the corresponding restriction functor $J^* : \mathcal{A} = Endof(Sets) \to Sets^{\underline{\mathbb{N}}}$, $F \mapsto F \circ J$, and its left adjoint (i.e. left Kan extension of J) $J_! : Sets^{\underline{\mathbb{N}}} \to \mathcal{A}$, defined by the usual formula $(J_!G)(X) = \varinjlim_{\underline{\mathbb{N}}/X} G(\mathbf{n})$, where $\underline{\mathbb{N}}/X$ is the category of all maps from standard finite sets into a set X. Then $J_!$ is fully faithful, and its essential image equals $\mathcal{A}_{alg}$, hence it induces an equivalence between $Sets^{\underline{\mathbb{N}}}$ and $\mathcal{A}_{alg}$, and $J^*|_{\mathcal{A}_{alg}}$ is the quasi-inverse equivalence. Moreover, J^* commutes with arbitrary limits of functors, and $J_!$ commutes with arbitrary inductive and finite projective limits, hence $\mathcal{A}_{alg} \subset \mathcal{A}$ is stable under these types of limits in $\mathcal{A}$.*

Proof. The existence of $J_!$ follows from the existence of inductive limits of sets, and the explicit formula for $(J_!G)(X)$ is well-known and can be either checked directly or found in SGA 4 I, for example. Full faithfulness of $J_!$ is a consequence of that of J, since $(J_!G)(\mathbf{m})$ is computed as the limit of $G(\mathbf{n})$ along the category $\underline{\mathbb{N}}/\mathbf{m}$ of all maps $\mathbf{n} \to \mathbf{m}$, which has a final object $\mathrm{id}_{\mathbf{m}}$, hence $(J_!G)(\mathbf{m}) \cong G(\mathbf{m})$, i.e. $G \to J^*J_!G$ is an isomorphism, hence $J_!$ is fully faithful. Notice that $\underline{\mathbb{N}}/X$ is a filtered category, and $\varinjlim_{\underline{\mathbb{N}}/X} \mathbf{n} \cong X$, hence for any algebraic endofunctor F we have $F(X) \cong \varinjlim_{\underline{\mathbb{N}}/X} F(\mathbf{n}) = (J_!J^*F)(X)$, i.e. any algebraic functor lies indeed in the essential image of $J_!$. To prove the converse we use the alternative description of $(J_!G)(X)$ in terms of inductive limits and finite products, given in Lemma **4.1.4** below, and the fact that filtered inductive limits in *Sets* commute with finite projective limits; this shows immediately that $J_!G$ is indeed algebraic. This fact also shows that $J_!$ commutes with finite projective limits, and it has to commute with arbitrary inductive limits since it has a right adjoint; this proves the remaining statements.

Lemma 4.1.4 *Given any functor $G : \underline{\mathbb{N}} \to Sets$, its left Kan extension $J_!G : Sets \to Sets$ can be computed as follows: $(J_!G)(X)$ is the cokernel* *of the pair of maps p, q from $H_1(X) := \bigsqcup_{\varphi:\mathbf{m}\to\mathbf{n}} G(\mathbf{m}) \times X^n$ to $H_0(X) := \bigsqcup_{n\geq 0} G(\mathbf{n}) \times X^n$, defined by the following two requirements:*

a) The restriction of p to the component of $H_1(X)$ indexed by $\varphi : \mathbf{m} \to \mathbf{n}$ is the map $\mathrm{id}_{G(\mathbf{m})} \times X^\varphi : G(\mathbf{m}) \times X^n \to G(\mathbf{m}) \times X^m$, where $X^\varphi = \mathbf{Hom}(\varphi, \mathrm{id}_X) : \mathbf{Hom}(\mathbf{n}, X) = X^n \to \mathbf{Hom}(\mathbf{m}, X) = X^m$ is the canonical map $(x_1, \ldots, x_n) \mapsto (x_{\varphi(1)}, \ldots, x_{\varphi(m)})$.

b) The restriction of q to the same component is the map $G(\varphi) \times \mathrm{id}_{X^n} : G(\mathbf{m}) \times X^n \to G(\mathbf{n}) \times X^n$.

Proof. Recall that *any* inductive limit $\varinjlim_{\mathcal{I}} G$ of a functor $G : \mathcal{I} \to \mathcal{C}$ can be written as the cokernel of two maps p and q from $\bigsqcup_{(\varphi:s\to t)\in \mathrm{Ar}\,\mathcal{I}} G(s)$ to $\bigsqcup_{s\in \mathrm{Ob}\,\mathcal{I}} G(s)$, the restrictions of which to the component indexed by some $\varphi : s \to t$ is the identity morphism of $G(s)$ (resp. the morphism $G(\varphi) : G(s) \to G(t)$). Now we apply this for the computation of $(J_!G)(X) = \varinjlim_{\underline{\mathbb{N}}/X} G(\mathbf{n})$. In this case $\mathcal{I} = \underline{\mathbb{N}}/X$, $\mathrm{Ob}\,\mathcal{I} = \{\text{maps } \psi : \mathbf{n} \to X\} = \sqcup_{n\geq 0} X^n = W(X)$, and $\mathrm{Ar}\,\mathcal{I} = \{\mathbf{m} \xrightarrow{\varphi} \mathbf{n} \xrightarrow{\psi} X\} = \sqcup_{\varphi:\mathbf{m}\to\mathbf{n}} X^n$. Taking the required sums along these index sets yields the objects $H_0(X)$ and $H_1(X)$ described in the statement of the lemma, and $p, q : H_1(X) \rightrightarrows H_0(X)$ also turn out to coincide with those mentioned in the statement.

4.1.5. Notice that the injective maps $\mathbf{n} \to X$ form a cofinal set in $\underline{\mathbb{N}}/X$, hence it is sufficient to take the inductive limit along only such maps in order to compute $(J_!G)(X)$. In particular, for any endofunctor F we have $(J_!J^*F)(X) = \varinjlim_{\text{finite } I\subset X} F(I) \to F(X)$. If X is finite, this map is an isomorphism; in general it is always *injective*, i.e. $J_!J^*F \to F$ is a *monomorphism*. This follows from the fact that any monomorphism with non-empty source splits in *Sets*, hence all maps $F(I) \to F(X)$ for non-empty finite subsets $I \subset X$ are injective, hence this is true for the filtered inductive limit of the $F(I)$. Let us denote $J_!J^*F$ by F_{alg}. Clearly, $F_{alg} \subset F$ is the largest algebraic subfunctor of F, and any morphism $F' \to F$ from an algebraic F' factorizes through F_{alg}. We also know that $F \mapsto F_{alg}$ commutes with arbitrary inductive and finite projective limits (cf. **4.1.3**). Moreover, arbitrary (infinite) projective limits can be computed in $\mathcal{A}_{alg}$ by first computing it (componentwise) in $\mathcal{A}$, and then taking the largest algebraic subfunctor; alternatively, we might first restrict everything to $\underline{\mathbb{N}}$ by applying J^*, compute (componentwise) the projective limit in $Sets^{\underline{\mathbb{N}}}$, and then extend resulting functor $\underline{\mathbb{N}} \to Sets$ to an algebraic endofunctor by means of $J_!$.

4.1.6. Observe that if Σ is a monad over *Sets*, then $\Sigma_{alg} \subset \Sigma$ is a *submonad* of Σ (cf. **3.3.3**). Indeed, Σ_{alg}^2 is algebraic, hence $\Sigma_{alg}^2 \to \Sigma^2 \xrightarrow{\mu} \Sigma$ factorizes through $\Sigma_{alg} \subset \Sigma$, and similarly $\varepsilon : \mathrm{Id}_{Sets} \to \Sigma$ factorizes through Σ_{alg} since Id_{Sets} is algebraic. Clearly, $\Sigma_{alg} \to \Sigma$ is universal among all homomorphisms from algebraic monads into Σ. In other words, the inclusion functor from the category of algebraic monads into the category of all monads admits a right adjoint $\Sigma \mapsto \Sigma_{alg}$, hence it commutes with arbitrary inductive limits of algebraic monads.

4.1.7. We see that the algebraicity of a monad Σ means that $\Sigma(X) = \bigcup_{\text{finite } I\subset X} \Sigma(I)$. If we think of $\Sigma(X)$ as "the set of all formal Σ-linear combinations of elements of X", this condition means that any such Σ-linear combination actually involves only finitely many elements of X. This explains why all monads defined by algebraic systems (in terms of some operations and relations between them), e.g. the monads W and $\Sigma_R = R$ for an associative ring R, or any of the monads listed in **3.4.12**, are actually algebraic in the sense of **4.1.1**. However, monads $\hat{\mathbb{Z}}_\infty$ (cf. **3.4.13**) and $\hat{\mathbb{Z}}_p$ (cf. **3.4.11**) are clearly *not* algebraic; their largest algebraic subfunctors (which turn out to be submonads) are $\mathbb{Z}_\infty$ and $\mathbb{Z}_p$, respectively.

4.1.8. Notice that these monads like $\hat{\mathbb{Z}}_\infty$ still have a similar property: any formal $\hat{\mathbb{Z}}_\infty$-linear combination of elements of a set X involves only at most countably many elements of X. This means that if we replace $\underline{\mathbb{N}}$ by the category of all ordinals $\leq \omega$, these monads will lie in the essential image of the corresponding left Kan extension $J_{\leq\omega,!}$. This leads to the idea of *cardinal filtration* on the category of all endofunctors, when we consider those endofunctors which lie in the essential image of $J_{\leq\alpha,!}$ or $J_{<\alpha,!}$ for some "small" cardinal α (e.g. belonging to the chosen universe $\mathcal{U}$). The set of all endofunctors which belong to some step of the cardinal filtration forms a full subcategory of $\mathcal{A}$, stable under $\otimes$ (i.e. composition of functors) and arbitrary projective and inductive limits (along small index categories). We call such endofunctors *accessible* (cf. SGA 4 I); clearly, their category $\mathcal{A}_{acc}$ is a very good (rigorous) replacement for $\mathcal{A} = \mathit{Endof}(\mathit{Sets})$, which contains all interesting endofunctors and at the same time avoids all set-theoretical complications (e.g. it is a $\mathcal{U}$-category).

4.1.9. Since J^* induces an equivalence between categories $\mathcal{A}_{alg}$ and $\mathit{Sets}^{\underline{\mathbb{N}}}$, and since it is nothing else than the restriction functor, we usually denote an algebraic endofunctor F and its restriction J^*F by the same letter F. We also often write $F(n)$ instead of $F(\mathbf{n})$ or $(J^*F)(\mathbf{n})$. In this sense an algebraic endofunctor F is something like a (co)simplicial object: we get a collection of sets $\{F(n)\}_{n\geq 0}$, together with some transition maps $F(\varphi) : F(m) \to F(n)$, defined for any map of finite sets $\varphi : \mathbf{m} \to \mathbf{n}$. In this context we think of $J_!F$ (denoted again by F) as some sort of extension of F to arbitrary sets.

4.1.10. In particular, we see that the category $\mathcal{A}_{alg} \cong \mathit{Sets}^{\underline{\mathbb{N}}}$ really makes sense, i.e. it is not "too large" (it is a $\mathcal{U}$-category in the language of universes), so in this way we avoid the set-theoretical complications of **3.2.6**.

4.2. (Operations with algebraic endofunctors.) Let's express some category-theoretic operations with algebraic endofunctors in terms of corresponding "cosimplicial objects".

4.2.1. We have already seen that an algebraic endofunctor F corresponds to a "cosimplicial set", i.e. to a collection of sets $\{F(n)\}_{n\geq 0}$, together with some maps $F(\varphi) : F(m) \to F(n)$, defined for any map of finite sets $\varphi : \mathbf{m} \to \mathbf{n}$, subject to usual conditions $F(\psi \circ \varphi) = F(\psi) \circ F(\varphi)$ and $F(\mathrm{id}_{\mathbf{n}}) = \mathrm{id}_{F(n)}$. Moreover, a natural transformation $\beta : F \to G$ from an algebraic endofunctor F into another endofunctor G is given by a collection of maps $\beta_n : F(n) \to G(n)$, such that $G(\varphi) \circ \beta_m = \beta_n \circ F(\varphi)$ for any $\varphi : \mathbf{m} \to \mathbf{n}$.

4.2.2. Consider an algebraic endofunctor Σ, two arbitrary sets X and Y, and some map $\alpha : \Sigma(X) \to Y$ (if $X = Y$, we say that any such map is a *pre-action of* Σ *on* X). Now recall the expression for $\Sigma(X)$ as $\mathrm{Coker}(p, q : H_1(X) \rightrightarrows H_0(X))$, given in **4.1.4**. It shows immediately that to give such an $\alpha : \Sigma(X) \to Y$ is equivalent to giving a family of maps $\{\alpha^{(n)} : \Sigma(n) \times X^n \to Y\}_{n\geq 0}$, subject to condition

$$\alpha^{(m)} \circ (\mathrm{id}_{\Sigma(m)} \times X^{\varphi}) = \alpha^{(n)} \circ (\Sigma(\varphi) \times \mathrm{id}_{X^n}) : \Sigma(m) \times X^n \to Y \text{ for all } \varphi : \mathbf{m} \to \mathbf{n} \tag{4.2.2.1}$$

We see that $\Sigma(n)$ parametrizes something like "n-ary operations from X to Y"; if we agree to write $[t]_\alpha(x_1, x_2, \ldots, x_n)$ instead of $\alpha^{(n)}(t; x_1, x_2, \ldots, x_n)$, we see that giving an $\alpha : \Sigma(X) \to Y$ is equivalent to giving a collection of maps ("operations") $[t]_\alpha : X^n \to Y$ for each "formal n-ary operation" $t \in \Sigma(n)$, $n \geq 0$, subject to the only condition

$$\big[(F(\varphi))(t)\big]_\alpha(x_1, x_2, \ldots, x_n) = [t]_\alpha(x_{\varphi(1)}, x_{\varphi(2)}, \ldots, x_{\varphi(m)})$$
$$\text{for any } t \in \Sigma(m),\ \varphi : \mathbf{m} \to \mathbf{n}, \text{ and } x_1, \ldots, x_n \in X. \tag{4.2.2.2}$$

When we are given a pre-action $\alpha : \Sigma(X) \to X$ on some set X, we sometimes write $[t]_X$ instead of $[t]_\alpha$. Note that a map $f : X \to Y$ is compatible with some pre-actions $\alpha : \Sigma(X) \to X$ and $\beta : \Sigma(Y) \to Y$ (i.e. $f \circ \alpha = \beta \circ \Sigma(f)$) iff $f\big([t]_\alpha(x_1, x_2, \ldots, x_n)\big) = [t]_\beta\big(f(x_1), f(x_2), \ldots, f(x_n)\big)$ for all choices of t and x_i.

4.2.3. Now suppose F, G and Σ are three algebraic endofunctors, and we want to describe all natural transformations $\alpha : \Sigma F \to G$. We know that this is equivalent to describing families of morphisms $\alpha_n : \Sigma F(n) \to G(n)$, functorial in n. On the other hand, we have just seen that any such α_n is given by a family of maps $\alpha_n^{(k)} : \Sigma(k) \times F(n)^k \to G(n)$, indexed by $k, n \geq 0$, subject to compatibility conditions with respect to maps $\varphi : \mathbf{m} \to \mathbf{n}$ and $\psi : \mathbf{k} \to \mathbf{l}$, similar to those considered above.

4.2.4. Let's describe maps $\alpha : \Xi\Sigma(X) \to Y$, where Ξ and Σ are algebraic endofunctors, and X and Y are two arbitrary sets. We know that α is determined by a collection of maps $\alpha^{(n)} : \Xi(n) \times \Sigma(X)^n \to Y$. On the other hand, functors $X \mapsto \Xi(n) \times \Sigma(X)^n$ are clearly algebraic again (for any fixed $n \geq 0$), hence $\alpha^{(n)}$ is in its turn given by a collection of maps $\alpha^{(n,m)} : \Xi(n) \times \Sigma(m)^n \times X^m \to Y$, subject to some compatibility conditions with respect to maps $\varphi : \mathbf{m} \to \mathbf{m}'$ and $\psi : \mathbf{n} \to \mathbf{n}'$.

4.2.5. This result generalizes to natural transformations $\alpha : \Xi\Sigma F \to G$, where Ξ, Σ, F and G are algebraic endofunctors. We see that such natural transformations are given by compatible families of maps $\{\alpha_n^{(k,m)} : \Xi(k) \times \Sigma(m)^k \times F(n)^m \to G(n)\}_{k,m,n\geq 0}$. Of course, similar descriptions can be obtained for longer compositions of algebraic endofunctors: a map $\alpha : \Sigma_1 \cdots \Sigma_s(X) \to Y$ is given by $\alpha^{(n_1,\ldots,n_s)} : \Sigma_1(n_1) \times \Sigma_2(n_2)^{n_1} \times \cdots \times \Sigma_s(n_s)^{n_{s-1}} \times X^{n_s} \to Y$, and a natural transformation $\alpha : \Sigma_1 \cdots \Sigma_s F \to G$ is given by $\alpha_n^{(n_1,\ldots,n_s)} : \Sigma_1(n_1) \times \Sigma_2(n_2)^{n_1} \times \cdots \times \Sigma_s(n_s)^{n_{s-1}} \times F(n)^{n_s} \to G(n)$. If we put $M(m,n;\Sigma) := \Sigma(m)^n$ and consider this as "the set of $m \times n$-matrices with entries in Σ", then the above maps can be seen as some "matrix multiplication rules" (notice the inverse order of arguments!) $M(n_1,n_0;\Sigma_1) \times M(n_2,n_1;\Sigma_2) \times \cdots \times M(n_s,n_{s-1};\Sigma_s) \times M(n,n_s;F) \to M(n,n_0;G)$; this observation will be used later to compare our approach with that of M. J. Shai Haran.

4.2.6. Of course, inductive and projective limits of algebraic functors are computed in terms of corresponding "cosimplicial sets" componentwise: we have $(\varinjlim \Sigma_\alpha)(n) = \varinjlim \Sigma_\alpha(n)$, and $(\varprojlim \Sigma_\alpha)(n) = \varprojlim \Sigma_\alpha(n)$. For example, $(\Sigma_1 \times_\Sigma \Sigma_2)(n) = \Sigma_1(n) \times_{\Sigma(n)} \Sigma_2(n)$. Notice that the formula $(\varinjlim \Sigma_\alpha)(X) = \varinjlim \Sigma_\alpha(X)$ is still valid for an arbitrary set X, while the similar formula for projective limits in general doesn't hold unless the limit is finite (cf. **4.1.5**).

4.2.7. In particular, the algebraic subfunctors F' of a given algebraic functor F are given by collections of subsets $\{F'(n) \subset F(n)\}_{n\geq 0}$, stable under all maps $F(\varphi) : F(m) \to F(n)$. Of course, these are exactly the subobjects of F in $\mathcal{A}_{alg}$. Notice that after we extend F' and F to endofunctors on *Sets*, we still have $F'(X) \subset F(X)$ for any set X. Another interesting consequence: an algebraic equivalence relation R on some algebraic functor F is given by a family of equivalence relations $R(n) \subset F(n) \times F(n)$, compatible with all maps $F(\varphi)$. In this case we can construct the quotient F/R, given by

$(F/R)(n) = F(n)/R(n)$. It is again an algebraic functor, and, if we extend everything to arbitrary sets X, we still have $(F/R)(X) = F(X)/R(X)$, and $F \to F/R$ is a strict epimorphism with kernel R both in $\mathcal{A}$ and $\mathcal{A}_{alg}$.

4.2.8. Notice that any morphism $\rho : F \to G$ of algebraic endofunctors has an algebraic kernel pair $R := F \times_G F$ as well as an algebraic image $I = \rho(F) \cong F/R$. Of course, these notions can be computed componentwise: $R(n) \subset F(n) \times F(n)$ is the equivalence relation defined by $\rho_n : F(n) \to G(n)$, and $F(n)/R(n) \cong \rho_n(F(n)) = I(n) \subset G(n)$. Moreover, these formulas extend to arbitrary sets X, i.e. $R(X) \subset F(X) \times F(X)$ is still the kernel of $\rho_X : F(X) \to G(X)$ and so on.

4.2.9. What about natural transformations $\varepsilon : \mathrm{Id}_{\mathit{Sets}} \to F$ from the identity functor into an arbitrary (algebraic) endofunctor F? Since $\mathrm{Id}_{\mathit{Sets}} = \mathrm{Hom}(\mathbf{1}, -)$, we see by Yoneda that such an ε is uniquely determined by an element $\boldsymbol{e} := \varepsilon_{\mathbf{1}}(1) \in F(1)$, which can be chosen arbitrarily. In this case we get $\varepsilon_X(x) = (F(\tilde{x}))(\boldsymbol{e})$ for any set X and any $x \in X$, where $\tilde{x} : \mathbf{1} \to X$ is the only map with image x. In particular, for an algebraic F the maps $\varepsilon_n : \mathbf{n} \to F(n)$ can be reconstructed from $\boldsymbol{e} \in F(1)$ in this way.

4.2.10. According to **4.1.4**, we see that for any algebraic endofuctor Σ and any set X we have a surjection $H_0(X) = \bigsqcup_{n \geq 0} \Sigma(n) \times X^n \twoheadrightarrow \Sigma(X)$. Consider its individual component $\Sigma(n) \times X^n \to \Sigma(X)$; clearly, it maps $(t; x_1, \ldots, x_n)$ into $(\Sigma(x))(t) \in \Sigma(X)$, where $x : \mathbf{n} \to X$ is the map $k \mapsto x_k$. Let's denote this element by $t(\{x_1\}, \ldots, \{x_n\})$. For example, if we take $X := \mathbf{n}$, $x := \mathrm{id}_{\mathbf{n}}$, $x_k = k$, we get $t(\{1\}, \ldots, \{k\}) = t \in \Sigma(\mathbf{n})$. Notice that *any* element of $\Sigma(X)$ can be written in form $t(\{x_1\}, \ldots, \{x_n\})$ for some $n \geq 0$; in fact, $\Sigma(X)$ consists of all such expressions modulo the equivalence relation $\equiv$ generated by $(\varphi_* t)(\{x_1\}, \ldots, \{x_n\}) \equiv t(\{x_{\varphi(1)}\}, \ldots, \{x_{\varphi(n)}\})$ for any $\varphi : \mathbf{m} \to \mathbf{n}$, $t \in \Sigma(m)$ and $x_1, \ldots, x_n \in X$, where we put $\varphi_* t := (\Sigma(\varphi))(t)$. Moreover, since the category $\underline{\mathbb{N}}/X$ is filtering, we see that two elements $t(\{x_1\}, \ldots, \{x_n\})$ and $t'(\{y_1\}, \ldots, \{y_m\})$ are equal in $\Sigma(X)$ iff we can find some $\varphi : \mathbf{n} \to \mathbf{p}$, $\psi : \mathbf{m} \to \mathbf{p}$ and some elements $z_1, \ldots, z_p \in X$, such that $\varphi_* t = \psi_* t'$, $x_i = z_{\varphi(i)}$, and $y_j = z_{\psi(j)}$. We can even assume these elements z_k to be pairwise distinct.

Notice that $\Sigma(f) : \Sigma(X) \to \Sigma(Y)$ maps $t(\{x_1\}, \ldots, \{x_n\}) \in \Sigma(X)$ into $t\big(\{f(x_1)\}, \ldots, \{f(x_n)\}\big)$, for any map $f : X \to Y$.

4.2.11. In particular, suppose we have a map $\alpha : \Sigma(X) \to Y$, described by the collection $\{\alpha^{(n)} : \Sigma(n) \times X^n \to Y\}$ as in **4.2.2**. By definition, $\alpha^{(n)}$ is just the composite map $\Sigma(n) \times X^n \to H_0(X) \twoheadrightarrow \Sigma(X) \xrightarrow{\alpha} Y$, hence α maps an element $t(\{x_1\}, \ldots, \{x_n\}) \in \Sigma(X)$ into $\alpha^{(n)}(t; x_1, \ldots, x_n) = [t]_\alpha(x_1, \ldots, x_n)$.

4.3. (Algebraic monads.) Now we want to combine together some of previous considerations to obtain a "naïve" description of an algebraic monad

$\Sigma = (\Sigma, \mu, \varepsilon)$ (i.e. an algebra in $\mathcal{A}_{alg}$, or equivalently, a monad over *Sets* with an algebraic underlying endofunctor) in terms of a sequence of sets $\{\Sigma(n)\}_{n\geq 0}$ and some maps between these sets and their products, subject to some conditions. This naïve approach has its advantages: for example, it immediately generalizes to the case of an arbitrary category $\mathcal{C}$, and it is completely clear how to apply left exact functors to such objects (e.g. pullback, direct image or global sections functors between topoi), or how to define sheaves of them over any site $\mathcal{S}$. However, we'll develop a (plain inner) monad interpretation in the topos case as well, since it also has its advantages.

4.3.1. So let's fix an algebraic monad $\Sigma = (\Sigma, \mu, \varepsilon)$ over *Sets*, and a set X with some Σ-action $\alpha : \Sigma(X) \to X$. We are going to obtain a "cosimplicial" description of these things.

First of all, an algebraic endofunctor Σ and natural transformations $\mu : \Sigma^2 \to \Sigma$ and $\varepsilon : \mathrm{Id}_{Sets} \to \Sigma$ are uniquely determined by the following data:

a) A collection of sets $\{\Sigma(n)\}_{n\geq 0}$ and maps $\Sigma(\varphi) : \Sigma(m) \to \Sigma(n)$, defined for any map $\varphi : \mathbf{m} \to \mathbf{n}$, subject to conditions $\Sigma(\mathrm{id}_{\mathbf{n}}) = \mathrm{id}_{\Sigma(n)}$ and $\Sigma(\psi \circ \varphi) = \Sigma(\psi) \circ \Sigma(\varphi)$ (cf. **4.2.1**).

b) A collection of "multiplication" or "evaluation" maps $\mu_n^{(k)} : \Sigma(k) \times \Sigma(n)^k \to \Sigma(n)$, subject to conditions $\mu_n^{(k)} \circ (\mathrm{id}_{\Sigma(k)} \times \Sigma(n)^{\varphi}) = \mu_n^{(k')} \circ (\Sigma(\varphi) \times \mathrm{id}_{\Sigma(n)^{k'}})$ for any $\varphi : \mathbf{k} \to \mathbf{k}'$, and $\mu_n^{(k)} \circ (\mathrm{id}_{\Sigma(k)} \times \Sigma(\psi)^k) = \mu_m^{(k)}$ for any $\psi : \mathbf{m} \to \mathbf{n}$ (cf. **4.2.3**). Usually we write $[t]_{\Sigma(n)}(x_1, x_2, \ldots, x_k)$ or even $t(x_1, x_2, \ldots, x_k)$ instead of $\mu_n^{(k)}(t; x_1, x_2, \ldots, x_k)$ for any $t \in \Sigma(k)$ and $x_1, \ldots, x_k \in \Sigma(n)$. In this case the above requirements can be rewritten as $[\varphi_*(t)]_{\Sigma(n)}(x_1, \ldots, x_{k'}) = [t]_{\Sigma(n)}(x_{\varphi(1)}, \ldots, x_{\varphi(k)})$ and $\psi_*\bigl([t]_{\Sigma(m)}(x_1, \ldots, x_k)\bigr) = [t]_{\Sigma(n)}(\psi_* x_1, \ldots, \psi_* x_k)$, where we write $\varphi_*(t)$ or $\varphi_* t$ instead of $(\Sigma(\varphi))(t)$. If we put $M(m, n; \Sigma) := \Sigma(m)^n$ ("the set of $m \times n$-matrices over Σ"), then $\left(\mu_n^{(k)}\right)^m : \Sigma(k)^m \times \Sigma(n)^k \to \Sigma(n)^m$ can be interpreted as a "matrix multiplication map" $M(k, m; \Sigma) \times M(n, k; \Sigma) \to M(n, m; \Sigma)$ (cf. **4.2.5**).

c) An element $\boldsymbol{e} \in \Sigma(1)$, called the *identity* of Σ. Then $\varepsilon_X : X \to \Sigma(X)$ maps any $x \in X$ into $(\Sigma(\tilde{x}))(\boldsymbol{e})$, where $\tilde{x} : \mathbf{1} \to X$ is the map with image x (cf. **4.2.9**). We usually denote the element $\varepsilon_X(x) \in \Sigma(X)$ by $\{x\}_X$ or $\{x\}$. In particular, we have the "basis elements" $\{k\} = \{k\}_{\mathbf{n}} \in \Sigma(n)$, $1 \leq k \leq n$. For example, $\{1\}_{\mathbf{1}} = \boldsymbol{e}$. Clearly, for any map $f : X \to Y$ and any $x \in X$ we have $f_*\{x\} = (\Sigma(f))(\varepsilon_X(x)) = \{f(x)\}$.

d) Similarly, a pre-action $\alpha : \Sigma(X) \to X$ is given by a collection of maps $\alpha^{(n)} : \Sigma(n) \times X^n \to X$, subject to relation (4.2.2.1), and we usually write $[t]_\alpha(x_1, \ldots, x_n)$ or $[t]_X(x_1, \ldots, x_n)$ or even $t(x_1, \ldots, x_n)$ instead of $\alpha^{(n)}(t; x_1, \ldots, x_n)$ (cf **4.2.2**). Notice that $\mu_X : \Sigma^2(X) \to \Sigma(X)$ defines a pre-action of Σ on $\Sigma(X)$, so the notation $[t]_{\Sigma(X)}(f_1, \ldots, f_n)$ makes sense for any $t \in \Sigma(n)$ and $f_1, \ldots, f_n \in \Sigma(X)$. If we take $X = \mathbf{n}$, we recover again the maps $\mu_n^{(k)}$ of b), so our notation $[t]_{\Sigma(n)}$ is at least consistent.

4.3.2. Now we want to express in terms of these data the monad axioms $\mu \circ (\varepsilon \star \Sigma) = \mathrm{id}_\Sigma = \mu \circ (\Sigma \star \varepsilon)$ and $\mu \circ (\mu \star \Sigma) = \mu \circ (\Sigma \star \mu)$, as well as the axioms $\alpha \circ \varepsilon_X = \mathrm{id}_X$ and $\alpha \circ \Sigma(\alpha) = \alpha \circ \mu_X$ for a pre-action $\alpha : \Sigma(X) \to X$ to be an action. Let's start with the action axioms:

a) Condition $\alpha \circ \varepsilon_X = \mathrm{id}_X$ is equivalent to $\alpha_1^{(1)}(\boldsymbol{e}; x) = x$ for any $x \in X$, i.e. $[\boldsymbol{e}]_X(x) = x$; in other words, $[\boldsymbol{e}]_X : X \to X$ has to be the identity map. This condition together with functoriality in $\mathbf{n}$ implies that $[\{k\}_{\mathbf{n}}]_X : X^n \to X$ is the projection onto the k-th component.

b) Condition $\alpha \circ \Sigma(\alpha) = \alpha \circ \mu_X$ translates into the commutativity of the following diagram, which can be thought of as a sort of "associativity of matrix multiplication" $M(k, 1; \Sigma) \times M(n, k; \Sigma) \times M(1, n; X) \to M(1, 1; X)$ (cf. **4.2.5**):

$$\begin{tikzcd} \Sigma(k) \times \Sigma(n)^k \times X^n \arrow[r, "\mathrm{id}_{\Sigma(k)} \times (\alpha^{(n)})^k"] \arrow[d, "\mu_n^{(k)} \times \mathrm{id}_{X^n}"] & \Sigma(k) \times X^k \arrow[d, "\alpha^{(k)}"] \\ \Sigma(n) \times X^n \arrow[r, "\alpha^{(n)}"] & X \end{tikzcd} \tag{4.3.2.1}$$

In our operational notation this can be written as follows:

$$\big[[t]_{\Sigma(n)}(t_1, \ldots, t_k)\big]_X(x_1, \ldots, x_n) = [t]_X\big([t_1]_X(x_1, \ldots, x_n), \ldots, [t_k]_X(x_1, \ldots)\big)$$
$$\text{for } t \in \Sigma(k),\ t_1, \ldots, t_k \in \Sigma(n),\ x_1, \ldots, x_n \in X \tag{4.3.2.2}$$

It is useful to think about $t \in \Sigma(n)$ as some sort of formal polynomial or formal operation with n arguments, and to interpret the above condition as some sort of substitution rule.

c) Recall that $f : X \to Y$ is compatible with given pre-actions $\alpha : \Sigma(X) \to X$ and $\beta : \Sigma(Y) \to Y$ iff $f \circ \alpha^{(n)} = \beta^{(n)} \circ (\mathrm{id}_{\Sigma(n)} \times f^n)$, i.e. $f\big([t]_X(x_1, \ldots, x_n)\big) = [t]_Y\big(f(x_1), \ldots, f(x_n)\big)$ (cf. **4.2.2**).

4.3.3. We continue by expressing the conditions for $\Sigma = (\Sigma, \mu, \varepsilon)$ to be a monad. Notice that these conditions are some equalities between natural transformations of algebraic endofunctors, hence it is enough to require these pairs of natural transformations to coincide on standard finite sets.

First of all, $\mu_n \circ \Sigma(\mu_n) = \mu_n \circ \mu_{\Sigma(n)}$ and $\mu_n \circ \varepsilon_{\Sigma(n)} = \mathrm{id}_{\Sigma(n)}$ are equivalent to saying that the pre-action $\mu_n : \Sigma^2(n) \to \Sigma(n)$ of Σ on $\Sigma(n)$ is actually an action, so we obtain conditions **4.3.2**a) and b) for $(X, \alpha) = (\Sigma(n), \mu_n)$, i.e. we have to replace $\alpha^{(k)}$ with $\mu_n^{(k)}$. We obtain the following two conditions:

a) $\mu_n^{(1)}(\boldsymbol{e}; t) = t$ for any $t \in \Sigma(n)$ and any $n \geq 0$. In other words, we require $[\boldsymbol{e}]_{\Sigma(n)}(t) = t$, i.e. $[\boldsymbol{e}]_{\Sigma(n)} = \mathrm{id}_{\Sigma(n)}$.

b) The associativity $\mu\circ(\Sigma\star\mu) = \mu\circ(\mu\star\Sigma)$ translates into some sort of "associativity of matrix multiplication" $M(k,1;\Sigma)\times M(n,k;\Sigma)\times M(m,n;\Sigma) \to M(m,1;\Sigma)$, or, if the reader prefers, $M(k,s;\Sigma)\times M(n,k;\Sigma)\times M(m,n;\Sigma) \to M(m,s;\Sigma)$, where we put $M(m,n;\Sigma) := \Sigma(m)^n$ (cf. **4.2.5**):

$$\begin{tikzcd} \Sigma(k)\times\Sigma(n)^k\times\Sigma(m)^n \arrow[r, "\mathrm{id}_{\Sigma(k)}\times(\mu_m^{(n)})^k"] \arrow[d, "\mu_n^{(k)}\times\mathrm{id}_{\Sigma(m)^n}"] & \Sigma(k)\times\Sigma(m)^k \arrow[d, "\mu_m^{(k)}"] \\ \Sigma(n)\times\Sigma(m)^n \arrow[r, "\mu_m^{(n)}"] & \Sigma(m) \end{tikzcd} \qquad (4.3.3.1)$$

Of course, this can be rewritten in form (4.3.2.2), where we put $X := \Sigma(m)$.

c) Now only the axiom $\mu_n\circ\Sigma(\varepsilon_n) = \mathrm{id}_{\Sigma(n)}$ remains. According to **4.2.10**, $\Sigma(\varepsilon_n)$ maps $t = t(\{1\},\ldots,\{n\}) \in \Sigma(n)$ into $t(\{\{1\}\},\ldots,\{\{n\}\}) \in \Sigma^2(n)$, and **4.2.11** shows that μ_n maps this element to $[t]_{\Sigma(n)}(\{1\},\ldots,\{n\})$, hence the last axiom boils down to

$$[t]_{\Sigma(n)}\bigl(\{1\}_{\mathbf{n}},\ldots,\{n\}_{\mathbf{n}}\bigr) = t \text{ for any } t\in\Sigma(n) \qquad (4.3.3.2)$$

This is something like the identity $F(T_1,T_2,\ldots,T_n) = F$ for any polynomial F from $\mathbb{Z}[T_1,T_2,\ldots,T_n]$ (actually, if we consider the monad defined by the category of commutative rings, we see that this identity is indeed a special case of (4.3.3.2)). Another possible interpretation: $I_n := (\{1\},\ldots,\{n\}) \in \Sigma(n)^n = M(n,n;\Sigma) =: M(n,\Sigma)$ is the "identity matrix", i.e. it induces the identity map $\Sigma(n)\to\Sigma(n)$.

d) Finally, let us express the conditions for a natural transformation $\rho : \Sigma\to\Xi$ to be a homomorphism of algebraic monads. Of course, ρ is given by a collection of maps $\rho_n : \Sigma(n)\to\Xi(n)$, such that $\Xi(\varphi)\circ\rho_m = \rho_n\circ\Sigma(\varphi)$ for any $\varphi : \mathbf{m}\to\mathbf{n}$. Compatibility with the unit $\rho\circ\varepsilon_\Sigma = \varepsilon_\Xi$ translates into $\rho_1(\boldsymbol{e}_\Sigma) = \boldsymbol{e}_\Xi$, and compatibility with the multiplication $\rho\circ\mu_\Sigma = \mu_\Xi\circ(\rho\star\rho)$ translates into $\rho_n\circ\mu_{\Sigma,n}^{(k)} = \mu_{\Xi,n}^{(k)}\circ(\rho_k\times\rho_n^k)$, or, equivalently, into

$$\rho_n\bigl([t]_{\Sigma(n)}(t_1,\ldots,t_k)\bigr) = [\rho_k(t)]_{\Xi(n)}\bigl(\rho_n(t_1),\ldots,\rho_n(t_k)\bigr)$$
$$\text{for any } t\in\Sigma(k),\ t_1,\ldots,t_k\in\Sigma(n) \qquad (4.3.3.3)$$

Sometimes we identify ρ with the "graded" map $\|\rho\| := \sqcup\rho_n : \|\Sigma\| = \bigsqcup_{n\geq 0}\Sigma(n) \to \|\Xi\| = \bigsqcup_{n\geq 0}\Xi(n)$, and write $\rho(t)$ or even ρt instead of $\rho_n(t)$ for some $t\in\Sigma(n)$.

4.3.4. Consider the case when we require all $\Sigma(n)$ to be modules over a commutative ring K, all $\Sigma(\varphi)$ to be K-linear, and all $\mu_n^{(k)} : \Sigma(k)\times\Sigma(n)^k \to \Sigma(n)$ to be K-polylinear. Then $\mu_n^{(k)}$ induces some $\tilde{\mu}_n^{(k)} : \Sigma(k)\otimes\Sigma(n)^{\otimes k} \to$

$\Sigma(n)$, and we can rewrite the above conditions for Σ to be an algebraic monad in terms of these maps $\tilde{\mu}_n^{(k)}$, thus obtaining nothing else than the definition of an *operad over* K (up to some minor points: in fact, we obtain an equivalent description of operads with compatible cosimplicial structure). Of course, the definition of an operad makes sense in any ACU $\otimes$-category $\mathcal{C} = (\mathcal{C}, \otimes)$; if we put $(\mathcal{C}, \otimes) := \mathbb{F}_\varnothing\text{-}Mod = (Sets, \times)$, we obtain back the definition of an algebraic monad, hence algebraic monads can be thought of as *(cosimplicial) operads over* $\mathbb{F}_\varnothing$.

4.3.5. Notice that $\mu_1^{(1)} : \Sigma(1) \times \Sigma(1) \to \Sigma(1)$ and $\boldsymbol{e} \in \Sigma(1)$ define a monoid structure on the underlying set $|\Sigma| := \Sigma(1)$ of an algebraic monad Σ, if we put $\lambda \cdot \mu = \lambda\mu := \mu_1^{(1)}(\lambda; \mu) = [\lambda]_{\Sigma(1)}(\mu) = [\lambda]_{|\Sigma|}(\mu)$; in particular, $[\lambda]_{|\Sigma|} : |\Sigma| \to |\Sigma|$ is just the left multiplication by λ map. Indeed, the associativity $(\lambda\mu)\nu = \lambda(\mu\nu)$ is a special case of **4.3.3**b), and $\boldsymbol{e}\lambda = \lambda$ and $\lambda\boldsymbol{e} = \lambda$ are special cases of **4.3.3**a) and c). Of course, this is exactly the monoid structure on $|\Sigma|$ considered in **3.4.9**. If $M = (M, \alpha)$ is a Σ-module, we recover again the monoid action of $|\Sigma|$ on M by taking $\alpha^{(1)} : \Sigma(1) \times M \to M$. Then $\lambda \cdot x = \lambda x := \alpha^{(1)}(\lambda; x) = [\lambda]_M(x)$ for any $\lambda \in |\Sigma|$, $x \in M$. That's why we usually denote the action of unary operations $\lambda \in |\Sigma|$ on an element $x \in M$ simply by λx.

4.3.6. We have already seen in **4.2.10** that $\Sigma(X)$ consists of all formal expressions of form $t(\{x_1\}, \ldots, \{x_n\})$, $n \geq 0$, $t \in \Sigma(n)$, $x_1, \ldots, x_n \in X$, modulo a certain equivalence relation. On the other hand, since we have agreed to write $\{x\}$ for $\varepsilon_X(x) \in \Sigma(X)$, and since μ_X is a canonical Σ-structure on $\Sigma(X)$, we see that $[t]_{\Sigma(X)}(\{x_1\}, \ldots, \{x_n\})$ actually means $\mu_X^{(n)}(t; \varepsilon_X(x_1), \ldots, \varepsilon_X(x_n))$. Since $\mu_X \circ \Sigma(\varepsilon_X) = \mathrm{id}_{\Sigma(X)}$, this expression is indeed equal to $t(\{x_1\}, \ldots, \{x_n\})$ of **4.2.10**, so our notation is consistent in this respect.

4.3.7. We see that a Σ-module structure $\alpha : \Sigma(X) \to X$ is given by a collection of "operation maps" $[t]_X : X^n \to X$, parametrized by $t \in \bigsqcup_{n \geq 0} \Sigma(n)$. These maps have to satisfy some relations like $[\boldsymbol{e}]_X = \mathrm{id}_X$, $[\varphi_* t]_X = [t]_X \circ X^\varphi$ for any $\varphi : \mathbf{m} \to \mathbf{n}$, i.e. $[\varphi_* t]_X(x_1, \ldots, x_n) = [t]_X(x_{\varphi(1)}, \ldots, x_{\varphi(m)})$, and the "compound operation" relations (4.3.2.2). Conversely, **4.3.2** shows that these relations are sufficient for a collection of maps $[t]_X$ to define a Σ-action on X. Since all these relations are algebraic, we see that a Σ-structure on a set X is an example of an algebraic structure, hence $\Sigma\text{-}Mod = Sets^\Sigma$ has all the properties of a category defined by an algebraic structure.

4.3.8. Given any set X, we can construct its "absolute endomorphism ring" $\mathrm{END}(X)$. It is an algebraic monad defined by $(\mathrm{END}(X))(n) := \mathbf{Hom}_{Sets}(X^n, X)$; we take $\boldsymbol{e}_\Xi$ to be equal to id_X, and $\mu_n^{(k)} : \mathbf{Hom}(X^k, X) \times \mathbf{Hom}(X^n, X)^k \cong \mathbf{Hom}(X^k, X) \times \mathbf{Hom}(X^n, X^k) \to \mathbf{Hom}(X^n, X)$ are the usual composition maps. The conditions of **4.3.3** obviously hold, and we have a canonical action of $\mathrm{END}(X)$ on X. Furthermore, the considerations of **4.3.7** show that giving an action $\alpha : \Sigma(X) \to X$ of an algebraic monad Σ on X is equivalent to giving a monad homomorphism ("representation") $\rho : \Sigma \to \mathrm{END}(X)$, and then the Σ-action on X can be recovered from the canonical $\mathrm{END}(X)$-action on X by means of scalar restriction along ρ.

We say that X is a *faithful* Σ*-module* if the corresponding homomorphism $\rho : \Sigma \to \mathrm{END}(X)$ is injective, i.e. if $[t]_X = [t']_X$ implies $t = t'$ for any $t, t' \in \Sigma(n)$ and any $n \geq 0$. In general case we denote the image $\rho(\Sigma)$ of ρ by Σ_X; clearly, this is the only strict quotient of Σ which acts faithfully on X.

4.3.9. (Terminology and notations.) Given any algebraic monad Σ, we call the elements of $\Sigma(0)$ *constants (of* Σ*)*, elements of $|\Sigma| = \Sigma(1)$ — *unary operations*, elements of $\Sigma(2)$ — *binary operations*, $\ldots$, elements of $\Sigma(n)$ — n*-ary operations (of* Σ*)*. We say that $\|\Sigma\| := \bigsqcup_{n\geq 0} \Sigma(n)$ is the set of all operations of Σ; if $t \in \Sigma(n)$, we say that *the arity of* t *is equal to* n, and sometimes write $t^{[n]}$ or $r(t) = n$ to emphasize this fact. Cleary, an operation t of arity n induces a map $[t]_X : X^n \to X$ on any Σ-module X. We often write $t_X(x_1, \ldots, x_n)$, $t(x_1, \ldots, x_n)$ or even $t\, x_1\, \ldots\, x_n$ instead of $[t]_X(x_1, \ldots, x_n)$. In particular, if $\lambda \in |\Sigma|$ is a unary operation, we write $\lambda_X(x)$ or λx instead of $[\lambda]_X(x)$; this is especially useful when λ is denoted by some sign like $-$; then we write $-x$ instead of $[-]_X(x)$, and $-_X : X \to X$ instead of $[-]_X$. Notice that $(\lambda t)(x_1, \ldots, x_n) = \lambda(t(x_1, \ldots, x_n))$ for any $\lambda \in |\Sigma|$, $t \in \Sigma(n)$ (this is a special case of (4.3.2.2)), so the notation $\lambda t(x_1, \ldots, x_n)$ is unambiguous.

Similarly, if a binary operation is denoted by some sign like $+$, $*$, $\times$ $\ldots$, which is usually written in infix form, we write $x_1 + x_2$ instead of $[+]_X(x_1, x_2)$; notice that $[+] = [+]_{\Sigma(2)}(\{1\}, \{2\}) = \{1\} + \{2\}$ in $\Sigma(2)$. We obey the usual precedence rules while using such notations, inserting parentheses when necessary; if an operation $+$ is associative, we write $x+y+z$ instead of $(x+y)+z$ or $x + (y + z)$. For example, any R-linear combination $t = (\lambda_1, \ldots, \lambda_n) \in \Sigma_R(n) = R^n$, where $\lambda_i \in R$, R is any associative ring (cf. **3.4.8**), can be safely written now in form $t = \lambda_1\{1\} + \lambda_2\{2\} + \cdots + \lambda_n\{n\}$, where $\lambda_i \in R = |\Sigma_R|$ are interpreted as unary operations of Σ_R, and $[+] = (1, 1) \in \Sigma_R(2)$ is of course a binary operation of Σ_R. In this situation we also have a unary operation $[-] := -1 \in |\Sigma_R|$, and a constant $0 \in \Sigma_R(0)$.

4.3.10. In particular, for any constant $c \in \Sigma(0)$ and any Σ-module X we get the *value* $c_X = [c]_X \in X$ of this constant in X. Of course, any Σ-homomorphism $f : X \to Y$ has to preserve all constants: $f(c_X) = c_Y$. For example, if $i : \mathbf{0} = \varnothing \to \mathbf{1}$ is the canonical embedding, then the unary operation $i_* c = c_{|\Sigma|} \in \Sigma(1)$ acts on any Σ-module X by mapping all elements of X into c_X, i.e. $[i_* c]_X(x) = c_X$. This shows that $\Sigma(i) : \Sigma(0) \to \Sigma(1)$ is *in-*

jective: indeed, $i_*c = i_*c'$ implies $[i_*c]_{\Sigma(0)} = [i_*c']_{\Sigma(0)}$, hence $c = [i_*c]_{\Sigma(0)}(x) = [i_*c']_{\Sigma(0)}(x) = c'$, where we can take any auxiliary element $x \in \Sigma(0)$ (e.g. $x = c$).

This fact, combined with the observation that any monomorphism (i.e. injective map) $f : X \to Y$ with non-empty source splits in *Sets* (i.e. admits a left inverse), hence $\Sigma(f) : \Sigma(X) \to \Sigma(Y)$ has to be injective when X is non-empty, shows that $\Sigma(f)$ is in fact injective even if X is empty. In particular, all maps $\Sigma(\varphi) : \Sigma(m) \to \Sigma(n)$ for injective $\varphi : \mathbf{m} \to \mathbf{n}$ are injective; if we put $\Sigma(\infty) := \Sigma(\mathbb{Z}_{\geq 0}) = \varinjlim_{n \geq 0} \Sigma(n) = \bigcup_n \Sigma(n)$, we see that $\Sigma(n)$ is a subset of $\Sigma(\infty)$, and these subsets form a filtration on $\Sigma(\infty)$, so we can combine all $\Sigma(n)$ into one large filtered set $\Sigma(\infty)$.

4.3.11. Notice that $\Sigma(0) = L_\Sigma(\varnothing)$ is an initial object of Σ-*Mod* $=$ *Sets*$^\Sigma$. We say that Σ is a *monad without constants* if $\Sigma(0) = \varnothing$; in this case the empty set is (the only) initial object of Σ-*Mod*, in particular, the empty set admits a (unique) Σ-action. We have already seen some examples of monads without constants like $\mathbb{F}_\varnothing$, $\mathbf{\Delta}$ or Aff_R (cf. **3.4.12**). We can also consider the submonad $W_+ \subset W$ given by $W_+(X) = W(X)$ if $X \neq \varnothing$, $W_+(\varnothing) = \varnothing$; this is also a monad without constants, and W_+-*Mod* is the category of semigroups.

4.3.12. If $\Sigma(0)$ consists of exactly one constant, usually denoted by 0, we say that Σ is a *monad with zero*. In this case $\Sigma(0) = \{0\}$ is both a initial and a final object, i.e. a *zero object* $0 := \Sigma(0)$ of Σ-*Mod*, and conversely, since $\Sigma(0)$ is always initial and $\mathbf{1}$ is always final in Σ-*Mod*, if Σ-*Mod* has a zero object, then Σ is a monad with zero. In this case any Σ-module X has a pointed element $0_X = 0$, which has to be respected by any Σ-homomorphism; in particular, X cannot be empty. Moreover, any set $\mathrm{Hom}_\Sigma(M, N)$ has a pointed element — the *zero homomorphism* $0_{MN} : M \to 0 \to N$. Almost all monads considered by us up to now have been monads with zero, with the exception of W_U and of those listed in **4.3.11**.

4.3.13. (Triviality of a monad.) Recall that the category of all monads over *Sets* admits a final object — the *trivial* or *final monad* $\mathbf{1}$, such that $\mathbf{1}(X)$ is the one-point set $\mathbf{1}$ for any set X. Clearly, this monad is algebraic, so it is a final object in the category of algebraic monads. We say that Σ is *trivial* if it is isomorphic to $\mathbf{1}$; in this case we write $\Sigma \cong \mathbf{1}$ or even $\Sigma = \mathbf{1}$. In some cases we write 0 or $\mathbf{0}$ instead of $\mathbf{1}$, since $\mathbf{1}$ is the monad defined by the trivial ring 0. Let's list some criteria for triviality of a monad:

Proposition. *The following conditions for an algebraic monad Σ are equivalent: (i) Σ is trivial; (ii) Σ-Mod is a trivial category, i.e. all its objects are final; (ii') The only sets which admit a Σ-structure are the one-element sets; (iii) $\varepsilon_X : X \to \Sigma(X)$ is* not *injective for at least one set X, and*

$\Sigma(0) \neq \varnothing$; *(iii') $\varepsilon_{\mathbf{2}} : \mathbf{2} \to \Sigma(2)$ is not injective, i.e. $\{1\}_{\mathbf{2}} = \{2\}_{\mathbf{2}}$, and* $\Sigma(0) \neq \varnothing$; *(iv) $\boldsymbol{e} \in \Sigma(1)$ comes from a constant $c \in \Sigma(0)$, i.e. $\boldsymbol{e} = [c]_{\Sigma(1)}$.*

Proof. (i)⇒(ii): Indeed, for any Σ-module X the action $\alpha : \Sigma(X) = \mathbf{1} \to X$ has to be surjective, and this is possible only if $X \cong \mathbf{1}$; (ii)⇔(ii') is evident. (ii')⇒(i): All sets $\Sigma(n)$ admit a Σ-structure, so we must have $\Sigma(n) \cong \mathbf{1}$, i.e. $\Sigma = \mathbf{1}$. The implications (i)⇒(iii')⇒(iii) are also evident. (iii)⇒(iii'): If we have $\varepsilon_X(x) = \varepsilon_X(y)$ for some $x \neq y \in X$, consider the map $f : X \to \mathbf{2}$, such that $f(x) = 1$ and $f(z) = 2$ otherwise; applying $\Sigma(f)$ to our equality we get (iii'). (iii')⇒(ii'): Since $\{1\}$ and $\{2\}$ act on any Σ-module X by means of corresponding projections $X^2 \to X$, we see that $\{1\} = \{2\}$ implies the equality of these projections, hence X consists of at most one element; it cannot be empty since $\Sigma(0) \neq \varnothing$. Now (i)⇒(iv) is again trivial, and (iv)⇒(ii') follows from the fact that if $\boldsymbol{e} = [c]_{|\Sigma|}$ for some constant c, then for any Σ-module X the identity map $\mathrm{id}_X = [\boldsymbol{e}]_X$ has to coincide with the constant map which maps all elements of X into the value $c_X \in X$ of the constant, hence $X = \{c_X\}$ is an one-element set.

4.4. (Algebraic submonads and strict quotients.) Projective limits of algebraic monads are computed in the category of algebraic endofunctors (cf. **3.1.8** and **4.1.5**), i.e. $(\varprojlim \Sigma_\alpha)(n) = \varprojlim(\Sigma_\alpha(n))$. In particular, $\rho : \Sigma' \to \Sigma$ is a monomorphism of algebraic monads iff all maps $\rho_n : \Sigma'(n) \to \Sigma(n)$ are injective. We see that the subobjects of Σ in the category of algebraic monads are exactly the *algebraic submonads* $\Sigma' \subset \Sigma$, i.e. those algebraic subfunctors which admit a (necessarily unique) monad structure compatible with that of Σ. According to **4.2.6**, such a subfunctor is determined by a collection of subsets $\Sigma'(n) \subset \Sigma(n)$, stable under maps $\Sigma(\varphi) : \Sigma(m) \to \Sigma(n)$ for all $\varphi : \mathbf{m} \to \mathbf{n}$. Clearly, an algebraic subfunctor $\Sigma' \subset \Sigma$ is a submonad iff $\boldsymbol{e} \in \Sigma'(1) \subset \Sigma(1)$ and $\mu_n^{(k)}\bigl(\Sigma'(k) \times \Sigma'(n)^k\bigr) \subset \Sigma'(n)$ for all n, $k \geq 0$. For example, if we put $\Sigma_+(0) := \varnothing$, $\Sigma_+(n) := \Sigma(n)$ for $n \geq 1$, we obtain an algebraic submonad $\Sigma_+ \subset \Sigma$, which is the largest algebraic submonad of Σ without constants. Notice that since an algebraic submonad $\Sigma' \subset \Sigma$ is completely determined by the subset $\|\Sigma'\| := \bigsqcup_{n\geq 0} \Sigma'(n) \subset \|\Sigma\|$, all algebraic submonads of a fixed monad Σ constitute a set.

4.4.1. Notice that the image $\rho(\Sigma) \subset \Xi$ of a homomorphism of algebraic monads $\rho : \Sigma \to \Xi$, computed in the category of algebraic functors (cf. **4.2.8** and **4.3.3**d)), is actually an algebraic submonad of Ξ, not just an algebraic subfunctor, and we obtain a decomposition $\rho : \Sigma \twoheadrightarrow \rho(\Sigma) \hookrightarrow \Xi$ of ρ into a surjective monad homomorphism followed by an inclusion of a submonad. In this case the kernel $R := \Sigma \times_\Xi \Sigma$ in the category of algebraic monads coincides with the same kernel computed in the category of (algebraic or all) endofunctors, hence $\Sigma \to \rho(\Sigma)$ is a strict epimorphism in all these categories, and $\rho(\Sigma) \cong \Sigma/R$. Recall that $(\rho(\Sigma))(n) = \rho_n(\Sigma(n)) \subset \Xi(n)$ (cf. **4.2.8**). We say that $\rho(\Sigma)$ is the *image* of $\rho : \Sigma \to \Xi$; clearly, $\Sigma \to \rho(\Sigma) \to \Xi$ is the (necessarily unique) decomposition of ρ into a strict epimorphism followed by a monomorphism.

4.4.2. (Submonad generated by a set of operations.) Given a monad Σ and any subset $U = \bigsqcup_{n\geq 0} U_n$ of the set $\|\Sigma\| = \bigsqcup_{n\geq 0} \Sigma(n)$ of all operations of Σ, we can consider the *submonad* Σ' *of* Σ*, generated by* U, i.e. the smallest submonad Σ' of Σ, which contains U (more precisely, such that $U \subset \|\Sigma'\|$). Such a submonad always exists: indeed, we can simply take the intersection $\bigcap_\alpha \Sigma_\alpha$ of all submonads $\Sigma_\alpha \subset \Sigma$ which contain U. Recall that projective limits of algebraic monads are computed componentwise, hence $\Sigma'(n) = (\bigcap_\alpha \Sigma_\alpha)(n) = \bigcap_\alpha \Sigma_\alpha(n) \subset \Sigma(n)$. We usually denote this submonad Σ' by $\langle U\rangle$ or $\mathbb{F}_\varnothing\langle U\rangle$; when U is a finite set $\{u_1, \ldots, u_n\}$, we also write $\mathbb{F}_\varnothing\langle u_1, u_2, \ldots, u_n\rangle$; if we want to emphasize the arities $r_i := r(u_i)$ of the u_i, we write $\mathbb{F}_\varnothing\langle u_1^{[r_1]}, \ldots, u_n^{[r_n]}\rangle$.

If $\langle U\rangle = \Sigma$, we say that U *generates* Σ, or that U *is a system of generators of* Σ; by definition, this means that any algebraic submonad of Σ containing U is necessarily equal to Σ. This implies the usual consequences: for example, if two homomorphisms $\rho_1, \rho_2 : \Sigma \to \Xi$ coincide on U, then they are equal, since $\mathrm{Ker}(\rho_1, \rho_2)$ is an algebraic submonad of Σ containing U. Applying this to $\Xi = \mathrm{END}(X)$ we see that a Σ-structure on a set X is completely determined by the collection of maps $[u]_X : X^{r(u)} \to X$ for $u \in U$.

4.4.3. (Explicit description of $\mathbb{F}_\varnothing\langle U\rangle$.) Let $\Sigma' = \mathbb{F}_\varnothing\langle U\rangle \subset \Sigma$ be the submonad of Σ generated by some set $U \subset \|\Sigma\|$. Notice that the subsets $\Sigma'(n) \subset \Sigma(n)$ have to satisfy the following two requirements:

1) All "variables" $\{k\}_{\mathbf{n}} \in \Sigma(n)$, $1 \leq k \leq n$, are contained in $\Sigma'(n)$.

2) If $u \in U_k$ is any generator of arity k, and $t_1, \ldots, t_k \in \Sigma'(n)$ for some $k \geq 0$, then $[u]_{\Sigma(n)}(t_1, \ldots, t_k)$ also belongs to $\Sigma'(n)$.

Conversely, consider the subsets $\Sigma''(n) \subset \Sigma(n)$ of operations which can be obtained by applying the above rules a finite number of times. Clearly, $U_n \subset \Sigma''(n) \subset \Sigma'(n)$. In fact, this collection actually defines a submonad of Σ containing U, hence $\Sigma' = \Sigma''$ and we obtain a description of $\Sigma' = \mathbb{F}_\varnothing\langle U\rangle$. Indeed, we have $\boldsymbol{e} = \{1\}_{\mathbf{1}} \in \Sigma''(1)$, and the substitution property "$[t]_{\Sigma(n)}(t_1, \ldots, t_k) \in \Sigma''(n)$ whenever $t \in \Sigma''(k)$, $t_i \in \Sigma''(n)$" is shown starting from requirement 2) by induction on the number of applications of rules 1) and 2) used for constructing t. The remaining property "$\varphi_* t = (\Sigma(\varphi))(t) \in \Sigma''(n)$ for any $t \in \Sigma''(m)$ and $\varphi : \mathbf{m} \to \mathbf{n}$" follows from the substitution property since $\varphi_* t = [t]_{\Sigma(n)}(\{\varphi(1)\}_{\mathbf{n}}, \ldots, \{\varphi(m)\}_{\mathbf{n}})$.

4.4.4. The above definitions admit a useful generalization. Suppose that we are given some algebraic monad homomorphism $\rho : \Sigma_0 \to \Sigma$, and some set $U \subset \|\Sigma\|$. Then we denote by $\Sigma_0\langle U\rangle$ the smallest algebraic submonad of Σ that contains both $\rho(\Sigma_0)$ and U; we say that this is *the submonad of* Σ *generated over* Σ_0 *by* U. Clearly, $\Sigma_0\langle U\rangle = \bigl\langle U \cup \|\rho(\Sigma_0)\|\bigr\rangle$, and $\rho : \Sigma_0 \to \Sigma$ factorizes through $\Sigma_0\langle U\rangle \subset \Sigma$. Of course, we write $\Sigma_0\langle u_1, \ldots, u_n\rangle$ when U is a finite set. This notion has similar properties to those considered above. For example, if Σ is generated by U over Σ_0, a Σ-structure on any set X is completely determined by its scalar restriction along $\rho : \Sigma_0 \to \Sigma$ together with the action of the generators $[u]_X : X^{r(u)} \to X$, $u \in U$.

4.4.5. (Finite generation and pre-unarity.) Given an algebraic monad homomorphism $\rho : \Sigma \to \Xi$, we say that Ξ *is finitely generated over* Σ if there is a finite set $U \subset \|\Xi\|$, such that $\Xi = \Sigma\langle U\rangle$. In this case we can choose a finite subsystem of generators $V_0 \subset V$ from any other system V of generators of Ξ over Σ: indeed, according to **4.4.3**, any operation $t \in \|\Xi\|$ can be expressed in terms of a finite number of operations from $\rho(\Sigma)$ and V; let's denote by $V_0(t)$ the finite set of involved operations from V, and put $V_0 := \bigcup_{u\in U} V_0(u)$.

We say that Ξ *is generated over* Σ *by operations of arity* $\leq r$, or that $\rho : \Sigma \to \Xi$ *is of arity* $\leq r$ if $\Xi = \Sigma\langle V\rangle$ for some subset $V \subset \|\Xi\|_{\leq r} := \bigsqcup_{n\leq r} \Xi(n) \subset \|\Xi\|$; clearly, in this case we can always take $V = \|\Xi\|_{\leq r}$. On the other hand, if at the same time Ξ is finitely generated over Σ, we can find a finite system of generators of arity $\leq r$.

Finally, we say that Ξ *is pre-unary over* Σ, or that Ξ *is a pre-unary extension of* Σ, or that $\rho : \Sigma \to \Xi$ *is a pre-unary homomorphism* if $\Xi = \Sigma\langle V\rangle$ for some set of unary operations $V \subset \Xi(1)$. In this case we can again take $V = |\Xi| = \Xi(1)$; if Ξ is finitely generated over Σ, we can choose a finite set of unary generators $V \subset |\Xi|$.

4.4.6. (Examples.) For example, $\mathbb{Z} = \Sigma_{\mathbb{Z}}$ is finitely generated (absolutely, i.e. over $\mathbb{F}_\varnothing$) by operations of arity ≤ 2: we can choose $0 \in \Sigma_{\mathbb{Z}}(0)$, $[-] := -1 \in \mathbb{Z} = \Sigma_{\mathbb{Z}}(1)$ and $[+] := (1, 1) \in \mathbb{Z}^2 = \Sigma_{\mathbb{Z}}(2)$ as a system of generators of $\mathbb{Z}$. Any ring R is pre-unary over $\mathbb{Z}$; we'll show later they are actually *unary* and that all unary extensions of $\mathbb{Z}$ come from associative rings. An associative ring R is (absolutely) finitely generated iff it is finitely generated over $\mathbb{Z}$ in the usual sense. Notice that $\mathbb{Z}$ and $\mathbb{Z}_\infty$ are *not* pre-unary extensions of $\mathbb{F}_1$ or $\mathbb{F}_{\pm 1}$, but are generated in arity ≤ 2. Another interesting example: while $R \times R$ $(= \Sigma_R \times \Sigma_R)$ is a unary extension of $R = \Sigma_R$ for any associative ring R, algebraic monad $\mathbb{Z}_\infty \times \mathbb{Z}_\infty$ is *not* a pre-unary extension of $\mathbb{Z}_\infty$.

4.4.7. (Strict quotients and equivalence relations.) Recall that the cokernels and the fibered products are computed in $\mathcal{A}_{alg}$ componentwise, hence a morphism $\rho : \Sigma \to \Xi$ in $\mathcal{A}_{alg} \cong Sets^{\underline{\mathbb{N}}}$ is a strict epimorphism iff all its components $\rho_n : \Sigma(n) \to \Xi(n)$ are strict epimorphisms in *Sets*, i.e. surjective maps. We see that the strict quotients of an algebraic endofunctor Σ are given by "surjective" maps $\rho : \Sigma \to \Xi$, i.e. natural transformations ρ, such that all $\rho_n : \Sigma(n) \to \Xi(n)$ are surjective, or equivalently, such that $\|\rho\| : \|\Sigma\| \to \|\Xi\|$ is surjective. Such (algebraic) strict quotients are in one-to-one correspondence with algebraic equivalence relations $R \subset \Sigma \times \Sigma$, given by some systems of equivalence relations $R(n) \subset \Sigma(n) \times \Sigma(n)$, compatible with all maps $\Sigma(\varphi)$ as usual (cf. **4.2.8**); in this case $\Xi = \Sigma/R$ and $\Xi(n) = \Sigma(n)/R(n)$.

Now suppose Σ has an algebraic monad structure. We say that its strict quotient Σ/R (in $\mathcal{A}_{alg}$) is *compatible* with the monad structure of Σ, or that equivalence relation R is *compatible* with the monad structure of Σ, if $\Xi = \Sigma/R$ admits a (necessarily unique) monad structure, for which $\Sigma \to \Sigma/R$ becomes a monad homomorphism.

Clearly, in this case Σ/R is a quotient of Σ modulo R in the category of algebraic monads as well, so $\Sigma \to \Sigma/R$ is still a strict epimorphism in this category. Conversely, an algebraic monad homomorphism $\rho : \Sigma \to \Xi$ is a strict epimorphism iff ρ is surjective, i.e. is of the form just discussed, and then $\Xi \cong \Sigma/R$ in both categories for some compatible algebraic equivalence relation R on Σ. To see this we just consider the decomposition of any algebraic monad homomorphism $\rho : \Sigma \to \Xi$ into a surjection $\tilde{\rho} : \Sigma \to \rho(\Sigma)$ and an embedding of a submonad $i : \rho(\Sigma) \to \Xi$ (cf. **4.4.1**), and $R := \Sigma \times_\Xi \Sigma$ is the kernel of both ρ and $\tilde{\rho}$, hence it is necessarily a compatible equivalence relation, and $\rho(\Sigma) \cong \Sigma/R$ both as a monad and as an algebraic endofunctor. This also shows that ρ is a strict epimorphism iff $\rho(\Sigma) = \Xi$, i.e. iff ρ is surjective.

4.4.8. When an algebraic equivalence relation $R \subset \Sigma \times \Sigma$ on an algebraic monad is compatible with the monad structure? First of all, it is given by a collection of equivalence relations $R(n) \subset \Sigma(n) \times \Sigma(n)$, compatible with maps $\varphi_* = \Sigma(\varphi) : \Sigma(m) \to \Sigma(n)$ for all $\varphi : \mathbf{m} \to \mathbf{n}$; in other words,

$$t \equiv_{R(m)} t' \text{ implies } \varphi_*(t) \equiv_{R(n)} \varphi_*(t') \quad \text{for any } \varphi : \mathbf{m} \to \mathbf{n}, \text{ and any } t,\, t' \in \Sigma(m). \tag{4.4.8.1}$$

Clearly, $\Xi := \Sigma/R$ admits a compatible monad structure iff all the maps $\Sigma(k) \times \Sigma(n)^k \xrightarrow{\mu_n^{(k)}} \Sigma(n) \to \Xi(n) = \Sigma(n)/R(n)$ factorize through the corresponding canonical projections $\Sigma(k) \times \Sigma(n)^k \to \Xi(k) \times \Xi(n)^k$. This means the following:

$$[t]_{\Sigma(n)}(t_1,\ldots,t_k) \equiv_{R(n)} [t']_{\Sigma(n)}(t'_1,\ldots,t'_k) \\ \text{whenever } t \equiv_{R(k)} t' \text{ and } t_i \equiv_{R(n)} t'_i \text{ for } 1 \leq i \leq k \quad (4.4.8.2)$$

Notice that this condition actually implies (4.4.8.1) in view of the formula $\varphi_* t = [t]_{\Sigma(n)}(\{\varphi(1)\},\ldots,\{\varphi(m)\})$, hence it is sufficient for the collection of equivalence relations $\{R(n) \subset \Sigma(n) \times \Sigma(n)\}_{n\geq 0}$ to define a compatible algebraic equivalence relation on Σ.

4.4.9. (Compatible equivalence relations generated by a set of equations.) First of all, notice that the intersection $R = \bigcap_\alpha R_\alpha$ of any family of compatible algebraic equivalence relations R_α on an algebraic monad Σ is again compatible. This can be either checked directly from (4.4.8.2) and $R(n) = \bigcap_\alpha R_\alpha(n)$, or deduced from the fact that R is the kernel of $\Sigma \to \prod_\alpha \Sigma/R_\alpha$.

This means that, given any set of "equations" $E = \bigsqcup_{n\geq 0} E_n$, $E_n \subset \Sigma(n) \times \Sigma(n)$, we can find the smallest compatible algebraic equivalence relation $R = \langle E\rangle$ on Σ, which contains E, by simply taking the intersection of all such equivalence relations. Alternatively, we might consider all statements of form "$t_1 \equiv t_2$", which can be obtained by a finite number of applications of the following rules:

0) $t \equiv t'$ if $(t,t') \in E$;

1) $t \equiv t$ for any $t \in \Sigma(n)$;

2) $t \equiv t'$ implies $t' \equiv t$;

3) $t \equiv t'$ and $t' \equiv t''$ imply $t \equiv t''$;

4) The substitution rule (4.4.8.2).

We say that $\langle E\rangle$ is *the (compatible algebraic) equivalence relation on Σ generated by (the equations, or relations from) E.*

Clearly, an algebraic monad homomorphism $\rho : \Sigma \to \Xi$ factorizes through $\Sigma \to \Sigma/\langle E\rangle$ iff the kernel $R := \Sigma \times_\Xi \Sigma$ contains E, i.e. iff all the equations from E are fulfilled in Ξ: $\rho_n(t) = \rho_n(t')$ whenever $(t,t') \in E_n$. Putting here $\Xi := \mathrm{END}(X)$ and taking into account the surjectivity of $\Sigma \to \Sigma/\langle E\rangle$, we see that $\Sigma/\langle E\rangle$-*Mod* can be identified with the full subcategory of Σ-*Mod*, consisting of all Σ-modules X, such that $[t]_X = [t']_X$ whenever $(t,t') \in E$.

When E is a finite set of pairs: $E = \{(t_1,u_1),\ldots,(t_s,u_s)\}$, we also write $\langle t_1 = u_1,\ldots,t_s = u_s\rangle$ instead of $\langle E\rangle$.

4.4.10. (Finitely generated and unary equivalence relations.) We introduce for compatible equivalence relations R on algebraic monads Σ and corresponding strict quotients $\Xi := \Sigma/R$ a terminology similar to that of **4.4.5**. For example, we say that R is *finitely generated*, or that Ξ *is obtained from* Σ *by imposing finitely many relations (or conditions, or equations)*, if R is generated by some finite set of equations E. We say that R is *generated in arity* $\leq k$, or *generated by equations or relations of arity* $\leq k$, or that Ξ *is obtained from* Σ *by imposing relations in arity* $\leq k$, if R is generated by some subset $E \subset \bigsqcup_{0\leq n\leq k} \Sigma(n) \times \Sigma(n)$; if R is also finitely generated, then such a set of equations can be chosen to be finite. Finally, we say that R is *unary* if it is generated by some subset of $\Sigma(1) \times \Sigma(1)$.

4.5. (Free algebraic monads.) Now we would like to construct a very important class of algebraic monads, namely, the *free algebraic monads.* For this purpose we fix some "graded set" $U = \bigsqcup_{n\geq 0} U_n$, consider the category of couples (Σ, φ), where Σ is an algebraic monad and $\varphi = \bigsqcup_{n\geq 0} \varphi_n : U \to \|\Sigma\|$ is a graded map of sets (i.e. essentially a collection of maps $\varphi_n : U_n \to \Sigma(n)$), and define the *free algebraic monad* $\mathbb{F}_\varnothing\langle U\rangle$ to be the initial object of this category.

4.5.1. By definition, we must have a graded map $j : U \to \|\mathbb{F}_\varnothing\langle U\rangle\|$, such that any graded map $\varphi : U \to \Sigma$ induces a uniquely determined monad homomorphism $\rho : \mathbb{F}_\varnothing\langle U\rangle \to \Sigma$, for which $\varphi = \|\rho\| \circ j$. In other words, monad homomorphisms $\rho : \mathbb{F}_\varnothing\langle U\rangle \to \Sigma$ are in one-to-one correspondence with graded maps $\varphi : U \to \|\Sigma\|$, i.e. with families of maps $\{\varphi_n : U_n \to \Sigma(n)\}_{n\geq 0}$. Applying this to $\Sigma = \mathrm{END}(X)$, we see that $\mathbb{F}_\varnothing\langle U\rangle$-*Mod* is equivalent to the category of sets X, equipped with an arbitrary family of maps $[u]_X : X^{r(u)} \to X$, defined for each $u \in U$, where $r : U \to \mathbb{Z}_{\geq 0}$ is the arity map (i.e. $r(u) = n$ iff $u \in U_n$).

4.5.2. Of course, we have to show the existence of free algebraic monads $\mathbb{F}_\varnothing\langle U\rangle$. For this purpose we construct such a monad as a certain submonad $W_{U,r}$ of the monad W_U of words with constants from U (cf. **3.4.6**). Recall that $W_U(X) = W(U \sqcup X)$ consists of all words in alphabet $U \sqcup X$, usually written in form $u_1\{x_1\}\{x_2\}u_2\{x_3\}$ (we omit braces around constants, i.e. letters from U). Informally speaking, $\mathbb{F}_\varnothing\langle U\rangle = W_{U,r} \subset W_U$ is constructed as the set of all valid expressions, constructed with the aid of operations from U from the variables from X, and written in the prefix notation. For example, if $r(u_1) = 1$ and $r(u_2) = 2$, then $\{x\}$, $u_1\{z\}$ and $u_2u_1\{x\}u_2\{y\}\{z\}$ are elements of $W_{U,r}(\{x, y, z\})$. We present a more formal construction below, which has the advantage of allowing a straightforward generalization to the topos case.

4.5.3. Let us denote by $\mathbb{S}$ the set of all maps φ from the set $\{-\infty\} \cup \mathbb{Z}_{\geq 0}$

into itself, which satisfy the following conditions: 1) $\varphi(-\infty) = -\infty$; 2) $\varphi(x) \neq -\infty$ for at least one x; 3) $\varphi(x+1) = \varphi(x)+1$ whenever $\varphi(x) \neq -\infty$. Clearly, this set is closed under composition and contains the identity map, hence $\mathbb{S}$ is a monoid. We call $\mathbb{S}$ the *syntax monoid*. Its elements φ are classified by pairs of integers r, $s \geq 0$, where r is the smallest $x \geq 0$ for which $\varphi(x) \geq 0$, and $s = \varphi(r)$. We denote the corresponding element of $\mathbb{S}$ by $G_{r,s}$. Notice that $G_{0,s}$ is the only element $\varphi \in \mathbb{S}$, such that $\varphi(0) = s$. In particular, $G_{0,0}$ is the identity of $\mathbb{S}$. We can compute $G_{r,s}G_{r',s'}$ explicitly: if $r \leq s'$, it is equal to $G_{r',s+s'-r}$, otherwise it equals $G_{r+r'-s',s}$.

4.5.4. Now we define the *verification maps* $d_X : W_U(X) = W(U \sqcup X) \to \mathbb{S}$ as follows. Recall that $W_U(X)$ is a free monoid generated by $U \sqcup X$, so d_X is completely determined by its values on generators $U \sqcup X$, if we also require it to be a monoid homomorphism. We put $d_X(u) := G_{r(u),1}$ for any $u \in U$, i.e. $d_X(u) = G_{n,1}$ for $u \in U_n$, and $d_X(x) := G_{0,1}$ for any $x \in X$; then $d_X(z_1z_2\dots z_s) = d_X(z_1)d_X(z_2)\dots d_X(z_s)$ for any $z_i \in U \sqcup X$.

Put $W_{U,r}(X) := d_X^{-1}(G_{0,1}) \subset W_U(X)$, i.e. this is the set of all words $t \in W_U$, such that $d_X(t) = G_{0,1}$, or equivalently such that $(d_X(t))(0) = 1$; we call such words t *valid expressions* or *terms* with respect to the set of operations U. Clearly, $d_Y \circ W_U(f) = d_X$ for any map of sets $f : X \to Y$, hence $(W_U(f))(W_{U,r}(X)) \subset W_{U,r}(Y)$, so $W_{U,r}$ is indeed a subfunctor of W_U, and even an algebraic subfunctor, since it commutes with filtered inductive limits.

4.5.5. We have to check that $W_{U,r} \subset W_U$ is indeed a submonad, i.e. that if we replace in some term t some variables $\{x_i\}$ (or even $\{t_i\}$) with some other terms t_i, then the resulting word $\tilde{t}$ is necessarily a term. This is clear, since the expression for $d(\tilde{t}) \in \mathbb{S}$ is obtained from that of $d(t)$ by replacing $d(\{x_i\})$ with $d(t_i)$; but $d(t_i) = G_{0,1} = d(\{x_i\})$ since all t_i are terms, hence $d(\tilde{t}) = d(t) = G_{0,1}$, hence $\tilde{t}$ is indeed a term.

4.5.6. (Structural induction.) Notice that we have some general rules for constructing new terms:

1) Any variable $x \in X$ defines a one-letter term $\{x\} \in W_{U,r}(X)$.

2) If $u \in U_n$ is an operation of arity n and $t_1, \dots, t_n$ are terms, then $u\,t_1 \dots t_n$ is also a term.

Conversely, one checks in the usual way that any term $t \in W_{U,r}(X)$ is necessarily non-empty, and that it is either of form 1), when the first letter of t belongs to X, or of form 2) with uniquely determined $u \in U$ and terms of smaller length $t_1, \dots, t_n \in W_{U,r}(X)$, when the first letter of t is an operation from U; moreover, if $t = u\,t_1 \ldots t_n$ for some operation $u \in U$ and some terms t_i, then necessarily $n = r(u)$, and such a list of t_i is unique.

The fact that any term of $W_{U,r}(X)$ can be uniquely written either in form 1) or 2) allows us to prove statements by induction on length of a term t, proving them first for case $t = \{x\}$, and then proving them for $t = u\,t_1 \ldots t_n$, assuming them to be already proved for all t_i; this sort of induction is called *structural induction*.

4.5.7. In particular, for any $u \in U_n$ we get an element $j(u) = j_n(u) := u\{1\}\{2\}\ldots\{n\} \in W_{U,r}(n)$, so we get a graded map $j : U \to \|W_{U,r}\|$. Since j is injective, we often identify U with the subset $j(U) \subset \|W_{U,r}\|$ and write $\langle u \rangle$ or even just u instead of $j(u)$. Notice that

$$\big[j(u)\big]_{W_{U,r}(X)}(t_1, t_2, \ldots, t_n) = u\,t_1\,t_2\,\ldots\,t_n \text{ for any } u \in U_n,\ t_i \in W_{U,r}(X). \tag{4.5.7.1}$$

This shows that $j(U) \subset \|W_{U,r}\|$ generates $W_{U,r}$ (cf. **4.4.3**).

4.5.8. Now let's check that $(W_{U,r}, j)$ satisfies the universal property required from $\mathbb{F}_\varnothing\langle U \rangle$, thus proving the existence of free algebraic monads. We have to show that for any graded map $\varphi : U \to \|\Sigma\|$ there is a unique monad homomorphism $\rho : W_{U,r} \to \Sigma$, such that $\varphi = \|\rho\| \circ j$, i.e. we require $\rho_n(j_n(u)) = \varphi_n(u)$. Taking (4.5.7.1) into account, we obtain

$$\rho_X(u\,t_1\,\ldots\,t_n) = [\varphi(u)]_{\Sigma(X)}\big(\rho_X(t_1), \ldots, \rho_X(t_n)\big)$$
$$\text{for any } u \in U_n,\ t_i \in W_{U,r}(X). \tag{4.5.8.1}$$

Since we must also have $\rho_X(\{x\}) = \{x\}$ for any $x \in X$, we prove by structural induction in $t \in W_{U,r}(X)$ the existence and uniqueness of such maps $\rho_X : W_{U,r}(X) \to \Sigma(X)$. They are clearly functorial in X, so we get a natural transformation of algebraic functors $\rho : W_{U,r} \to \Sigma$. It remains to prove that ρ is indeed a monad homomorphism; we have to check (4.3.3.3) for this, and this is easily shown by structural induction in t, using (4.5.8.1) for the induction step.

4.5.9. We have already seen that $\mathbb{F}_\varnothing\langle U \rangle = W_{U,r}$ is generated by $U \cong j(U) \subset \|W_{U,r}\|$. This justifies the notation $\mathbb{F}_\varnothing\langle U \rangle$. Moreover, if we have any subset $U \subset \|\Sigma\|$, it induces a canonical homomorphism $\rho : \mathbb{F}_\varnothing\langle U \rangle \to \Sigma$, and its image is exactly the submonad Σ' of Σ generated by U, which has been also denoted before by $\mathbb{F}_\varnothing\langle U \rangle$. In particular, U generates Σ iff $\rho : \mathbb{F}_\varnothing\langle U \rangle \to \Sigma$ is surjective, i.e. a strict epimorphism. If it is an isomorphism, we say that U *freely generates* Σ, or that U *is a system of free generators of* Σ. For example, the free algebraic monad $\mathbb{F}_\varnothing\langle U \rangle$ is freely generated by $j(U) \cong U$.

4.5.10. In general we tend to use capital letters for the elements of U when Σ is freely generated by U, to distinguish this situation from the case when U is just any system of generators of Σ. Of course, we follow the conventions of **4.4.2** when U is a finite set, thus writing for example $\mathbb{F}_\varnothing\langle N^{[0]}, T^{[1]}, U^{[2]}\rangle$ to denote the free monad generated by one constant, one unary, and one binary generator.

4.5.11. (Algebraic monads from algebraic systems.) Now suppose given a graded set of operations $U = \sqcup_{n\geq 0} U_n$, or equivalently, a set U together with an arity map $r : U \to \mathbb{Z}_{\geq 0}$, and any set of "equations" or "relations" $E \subset \|\mathbb{F}_\varnothing\langle U\rangle \times \mathbb{F}_\varnothing\langle U\rangle\|$. Then we can construct the compatible algebraic equivalence relation $\langle E\rangle$ on $\mathbb{F}_\varnothing\langle U\rangle$ generated by E, and consider the quotient $\Sigma := \mathbb{F}_\varnothing\langle U\rangle/\langle E\rangle$. This quotient will be denoted $\mathbb{F}_\varnothing\langle U|E\rangle$ or even $\langle U|E\rangle$; when U and E are finite we adopt the conventions of **4.4.2** and **4.4.9**, thus writing expressions like $\mathbb{F}_\varnothing\langle 0^{[0]}, \zeta^{[1]} \,|\, \zeta 0 = 0, \zeta^n\{1\} = \{1\}\rangle$ (this is actually $\mathbb{F}_{1^n}$) or $\mathbb{F}_\varnothing\langle U^{[2]} \,|\, U\{1\}\{2\} = U\{2\}\{1\}, U\{1\}U\{2\}\{3\} = UU\{1\}\{2\}\{3\}\rangle$. We usually prefer to replace the free variables like $\{1\}$, $\{2\}$, $\ldots$, with some (arbitrarily chosen) letters like x, y, $\ldots$, and of course we use the infix notation and parentheses when appropriate, thus writing the second of the above examples in form $\mathbb{F}_\varnothing\langle +^{[2]} \,|\, x+y = y+x,\ x+(y+z) = (x+y)+z\rangle$. Another useful convention: we often write unary equations without naming the free variable $\boldsymbol{e} = \{1\}_{\mathbf{1}}$ explicitly, using the monoid structure of $|\Sigma| = \Sigma(1)$ instead. Thus we write $\zeta^n = \boldsymbol{e}$ instead of $\zeta\zeta\cdots\zeta\{1\} = \{1\}$, and $\varphi^2 = \varphi + \boldsymbol{e}$ instead of $\varphi\varphi\{1\} = \varphi\{1\}+\{1\}$; we can also write $-^2 = \boldsymbol{e}$ instead of $-(-x) = x$ (here ζ, φ and $-$ are unary, and $+$ is binary). This is possible because of the associativity relation $t(u_1,\ldots,u_k)\cdot x = t(u_1(x),\ldots,u_k(x))$, true for any $t \in \Sigma(k)$, $u_i \in |\Sigma|$, and for any x from a Σ-module X, e.g. for $\boldsymbol{e} \in |\Sigma|$.

We call such couples (U, E) as above *algebraic systems*, and $\mathbb{F}_\varnothing\langle U|E\rangle$ is the *algebraic monad defined by algebraic system* (U, E). If $\Sigma \cong \mathbb{F}_\varnothing\langle U|E\rangle$, we say that (U, E) is a *presentation of algebraic monad* Σ. Of course, in this case the image of U in Σ generates Σ, and all relations from E are fulfulled in Σ, and all other relations between operations of Σ can be deduced from E, so we have indeed something very similar to the usual description of an algebra in terms of a list of generators and relations.

Clearly, algebraic monad homomorphisms $\rho : \mathbb{F}_\varnothing\langle U|E\rangle \to \Xi$ are in one-to-one correspondence with graded maps $\varphi : U \to \|\Xi\|$, such that the image under ρ of any equation from E is fulfilled in Ξ. Taking here $\Xi = \mathrm{END}(X)$ for some set X, we see that $\mathbb{F}_\varnothing\langle U|E\rangle$-*Mod* is exactly the category of sets X with an algebraic structure of species defined by the algebraic system (U, E). In this way the study of any algebraic structure is reduced to the study of modules over some algebraic monad. Of course, different algebraic systems can correspond to isomorphic algebraic monads.

4.5.12. Conversely, any algebraic monad Σ admits some presentation (U, E), since one can always take for U any system of generators of Σ (e.g. $U = \|\Sigma\|$), and then take for E any system of generating equations for the equivalence relation R on the free monad $\mathbb{F}_\varnothing\langle U\rangle$, defined by the canonical surjection $\mathbb{F}_\varnothing\langle U\rangle \to \Sigma$ (we can always take $E = R$). In this sense the study of categories of sets equipped with some algebraic structures, i.e. *universal algebra*, is nothing else than the study of categories of modules over algebraic monads. For example, we see immediately that the forgetful functors from any of these categories into *Sets* is monadic. However, the category of algebraic monads themselves is more convenient for different category-theoretic operations (e.g. computation of projective limits) than the category of algebraic systems (since even the definition of a morphism of algebraic systems is quite difficult to handle when written without monads).

4.5.13. The above constructions generalize to the case when we replace $\mathbb{F}_\varnothing$ with an arbitrary algebraic monad Σ_0. For example, for any graded set we can construct an algebraic monad $\Sigma_0\langle U\rangle$ over Σ_0 (i.e. a homomorphism of algebraic monads $\Sigma_0 \to \Sigma_0\langle U\rangle$), which is an initial object in the category of triples (Σ, ρ, φ), consisting of an (algebraic) monad Σ, a homomorphism $\rho : \Sigma_0 \to \Sigma$, and a graded map $\varphi : U \to \|\Sigma\|$. Then the category of $\Sigma_0\langle U\rangle$-modules consists exactly of Σ_0-modules X together with arbitrarily chosen maps $[u]_X : X^{r(u)} \to X$ for all $u \in U$. This actually shows that the canonical map $j : U \to \|\Sigma_0\langle U\rangle\|$ is injective unless $\Sigma_0 = \mathbf{1}$ or $\Sigma_0 \cong \mathbf{1}_+ \subset \mathbf{1}$.

We have to show the existence of such free monads $\Sigma_0\langle U\rangle$, but this is simple: take any presentation (U', E') of Σ_0 and put $\Sigma_0\langle U\rangle := \mathbb{F}_\varnothing\langle U' \sqcup U | E'\rangle$. Then everything follows from the universal properties of all constructions involved, or from the observation that in this way we get a correct category of $\Sigma_0\langle U\rangle$-modules, using **3.3.21** to obtain monad homomorphisms from functors between categories of modules.

4.5.14. Of course, we can construct a monad $\Sigma_0\langle U|E\rangle = \Sigma_0\langle U\rangle/\langle E\rangle$ over an algebraic monad Σ_0, starting from an arbitrary graded set U and an arbitrary set of relations $E \subset \|\Sigma_0\langle U\rangle^2\|$. It has properties similar to those considered before for $\Sigma_0 = \mathbb{F}_\varnothing$, and we introduce similar terminology and notations. For example, given a monad Σ over Σ_0, i.e. a homomorphism of algebraic monads $\rho : \Sigma_0 \to \Sigma$, we say that (U, E) is a *presentation of* Σ *over* Σ_0 if $\Sigma \cong \Sigma_0\langle U|E\rangle$. Of course, such a presentation always exists; it can found by the same reasoning as in **4.5.12**.

4.5.15. (Finite presentation and unarity.) Given any algebraic monad Σ over Σ_0, or equivalently a homomorphism $\rho : \Sigma_0 \to \Sigma$, we say that Σ *is* *finitely presented over* Σ_0, or that ρ *is of finite presentation*, if there exists a presentation $\Sigma \cong \Sigma_0\langle U|E\rangle$ with both U and E finite. We say that Σ *is generated in arity* $\leq r$ *with relations in arity* $\leq s$ *over* Σ_0 if we can find a presentation with all operations in U of arity $\leq r$ and all relations in E of arity $\leq s$. If $r \leq s$, we can replace U by any other system U' of generators of arity $\leq r$, since all the relations expressing operations from U in terms of new generators from U' will be in arity $\leq r \leq s$.

Finally, we say that Σ is *unary* over Σ_0, if U can be chosen inside $|\Sigma| = \Sigma(1)$, and E inside $|\Sigma_0\langle U\rangle^2|$. If we have another set U' of unary generators of such a Σ over Σ_0, then the induced equivalence relation on $\Sigma_0\langle U'\rangle$ will be necessarily unary, since we can replace in all unary equations from E the operations from U with their expressions in terms of unary operations from U' (and maybe some operations of other arities from Σ_0, but this doesn't affect anything).

4.5.16. (Examples.) We know that $\mathbb{F}_1 = \mathbb{F}_\varnothing\langle 0^{[0]}\rangle$, so it is finitely presented over $\mathbb{F}_\varnothing$, but of course not unary. Similarly, $\mathbb{F}_{\pm 1} = \mathbb{F}_1\langle -^{[1]} \mid -(-x) = x, -0 = 0\rangle = \mathbb{F}_\varnothing\langle 0^{[0]}, -^{[1]} \mid -(-x) = x, -0 = 0\rangle$ is finitely presented over both $\mathbb{F}_\varnothing$ and $\mathbb{F}_1$, and unary over $\mathbb{F}_1$ (since we might replace the equation for constants $-0 = 0$ with unary equation $-0_{\mathbb{F}_1(1)}\{1\} = 0_{\mathbb{F}_1(1)}\{1\}$), but not over $\mathbb{F}_\varnothing$. A more interesting example: $\mathbb{Z}$ is finitely presented over $\mathbb{F}_\varnothing$, since $\mathbb{Z} = \mathbb{F}_\varnothing\big\langle 0^{[0]}, -^{[1]}, +^{[2]} \mid x+y = y+x,\ x+(-x) = 0,\ x+0 = x,\ (x+y)+z = x+(y+z)\big\rangle$. It is also finitely presented over $\mathbb{F}_1$ and $\mathbb{F}_{\pm 1}$, but not unary over any of these monads.

4.5.17. (Non-commutative tensor products.) Presentations of algebraic monads can be used to show existence of "non-commutative tensor products" $\Sigma_1 \boxtimes_\Sigma \Sigma_2$, i.e. pushouts of pairs of morphisms $\rho_i : \Sigma \to \Sigma_i$ in the category of algebraic monads. Indeed, we have just to take any presentations $\Sigma_i = \Sigma\langle U_i|E_i\rangle$ of Σ_i, and put $\Sigma_1 \boxtimes_\Sigma \Sigma_2 := \Sigma\langle U_1, U_2|E_1, E_2\rangle = \Sigma\langle U_1 \sqcup U_2|E_1 \sqcup E_2\rangle$. Alternatively, we might take $\Sigma_1\langle U_2|\rho_{1,*}(E_2)\rangle$, where $\rho_{1,*} : \|\Sigma_0\langle U_2\rangle^2\| \to \|\Sigma_1\langle U_2\rangle^2\|$ is the canonical map induced by ρ_1. Again, the required universal property of $\Sigma_1 \boxtimes_\Sigma \Sigma_2$ follows from the universal properties of monads of form $\Sigma\langle U|E\rangle$. We see that $\big(\Sigma_1 \boxtimes_\Sigma \Sigma_2\big)$-*Mod* is isomorphic to the category of sets X, equipped with both a Σ_1-structure and a Σ_2-structure, restricting to the same Σ-structures. Actually, since this is a category of algebraic systems, we could use it to define $\Sigma_1 \boxtimes_\Sigma \Sigma_2$ as the monad defined by the forgetful functor from this category, applying **3.3.21** to construct monad homomorphisms from functors between categories of modules.

4.5.18. (Finite presentation, unarity and pushouts.) Of course, the notions of finite generation and presentation have the usual properties with respect to pushouts, i.e. NC-tensor products. They are proved essentially in the usual way, well-known for usual associative and especially commutative algebras, so we just list them in the order they can be proved, omitting the proofs themselves.

- Finitely generated and finitely presented homomorphisms of algebraic monads are stable under pushouts.
- A strict epimorphism $\Sigma \twoheadrightarrow \Xi$ is of finite presentation iff its kernel $R = \Sigma \times_\Xi \Sigma$ is finitely generated as an equivalence relation on Σ.
- For any homomorphism $\rho : \Sigma \to \Xi$ the codiagonal (or multiplication) map $\nabla : \Xi \boxtimes_\Sigma \Xi \to \Xi$ is a strict epimorphism (i.e. a surjection); if ρ is of finite type (i.e. if Ξ is finitely generated over Σ), then ∇ is of finite presentation.
- In the situation $\Sigma \to \Sigma' \to \Xi$, if Ξ is finitely presented over Σ, and Σ' is finitely generated over Σ, then Ξ is finitely presented over Σ'.
- In the same situation if Ξ is finitely generated over Σ, then it is finitely generated over Σ'.

Moreover, we have similar statements about unary and pre-unary homomorphisms:

- Pre-unary and unary homomorphisms are stable under pushouts.
- A strict epimorphism is always pre-unary; it is unary iff its kernel is a unary equivalence relation (cf. **4.4.10**).
- If Ξ is pre-unary over Σ, then the codiagonal map $\nabla : \Xi \boxtimes_\Sigma \Xi \to \Xi$ is unary.
- In the situation $\Sigma \to \Sigma' \to \Xi$, if Ξ is unary over Σ, and Σ' is pre-unary over Σ, then Ξ is unary over Σ'.
- In the same situation, if Ξ is pre-unary over Σ, the same is true over Σ'.

4.5.19. (Arbitrary inductive limits of algebraic monads.) Since $\mathbb{F}_\varnothing$ is an initial object of the category of algebraic monads, we see that we have coproducts $\Sigma_1 \boxtimes \Sigma_2 := \Sigma_1 \boxtimes_{\mathbb{F}_\varnothing} \Sigma_2$ and cokernels of pairs of morphisms in this category as well. Since filtered inductive limits of algebraic monads also exist (they can be computed componentwise), we deduce first the existence of infinite coproducts, and then the existence of arbitrary (small) inductive limits of algebraic monads. So presentations and algebraic systems enable us to compute inductive limits of algebraic monads, while projective limits and submonads are better described in terms of algebraic monads themselves (cf. **4.4**).

4.5.20. (Algebraic monads as algebraic structures.) We have seen that the forgetful functor $\Sigma \mapsto \|\Sigma\|$ from the category of algebraic monads into the category of $\mathbb{Z}_{\geq 0}$-graded sets $Sets_{/\mathbb{Z}_{\geq 0}}$ admits a left adjoint $U \mapsto \mathbb{F}_\varnothing\langle U\rangle$, hence we get a monad $\mathbb{M}$ over $Sets_{/\mathbb{Z}_{\geq 0}}$. Moreover, $\mathbb{M} : U \mapsto \|\mathbb{F}_\varnothing\langle U\rangle\|$ clearly commutes with filtered inductive limits of graded sets, so it is something like an algebraic monad over $Sets_{/\mathbb{Z}_{\geq 0}}$. On the other hand, we have seen in **4.3.3** that the structure of an algebraic monad on a graded set $\|\Sigma\|$ is itself algebraic, so we might expect the forgetful functor $\Sigma \to \|\Sigma\|$ to be monadic, i.e. the category of algebraic monads to be equivalent to $(Sets_{/\mathbb{Z}_{\geq 0}})^{\mathbb{M}}$, and this is indeed the case.

In fact, we might fix some set S from the very beginning, and consider monads Σ over the category $Sets_{/S}$ of S-graded sets $X = \bigsqcup_{s\in S} X_s$, or equivalently, of maps $X \xrightarrow{r} S$. Algebraic monads and algebraic endofunctors $\Sigma : Sets_{/S} \to Sets_{/S}$ are defined by the same requirement to commute with filtered inductive limits; they are completely determined by their restriction to the category $\underline{\mathbb{N}}_{/S}$ of finite S-graded sets. Since $\mathrm{Ob}(\underline{\mathbb{N}}_{/S}) \cong \bigsqcup_{n\geq 0} S^n = W(S)$, we can index objects of $\underline{\mathbb{N}}_{/S}$ by words or sequences $(s_1, \ldots, s_n)$ in alphabet S. We see that an algebraic endofunctor Σ over $Sets_{/S}$ is given by a collection of S-graded sets $\Sigma(s_1, \ldots, s_n) = \bigsqcup_{t\in S} \Sigma(s_1, \ldots, s_n)_t$, parametrized by $(s_1, \ldots, s_n) \in W(S)$, and some maps between these sets. A pre-action α of Σ on some S-graded set is given then by a collection of maps $\Sigma(s_1, \ldots, s_n)_t \times X_{s_1} \times \cdots \times X_{s_n} \to X_t$, and a monad structure on Σ is given by a collection of graded identity elements $\boldsymbol{e}_s \in \Sigma(s)_s$, $s \in S$, and pre-actions of Σ on all S-graded sets $\Sigma(s_1, \ldots, s_n)$. Of course, all these data are subject to some compatibility conditions similar to those considered before, but more cumbersome to write down, so we decided not to adopt this approach from the very beginning and treat the simplest case, even if almost all statements and constructions generalize to the S-graded case.

Notice that the underlying set $\|\Sigma\| = \bigsqcup \Sigma(s_1, \ldots, s_n)_t$ of an algebraic monad Σ over $Sets_{/S}$ is itself $W(S) \times S$-graded, so the category of such monads can be described itself as a category of modules over some monad $\mathbb{M}_S$ on the category of $W(S) \times S$-graded sets. This observation allows one in principle to deduce properties of categories of algebraic monads from the properties of categories of modules over algebraic monads, at the cost of making everything less explicit.

4.5.21. (Graded algebraic monads.) One might expect that in the case when S is a commutative group or at least a commutative monoid, algebraic monads over $Sets_{/S}$ are something like S-graded algebraic monads (over $Sets$). However, this is usually not the case. For example, the degree translation functors $T_t : X \mapsto X(t)$, $X(t)_s = X_{s+t}$, are usually expected to transform graded modules into graded modules, but in our situation we don't obtain a Σ-structure on $X(t)$ from a Σ-structure on X unless we are given a compatible family of morphisms $\Sigma(s_1, \ldots, s_n)_s \to \Sigma(s_1+t, \ldots, s_n+t)_{s+t}$ for each $t \in S$; if S is a group, then all this morphisms clearly have to be isomorphisms.

For example, we can construct some canonical maps $\theta : \mathbb{M}(r_1, \ldots, r_m)_n = \mathbb{F}_\varnothing\bigl\langle\langle 1\rangle^{[r_1]}, \ldots, \langle m\rangle^{[r_m]}\bigr\rangle(n) \to \mathbb{M}(r_1+1, \ldots, r_m+1)_{n+1}$, $t \mapsto \tilde{t}$ by structural induction in t: we map $\{k\}_{\mathbf{n}}$ into $\{k\}_{\mathbf{n+1}}$, and $t = \langle k\rangle\, t_1 \ldots t_{r_k}$ into $\tilde{t} := \langle k\rangle\, \tilde{t}_1, \ldots, \tilde{t}_{r_k}\, \{n+1\}_{\mathbf{n+1}}$, i.e. we add an extra argument $\{n+1\}$ to each operation $\langle k\rangle$. Of course, these maps θ are injective, but not bijective, so $\mathbb{M}$ is not $\mathbb{Z}$-graded; however, we can take the inductive limit along all these maps, thus obtaining a $\mathbb{Z}$-graded monad $\mathbb{M}_+$ and a monomorphism $\rho : \mathbb{M} \to \mathbb{M}_+$. Then the $\mathbb{M}_+$-modules are something like algebraic monads, but they admit degree translation in both directions, and we obtain functors ρ^* and ρ_* between such things and usual algebraic monads.

4.6. (Modules over an algebraic monad.) Let's fix an algebraic monad Σ over $\mathcal{C} = \mathit{Sets}$. We want to study some basic properties of the category of Σ-modules $\Sigma\text{-}\mathit{Mod} = \mathcal{C}^\Sigma$, as well as the categories of (left, right or two-sided) Σ-modules in $\mathcal{A}_{alg}$. These latter categories of "algebraic modules" over Σ actually behave themselves more like complexes of modules over an ordinary ring, while $\Sigma\text{-}\mathit{Mod}$ is a close counterpart of category of modules over an ordinary (associative) ring. In particular, we are going to prove all properties listed and used before for $\Sigma = \mathbb{Z}_\infty$ in **2.14.12**.

4.6.1. (Projective limits.) Of course, arbitrary projective limits $\varprojlim M_\alpha$ exist in the category $\Sigma\text{-}\mathit{Mod} = \mathit{Sets}^\Sigma$ of modules over an algebraic monad Σ, and they are essentially computed in the category of sets (cf. **3.1.12**). If we identify $(\varprojlim M_\alpha)^n$ with $\varprojlim M_\alpha^n$, then $[t]_M : M^n \to M$ is identified with $\varprojlim\, [t]_{M_\alpha}$, for any $t \in \Sigma(n)$.

4.6.2. (Submodules.) In particular, $f : N \to M$ is a monomorphism in $\Sigma\text{-}\mathit{Mod}$ iff it is a monomorphism in $\mathcal{C} = \mathit{Sets}$, i.e. an injective map. Therefore, the subobjects in $\Sigma\text{-}\mathit{Mod}$ of a Σ-module $M = (M, \eta_M)$ are given by the *submodules* N of M, i.e. those subsets $N \subset M$, which admit a (necessarily unique) Σ-structure, compatible with that of M, i.e. such that the inclusion $i : N \to M$ is a Σ-homomorphism (cf. **3.1.12**). Clearly, $N \subset M$ is a submodule of M iff the image of $\Sigma(N) \stackrel{\Sigma(i)}{\to} \Sigma(M) \stackrel{\eta_M}{\to} M$ is contained in N; in this case it has to be equal to N since $\eta_M \circ \varepsilon_M = \mathrm{id}_M$.

Since $\Sigma(N)$ consists of expressions $t(\{x_1\}, \ldots, \{x_n\})$ with $t \in \Sigma(n)$ and $x_i \in N$ (cf. **4.2.10**), and $\eta_M : \Sigma(M) \to M$ maps such an expression into $[t]_M(x_1, \ldots, x_n)$, we see that $N \subset M$ is a submodule of M iff $[t]_M(N^n) \subset N$ for any $n \geq 0$, $t \in \Sigma(n)$, i.e. N has to be stable under all operations of Σ.

Actually, the set of all operations of Σ, under which N is stable, forms an algebraic submonad of Σ for any subset $N \subset M$, hence it is sufficient to check the stability of N under a set of generators of Σ.

4.6.3. (Module structure on sets of maps.) Given any set S and any Σ-module M, we obtain a canonical Σ-structure on the set $H := \mathrm{Hom}(S, M) = \mathrm{Hom}_{Sets}(S, M)$ of all maps from S to M (cf. **3.5.15**). This structure is nothing else than the product Σ-structure on $\mathrm{Hom}_{Sets}(S, M) \cong M^S$; the description of projective limits of Σ-modules given in **4.6.1** shows that all operations $t \in \Sigma(n)$ act on maps $S \to M$ pointwise, i.e.

$$\bigl([t]_{\mathrm{Hom}(S,M)}(f_1, \ldots, f_n)\bigr)(s) = [t]_M\bigl(f_1(s), \ldots, f_n(s)\bigr) \\ \text{for any } t \in \Sigma(n), \text{ any maps } f_i : S \to M \text{ and any } s \in S. \tag{4.6.3.1}$$

When N is another Σ-module, $\mathrm{Hom}_\Sigma(N, M)$ is a subset of $\mathrm{Hom}(N, M)$, but in general this is not a Σ-submodule of $\mathrm{Hom}(N, M)$ with respect to the Σ-structure just considered, at least if we don't suppose Σ to be commutative (cf. **5.3.1**).

4.6.4. (Image of a Σ-homomorphism.) Let $f : M = (M, \eta_M) \to N = (N, \eta_N)$ be an arbitrary Σ-homomorphism. Let's denote by $I := f(M) \subset N$ its image as a map of sets, so we get the canonical decomposition $f : M \xrightarrow{\pi} I \xrightarrow{i} N$ of f into a surjection followed by an embedding. Notice that any epimorphism in *Sets* admits a section, hence it is respected by Σ, and we have seen in **4.3.10** that Σ preserves injectivity of maps as well, so $\Sigma(f) = \Sigma(i) \circ \Sigma(\pi)$ is the canonical decomposition of $\Sigma(f)$. Now $\eta_N \circ \Sigma(i) \circ \Sigma(\pi) = i \circ \pi \circ \eta_M$ since f is a Σ-homomorphism, i.e. the outer circuit of the following diagram is commutative:

$$\begin{tikzcd}
\Sigma(M) \arrow[r, "\Sigma(\pi)"'] \arrow[rr, bend left, "\Sigma(f)"] \arrow[d, "\eta_M"] & \Sigma(I) \arrow[r, "\Sigma(i)"'] \arrow[d, dashed, "\eta_I"] & \Sigma(N) \arrow[d, "\eta_N"] \\
M \arrow[r, "\pi"] \arrow[rr, bend right, "f"'] & I \arrow[r, "i"] & N
\end{tikzcd} \tag{4.6.4.1}$$

The existence of the middle arrow and the commutativity of this diagram now follow from the fact that the rows are the canonical decompositions of $\Sigma(f)$ and of f. Existence of η_I shows that I is a submodule of N, with Σ-structure given by η_I, and the commutativity of the diagram shows that both $\pi : M \to I$ and $i : I \to N$ are Σ-homomorphisms. In this way we see that any Σ-homomorphism can be uniquely decomposed into a surjective homomorphism followed by an embedding of a submodule.

4.6.5. (Free Σ-modules.) Recall that $\mu_S : \Sigma^2(S) \to \Sigma(S)$ is a Σ-structure on $\Sigma(S)$, thus defining a Σ-module $L_\Sigma(S) = (\Sigma(S), \mu_S)$, which will be usually denoted by $\Sigma(S)$. Moreover, we have seen in **3.3.8** that L_Σ is a left adjoint to the forgetful functor $\Gamma_\Sigma : \Sigma\text{-}\mathit{Mod} \to \mathit{Sets}$, i.e. there is a canonical bijection $\mathrm{Hom}_\Sigma(\Sigma(S), M) \cong \mathrm{Hom}_{\mathit{Sets}}(S, M)$ between Σ-homomorphisms $f : \Sigma(S) \to M$ and arbitrary maps $f^\flat : S \to M$. Of course, we have a canonical embedding $\varepsilon_S : S \to \Sigma(S)$, and the above correspondence is given by $f^\flat = f \circ \varepsilon_S$. We say that a Σ-module M is *free*, if it is isomorphic to some $\Sigma(S)$ (cf. **3.3.9**); the image of S in M is called a *system of free generators of* M. Finally, we say that M is *free of (finite) rank* n if it is isomorphic to $\Sigma(n)$.

4.6.6. (Matrices.) In **3.3.13** and **3.3.16** we have constructed a certain category $\mathcal{C}_\Sigma = \mathit{Sets}_\Sigma$ and a fully faithful functor $Q_\Sigma : \mathit{Sets}_\Sigma \to \Sigma\text{-}\mathit{Mod}$, which transforms a set S into the corresponding free module $(\Sigma(S), \mu_S)$. Clearly, the essential image of Q_Σ consists of all free Σ-modules, and Q_Σ induces an equivalence of categories Sets_Σ and the category of free Σ-modules. Recall that $\mathrm{Hom}_{\mathcal{C}_\Sigma}(S, T) = \mathrm{Hom}_{\mathcal{C}}(S, \Sigma(T)) \cong \mathrm{Hom}_\Sigma(\Sigma(S), \Sigma(T))$.

We also have the full subcategory $\underline{\mathbb{N}}_\Sigma \subset \mathit{Sets}_\Sigma$, given by the standard finite sets and corresponding free modules; of course, the essential image under Q_Σ of this category consists of all free Σ-modules of finite rank. We see that $\mathrm{Hom}_{\underline{\mathbb{N}}_\Sigma}(\mathbf{n}, \mathbf{m}) = \mathrm{Hom}_{\mathit{Sets}}(\mathbf{n}, \Sigma(m)) = \Sigma(m)^n \cong \mathrm{Hom}_\Sigma(\Sigma(n), \Sigma(m))$. This is the reason why we put $M(m, n; \Sigma) := \Sigma(m)^n$ and call this "the set of $m \times n$-matrices with entries in Σ" (cf. **4.3.1**,b)). The composition of homomorphisms $\mathrm{Hom}_\Sigma(\Sigma(n), \Sigma(m)) \times \mathrm{Hom}_\Sigma(\Sigma(k), \Sigma(n)) \to \mathrm{Hom}_\Sigma(\Sigma(k), \Sigma(m))$ defines the "matrix multiplication" $M(m, n; \Sigma) \times M(n, k; \Sigma) \to M(m, k; \Sigma)$, which is essentially given by the maps $\mu_m^{(n)} : \Sigma(n) \times \Sigma(m)^n \to \Sigma(m)$, up to a permutation of arguments (cf. **4.3.3**,b) and **3.3.13**). We have the "identity matrices" $I_n = (\{1\}_{\mathbf{n}}, \ldots, \{n\}_{\mathbf{n}}) \in M(n, n; \Sigma)$, which correspond to $\mathrm{id}_{\Sigma(n)}$ (cf. **4.3.3**,c)). Moreover, if we put $M(1, n; X) := X^n$ for any set X ("the set of rows over X"), then a Σ-action α on X corresponds to a family of maps $M(n, k; \Sigma) \times M(1, n; X) \to M(1, k; X)$, essentially given by the $\alpha^{(n)} : \Sigma(n) \times X^n \to X$ (cf. **4.3.2**,b)), which can be understood as some sort of multiplication of a row by a matrix.

4.6.7. (Invertible matrices.) Of course, we have the groups of *invertible matrices* $GL_n(\Sigma) = GL(n, \Sigma) \subset M(n, n; \Sigma)$, which correspond to $\mathrm{Aut}_\Sigma(\Sigma(n)) \subset \mathrm{End}_\Sigma(\Sigma(n))$. Clearly, a square matrix $f = (f_1, f_2, \ldots, f_n)$, $f_i \in \Sigma(n)$ is invertible iff there is another square matrix $g = (g_1, \ldots, g_n)$, such that $f_i(g_1, \ldots, g_n) = \{i\}_{\mathbf{n}}$ and $g_i(f_1, \ldots, f_n) = \{i\}_{\mathbf{n}}$ for all $1 \leq i \leq n$.

4.6.8. (Initial object and ideals.) Notice that $\mathbf{0} := \Sigma(0)$ is the initial object of $\Sigma\text{-}\mathit{Mod}$. If Σ is a monad without constants, $\Sigma(0)$ is actually an empty set, so we prefer to denote it by $\varnothing$ and call it the *empty* Σ*-module*; when Σ is a monad with zero, then $\mathbf{0} = \Sigma(0)$ is a one-element set, hence a zero (both initial and final) object of Σ-*Mod*. Any module M contains a smallest submodule, equal to the image of $\Sigma(0) \to M$; it is called the *initial submodule* of M, or the *empty* or *zero submodule* of M, and it is denoted by $\varnothing_M$, $\varnothing$ or 0 (depending on the situation). In any case it is a quotient of $\Sigma(0)$; if Σ has at most one constant, then it is always isomorphic to $\Sigma(0)$.

On the other hand, $\Sigma(0) \to \Sigma(1)$ is always injective (cf. **4.3.10**), so the set of all *ideals* (i.e. Σ-submodules of $|\Sigma| = \Sigma(1)$) has a smallest element — the *initial, empty* or *zero ideal*, which is always isomorphic to the initial object $\Sigma(0)$.

4.6.9. (Submodule generated by a subset.) If $M = (M, \eta_M)$ is a Σ-module, any subset $S \subset M$ induces a Σ-homomorphism $f : \Sigma(S) \to M$; according to **4.6.4**, the image $\langle S\rangle := f(\Sigma(S)) \subset M$ is a submodule of M, clearly containing S. We claim that $\langle S\rangle$ *is the smallest submodule of* M *containing* S*, i.e. the submodule of* M*, generated by* S. Indeed, if a submodule $N \subset M$ contains S, then $\langle S\rangle = f(\Sigma(S)) = \eta_M(\Sigma(S)) \subset \eta_M(\Sigma(N)) = N$ (here we have identified $\Sigma(S)$ and $\Sigma(N)$ with their images in $\Sigma(M)$). Since $\Sigma(S)$ consists of all expressions $t(\{z_1\}, \ldots, \{z_n\})$ with $t \in \Sigma(n)$ and $z_i \in S$ (cf. **4.2.10**), we see that $\langle S\rangle$ consists of all elements of the form $t_M(z_1, \ldots, z_n) \in M$ with $t \in \Sigma(n)$ and $z_i \in S$. In particular, any element of $\langle S\rangle$ can be expressed in terms of a finite subset $S_0 \subset S$, i.e. belongs to some $\langle S_0\rangle$ with finite $S_0 \subset S$.

4.6.10. (Finitely generated modules.) Of course, we say that $S \subset M$ *generates* M if $\langle S\rangle = M$, i.e. if $f : \Sigma(S) \to N$ is surjective. In this case we also say that S *is a system of generators of* M. If M admits a finite system of generators, i.e. if there exists a surjective homomorphism $\Sigma(n) \to M$, then we say that M *is finitely generated* or *of finite type.* In this case any system T of generators of M contains a finite subsystem $T_0 \subset M$ of generators of M, since the finite set S is contained in $\langle T_0\rangle$ for a finite $T_0 \subset T$.

4.6.11. (Compatible equivalence relations and strict quotients.) We say that an equivalence relation $R \subset M \times M$ on the underlying set of a Σ-module $M = (M, \eta_M)$ is *compatible* with its Σ-structure, or that the quotient set M/R is *compatible* with the Σ-structure of M, if M/R admits a (necessarily unique) Σ-structure, for which $\pi : M \to M/R$ becomes a homomorphism. Clearly, in this case $R = M \times_{M/R} M$ is the kernel of π, and $\pi : M \to M/R$ is the cokernel of $R \rightrightarrows M$ both in *Sets* and in Σ-*Mod*, hence M/R is a strict quotient of M. Conversely, the kernel $R = M \times_N M$ of any Σ-homomorphism $f : M \to N$ is always compatible with the Σ-structure on M, since the image $f(M) \cong M/R$ admits a Σ-structure (cf. **4.6.4**), so if f is a strict epimorphism, we must have $N \cong M/R$ in Σ-*Mod*, hence in *Sets* as well because of the compatibility of R. We see that strict epimorphisms of Σ-*Mod* are precisely the surjective Σ-homomorphisms, and that the strict quotients of M in Σ-*Mod* are exactly the quotients M/R of M modulo compatible equivalence relations R; in particular, strict quotients of M constitute a set. Observe that in **4.4.7** we have obtained similar results for the category of algebraic monads; one might expect this because of **4.5.20**.

Notice that R is compatible with the Σ-structure on M iff all maps $M^n \xrightarrow{[t]_M} M \xrightarrow{\pi} M/R$ factorize through $M^n \to (M/R)^n$. This means the following:

$$[t]_M(x_1, \ldots, x_n) \equiv_R [t]_M(y_1, \ldots, y_n) \text{ whenever } x_i \equiv_R y_i \text{ for all } 1 \leq i \leq n,$$
$$\text{for any } t \in \Sigma(n),\ x_1,\ \ldots,\ x_n,\ y_1,\ \ldots,\ y_n \in M \quad (4.6.11.1)$$

Of course, it is enough to require this for t from a system of generators of Σ.

4.6.12. (Equivalence relation generated by a set of equations.) Notice that the intersection $R = \bigcap R_\alpha$ of any family of compatible equivalence relations on a Σ-module M is again a compatible equivalence relation, since R is the kernel of $M \to \prod_\alpha M/R_\alpha$. This means that for any set of equations (or relations) E on M, i.e. for any subset $E \subset M \times M$, we can find the smallest compatible equivalence relation $\langle E \rangle$ containing E, simply by taking the intersection of all such relations. We have an alternative description of $R = \langle E \rangle$: namely, $x \equiv_R y$ iff this relation can be obtained after a finite number of applications of rules 0) $x \equiv y$ for any $(x, y) \in E$; 1) $x \equiv x$; 2) $x \equiv y$ implies $y \equiv x$; 3) $x \equiv y$ and $y \equiv z$ imply $x \equiv z$; 4) the substitution rule (4.6.11.1).

Since the kernel R' of any Σ-homomorphism $f : M \to N$ is a compatible equivalence relation, it contains E iff it contains $\langle E \rangle$, i.e. iff f factorizes through $\pi : M \to M/\langle E \rangle$. In other words, π is universal among all Σ-homomorphisms $f : M \to N$, such that $f(x) = f(y)$ for any $(x, y) \in E$.

4.6.13. (Cokernels of pairs of morphisms.) Suppose M is a Σ-module, and $p,\ q : S \rightrightarrows M$ are two maps of sets. Consider the category of all Σ-homomorphisms $f : M \to N$, such that $f \circ p = f \circ q$. We claim that it has an initial object: indeed, $f \circ p = f \circ q$ iff the kernel of f contains the set $E = (p, q)(S)$ of all pairs $(p(s), q(s))$, hence $M \to M/\langle E \rangle$ has the required universal property. This applies in particular when S is a Σ-module, and p and q are Σ-homomorphisms; we see that *cokernels of pairs of morphisms exist in* Σ-*Mod*.

4.6.14. (Finite direct sums.) Let's show that *finite direct sums*, i.e. *coproducts* exist in Σ-*Mod*. Since this category has an initial object $\Sigma(0)$, we have to show the existence of the direct sum $M_1 \oplus M_2$ of two Σ-modules. First of all, if both M_1 and M_2 are free, then their direct sum exists since $\Sigma(S_1 \sqcup S_2)$ satisfies the universal property required from $\Sigma(S_1) \oplus \Sigma(S_2)$. Next, we know that any M_i can be written as a cokernel of a pair of morphisms p_i, $q_i : \Sigma(E_i) \rightrightarrows \Sigma(S_i)$ between two free modules (cf. **3.3.20**). Since inductive limits commute with other inductive limits, we see that $M_1 \oplus M_2 \cong \mathrm{Coker}(p_1, q_1) \oplus \mathrm{Coker}(p_2, q_2) \cong \mathrm{Coker}\bigl(p_1 \oplus p_2, q_1 \oplus q_2 : \Sigma(E_1 \sqcup E_2) \rightrightarrows \Sigma(S_1 \sqcup S_2)\bigr)$. Now the last cokernel is representable (cf. **4.6.13**), hence $M_1 \oplus M_2$ is representable as well.

4.6.15. (Elements of direct sums.) Notice that if S generates M, and T generates N, then $S \sqcup T$ generates $M \oplus N$; in particular, if both M and N are finitely generated, $M \oplus N$ is also finitely generated. This also means that $M \sqcup N$ generates $M \oplus N$; taking **4.6.9** into account, we see that any element of $M \oplus N$ can be written in form $t(x_1, \ldots, x_n, y_1, \ldots, y_m)$ for some m, $n \geq 0$, $t \in \Sigma(n)$, $x_i \in M$ and $y_j \in N$; we can even assume $x_i \in S$ and $y_j \in T$.

Contrary to the case of modules over an associative ring, we cannot express in general any element of $M \oplus N$ in form $t'(x, y)$ for some $t' \in \Sigma(2)$, $x \in M$ and $y \in N$. However, if such a statement is true for all direct sum decompositions of form $\Sigma(n+m) = \Sigma(n) \oplus \Sigma(m)$, i.e. if any $t \in \Sigma(n+m)$ can be represented in form $t = t'\bigl(t_1(\{1\}, \ldots, \{n\}), t_2(\{n+1\}, \ldots, \{n+m\})\bigr)$ for some $t' \in \Sigma(2)$, $t_1 \in \Sigma(n)$ and $t_2 \in \Sigma(m)$, then our description of arbitrary elements of $M \oplus N$ together with the associativity relations shows that any element of $M \oplus N$ can be written in form $t'(x, y)$ with $x \in M$, $y \in N$ in this case, i.e. the validity of the statement for $\Sigma(n + m) = \Sigma(n) \oplus \Sigma(m)$, for all n, $m \geq 0$, implies its validity in general.

This remark is applicable in particular to $\mathbb{Z}_\infty$, $\mathbb{Z}_{(\infty)}$ and $\bar{\mathbb{Z}}_\infty$, since any formal octahedral combination $t = \lambda_1 x_1 + \cdots + \lambda_n x_n + \mu_1 y_1 + \cdots + \mu_m y_m$, $\sum_i |\lambda_i| + \sum_j |\mu_j| \leq 1$, can be re-written as $\lambda x + \mu y$ with x an octahedral combination of x_i and y an octahedral combination of y_j. Indeed, put $\lambda := \sum_i |\lambda_i|$, $\mu := \sum_j |\mu_j|$. If $\lambda = 0$ or $\mu = 0$, the statement is trivial; otherwise we put $x := \sum_i (\lambda^{-1}\lambda_i) x_i$ and $y := \sum_j (\mu^{-1}\mu_j) y_j$.

4.6.16. (Filtered inductive limits.) *Filtered inductive limits* $\varinjlim M_\alpha$ *of* Σ-*modules exist, and they can be computed in Sets.* Indeed, since Σ is algebraic, it commutes with filtered inductive limits, hence $\Sigma(\varinjlim M_\alpha) \cong \varinjlim \Sigma(M_\alpha)$. Now put $M := \varinjlim M_\alpha$ (in *Sets*), and $\eta_M := \varinjlim \eta_{M_\alpha} : \Sigma(M) \to M$, where $\eta_{M_\alpha} : \Sigma(M_\alpha) \to M_\alpha$ is the Σ-structure of M_α.

4.6.17. (Arbitrary inductive limits.) Notice that arbitrary sums $\bigoplus_{i \in I} M_i$ exist in Σ-*Mod*, since such a sum can be written as the filtered inductive limit of $\bigoplus_{i \in J} M_i$ for all finite $J \subset I$; one could also use directly the same reasoning as in **4.6.14**. Since cokernels of pairs of morphisms also exist in Σ-*Mod* (cf. **4.6.13**), we see that *arbitrary inductive limits exist in* Σ-*Mod.*

4.6.18. (Lattices of submodules and of strict quotients.) Existence of arbitrary sums and products, together with existence of images of homomorphisms (cf. **4.6.4**), shows that any family of subobjects N_α of M has both an infimum and a supremum in the ordered set of subobjects of M: one simply computes $\inf N_\alpha$ as $\bigcap N_\alpha$, and $\sup N_\alpha$, denoted also by $\sum_\alpha N_\alpha$, is the image of the canonical homomorphism $\bigoplus N_\alpha \to M$. Similar statements are true for strict quotients of M as well.

4.6.19. (Scalar extension.) Combining **4.6.17** and **3.3.19**, we see that for any homomorphism of algebraic monads $\rho : \Sigma \to \Xi$ the scalar restriction functor $\rho^* : \Xi\text{-}\mathit{Mod} \to \Sigma\text{-}\mathit{Mod}$ admits a left adjoint — the *scalar extension* or *base change* functor $\rho_* : \Sigma\text{-}\mathit{Mod} \to \Xi\text{-}\mathit{Mod}$. We denote $\rho_* M$ also by $\Xi \otimes_\Sigma M$; however, this notation still has to be justified. Of course, in situation $\Sigma \xrightarrow{\rho} \Xi \xrightarrow{\sigma} \Lambda$ we have $(\sigma\rho)^* = \rho^*\sigma^*$ and $(\sigma\rho)_* \cong \sigma_*\rho_*$, so $\Lambda \otimes_\Xi (\Xi \otimes_\Sigma M)$ can be identified with $\Lambda \otimes_\Sigma M$, i.e. we have some sort of "associativity". Also note that $\mathrm{Hom}_\Xi(\Xi \otimes_\Sigma M, N) \cong \mathrm{Hom}_\Sigma(M, \rho^* N)$, and $\Xi \otimes_\Sigma \Sigma(S) \cong \Xi(S)$. This is actually a special case of the "associativity", since $\Sigma \otimes_{\mathbb{F}_\varnothing} S = \Sigma(S)$ for any set S. We see that $\otimes_{\mathbb{F}_\varnothing}$ corresponds to the left $\otimes$-action of $\mathcal{A}_{alg} \subset \mathcal{A}$ on $\mathcal{C}$, denoted by $\otimes$ or $\oslash$ in **3.2.5**. Of course, ρ_* commutes with arbitrary inductive limits, and in particular it is right exact; for example, $\Xi \otimes_\Sigma (M \oplus M') \cong (\Xi \otimes_\Sigma M) \oplus (\Xi \otimes_\Sigma M')$. Similarly, the scalar restriction functor ρ^* commutes with arbitrary projective limits and in particular it is left exact, but in general not (right) exact, contrary to what one might expect. In fact, we'll show in the next chapter that *when Σ is commutative, ρ^* is (right) exact iff $\rho : \Sigma \to \Xi$ is unary*, and in this case ρ^* commutes with arbitrary inductive limits, and even admits a right adjoint $\rho^!$.

4.6.20. (Some counterexamples.) Up to now we have seen that Σ-*Mod* behaves in most respects like the category of (left) modules over an associative ring. However, some properties do not extend to this case. Consider for example the algebraic monad Σ, such that Σ-*Mod* is the category of commutative rings. Then $\mathbb{Z} \to \mathbb{Q}$ is both a monomorphism and an epimorphism in Σ-*Mod*, but not a strict epimorphism and not an isomorphism. Also note that the direct sums lose most of their properties: for example, direct sum of two monomorphisms need not be a monomorphism, since both $\mathbb{Z} \to \mathbb{Q}$ and $\mathrm{id}_{\mathbb{Z}/2\mathbb{Z}}$ are injective, but their coproduct in the category of commutative rings, i.e. the tensor product over $\mathbb{Z}$, is not injective. Direct sums don't commute with finite direct products as well, since this is not true even in $\mathbb{F}_\varnothing\text{-}\mathit{Mod} = \mathit{Sets}$. On the other hand, filtered inductive limits are still left exact, since both filtered inductive limits and finite projective limits of Σ-modules are computed in *Sets*.

4.6.21. (Finite presentation.) We say that a compatible equivalence relation R on a Σ-module M is *finitely generated*, if it is generated by a finite set of equations $E \subset M \times M$. Clearly, R is finitely generated iff $\pi : M \to M/R$ is the universal coequalizer of a pair of maps $p', q' : \mathbf{n} \rightrightarrows M$ (cf. **4.6.13**), or equivalently, if $\pi : M \to M/R$ is the cokernel of a pair of morphisms $p, q : \Sigma(n) \rightrightarrows M$.

Given a module M, we say that a pair of sets (S, E), where $S \subset M$ and $E \subset \Sigma(S) \times \Sigma(S)$ is a *presentation* of M if $f : \Sigma(S) \to M$ is surjective and the kernel of f is generated by E; then $M \cong \Sigma(S)/\langle E \rangle$. Equivalently, we can say that a presentation of M is a pair of maps of sets $p', q' : E \rightrightarrows \Sigma(S)$, or a pair of homomorphisms $p, q : \Sigma(E) \rightrightarrows \Sigma(S)$, such that M is the cokernel in Σ-*Mod* of this pair of maps.

We say that M is *of finite presentation*, or *finitely presented*, if it admits a finite presentation (S, E), i.e. if it is a coequalizer of a pair of maps of sets $p', q' : \mathbf{m} \rightrightarrows \Sigma(n)$, or equivalently, if it is a cokernel of a pair of homomorphisms $p, q : \Sigma(m) \rightrightarrows \Sigma(n)$ between free modules of finite rank. Of course, such a pair of morphisms can be given by two matrices $P, Q \in \Sigma(n)^m = M(n, m; \Sigma)$.

Notice that the isomorphism classes of all finitely generated (resp. finitely presented) modules over a fixed algebraic monad Σ constitute a set, since any such module is isomorphic to a strict quotient of some $\Sigma(n)$. In other words, the category of finitely generated (resp. finitely presented) modules is equivalent to a small category.

These notions of finite presentation and of finite type have all the usual properties, which can be shown essentially in the same way as over associative rings. For example, the cokernel of a pair of homomorphisms from a finitely generated module into a finitely presented module is itself finitely presented; conversely, the kernel of a surjective homomorphism from a finitely generated module to a finitely presented module is finitely generated as a compatible equivalence relation. Another simple fact: any finite inductive limit (e.g. finite sum) of finitely generated (resp. finitely presented) modules is itself finitely generated (resp. finitely presented).

4.6.22. Any Σ-module is a filtered union of its finitely generated submodules, hence it is a filtered inductive limit of a system of finitely generated modules with injective transition morphisms. Moreover, *any Σ-module M is a filtered inductive limit of finitely presented Σ-modules.* To show this we consider the category $\mathcal{D}$ of finitely presented Σ-modules (or any its small subcategory, equivalent to the whole of $\mathcal{D}$), and observe that the category $\mathcal{D}/M$ of homomorphisms $N \to M$ from finitely presented modules into M is filtering, being stable under finite inductive limits, and $\varinjlim_{\mathcal{D}/M} N \cong M$.

4.6.23. (Direct factors and projective modules.) Recall that N is a *direct* *factor* of an object M of some category (e.g. Σ-*Mod*) if we have two morphisms $i : N \to M$ and $\sigma : M \to N$, such that $\sigma i = \mathrm{id}_N$. Then i is a strict monomorphism, and σ is a strict epimorphism, so N can be considered both as a submodule and as a strict quotient of M. Moreover, in this situation $p := i\sigma \in \mathrm{End}_\Sigma(M)$ is an idempotent, i.e. $p^2 = p$, $i : N \to M$ is the kernel of $p, \mathrm{id}_M : M \rightrightarrows M$, and $\sigma : M \to N$ is the cokernel of the same pair. Conversely, if $p^2 = p$ is an idempotent in $\mathrm{End}(M)$, then we might reconstruct N either as the kernel or as the cokernel of p and id_M (if any of these exists in the category under consideration; of course, they both exist in Σ-*Mod*), thus obtaining a direct factor of M. In this way we obtain a bijection between direct factors of M and idempotents in $\mathrm{End}(M)$. Notice that the property of being a direct factor is preserved by all functors, and since direct factors can be described both as kernels and as cokernels, we see that a direct factor of a left (resp. right) exact functor is again left (resp. right) exact. Another consequence: since $N = \mathrm{Coker}(p, \mathrm{id}_M : M \rightrightarrows M)$, if M is finitely generated (resp. finitely presented), the same is true for any its direct factor N.

Let's apply this to *projective* modules P, i.e. such Σ-modules P, for which $\mathrm{Hom}_\Sigma(P, -)$ preserves strict epimorphisms, i.e. transforms surjective homomorphisms into surjective maps. Clearly, any free Σ-module $\Sigma(S)$ is projective, since $\mathrm{Hom}_\Sigma(\Sigma(S), -) = \mathrm{Hom}_{Sets}(S, -)$ (if S is infinite, we need the axiom of choice here), and any direct factor of a projective module is projective, so we see that *direct factors of free modules are projective.* Conversely, if P is projective, and S is any system of generators of P, then the surjection $\sigma : \Sigma(S) \to P$ admits a section, hence P is a direct factor of a free module $\Sigma(S)$.

4.6.24. Given any Σ-module M and any *filtered* inductive limit $N = \varinjlim N_\alpha$, we have a canonical map $i = i_M : \varinjlim \mathrm{Hom}_\Sigma(M, N_\alpha) \to \mathrm{Hom}_\Sigma(M, \varinjlim N_\alpha)$. We claim that *if M is finitely presented (resp. finitely generated), then i is always bijective (resp. injective).* Indeed, in this case we can find a finite presentation of M, i.e. a right exact sequence $\Sigma(m) \rightrightarrows \Sigma(n) \to M$ (resp. a surjective homomorphism $\Sigma(n) \to M$). Consider the following diagram (resp. its left square):

$$\begin{tikzcd}
\varinjlim \mathrm{Hom}_\Sigma(M, N_\alpha) \arrow[r] \arrow[d, "i"] & \varinjlim \mathrm{Hom}_\Sigma(\Sigma(n), N_\alpha) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "i_{\Sigma(n)}"] & \varinjlim \mathrm{Hom}_\Sigma(\Sigma(m), N_\alpha) \arrow[d, "i_{\Sigma(m)}"] \\
\mathrm{Hom}_\Sigma(M, \varinjlim N_\alpha) \arrow[r] & \mathrm{Hom}_\Sigma(\Sigma(n), \varinjlim N_\alpha) \arrow[r, shift left] \arrow[r, shift right] & \mathrm{Hom}_\Sigma(\Sigma(m), \varinjlim N_\alpha)
\end{tikzcd} \tag{4.6.24.1}$$

Its rows are left exact (resp. the horizontal arrows of the left square are injective), and the vertical arrows $i_{\Sigma(n)}$ and $i_{\Sigma(m)}$ are isomorphisms, since

$\mathrm{Hom}_\Sigma(\Sigma(n), N) \cong N^n$, and filtered inductive limits commute with finite products in *Sets*. This implies the bijectivity (resp. injectivity) of $i = i_M$.

Conversely, *if for some Σ-module M the functor* $\mathrm{Hom}_\Sigma(M, -)$ *commutes with filtered inductive limits, then M is finitely presented.* Indeed, write M as a filtered inductive limit $\varinjlim N_\alpha$ of some finitely presented modules (cf. **4.6.22**), and denote by $j_\alpha : N_\alpha \to M$ the canonical homomorphisms $N_\alpha \to \varinjlim N_\alpha \cong M$. We see that $\mathrm{id}_M \in \mathrm{Hom}_\Sigma(M, M) \cong \mathrm{Hom}_\Sigma(M, \varinjlim N_\alpha)$ has to come from some $\sigma \in \mathrm{Hom}_\Sigma(M, N_\alpha)$, i.e. $j_\alpha \circ \sigma = \mathrm{id}_M$. This means that M is a direct factor of a finitely presented module N_α, hence M is finitely presented itself.

4.7. (Categories of algebraic modules.) Now we would like to prove some basic properties of *algebraic modules over an algebraic monad* Σ, i.e. the (left or right) Σ-modules in $\mathcal{A}_{alg} \subset \mathcal{A}$.

4.7.1. (Left Σ-modules.) Let's begin with the case of a left algebraic Σ-module F, i.e. an algebraic endofunctor F, together with a Σ-action $\alpha : \Sigma F \to F$, such that $\alpha \circ (\varepsilon \star F) = \mathrm{id}_F$ and $\alpha \circ (\Sigma \star \alpha) = \alpha \circ (\mu \star F)$ (cf. **3.3.5**). We know that F is given by a collection of sets $\{F(n)\}_{n \geq 0}$ and of maps $F(\varphi) : F(m) \to F(n)$ for each $\varphi : \mathbf{m} \to \mathbf{n}$ (cf. **4.2.1**), and $\alpha : \Sigma F \to F$ is given by maps $\alpha_n^{(k)} : \Sigma(k) \times F(n)^k \to F(n)$, subject to certain compatibility conditions (cf. **4.2.3**). Recall that we put $[t]_{F(n)}(x_1, \ldots, x_k) = \alpha_n^{(k)}(t; x_1, \ldots, x_n)$ for any $t \in \Sigma(k)$ and any $x_i \in F(n)$.

Now we can write down the conditions for such a collection of data to define a left Σ-module F. Actually, equality of natural transformations of algebraic functors can be checked on standard finite sets, so we end up with requiring the $\{\alpha_n^{(k)} : \Sigma(k) \times F(n)^k \to F(n)\}_{k \geq 0}$ to define a Σ-module structure on each $F(n)$, and requiring $F(\varphi) : F(m) \to F(n)$ to be a Σ-homomorphism for each $\varphi : \mathbf{m} \to \mathbf{n}$.

We see that a left algebraic Σ-module F is essentially the same thing as a functor $F : \underline{\mathbb{N}} \to \Sigma\text{-}\mathit{Mod} = \mathcal{C}^\Sigma$, $\mathbf{n} \mapsto F(n)$. We could deduce this result directly from **4.1.3** and **3.3.7**: indeed, we know that the restriction to $\mathcal{A}_{alg} \subset \mathcal{A} = \mathit{Funct}(\mathcal{C}, \mathcal{C})$ of the restriction functor $J^* : \mathit{Funct}(\mathcal{C}, \mathcal{C}) \to \mathit{Funct}(\underline{\mathbb{N}}, \mathcal{C})$ is an equivalence of categories (cf. **4.1.3**), clearly compatible with the left $\oslash$-action of $\mathcal{A}_{alg}$ on $\mathcal{A}_{alg} \subset \mathcal{A}$ and on $\mathit{Funct}(\underline{\mathbb{N}}, \mathcal{C})$, hence $\mathcal{A}_{alg}^\Sigma \cong \mathit{Funct}(\underline{\mathbb{N}}, \mathcal{C})^\Sigma \cong \mathit{Funct}(\underline{\mathbb{N}}, \mathcal{C}^\Sigma) = \Sigma\text{-}\mathit{Mod}^{\underline{\mathbb{N}}}$ by **3.3.7**.

4.7.2. (Matrix interpretation.) Let F be an algebraic left Σ-module. Then we put $M(n, m; F) := F(n)^m$, and interpret this as "the set of $n \times m$-matrices with entries in F" as before. The action maps $\alpha_n^{(k)} : \Sigma(k) \times F(n)^k \to F(n)$ can be interpreted as some matrix multiplication maps $M(k, m; \Sigma) \times M(n, k; F) \to M(n, m; F)$ (with opposite order of arguments), and then the associativity and unit relations for α can be understood as associativity of matrix multiplication $M(m,s;\Sigma)\times M(k,m;\Sigma)\times M(n,k;F)\to M(n,s;F)$, and the requirement for $I_k=(\{1\},\ldots,\{k\})\in M(k,k;\Sigma)$ to act identically on $M(n,k;F)=F(n)^k$.

4.7.3. (Monad of endomorphisms of an algebraic endofunctor.) If F is an algebraic left Σ-module, any $t\in\Sigma(k)$ induces a family of compatible maps $[t]_{F(n)}:F(n)^k\to F(n)$, i.e. a natural transformation $[t]_F:F^k\to F$. These natural transformations have to satisfy a substitution property similar to that discussed in **4.3.7**, and conversely, once we have a family of such natural transformations, satisfying this property as well as the requirement for $[\{s\}_{\mathbf{n}}]_F:F^k\to F$ to coincide with the projection onto the k-th component, we obtain a left Σ-structure on F.

This enables us to repeat the reasoning of **4.3.8**, and define for any algebraic endofunctor F its "absolute endomorphism ring" $\mathrm{END}(F)$ by putting $(\mathrm{END}(F))(n):=\mathrm{Hom}_{\mathcal{A}}(F^n,F)$, with the multiplication of $\mathrm{END}(F)$ given by composition of natural transformations. Then $\mathrm{END}(F)$ canonically acts on F from the left, i.e. we have a natural transformation $\mathrm{END}(F)\otimes F\to F$, and giving a left Σ-module structure on F is equivalent to giving a homomorphism of algebraic monads $\rho:\Sigma\to\mathrm{END}(F)$; then the left Σ-structure on F is recovered from the canonical $\mathrm{END}(F)$-structure by means of scalar restriction along ρ.

Similarly to **4.3.8**, we denote by Σ_F the image of Σ in $\mathrm{END}(F)$, and say that F is a *faithful* algebraic left Σ-module if $\rho:\Sigma\to\mathrm{END}(F)$ is injective. For example, Σ is always a faithful module over itself.

4.7.4. (Limits, subobjects, strict quotients…) Once the equivalence of categories $\mathcal{A}_{alg}^{\Sigma}\cong \mathit{Funct}(\underline{\mathbb{N}},\Sigma\text{-}\mathit{Mod})$ is established, we can deduce properties of this category from those of Σ-*Mod*. For example, arbitrary projective and inductive limits exist in $\mathcal{A}_{alg}^{\Sigma}$, and they are computed componentwise, e.g. $(\varinjlim F_\alpha)(n)=\varinjlim F_\alpha(n)$, the forgetful functor $\mathcal{A}_{alg}^{\Sigma}\to\mathcal{A}_{alg}$ is left exact and commutes with filtered inductive limits, and so on. In particular, subobjects F' of F are given by families of submodules $F'(n)\subset F(n)$, compatible with all maps $F(\varphi)$, strict quotients of F are parametrized by compatible equivalence relations $R\subset F\times F$, which can be interpeted as families of compatible equivalence relations $R(n)\subset F(n)\times F(n)$, and the strict quotient F/R is computed componentwise: $(F/R)(n)=F(n)/R(n)$.

4.7.5. (Scalar restriction and base change.) Recall that any homomorphism of algebraic monads $\rho:\Sigma\to\Xi$ induces scalar restriction functors $\rho^*:\Xi\text{-}\mathit{Mod}=\mathcal{C}^{\Xi}\to\Sigma\text{-}\mathit{Mod}$ and $\rho^*_{\mathcal{A}}:\mathcal{A}^{\Xi}\to\mathcal{A}^{\Sigma}$; identifying $\mathcal{A}^{\Sigma}$ with $\mathit{Funct}(\mathcal{C},\mathcal{C}^{\Sigma})$ (cf. **3.3.7**), we see that this functor $\mathcal{A}^{\Xi}\to\mathcal{A}^{\Sigma}$ is essentially given by $F \mapsto \rho^* \circ F$, for any $F : \mathcal{C} \to \mathcal{C}^\Xi$. When we restrict our attention to full subcategories of algebraic left modules $\mathcal{A}^\Sigma_{alg} \subset \mathcal{A}^\Sigma$, $\mathcal{A}^\Sigma_{alg} \cong \mathit{Funct}(\underline{\mathbb{N}}, \Sigma\text{-}\mathit{Mod})$, and similarly for $\mathcal{A}^\Xi_{alg}$, we see that the scalar restriction functor $\mathcal{A}^\Xi_{alg} \cong \mathit{Funct}(\underline{\mathbb{N}}, \Xi\text{-}\mathit{Mod}) \to \mathcal{A}^\Sigma_{alg}$ is given again by $F \mapsto \rho^* \circ F$, i.e. the scalar restriction of left algebraic modules is computed componentwise. Since $\rho^* : \Xi\text{-}\mathit{Mod} \to \Sigma\text{-}\mathit{Mod}$ admits a left adjoint ρ_* (cf. **4.6.19**), we see that the scalar restriction functor between categories of algebraic left modules admits a left adjoint as well, given by $F \mapsto \rho_* \circ F$. Of course, we say that $\rho_* F$ is the *scalar extension* or *base change* of F with respect to $\rho : \Sigma \to \Xi$, and denote $\rho_* F$ also by $\Xi \otimes_\Sigma F$; we have just seen that $(\rho_* F)(n) = \rho_*(F(n))$, or $(\Xi \otimes_\Sigma F)(n) = \Xi \otimes_\Sigma F(n)$.

4.7.6. (Kan extensions, skeleta and coskeleta.) Since $\mathcal{A}^\Sigma_{alg} \cong \mathit{Funct}(\underline{\mathbb{N}}, \mathcal{C}^\Sigma) = (\Sigma\text{-}\mathit{Mod})^{\underline{\mathbb{N}}}$, and arbitrary inductive and projective limits exist in $\Sigma\text{-}\mathit{Mod}$, we see that any functor $K : \mathcal{I} \to \underline{\mathbb{N}}$ admits both left and right Kan extensions $K_!$ and $K_* : (\Sigma\text{-}\mathit{Mod})^{\mathcal{I}} \to \mathcal{A}^\Sigma_{alg}$, i.e. both left and right adjoints to $K^* : F \mapsto F \circ K$. Usually we apply this for the embeddings $\mathcal{I} \to \underline{\mathbb{N}}$ of some subcategories $\mathcal{I}$ of $\underline{\mathbb{N}}$. For example, if we take for $\mathcal{I}$ the subcategory of $\underline{\mathbb{N}}$, which contains all objects of $\underline{\mathbb{N}}$, but only the identity morphisms, we see that the functor K^* which maps any left algebraic Σ-module F into the collection of Σ-modules $\{F(n)\}_{n \geq 0}$ (without any transition morphisms), admits both a left and a right adjoint, and we can compute these adjoints by usual formulas for Kan extensions. For example, $K_! G$ is given by $(K_! G)(n) = \varinjlim_{\mathcal{I}/\mathbf{n}} G(i)$, where $\mathcal{I}/\mathbf{n}$ is the category of couples (i, φ), $i \in \operatorname{Ob} \mathcal{I}$, $\varphi : K(i) \to \mathbf{n}$; in our case $K_!$ transforms a collection of Σ-modules $M = \{M_n\}_{n \geq 0}$, into $K_! M$, given by $(K_! M)(n) := \bigoplus_{\varphi : \mathbf{m} \to \mathbf{n}} M_m$, and similarly, $(K_* M)(n) = \prod_{\varphi : \mathbf{n} \to \mathbf{m}} M_m$. Notice that left Kan extensions commute with base change, while right Kan extensions commute with scalar restriction.

Another important example: when $\mathcal{I} = \{\mathbf{1}\} \subset \underline{\mathbb{N}}$, we see that $F \mapsto F(1)$, $\mathcal{A}^\Sigma_{alg} \to \Sigma\text{-}\mathit{Mod}$, admits both a left and a right adjoint; they are given by $(K_! M)(n) = \bigoplus_{\mathbf{n}} M = M^{(n)}$ (the direct sum of n copies of M), and $(K_* M)(n) = M$ for any $n \geq 0$.

Finally, let's take for $\mathcal{I}$ the full subcategory $\underline{\mathbb{N}}_{\leq n}$, consisting of all objects $\{\mathbf{m}\}_{m \leq n}$; in this manner we obtain the *skeleta* $\operatorname{sk}_n := K_{\leq n,!} K^*_{\leq n}$ and the *coskeleta* functors $\operatorname{cosk}_n := K_{\leq n,*} K^*_{\leq n}$, computed almost in the same way as for cosimplicial objects (remember, $\mathcal{A}^\Sigma_{alg} = (\Sigma\text{-}\mathit{Mod})^{\underline{\mathbb{N}}}$ is something like the category of cosimplicial Σ-modules, but not exactly, since we allow all maps between finite sets in $\underline{\mathbb{N}}$, not just the non-decreasing ones, and consider the empty set $\mathbf{0} \in \operatorname{Ob} \underline{\mathbb{N}}$ as well.)

4.7.7. (Free algebraic left Σ-modules.) An important consequence is that the forgetful functor $F \mapsto \|F\| = \bigsqcup_{n \geq 0} F(n)$ from the category of algebraic left Σ-modules into the category of graded sets admits a left adjoint. Indeed, we can decompose this functor into two functors: the first maps F into the collection of Σ-modules $\{F(n)\}_{n\geq 0}$, and it admits both a left and a right adjoint by the theory of Kan extensions, and the second maps such a collection $\{M_n\}$ into the collection of underlying sets of M_n; it also admits a left adjoint, given by $S = \bigsqcup_{n\geq 0} S_n \mapsto \{\Sigma(S_n)\}_{n\geq 0}$.

4.7.8. (Right Σ-modules.) Now we would like to obtain similar descriptions for the category $\mathcal{A}_{alg,\Sigma}$ of right algebraic Σ-modules, i.e. of algebraic endofunctors G, equipped with a right Σ-action $\beta : G\Sigma \to G$. Of course, G itself is given by a collection of sets $\{G(n)\}_{n\geq 0}$ and of maps $G(\varphi) : G(m) \to G(n)$, defined for each $\varphi : \mathbf{m} \to \mathbf{n}$ (cf. **4.2.1**), and β is given by a collection of maps $\beta_n^{(k)} : G(k) \times \Sigma(n)^k \to G(n)$. We can put $M(n,m;G) := G(n)^m$ as usual, and interpet the above maps as some matrix multiplication rules $M(k,s;G) \times M(n,k;\Sigma) \to M(n,s;G)$. Then the requirements for β to be a right Σ-action translate into some "associativity of matrix multiplication" $M(k,s;G)\times M(n,k;\Sigma)\times M(m,n;\Sigma) \to M(m,s;G)$ and into the requirement for the identity matrix $I_n \in \Sigma(n)^n$ to act identically on $G(n)$; the verification is similar to that of **4.3.3**,b) and c).

Notice that $\beta_n^{(k)} : G(k) \times \Sigma(n)^k \to G(n)$ can be interpreted as a map from $\Sigma(n)^k \cong \mathrm{Hom}_\Sigma(\Sigma(k), \Sigma(n))$ into $\mathrm{Hom}_{Sets}(G(k), G(n))$, so we essentially obtain a functor $\tilde{G}$ from the category $\underline{\mathbb{N}}_\Sigma$ of standard free Σ-modules of finite rank into the category of sets. We'll check in a moment that this is indeed a functor, but this can be also seen directly in terms of the "associativity of matrix multiplication" just discussed.

4.7.9. (Alternative description of right modules.) Recall that in **3.3.13** we have shown that the category $\mathcal{A}_\Sigma$ of right Σ-modules in $\mathcal{A} = \mathit{Funct}(\mathcal{C}, \mathcal{C})$ is equivalent to $\mathit{Funct}(\mathcal{C}_\Sigma, \mathcal{C})$, where $\mathcal{C}_\Sigma$ has the same objects as $\mathcal{C}$, but morphisms given by $\mathrm{Hom}_{\mathcal{C}_\Sigma}(X,Y) = \mathrm{Hom}_\mathcal{C}(X, \Sigma(Y)) \cong \mathrm{Hom}_\Sigma(\Sigma(X), \Sigma(Y))$, so $\mathcal{C}_\Sigma$ is essentially the category of free Σ-modules. We have also constructed a fully faithful functor $Q_\Sigma : \mathcal{C}_\Sigma \to \mathcal{C}^\Sigma = \Sigma\text{-}\mathit{Mod}$, that transforms X into the free module $L_\Sigma(X) = (\Sigma(X), \mu_X)$, and a functor $I_\Sigma : \mathcal{C} \to \mathcal{C}_\Sigma$ with a canonical right Σ-action; then the equivalence $\mathit{Funct}(\mathcal{C}_\Sigma, \mathcal{C}) \cong \mathit{Funct}(\mathcal{C}, \mathcal{C})_\Sigma = \mathcal{A}_\Sigma$ is given by $I_\Sigma^* : G \mapsto G \circ I_\Sigma$, with the right Σ-structure on $G \circ I_\Sigma$ induced by that of I_Σ.

In this way the full subcategory $\mathcal{A}_{alg,\Sigma} \subset \mathcal{A}_\Sigma$ of algebraic right Σ-modules corresponds to a certain full subcategory of $\mathit{Funct}(\mathcal{C}_\Sigma, \mathcal{C})$. We claim that *the full subcategory of* $\mathit{Funct}(\mathcal{C}_\Sigma, \mathcal{C})$*, equivalent to* $\mathcal{A}_{alg,\Sigma}$*, coincides with the essential image of the left Kan extension* $J_{\Sigma,!} : \mathit{Funct}(\underline{\mathbb{N}}_\Sigma, \mathcal{C}) \to \mathit{Funct}(\mathcal{C}_\Sigma, \mathcal{C})$*, where* $\underline{\mathbb{N}}_\Sigma$ *is the full subcategory of* $\mathcal{C}_\Sigma$ *with standard finite sets for objects (i.e.* $\underline{\mathbb{N}}_\Sigma$ *is essentially the category of "standard free* Σ*-modules of finite rank"),* *and $J_\Sigma : \underline{\mathbb{N}}_\Sigma \to \mathcal{C}_\Sigma$ is the natural embedding. The quasi-inverse equivalence is given by the restriction to $\mathcal{A}_{alg,\Sigma}$ of the restriction functor J_Σ^*.*

Since arbitrary inductive limits exist in $Funct(\mathcal{C}_\Sigma, \mathcal{C})$, the left Kan extension $J_{\Sigma,!}$, i.e. the left adjoint to J_Σ^*, exists for general reasons; it is fully faithful since J_Σ is fully faithful itself, so we have $J_\Sigma^* J_{!,\Sigma}(G') \cong G'$ for any functor $G' : \underline{\mathbb{N}}_\Sigma \to \mathcal{C}$. Now we have to check that some functor $G : \mathcal{C}_\Sigma \to \mathcal{C}$ is algebraic, i.e. the underlying functor $I_\Sigma^*(G) = G \circ I_\Sigma : \mathcal{C} \to \mathcal{C}$ of the corresponding right Σ-module commutes with arbitrary inductive limits, iff G is isomorphic to some $J_{\Sigma,!}(G')$. This follows now from **4.1.3** and lemma **4.7.10** below, once we take into account $J_\Sigma \circ \bar{I}_\Sigma = I_\Sigma \circ J$, and the conservativity of I_Σ^*. Indeed, if $G = J_{\Sigma,!}(G')$, we get $I_\Sigma^*(G) \cong J_! \bar{I}_\Sigma^*(G')$ from **4.7.10**, so $I_\Sigma^*(G)$ is algebraic by **4.1.3**. Conversely, if $I_\Sigma^*(G)$ is algebraic for some $G : \mathcal{C}_\Sigma \to \mathcal{C}$, we obtain $I_\Sigma^*(G) \cong J_! J^* I_\Sigma^*(G) = J_! \bar{I}_\Sigma^* J_\Sigma^*(G) \cong I_\Sigma^* J_{\Sigma,!} J_\Sigma^*(G)$, hence $J_{\Sigma,!} J_\Sigma^*(G) \to G$ is an isomorphism because of the conservativity of I_Σ^*.

Lemma 4.7.10 *In the above notations $I_\Sigma^* J_{\Sigma,!}(G') \cong J_! \bar{I}_\Sigma^*(G')$, where $\bar{I}_\Sigma : \underline{\mathbb{N}} \to \underline{\mathbb{N}}_\Sigma$ is the restriction of $I_\Sigma : \mathcal{C} \to \mathcal{C}_\Sigma$ to $\underline{\mathbb{N}}$, and $G' : \underline{\mathbb{N}}_\Sigma \to \mathcal{C}$ is an arbitrary functor. Moreover, both inductive limits used in computation of left Kan extensions $J_{\Sigma,!}$ and $J_!$ are in fact filtering.*

Proof. First of all, a canonical morphism $\kappa_{G'} : J_! \bar{I}_\Sigma^*(G') \to I_\Sigma^* J_{\Sigma,!}(G')$ is constructed by adjointness from $\bar{I}_\Sigma^*(G') \xrightarrow{\sim} J^* I_\Sigma^* J_{\Sigma,!}(G') = \bar{I}_\Sigma^* J_\Sigma^* J_{\Sigma,!}(G')$, obtained in its turn by applying $\bar{I}_\Sigma$ to the adjointness morphism $G' \to J_\Sigma^* J_{\Sigma,!}(G')$, which is in fact an isomorphism, $J_{\Sigma,!}$ being fully faithful. We have to show that $\kappa_{G'}$ is also an isomorphism; for this we write down the expressions for left Kan extensions in terms of inductive limits: $(J_! \bar{I}_\Sigma^*(G'))(S) = \varinjlim_{\underline{\mathbb{N}}/S} G'(I_\Sigma(\mathbf{n}))$, and $(I_\Sigma^* J_{\Sigma,!}(G'))(S) = \varinjlim_{\underline{\mathbb{N}}_\Sigma/I_\Sigma(S)} G'$. We see that in the first case we compute the inductive limit of $G'(I_\Sigma(\mathbf{n}))$ along the category $\underline{\mathbb{N}}/S$ of all maps $i : \mathbf{n} \to S$, i.e. essentially the limit of G' along the category $I_\Sigma(\underline{\mathbb{N}}/S)$ of all morphisms $I_\Sigma(i) : I_\Sigma(\mathbf{n}) \to I_\Sigma(S)$; if we identify $\mathcal{C}_\Sigma$ with a full subcategory of Σ-*Mod* by means of Q_Σ, we see that the first index category consists of all morphisms $\Sigma(n) \to \Sigma(S)$ of form $\Sigma(i)$ for some $i : \mathbf{n} \to S$. Now observe that the second inductive limit is computed for the same functor G', but along the category $\underline{\mathbb{N}}_\Sigma/I_\Sigma(S)$, which essentially consists of *all* Σ-homomorphisms $\Sigma(n) \to \Sigma(S)$.

So we have to show that the natural embedding $I_\Sigma(\underline{\mathbb{N}}/S) \to \underline{\mathbb{N}}_\Sigma/I_\Sigma(S)$ induces an isomorphism of inductive limits along these index categories, i.e. that the subcategory of homomorphisms of form $\Sigma(i) : \Sigma(n) \to \Sigma(S)$ is *cofinal* in the category of all Σ-homomorphisms $\Sigma(n) \to \Sigma(S)$.

First of all, notice that $\Sigma(S) = \varinjlim_{S_0 \subset S} \Sigma(S_0)$, where the limit is taken along all finite $S_0 \subset S$, since Σ is algebraic, hence $\mathrm{Hom}_\Sigma(\Sigma(n), \Sigma(S)) \cong$

$\Sigma(S)^n \cong \varinjlim_{S_0\subset S} \Sigma(S_0)^n \cong \varinjlim_{S_0\subset S} \operatorname{Hom}_\Sigma(\Sigma(n), \Sigma(S_0))$. This shows that any Σ-homomorphism $f : \Sigma(n) \to \Sigma(S)$ factorizes through $\Sigma(S_0) \subset \Sigma(S)$ for some finite $S_0 \subset S$; replacing S_0 by an isomorphic standard finite set $\mathbf{m}$, we see that any $f : \Sigma(n) \to \Sigma(S)$ factorizes through some $\Sigma(i) : \Sigma(m) \to \Sigma(S)$ for some *injective* $i : \mathbf{m} \to S$.

Now this remark implies immediately that such homomorphisms form a cofinal set in $\underline{\mathbb{N}}_\Sigma/I_\Sigma(S)$, and that both these categories are in fact filtering.

4.7.11. (Algebraic bimodules.) Now suppose we are given two algebraic monads Σ and Λ, and we want to describe the category $\mathcal{A}^\Lambda_{alg,\Sigma} \subset \mathcal{A}^\Lambda_\Sigma$ of *algebraic* (Λ, Σ)*-bimodules*, i.e. of algebraic functors $G : \mathcal{C} \to \mathcal{C}$, equipped both with a left Λ-action $\alpha : \Lambda G \to G$ and a right Σ-action $\beta : G\Sigma \to G$, these two actions being required to commute: $\alpha \circ (\Lambda \star \beta) = \beta \circ (\alpha \star \Sigma)$.

Recall that the category of all (Σ, Λ)-bimodules is equivalent to the category of functors $\mathit{Funct}(\mathcal{C}_\Sigma, \mathcal{C}^\Lambda)$, since $\mathcal{A}_\Sigma \cong \mathit{Funct}(\mathcal{C}_\Sigma, \mathcal{C})$ by **3.3.13**, and this equivalence is compatible with the left $\oslash$-actions of $\mathcal{A}$, so we get $\mathcal{A}^\Lambda_\Sigma = (\mathcal{A}_\Sigma)^\Lambda \cong \mathit{Funct}(\mathcal{C}_\Sigma, \mathcal{C})^\Lambda \cong \mathit{Funct}(\mathcal{C}_\Sigma, \mathcal{C}^\Lambda)$ by **3.3.7**.

Similarly, the category $\mathcal{A}_{alg,\Sigma} \subset \mathcal{A}_\Sigma$ is equivalent to $\mathit{Funct}(\underline{\mathbb{N}}_\Sigma, \mathcal{C})$ by **4.7.9**, and this equivalence is again compatible with the left $\oslash$-action of $\mathcal{A}$, hence $\mathcal{A}^\Lambda_{alg,\Sigma} \cong \mathit{Funct}(\underline{\mathbb{N}}_\Sigma, \mathcal{C})^\Lambda \cong \mathit{Funct}(\underline{\mathbb{N}}_\Sigma, \mathcal{C}^\Lambda) = \mathit{Funct}(\underline{\mathbb{N}}_\Sigma, \Lambda\text{-}\mathit{Mod})$. In the "cosimplicial set" setting this means that Λ controls the category Λ-*Mod* where our "cosimplicial objects" take values, and Σ affects the "index category" $\underline{\mathbb{N}}$, replacing the category of standard finite sets $\underline{\mathbb{N}}$ with a richer category $\underline{\mathbb{N}}_\Sigma$. It might be interesting, for example, to consider the case $\Lambda = \Sigma = \mathbf{\Delta}$.

Of course, if we take $\Lambda = \mathbb{F}_\varnothing$, then $\mathcal{A}^{\mathbb{F}_\varnothing}_{alg,\Sigma} = \mathcal{A}_{alg,\Sigma}$, so all the properties proved below for algebraic bimodules can be specialized to the case of algebraic right modules.

4.7.12. (Matrix description of algebraic bimodules.) Of course, an algebraic bimodule (G, α, β) with $\alpha : \Lambda G \to G$ and $\beta : G\Sigma \to G$ can be described in terms of a collection of sets $\{G(n)\}_{n\geq 0}$, maps $G(\varphi) : G(m) \to G(n)$ for each $\varphi : \mathbf{m} \to \mathbf{n}$, and some maps $\alpha^{(k)}_n : \Lambda(k) \times G(n)^k \to G(n)$ and $\beta^{(k)}_n : G(k) \times \Sigma(n)^k \to G(n)$, subject to all conditions listed before for left and right algebraic modules, as well as a certain commutativity relation for α and β. We have a matrix interpretation as before; then this commutativity relation can be interpreted again as some associativity of matrix multiplication $M(k, s; \Lambda) \times M(n, k; G) \times M(m, n; \Sigma) \to M(m, s; G)$. For example, if G is an algebraic Σ-bimodule, i.e. a (Σ, Σ)-bimodule, then we obtain some multiplication maps of matrices over G with matrices over Σ from both sides, with products being matrices over G.

4.7.13. (Operations with algebraic bimodules.) Equivalence of categories $\mathcal{A}^{\Lambda}_{alg,\Sigma} \cong \mathit{Funct}(\underline{\mathbb{N}}_\Sigma, \Lambda\text{-}\mathit{Mod})$ shows that arbitrary inductive and projective limits, subobjects, strict quotients, and so on, can be defined and computed componentwise in Λ-*Mod*, similarly to **4.7.4**. Moreover, any algebraic monad homomorphism $\sigma : \Lambda \to \Lambda'$ induces a scalar restriction functor $\sigma^* : \mathcal{A}^{\Lambda'}_{alg,\Sigma} \to \mathcal{A}^{\Lambda}_{alg,\Sigma}$, and its left adjoint, the scalar extension or base change functor σ_*, both of which can be computed componentwise by composing functors from $\underline{\mathbb{N}}_\Sigma$ into Λ-*Mod* or Λ'-*Mod* with scalar restriction or extension functors between Λ-*Mod* and Λ'-*Mod*. Therefore, we can fix Λ and study what happens when we change Σ.

4.7.14. (Scalar restriction and extension.) Now let's take some algebraic monad homomorphism $\rho : \Sigma \to \Xi$, and study the scalar restriction functor $\rho^* : \mathcal{A}^{\Lambda}_{alg,\Xi} \to \mathcal{A}^{\Lambda}_{alg,\Sigma}$. We know that $\rho^* : \mathcal{A}^{\Lambda}_{\Xi} \cong \mathit{Funct}(\mathcal{C}_\Xi, \mathcal{C}^\Lambda) \to \mathcal{A}^{\Lambda}_{\Sigma} \cong \mathit{Funct}(\mathcal{C}_\Sigma, \mathcal{C}^\Lambda)$ is given by pre-composing functors $\mathcal{C}_\Xi \to \mathcal{C}^\Lambda$ with a certain "base change" functor $\rho_* : \mathcal{C}_\Sigma \to \mathcal{C}_\Xi$ (cf. **3.3.17**). This implies that $\rho^* : \mathcal{A}^{\Lambda}_{alg,\Xi} \cong \mathit{Funct}(\underline{\mathbb{N}}_\Xi, \mathcal{C}^\Lambda) \to \mathcal{A}^{\Lambda}_{alg,\Sigma} \cong \mathit{Funct}(\underline{\mathbb{N}}_\Sigma, \mathcal{C}^\Lambda)$ is given by pre-composing functors $\underline{\mathbb{N}}_\Xi \to \mathcal{C}^\Lambda$ with the restriction $\bar{\rho}_* : \underline{\mathbb{N}}_\Sigma \to \underline{\mathbb{N}}_\Xi$ of ρ_* to $\underline{\mathbb{N}}_\Sigma$, i.e. ρ^* is essentially equal to $(\bar{\rho}_*)^*$.

Now we see that $\rho_* = (\bar{\rho}_*)^*$ admits both a left and a right adjoint, namely, the Kan extensions $\rho_! := (\bar{\rho}_*)_!$ and $\rho_? := (\bar{\rho}_*)_* : \mathit{Funct}(\underline{\mathbb{N}}_\Sigma, \Lambda\text{-}\mathit{Mod}) \to \mathit{Funct}(\underline{\mathbb{N}}_\Xi, \Lambda\text{-}\mathit{Mod})$, computed by means of appropriate inductive and projective limits in Λ-*Mod*.

4.7.15. (Tensor products of bimodules.) Given an algebraic (Ξ, Σ)-bimodule M and an algebraic (Σ, Λ)-bimodule N, we would like to construct a new algebraic (Ξ, Λ)-bimodule $M \otimes_\Sigma N$, and similarly, for any Σ-module X we would like to obtain a Ξ-module $M \otimes_\Sigma X$. Interpreting all bimodules involved as functors between certain categories, we see that all we have to do is to extend in some way $M : \underline{\mathbb{N}}_\Sigma \to \Xi\text{-}\mathit{Mod}$ to a functor $\tilde{M} : \Sigma\text{-}\mathit{Mod} \to \Xi\text{-}\mathit{Mod}$; then we'll put $M \otimes_\Sigma N := \tilde{M} \circ N$ and $M \otimes_\Sigma X := \tilde{M}(X)$. Since the canonical functor $Q'_\Sigma : \underline{\mathbb{N}}_\Sigma \to \Sigma\text{-}\mathit{Mod}$ is fully faithful (with the essential image equal to the subcategory of free Σ-modules of finite rank), we have a natural choice of $\tilde{M}$, given by the left Kan extension: $\tilde{M} := Q'_{\Sigma,!}(M)$. This defines our tensor product $M \otimes_\Sigma N$.

Clearly, $M \mapsto \tilde{M} = Q'_{\Sigma,!} M$ commutes with arbitrary inductive limits in M, hence the same can be said about $M \otimes_\Sigma N$. Another useful observation: since $Q'_{\Sigma,!} = Q_{\Sigma,!} \circ J_{\Sigma,!}$, we see that we might first extend $M : \underline{\mathbb{N}}_\Sigma \to \mathcal{C}^\Lambda$ to a functor $M' : \mathcal{C}_\Sigma \to \mathcal{C}^\Lambda$ by means of $J_{\Sigma,!}$, and then extend this M' to $\tilde{M} : \mathcal{C}^\Sigma \to \mathcal{C}^\Lambda$. Since $M = M'$ is the algebraic functor corresponding to M (cf. **4.7.9**), we see that for any set S we have $\tilde{M}(\Sigma(S)) = M(S)$, i.e. $M \otimes_\Sigma \Sigma(S) = M(S)$.

From this and similar observations we deduce that all scalar restrictions and extensions with respect to right and left algebraic module structures can be written in terms of this tensor product in the usual way. For example, if $\rho : \Sigma \to \Xi$ is a homomorphism, N is any Σ-module, and P is any Ξ-module, then the scalar extension $\rho_* N$ is isomorphic to $\Xi \otimes_\Sigma N$, where Ξ is considered as a (Ξ, Σ)-bimodule, and the scalar restriction $\rho^* P$ can be identified with $\Xi \otimes_\Xi P$, where this time Ξ is considered as a (Σ, Ξ)-bimodule.

4.8. (Addition. Hypoadditivity and hyperadditivity.) We have seen that the multiplication on the underlying set $|\Sigma|$ of an algebraic monad Σ is always defined, so in some sense multiplication is more fundamental than addition. Now we would like to study different versions of addition and additivity of algebraic monads.

We fix some algebraic monad Σ and a constant $0 \in \Sigma(0)$. Since $\mathbb{F}_1 = \mathbb{F}_\varnothing\langle 0^{[0]}\rangle$, we see that this is equivalent to fixing some $\rho : \mathbb{F}_1 \to \Sigma$, i.e. Σ is an algebraic monad over $\mathbb{F}_1$. In most cases Σ is supposed to be an *algebraic monad with zero*, i.e. with exactly one constant (cf. **4.3.12**), so we have only one way of choosing $0 \in \Sigma(0)$.

4.8.1. (Comparison maps.) Once the zero constant $0 \in \Sigma(0)$ is fixed, we can define the *comparison maps* $\pi_n = \pi_{\Sigma,n} : \Sigma(n) \to \Sigma(1)^n = |\Sigma|^n$ as follows. The k-th component π_n^k of π_n is given by $\pi_n^k : t \mapsto t(0_{\Sigma(1)}, \ldots, \{1\}_{\mathbf{1}}, \ldots, 0_{\Sigma(1)})$ for any $t \in \Sigma(n)$, i.e. we substitute $0 = 0_{\Sigma(1)} \in \Sigma(1)$ for all formal variables $\{i\}_{\mathbf{n}}$ in $t = t(\{1\}_{\mathbf{n}}, \ldots, \{n\}_{\mathbf{n}})$, with the only exception of $\{k\}_{\mathbf{n}}$.

If we are given a monad homomorphism $\rho : \Sigma \to \Xi$, and if we choose the constant $\rho_0(0) \in \Xi(0)$ to construct $\pi_{\Xi,n}$, then we get $\pi_{\Xi,n} \circ \rho_n = |\rho|^n \circ \pi_{\Sigma,n}$, i.e. the comparison maps are functorial in Σ.

Notice that the choice of $0 \in \Sigma(0)$ gives us a homomorphism $\mathbf{1} \to \Sigma(0)$ from the final to the initial Σ-module (when Σ is a monad with zero, this is an isomorphism), so for any direct sum $M \oplus N$ in Σ-*Mod* we get canonical homomorphisms $M \oplus N \to M \oplus \mathbf{1} \to M \oplus \Sigma(0) \cong M$ and $M \oplus N \to N$, which define together a *comparison homomorphism* $\pi_{M,N} : M \oplus N \to M \times N$. Of course, this construction can be generalized to arbitrary finite and even infinite sums and products, thus defining comparison homomorphisms $\bigoplus_i M_i \to \prod_i M_i$.

For example, the comparison homomorphism for n copies of $\Sigma(1)$ is a homomorphism $\Sigma(n) = \Sigma(1) \oplus \cdots \oplus \Sigma(1) \to \Sigma(1)^n$, which is exactly the comparison map π_n considered before.

Definition 4.8.2 *(Addition and pseudoaddition.) We say that a binary operation* $[+] \in \Sigma(2)$ *is a* pseudoaddition *(with respect to constant* $0 \in \Sigma(0)$*, when* Σ *has more than one constant), if* $\pi_2([+]) = (\boldsymbol{e}, \boldsymbol{e}) \in \Sigma(1)^2$*, in other* *words, if* $\{1\} + 0 = \{1\} = 0 + \{1\}$ *in* $|\Sigma|$*, i.e. if* $x + 0 = x = 0 + x$ *for any* Σ*-module* X *and any* $x \in X$*. We say that* $[+]$ *is* the addition of Σ *if* $[+]$ *is the only pseudoaddition of an algebraic monad with zero* Σ*. In this case we say that* Σ is a monad with addition.

Of course, any monad homomorphism $\rho : \Sigma \to \Xi$ transforms a pseudoaddition into a pseudoaddition, and if both Σ and Ξ are monads with addition, then any $\rho : \Sigma \to \Xi$ respects this addition.

We can define pseudoadditions of higher arities $t \in \Sigma(n)$ by requiring $\pi_n(t) = (\boldsymbol{e}, \boldsymbol{e}, \ldots, \boldsymbol{e}) \in \Sigma(1)^n$; if we have a binary pseudoaddition $[+]$, we can construct pseudoadditions of all higher arities by taking $(\cdots((\{1\} + \{2\}) + \{3\}) + \cdots) + \{n\}$.

4.8.3. (Alternative definition of addition.) Arguably better definitions of addition and zero are these: a *zero* is simply any *central* constant in an algebraic monad Σ in the sense of **5.1.1** (such central constant is necessarily unique, cf. **5.1.7**), and an *addition* in an algebraic monad with zero Σ is any central pseudoaddition (it is automatically the only pseudoaddition of Σ by **5.1.8**, i.e. is an addition in the sense of **4.8.2**). The only reason why we don't adopt and study these stronger definitions here is that they require the commutativity notions of the next Chapter, so we postpone a more detailed study of the relationship between (pseudo)addition and commutativity until **5.1.8**.

Definition 4.8.4 *We say that an algebraic monad* Σ *is* hypoadditive *(resp.* additive, hyperadditive*) if all comparison maps* $\pi_n : \Sigma(n) \to \Sigma(1)^n$ *are injective (resp. bijective, surjective).*

Clearly, Σ is additive iff it is both hypoadditive and hyperadditive. Notice that additivity and hypoadditivity imply that $\pi_0 : \Sigma(0) \to \mathbf{1}$ is injective, so Σ has at most one constant, so it has to be a monad with zero, and we don't need to specify our choice of $0 \in \Sigma(0)$ while discussing hypoadditivity or additivity.

Of course, there can be monads which are neither hypoadditive nor hyperadditive; some of them, like $\mathbb{F}_\infty$, will play important role in the sequel. Notice that any submonad Σ' of a hypoadditive monad Σ, containing the zero of Σ, is automatically hypoadditive, and similarly, any strict quotient of a hyperadditive monad is hyperadditive; this follows immediately from the functoriality of comparison maps.

4.8.5. If Σ is additive, then it has an addition $[+] = \pi_2^{-1}(\boldsymbol{e}, \boldsymbol{e})$, i.e. Σ is then a monad with addition. Conversely, if Σ is hypoadditive and admits a pseudoaddition $[+]$, it is necessarily unique (since π_2 is injective), so Σ

becomes a monad with addition; moreover, this addition is automatically commutative, since $\pi_2(\{2\}+\{1\}) = (\boldsymbol{e}, \boldsymbol{e}) = \pi_2(\{1\}+\{2\})$, and associative, since both $(\{1\}+\{2\})+\{3\}$ and $\{1\}+(\{2\}+\{3\})$ are mapped into $(\boldsymbol{e}, \boldsymbol{e}, \boldsymbol{e})$ by π_3. We see that $\{1\}+\{2\}+\cdots+\{n\}$ is the unique pseudoaddition of arity n, and for any choice of unary operations $\lambda_1, \ldots, \lambda_n \in |\Sigma|$ we get an n-ary operation $\lambda_1\{1\}+\cdots+\lambda_n\{n\} \in \Sigma(n)$, mapped into $(\lambda_1, \ldots, \lambda_n)$ by π_n, so *a hypoadditive monad Σ with pseudoaddition* $[+]$ *is in fact additive*, and in this case any n-ary operation $t \in \Sigma(n)$ can be uniquely written in form $\lambda_1\{1\}+\cdots+\lambda_n\{n\}$ with $\lambda_i \in |\Sigma|$.

One shows, essentially in the same way, that *an algebraic monad Σ is hyperadditive iff it admits a pseudoaddition*. Indeed, if Σ is hyperadditive, it admits a pseudoaddition, $\pi_2 : \Sigma(2) \to |\Sigma|^2$ being surjective; conversely, if $[+]$ is a pseudoaddition, then for any $\lambda = (\lambda_1, \ldots, \lambda_n) \in |\Sigma|^n$ we can construct an n-ary operation $(\cdots((\lambda_1\{1\}+\lambda_2\{2\})+\lambda_3\{3\})+\cdots)+\lambda_n\{n\}$, mapped into λ by π_n.

4.8.6. (Additive monads and semirings.) This implies that an additive monad Σ is generated (over $\mathbb{F}_\varnothing$) by its only constant $0 \in \Sigma(0)$, the addition $[+] \in \Sigma(2)$, and the set of its unary operations $\Sigma(1)$. We have also some relations: $0+0=0$, the commutativity $\{1\}+\{2\}=\{2\}+\{1\}$ and the associativity $(\{1\}+\{2\})+\{3\} = \{1\}+(\{2\}+\{3\})$ of the addition, $\lambda \cdot 0 = 0$ for any $\lambda \in |\Sigma|$; the addition $[+]_{|\Sigma|} : |\Sigma|^2 \to |\Sigma|$ gives a family of relations of form $\lambda = \lambda' + \lambda''$ for all λ', $\lambda'' \in |\Sigma|$, we have also a family of relations $\lambda = \lambda'\lambda''$ coming from the monoid structure of $|\Sigma|$, and finally, we have relations $\lambda(\{1\}+\{2\}) = \lambda\{1\}+\lambda\{2\}$ for any $\lambda \in |\Sigma|$, since both sides are mapped by π_2 into (λ, λ). Notice that this list of generators and relations is a presentation of Σ, since these relations are sufficient to rewrite any $t \in \Sigma(n)$ in form $\lambda_1\{1\}+\cdots+\lambda_n\{n\}$.

Conversely, if we have any algebraic monad Σ generated by such a list of generators and relations, i.e. if we are given some set $|\Sigma|$ together with a constant $0 \in |\Sigma|$ and two binary operations $+, \times : |\Sigma|^2 \to |\Sigma|$, subject to a family of relations which turn out to be exactly the axioms for a semiring structure $(0, +, \times)$ on $|\Sigma|$, then Σ is automatically additive, since the list of relations given above allows us to write any $t \in \Sigma(n)$ in form $\lambda_1\{1\}+\cdots+\lambda_n\{n\}$ with $\lambda_i \in |\Sigma|$, and Σ-*Mod* is the category of $|\Sigma|$-modules, i.e. additively written commutative monoids with a biadditive action of $|\Sigma|$.

We see that additive algebraic monads Σ are in one-to-one correspondence with semirings $|\Sigma|$.

4.8.7. (Additive monads and rings.) In particular, any monad $R = \Sigma_R$, defined by a ring R, is additive. Conversely, if Σ is additive, and if we have a unary operation $[-] \in \Sigma(1)$, such that $[-]^2 = \boldsymbol{e}$ (giving such a $[-]$ is equivalent to giving a homomorphism $\mathbb{F}_{\pm 1} = \mathbb{F}_1\langle -^{[1]} | -^2 = \boldsymbol{e},\, -0 = 0\rangle \to \Sigma$), and such that $\{1\} + (-\{1\}) = 0$ (this condition doesn't follow from the others), then $R := |\Sigma|$ is a ring, and $\Sigma \cong \Sigma_R$.

Proposition 4.8.8 *(Criteria of additivity.) The following conditions for an algebraic monad Σ are equivalent: (i) Σ is additive; (ii) All comparison maps $\pi_n : \Sigma(n) \to \Sigma(1)^n$ are isomorphisms; (iii) All comparison homomorphisms $\pi_{M,N} : M \oplus N \to M \times N$ are isomorphisms; (iii') All comparison homomorphisms $\bigoplus_{i=1}^k M_i \to \prod_{i=1}^k M_i$ are isomorphisms; (iv) Comparison homomorphisms $\pi_{\Sigma(n),\Sigma(1)} : \Sigma(n+1) \cong \Sigma(n) \oplus \Sigma(1) \to \Sigma(n) \times \Sigma(1)$ are isomorphisms for all $n \geq 0$; (v) It is possible to introduce on each* $\mathrm{Hom}_\Sigma(M,N)$ *a commutative monoid structure (usually written in additive form), such that all composition maps* $\mathrm{Hom}_\Sigma(N,P) \times \mathrm{Hom}_\Sigma(M,N) \to \mathrm{Hom}_\Sigma(M,P)$ *become biadditive; (vi) Σ-Mod is equivalent to the category of modules over a semiring R.*

Proof. (i)⇔(ii) is true by definition, and (ii)⇔(iv) is shown by induction in $n \geq 0$, using $\pi_{n+1} = (\pi_n \times \mathrm{id}_{\Sigma(1)}) \circ \pi_{\Sigma(n),\Sigma(1)}$. Now (iii)⇔(iii') by an easy induction in k, and (iii)⇒(iv) is evident. We know (i)⇒(vi) already. Let's show (vi)⇒(v). When we have (vi), we can define an addition on each $\mathrm{Hom}_\Sigma(M,N) \cong \mathrm{Hom}_R(M',N')$ pointwise, where M' and N' are the R-modules corresponding to M and N: $(\varphi_1 + \varphi_2)(x) := \varphi_1(x) + \varphi_2(x)$; it is immediate that sum of two R-homomorphisms is an R-homomorphism again, that this addition on $\mathrm{Hom}_R(M',N') \cong \mathrm{Hom}_\Sigma(M,N)$ is commutative and associative, with the zero map $M' \to 0 \to N'$ for zero, and that the composition maps are biadditive with respect to this addition, so we obtain (vi)⇒(v).

It remains to show (v)⇒(iii). First of all, (v) implies that the final object $\mathbf{1}$ of Σ-*Mod* is an initial object as well, i.e. Σ-*Mod* admits a zero object. Indeed, for any M the set $\mathrm{Hom}_\Sigma(\mathbf{1}, M)$ is a monoid under addition, so it cannot be empty; on the other hand, $\mathrm{End}_\Sigma(\mathbf{1})$ consists of exactly one element $\mathrm{id}_{\mathbf{1}}$, so it has to be the zero of this additive monoid, and biadditivity of $\mathrm{Hom}_\Sigma(\mathbf{1}, M) \times \mathrm{End}_\Sigma(\mathbf{1}) \to \mathrm{Hom}_\Sigma(\mathbf{1}, M)$ implies that for any $\varphi : \mathbf{1} \to M$ we have $\varphi = \varphi \circ \mathrm{id}_{\mathbf{1}} = \varphi \circ 0_{\mathbf{1}} = 0$, so $\mathrm{Hom}_\Sigma(\mathbf{1}, M) = 0$, $\mathbf{1}$ is the zero object of Σ-*Mod*, and for any M and N the composite homomorphism $M \to \mathbf{1} \to N$ is the zero of $\mathrm{Hom}_\Sigma(M,N)$. In particular, we have shown that $\Sigma(0) \cong \mathbf{1}$, i.e. Σ is a monad with zero.

Now let's take any two Σ-modules M_1 and M_2 and prove that $\pi_{M_1,M_2} : M_1 \oplus M_2 \to M_1 \times M_2$ is an isomorphism. Let's denote the projections $M_1 \times M_2 \to M_i$ by p_i, and the composite maps $M_i \to M_1 \oplus M_2 \to M_1 \times M_2$ by λ_i. These maps are defined by $p_i\lambda_i = \mathrm{id}_{M_i}$, $p_i\lambda_{3-i} = 0$, and π_{M_1,M_2}

is an isomorphism iff $M_1 \times M_2$ together with these maps λ_i satisfies the universal property for $M_1 \oplus M_2$. Notice that $\lambda_1 p_1 + \lambda_2 p_2 = \mathrm{id}_{M_1 \times M_2}$, since $p_i(\lambda_1 p_1 + \lambda_2 p_2) = (p_i\lambda_i)p_i + 0 \cdot p_{3-i} = p_i$. Now for any $f : M_1 \times M_2 \to N$ we have $f = f(\lambda_1 p_1 + \lambda_2 p_2) = f_1 p_1 + f_2 p_2$, so f is completely determined by its "components" $f_i := f\lambda_i : M_i \to N$. Conversely, given any $f_i : M_i \to N$, we can define $f : M_1 \times M_2 \to N$ by $f := f_1 p_1 + f_2 p_2$, and then clearly $f\lambda_i = f_i$, so $M_1 \times M_2$ is indeed the direct sum of M_1 and M_2, and π_{M_1,M_2} is an isomorphism.

This proposition, together with some more similar reasoning, implies the following corollary:

Corollary 4.8.9 *The following conditions for an algebraic monad Σ are equivalent: (i) $\Sigma \cong \Sigma_R$ for some associative ring R; (ii) Σ is additive, and there is a unary operation $[-] \in \Sigma(1)$, such that $\{1\} + (-\{1\}) = 0$; (iii) Σ-Mod is equivalent to the category of modules over an associative ring R; (iv) Σ-Mod is additive; (v) Σ-Mod is abelian.*

4.8.10. (Hypoadditivity.) Some of the above conditions generalize to the hypoadditive case. For example, Σ *is hypoadditive iff all comparison maps* $\Sigma(n+1) \cong \Sigma(n) \oplus \Sigma(1) \to \Sigma(n) \times \Sigma(1)$ *are injective.* One might expect all comparison maps $M \oplus N \to M \times N$ to be injective for a hypoadditive Σ; however, this is not true, as shown by the following example. Let's take $\Sigma := \mathbb{Z}_\infty$; since $\mathbb{Z}_\infty$ is a submonad of an additive monad $\mathbb{R}$, containing the zero constant of $\mathbb{R}$, it is hypoadditive. Consider the equivalence relation R on $\mathbb{Z}_\infty(2) = \{\lambda\{1\} + \mu\{2\} \mid |\lambda| + |\mu| \leq 1\}$, which identifies all interior points of $\mathbb{Z}_\infty(2)$, i.e. all elements with $|\lambda| + |\mu| < 1$, with zero. It is easy to see that R is compatible with the $\mathbb{Z}_\infty$-structure of $\mathbb{Z}_\infty(2)$, so we can compute the strict quotient $M := \mathbb{Z}_\infty(2)/R$, and let's denote by π the canonical projection $\mathbb{Z}_\infty(2) \to M$. Clearly, $M \oplus M$ is a strict quotient of $\mathbb{Z}_\infty(4) = \mathbb{Z}_\infty(2) \oplus \mathbb{Z}_\infty(2)$; let's denote the projection by π' and its kernel by R'. It is easy to check that R' identifies all interior points of $\mathbb{Z}_\infty(4)$ together, and doesn't identify any boundary points. In particular, $x := \pi'(\frac{1}{2}\{1\}+\frac{1}{2}\{3\})$ and $y := \pi'(\frac{1}{2}\{2\} + \frac{1}{2}\{4\})$ are distinct in $M \oplus M$, but the comparison map $M \oplus M \to M \times M$ maps both of them into $(0, 0)$, so it cannot be injective.

4.8.11. (Importance of hypoadditivity.) Notice that *any homomorphism $\rho : \Sigma \to \Xi$ of hypoadditive monads is completely determined by the map $|\rho| = \rho_1 : |\Sigma| \to |\Xi|$.* Indeed, the functoriality of comparison maps $\pi_{\Xi,n} \circ \rho_n = |\rho|^n \circ \pi_{\Sigma,n}$ shows that ρ_n is completely determined by $|\rho|$ when $\pi_{\Xi,n}$ is injective. This can be seen as a manifestation of the following fact: *most properties of a hypoadditive monad Σ can be seen at the level of* $|\Sigma|$. Conversely, there are some non-hypoadditive monads, like $\mathrm{Aff}_R \subset R$ (cf. **3.4.12**,g)), which are highly non-trivial (Aff_R-*Mod* is the category of “affine spaces over R”), but still have $|\,\mathrm{Aff}_R\,| = \mathbf{1}$, so almost all structure of Aff_R is outside of $|\,\mathrm{Aff}_R\,|$. This remark will have important consequences in our theory of spectra of generalized rings, i.e. commutative algebraic monads.

4.8.12. (Examples.) a) Of course, all monads Σ_R, defined by some associative ring R, are additive, hence all their submonads $\Sigma' \subset R$, containing the zero constant of Σ_R, are hypoadditive. This applies to almost all examples from **3.4.12**. In particular, $\mathbb{Z}_\infty$, $\bar{\mathbb{Z}}_\infty$, $\mathbb{Z}_{(\infty)}$, $\mathbb{F}_1$ and $\mathbb{F}_{\pm 1}$ are hypoadditive. However, $\mathrm{Aff}_R \subset R$, $\mathbf{\Delta} \subset \mathbb{R}$ and $\mathbb{F}_\varnothing \subset \mathbb{F}_1$ are not hypoadditive, since all of them are monads without constants.

b) Consider the algebraic monad Σ, defined by the category of commutative rings. Then $\Sigma(n) \cong \mathbb{Z}[T_1, \ldots, T_n]$. If we fix some constant $c \in \Sigma(0) = \mathbb{Z}$, say, $c = 0$, then the comparison maps look like $\pi_2 : F(X, Y) \mapsto (F(T, 0), F(0, T))$. Clearly, all these maps are surjective, so Σ is hyperadditive. It admits a lot of different pseudoadditions like $X + Y$, $X + Y + XY$ or even $X + Y + X^2Y$; this shows that a pseudoaddition need not be commutative or associative.

c) We'll see later that most non-commutative tensor products like $\mathbb{Z} \boxtimes_{\mathbb{F}_1} \mathbb{Z}$ are hyperadditive. This means that we should not expect to read all properties of such monads from $\Sigma(1)$.

d) All additive monads are generated in arity ≤ 2, and the same is true for all hypoadditive monads listed in a). Actually, all hypoadditive monads we encountered so far are generated in arity ≤ 2, and can be embedded into some additive monads. However, it is not clear why this should be true in general. Consider the hypoadditive submonad $\mathbb{Z}_{\sqrt{\infty}} \subset \mathbb{R}$, such that $\mathbb{Z}_{\sqrt{\infty}}(n)$ consists of all formal linear combinations $\lambda_1\{1\} + \cdots + \lambda_n\{n\}$ with $\lambda_i \in \mathbb{R}$ and $\sum_i \sqrt{|\lambda_i|} \leq 1$. It is easy to see that such linear combinations are stable under substitution, so $\mathbb{Z}_{\sqrt{\infty}}$ is indeed a hypoadditive submonad of $\mathbb{R}$. However, it still turns out to be generated in arity ≤ 2. Another example is given by $A_3 := \mathbb{Z}_\infty \cap \mathbb{Z}[1/3] \subset \mathbb{Q}$, which is hypoadditive and generated over $\mathbb{F}_{\pm 1}$ by the ternary “averaging operation” $s_3 := (1/3)\{1\} + (1/3)\{2\} + (1/3)\{3\}$, but can be shown not to be generated in arity ≤ 2.

4.8.13. Let's construct now a monad with zero, which is neither hypoadditive nor hyperadditive. Consider for this the equivalence relation R on $\mathbb{Z}_\infty(1) = [-1, 1]$, which identifies all interior points of $\mathbb{Z}_\infty(1)$ with zero. We know that R is compatible with the $\mathbb{Z}_\infty$-structure on $\mathbb{Z}_\infty(1)$, so we obtain a strict quotient $Q = \mathbb{Z}_\infty/R \cong \{-1, 0, 1\}$ of $\mathbb{Z}_\infty(1)$, which has been denoted by $\mathbb{F}_\infty$ in **2.14.13**. The $\mathbb{Z}_\infty$-structure on Q defines a homomorphism of algebraic monads $\rho : \mathbb{Z}_\infty \to \mathrm{END}(Q)$, cf. **4.3.8**; let's denote its image by $\mathbb{F}_\infty$. Clearly,

$\mathbb{F}_\infty$ is a strict quotient of $\mathbb{Z}_\infty$, and two elements t, $t' \in \mathbb{Z}_\infty(n)$ have the same image in $\mathbb{F}_\infty(n)$ iff the induced maps $[t]_Q$ and $[t']_Q : Q^n \to Q$ coincide. In particular, all $\mathbb{F}_\infty(n) \subset \mathrm{END}(Q)(n) = \mathrm{Hom}_{Sets}(Q^n, Q)$ are finite. It is easy to see that all elements $t = \lambda_1\{1\} + \cdots + \lambda_n\{n\} \in \mathbb{Z}_\infty(n)$ with $\sum_i |\lambda_i| < 1$ are identified with 0 in $\mathbb{F}_\infty(n)$, and all remaining elements are identified iff their sequences of signs $(\mathrm{sgn}(\lambda_1), \ldots, \mathrm{sgn}(\lambda_n))$ coincide, so we can write corresponding elements of $\mathbb{F}_\infty(n)$ in form $+?\{1\}+?\{2\}-?\{3\}+?\{5\}$, where the question marks replace arbitrary positive real numbers with sum one; clearly, elements of $\mathbb{F}_\infty(n)$ correspond to faces of the standard octahedron $\mathbb{Z}_\infty(n)$. In particular, $\mathrm{card}\, \mathbb{F}_\infty(n) = 3^n = \mathrm{card}\, |\mathbb{F}_\infty|^n$; however, the comparison maps $\pi_n : \mathbb{F}_\infty(n) \to |\mathbb{F}_\infty|^n$ are not bijective, since they map all elements different from $\pm\{i\}$ for $1 \leq i \leq n$ into zero.

4.8.14. Notice that $\mathbb{F}_\infty$ is generated by one constant $0^{[0]}$, one unary operation $-^{[1]}$, and one binary operation $*^{[2]} := +?\{1\}+?\{2\}$. We have some relations between them, like $-0 = 0$, $0 * 0 = 0$, $-(-x) = x$, $x * 0 = 0$, $x * x = x$, $x*(-x) = 0$, $(-x)*(-y) = -(x*y)$, $x*y = y*x$ and $(x*y)*z = x*(y*z)$. This system of generators and relations constitutes a presentation of $\mathbb{F}_\infty$, since it already enables us to represent any non-zero element of $\Sigma(n)$, where Σ is the generalized ring generated by these operations and relations, in form $\pm\{i_1\} * \cdots * \pm\{i_k\}$ with some $1 \leq i_1 < \cdots < i_k \leq n$, hence the surjection $\Sigma \twoheadrightarrow \mathbb{F}_\infty$ is an isomorphism, and $\mathbb{F}_\infty = \Sigma$ is finitely presented (over $\mathbb{F}_\varnothing$).

4.9. (Algebraic monads over a topos.) Most of the results of this chapter generalize, when properly understood, to the topos case. We start with a naïve approach:

4.9.1. (Algebraic monads over an arbitrary category.) Recall that in **4.3.3** we have described an algebraic monad Σ over *Sets* as a sequence of sets $\{\Sigma(n)\}_{n\geq 0}$, together with some maps $\Sigma(\varphi) : \Sigma(m) \to \Sigma(n)$ for each $\varphi : \mathbf{m} \to \mathbf{n}$, an element $\boldsymbol{e} \in \Sigma(1)$, and some multiplication maps $\mu_n^{(k)} : \Sigma(k) \times \Sigma(n)^k \to \Sigma(n)$, subject to some identities between certain maps, constructed from these maps by means of finite products and composition. Of course, this definition generalizes immediately to an arbitrary cartesian category $\mathcal{C}$, if we require all $\Sigma(n)$ to be objects of $\mathcal{C}$ instead of being sets, all $\Sigma(\varphi)$ and $\mu_n^{(k)}$ to be morphisms in $\mathcal{C}$, and choose some $\boldsymbol{e} \in \Gamma_\mathcal{C}(\Sigma(1)) = \mathrm{Hom}_\mathcal{C}(e_\mathcal{C}, \Sigma(1))$, where $e_\mathcal{C}$ is the final object of $\mathcal{C}$. Given any such "algebraic monad" over $\mathcal{C}$, we can define its action on a object X of $\mathcal{C}$ as a collection of morphisms $\alpha^{(k)} : \Sigma(k) \times X^k \to X$, satisfying the conditions of **4.3.2**; this defines *the category* $\mathcal{C}^\Sigma = \Sigma$*-Mod of* Σ*-modules in* $\mathcal{C}$. Furthermore, we can define left and right algebraic modules, algebraic monad homomorphisms and so on simply by copying their descriptions in terms of sets or sequences of sets and maps between products of these sets. Notice that “algebraic endofunctors over $\mathcal{C}$” are nothing else than functors $\underline{\mathbb{N}} \to \mathcal{C}$.

In fact, we can generalize these definitions even to the case when involved finite products are not representable in $\mathcal{C}$, by identifying $\mathcal{C}$ with a full subcategory of the category $\hat{\mathcal{C}} = \mathit{Funct}(\mathcal{C}^0, \mathit{Sets})$ of presheaves of sets. Notice that giving an algebraic monad in $\hat{\mathcal{C}}$ is equivalent to giving a presheaf of algebraic monads over $\mathcal{C}$, i.e. a functor from $\mathcal{C}^0$ to the category of algebraic monads over *Sets*.

4.9.2. (Application of left exact functors.) The advantage of this approach is that we can apply any left exact functor $h : \mathcal{C} \to \mathcal{C}'$ to any of these collections of data. For example, if Σ is an algebraic monad over $\mathcal{C}$, then $h\Sigma$, defined by $(h\Sigma)(n) := h\Sigma(n)$, is an algebraic monad over $\mathcal{C}'$, and for any Σ-module X we obtain a canonical $h\Sigma$-structure on hX. In this way any such left exact functor $h : \mathcal{C} \to \mathcal{C}'$ induces a functor $h = h^\Sigma : \mathcal{C}^\Sigma \to \mathcal{C}'^{h\Sigma}$ between corresponding categories of modules.

4.9.3. (Application to topoi.) This is especially useful for the left exact functors that arise from morphisms of topoi $f : \mathcal{E}' \to \mathcal{E}$, i.e. the pullback functor $f^* : \mathcal{E} \to \mathcal{E}'$ and the direct image functor $f_* : \mathcal{E}' \to \mathcal{E}$. In this way f^* induces a functor $f^{*,\Sigma} : \mathcal{E}^\Sigma \to \mathcal{E}'^{f^*\Sigma}$ for any algebraic monad Σ over $\mathcal{E}$, and f_* induces a functor $f_*^{\Sigma'} : \mathcal{E}'^{\Sigma'} \to \mathcal{E}^{f_*\Sigma'}$ for any algebraic monad Σ' over $\mathcal{E}'$. Moreover, canonical adjointness morphisms $\Sigma \to f_*f^*\Sigma$ and $f^*f_*\Sigma' \to \Sigma'$ are easily seen to be monad homomorphisms, so we can combine $f_*^{f^*\Sigma}$ with the scalar restriction along $\Sigma \to f_*f^*\Sigma$, thus obtaining a functor $\mathcal{E}'^{f^*\Sigma} \to \mathcal{E}^\Sigma$.

We can apply this in partucular to the canonical morphism $q : \mathcal{E} \to \mathit{Sets}$ from an arbitrary topos $\mathcal{E}$ into the point topos *Sets*. Then $q_*X = \Gamma_\mathcal{E}(X) = \operatorname{Hom}_\mathcal{E}(e_\mathcal{E}, X)$ for any $X \in \operatorname{Ob}\mathcal{E}$, i.e. q_* is the functor of global sections, and $q^* : S \mapsto S_\mathcal{E} = \bigsqcup_{s\in S} e_\mathcal{E}$ is the “functor of constant objects”. We see that for any algebraic monad Σ over $\mathcal{E}$ we have its global sections monad $\Gamma_\mathcal{E}(\Sigma)$ over *Sets*, that any algebraic monad Ξ over *Sets* defines a constant algebraic monad $\Xi_\mathcal{E} := q^*\Xi$ over $\mathcal{E}$, and so on. Notice that we could also apply to Σ the functor of sections $\Gamma_S : X \mapsto \operatorname{Hom}_\mathcal{E}(S, X)$ over any object $S \in \operatorname{Ob}\mathcal{E}$.

Another application: if $p : \mathit{Sets} \to \mathcal{E}$ is any point of $\mathcal{E}$, and $p^* : X \mapsto X_p$ is the corresponding fiber functor, then we can compute the fiber at p of any algebraic monad Σ over $\mathcal{E}$ by putting $\Sigma_p := p^*\Sigma$.

Finally, if $\mathcal{S}$ is a (small) site, we have the topos of presheaves $\hat{\mathcal{S}}$ and the topos of sheaves $\tilde{\mathcal{S}}$, the inclusion functor $i : \tilde{\mathcal{S}} \to \hat{\mathcal{S}}$, and its left adjoint, the “sheafification functor” $a : \hat{\mathcal{S}} \to \tilde{\mathcal{S}}$. Both of them are left exact, and $ai \cong \operatorname{Id}_{\tilde{\mathcal{S}}}$ since i is fully faithful; this allows us to treat any algebraic monad Σ in $\tilde{\mathcal{S}}$, i.e. a sheaf of algebraic monads over $\mathcal{S}$, as a presheaf of monads $i\Sigma$, and conversely, to “sheafify” any presheaf of algebraic monads, obtaining a sheaf of algebraic monads, i.e. an algebraic monad in $\tilde{\mathcal{S}}$.

4.9.4. (Monadicity of forgetful functors.) Given any algebraic monad Σ over a topos $\mathcal{E}$, we have a forgetful functor $\Gamma_\Sigma : \mathcal{E}^\Sigma \to \mathcal{E}$ (not to be confused with the global sections functor $\Gamma_\mathcal{E} : \mathcal{E} \to \mathit{Sets}$). We claim that *$\Gamma_\Sigma : \mathcal{E}^\Sigma \to \mathcal{E}$ is monadic, i.e. it admits a left adjoint $L_\Sigma : \mathcal{E} \to \mathcal{E}^\Sigma$, and induced functor $\mathcal{E}^\Sigma \to \mathcal{E}^{\tilde{\Sigma}}$, where $\tilde{\Sigma} := \Gamma_\Sigma L_\Sigma$ is a monad over $\mathcal{E}$, is an equivalence of categories.* We are going to show this in several steps.

4.9.5. First of all, let's show the existence of a left adjoint $L_{\tilde{\Sigma}}$ to Γ_Σ. Any topos $\mathcal{E}$ is equivalent to the category $\tilde{\mathcal{S}}$ of presheaves on some site $\mathcal{S}$, which can be chosen to be small. Consider first the category of presheaves $\hat{\mathcal{S}} \supset \tilde{\mathcal{S}}$. The collection of sheaves $\Sigma = \{\Sigma(n)\}$ is an "algebraic monad" over $\hat{\mathcal{S}}$ as well, so we have a forgetful functor $\Gamma_{\hat{\Sigma}} : \hat{\mathcal{S}}^\Sigma \to \hat{\mathcal{S}}$. On the other hand, Σ defines an algebraic monad $\Sigma_{[S]}$ over *Sets* for any object $S \in \operatorname{Ob}\mathcal{S}$, once we apply to Σ the left exact functor $F \mapsto F(S)$. We see that a Σ-structure α on a presheaf $F \in \operatorname{Ob}\hat{\mathcal{S}}$, given by a collection of morphisms of presheaves $\alpha^{(k)} : \Sigma(k) \times F^k \to F$, is nothing else than a collection of $\Sigma_{[S]}$-structures on each set $F(S)$, compatible with the "restriction maps" $F(S) \to F(S')$ and $\Sigma_{[S]} \to \Sigma_{[S']}$ for any morphism $\varphi : S' \to S$ in $\mathcal{S}$. Now we can define a left adjoint $L_{\hat{\Sigma}}$ to $\Gamma_{\hat{\Sigma}}$ by putting $(L_{\hat{\Sigma}}F)(S) := L_{\Sigma_{[S]}}(F(S))$ for any presheaf of sets F. Clearly, $\hat{\Sigma} := \Gamma_{\hat{\Sigma}} L_{\hat{\Sigma}}$ is given by $(\hat{\Sigma}F)(S) := \Sigma_{[S]}(F(S))$. The monadicity of $\Gamma_{\hat{\Sigma}} : \hat{\mathcal{S}}^\Sigma \to \hat{\mathcal{S}}$ follows now pointwise from the monadicity of all $\Gamma_{\Sigma_{[S]}}$.

We extend these results from $\hat{\mathcal{S}}$ to $\tilde{\mathcal{S}}$ in the usual way by putting $L_{\tilde{\Sigma}} := aL_{\hat{\Sigma}}i$ and $\tilde{\Sigma} := \Gamma_{\tilde{\Sigma}} L_{\tilde{\Sigma}} = a\hat{\Sigma}i$, where $a : \hat{\mathcal{S}} \to \tilde{\mathcal{S}}$ and $i : \tilde{\mathcal{S}} \to \hat{\mathcal{S}}$ denote the "sheafification" and inclusion functors as usual; in the first expression a actually denotes the extension $a^\Sigma : \hat{\mathcal{S}}^{i\Sigma} \to \tilde{\mathcal{S}}^{ai\Sigma} = \tilde{\mathcal{S}}^\Sigma$; it exists because of the left exactness of a and full faithfullness of i. Now the adjointness of $L_{\tilde{\Sigma}}$ and $\Gamma_{\tilde{\Sigma}}$ is immediate: $\operatorname{Hom}_{\tilde{\mathcal{S}}^\Sigma}(L_{\tilde{\Sigma}}F, G) = \operatorname{Hom}_{\tilde{\mathcal{S}}^\Sigma}(aL_{\hat{\Sigma}}iF, G) = \operatorname{Hom}_{\hat{\mathcal{S}}^\Sigma}(L_{\hat{\Sigma}}iF, iG) = \operatorname{Hom}_{\hat{\mathcal{S}}}(iF, \Gamma_{\hat{\Sigma}}iG) = \operatorname{Hom}_{\hat{\mathcal{S}}}(iF, i\Gamma_{\tilde{\Sigma}}G) = \operatorname{Hom}_{\tilde{\mathcal{S}}}(F, \Gamma_{\tilde{\Sigma}}G)$.

It remains to show the monadicity of $\Gamma_{\tilde{\Sigma}}$; however, we postpone this proof until **4.9.11**.

4.9.6. (Algebraic endofunctors on a topos.) Recall that an "algebraic endofunctor" Σ on a topos $\mathcal{E}$ is given by a collection of objects $\{\Sigma(n)\}_{n\geq 0}$ and morphisms $\Sigma(\varphi) : \Sigma(m) \to \Sigma(n)$ for each $\varphi : \mathbf{m} \to \mathbf{n}$, i.e. by a functor $\Sigma \in \operatorname{Ob}\mathcal{E}^{\underline{\mathbb{N}}} = \operatorname{Ob}\mathit{Funct}(\underline{\mathbb{N}}, \mathcal{E})$. We would like to extend Σ to a "true" endofunctor $\tilde{\Sigma} : \mathcal{E} \to \mathcal{E}$, such that any morphism $\alpha : \tilde{\Sigma}(X) \to Y$ would be given by a collection of morphisms $\alpha^{(k)} : \Sigma(k) \times X^k \to Y$ as in **4.2.2**. Clearly, this requirement determines $\tilde{\Sigma}(X)$ uniquely up to a unique isomorphism, and we see that $\tilde{\Sigma}(X)$ can be constructed in the same way as in **4.1.4**, i.e.

$\tilde{\Sigma}(X) = \mathrm{Coker}(p, q : H_1(X) \rightrightarrows H_0(X))$, where $H_0(X) = \bigsqcup_{n\geq 0} \Sigma(n) \times X^n$ and $H_1(X) = \bigsqcup_{\varphi:\mathbf{m}\to\mathbf{n}} \Sigma(m) \times X^n$.

When $\mathcal{E} = \tilde{\mathcal{S}}$, we might first construct an extension of Σ to $\hat{\Sigma} : \hat{\mathcal{S}} \to \hat{\mathcal{S}}$ with the correct universal property in $\hat{\mathcal{S}}$; clearly, $\hat{\Sigma}F$ is still computed by the same formulas as above in the category of presheaves, and all limits in this category are computed componentwise, so we have $(\hat{\Sigma}F)(S) = \Sigma_{[S]}(F(S))$, where $\Sigma_{[S]}$ is the algebraic endofunctor over *Sets*, constructed from the collection of sets $(\Sigma(n))(S)$ as before. Then we can put $\tilde{\Sigma} := a\hat{\Sigma}i : \tilde{\mathcal{S}} \to \tilde{\mathcal{S}}$; clearly, $\mathrm{Hom}_{\tilde{\mathcal{S}}}(\tilde{\Sigma}F, G) \cong \mathrm{Hom}_{\hat{\mathcal{S}}}(\hat{\Sigma}iF, iG)$ can be still described as in **4.2.2**, so we get an alternative construction of the extension $\tilde{\Sigma}$. In particular, our notations are compatible with those of **4.9.5**. Moreover, we see that the canonical morphism $a\hat{\Sigma} \to a\hat{\Sigma}ia = \tilde{\Sigma}a$ is an isomorphism. This follows from the fact that $\hat{\Sigma}F = \mathrm{Coker}(H_1(F) \rightrightarrows H_0(F))$ is computed for any presheaf F by means of inductive limits and finite products, and $a : \hat{\mathcal{S}} \to \tilde{\mathcal{S}}$ commutes with these types of limits.

This observation can be used to compute the values of $\tilde{\Sigma}$ on finite constant sheaves $\mathbf{n}_{\mathcal{E}}$: since $\mathbf{n}_{\mathcal{E}} = a\mathbf{n}_{\hat{\mathcal{S}}}$, and $\mathbf{n}_{\hat{\mathcal{S}}}$ is the constant presheaf $S \mapsto \mathbf{n}$, we see that $\tilde{\Sigma}(\mathbf{n}_{\tilde{\mathcal{S}}}) = a\hat{\Sigma}(\mathbf{n}_{\hat{\mathcal{S}}}) = \Sigma(n)$, since $\hat{\Sigma}(\mathbf{n}_{\hat{\mathcal{S}}})$ is the presheaf $S \mapsto \Sigma_{[S]}(\mathbf{n}) = \Sigma_{[S]}(n) = (\Sigma(n))(S)$, equal to $\Sigma(n)$. One checks similarly that $\Sigma(\varphi) = \tilde{\Sigma}(\varphi_{\mathcal{E}})$ for any map of finite sets $\varphi : \mathbf{m} \to \mathbf{n}$.

4.9.7. (Universal description of extensions.) We have just constructed a functor $\tilde{J}_{\mathcal{E}} : \mathcal{E}^{\underline{\mathbb{N}}} = \mathit{Funct}(\underline{\mathbb{N}}, \mathcal{E}) \to \mathit{Endof}(\mathcal{E})$, $\Sigma \mapsto \tilde{\Sigma}$, and a restriction functor $J^*_{\mathcal{E}} : \mathit{Endof}(\mathcal{E}) \to \mathcal{E}^{\underline{\mathbb{N}}}$ in the opposite direction, where $J_{\mathcal{E}} : \underline{\mathbb{N}} \to \mathcal{E}$ is given by $\mathbf{n} \mapsto \mathbf{n}_{\mathcal{E}}$, such that $J^*_{\mathcal{E}}\tilde{J}_{\mathcal{E}} \cong \mathrm{Id}_{\mathcal{E}^{\underline{\mathbb{N}}}}$. This implies that $\tilde{J}_{\mathcal{E}}$ is faithful, so it induces an equivalence between $\mathcal{E}^{\underline{\mathbb{N}}}$ and some subcategory $\mathit{Endof}_{alg}(\mathcal{E})$ of $\mathit{Endof}(\mathcal{E})$, which merits to be called *the category of algebraic endofunctors on* $\mathcal{E}$, and the quasi-inverse equivalence is given by the restriction of $J^*_{\mathcal{E}}$. In this respect everything is quite similar to **4.1.4**; however, $\tilde{J}_{\mathcal{E}}$ cannot be the left Kan extension $J_{\mathcal{E},!}$ of $J_{\mathcal{E}}$, since $J_{\mathcal{E}}$ is in general not fully faithful, so we cannot expect $J^*_{\mathcal{E}}J_{\mathcal{E},!} \cong \mathrm{Id}$. However, a similar description can be obtained if we replace $\mathit{Endof}(\mathcal{E})$ with the category of *(plain) inner endofunctors* $\mathit{InnEndof}(\mathcal{E})$ (cf. **3.5.8**):

Proposition 4.9.8 *Denote by $J_{\mathcal{E}} : \underline{\mathbb{N}} \to \mathcal{E}$ the canonical functor, and consider the restriction functor $J^*_{\mathcal{E}} : \mathit{InnEndof}(\mathcal{E}) \to \mathcal{E}^{\underline{\mathbb{N}}}$. This restriction functor admits a left adjoint $\tilde{J}_{\mathcal{E}} : \mathcal{E}^{\underline{\mathbb{N}}} \to \mathit{InnEndof}(\mathcal{E})$, such that the composite functor $\mathcal{E}^{\underline{\mathbb{N}}} \to \mathit{InnEndof}(\mathcal{E}) \to \mathit{Endof}(\mathcal{E})$ coincides with the extension functor $\tilde{J}_{\mathcal{E}} : \Sigma \mapsto \tilde{\Sigma}$ constructed before. Moreover, $\tilde{J}_{\mathcal{E}} : \mathcal{E}^{\underline{\mathbb{N}}} \to \mathit{InnEndof}(\mathcal{E})$ is fully faithful, i.e. $J^*_{\mathcal{E}}\tilde{J}_{\mathcal{E}} \cong \mathrm{Id}$, so the category $\mathcal{E}^{\underline{\mathbb{N}}}$ of "algebraic endofunctors" over $\mathcal{E}$ is equivalent to the essential image of $\tilde{J}_{\mathcal{E}}$, a full subcategory $\mathit{InnEndof}_{alg}(\mathcal{E})$*

of InnEndof$(\mathcal{E})$*, which will be called* the category of algebraic (plain) inner endofunctors on $\mathcal{E}$. *In particular, Endof*$_{alg}(\mathcal{E}) \subset$ *Endof*$(\mathcal{E})$ *is equivalent to this category, hence any algebraic endofunctor on* $\mathcal{E}$ *admits a canonical (algebraic) plain inner structure.*

Proof. First of all, let's construct a canonical plain inner structure on each endofunctor $\tilde{\Sigma} : \mathcal{E} \to \mathcal{E}$, obtained by extending some $\Sigma : \underline{\mathbb{N}} \to \mathcal{E}$; this would define a functor $\tilde{J}_{\mathcal{E}} : \mathcal{E}^{\underline{\mathbb{N}}} \to \mathit{InnEndof}(\mathcal{E})$, such that $J^*_{\mathcal{E}}\tilde{J}_{\mathcal{E}} \cong \mathrm{Id}$. So we need to construct a functorial family of morphisms $\alpha_{X,Y} : X \times \tilde{\Sigma}(Y) \to \tilde{\Sigma}(X \times Y)$, defined for any X, $Y \in \operatorname{Ob}\mathcal{E}$, satisfying the conditions of **3.5.5**. Since $\tilde{\Sigma}(Y) = \operatorname{Coker}(H_1(Y) \rightrightarrows H_0(Y))$, and similarly for $\tilde{\Sigma}(X \times Y)$, we get $X \times \tilde{\Sigma}(Y) = \operatorname{Coker}(X \times H_1(Y) \rightrightarrows X \times H_0(Y))$, since $j^*_X : S \mapsto X \times S$ commutes with arbitrary inductive limits, having a right adjoint $\mathbf{Hom}_{\mathcal{E}}(X, -)$, so we see that it is sufficient to define some morphisms $X \times H_0(Y) = X \times \bigsqcup_{n \geq 0}(\Sigma(n) \times Y^n) = \bigsqcup_{n\geq 0}(X \times \Sigma(n) \times Y^n) \to H_0(X \times Y) = \bigsqcup_{n\geq 0}(\Sigma(n) \times (X \times Y)^n)$. Of course, we can construct such a morphism by defining its n-th component $X \times \Sigma(n) \times Y^n \to \Sigma(n) \times (X \times Y)^n$ with the aid of the diagonal morphism $X \to X^n$. We define $X \times H_1(Y) \to H_1(X \times Y)$ in a similar way, thus obtaining a morphism $\alpha_{X,Y} : X \times \tilde{\Sigma}(Y) \to \tilde{\Sigma}(X \times Y)$ after taking cokernels. This construction is clearly functorial in X and Y; in order to check the remaining conditions for α we simply observe that in case $\mathcal{E} = \tilde{\mathcal{S}}$ we might construct α first on the category of presheaves $\hat{\mathcal{S}}$, by considering the unique plain inner structure on each $\Sigma_{[S]}$ over *Sets*, and then "sheafify" this plain inner structure on $\hat{\Sigma}$, thus obtaining a plain inner structure on $\tilde{\Sigma}$, easily seen to coincide with one constructed before in an invariant fashion.

We already know that $J^*_{\mathcal{E}}\tilde{\Sigma} \cong \Sigma$, i.e. $J^*_{\mathcal{E}}\tilde{J}_{\mathcal{E}} \cong \mathrm{Id}$, so it remains to show that $\tilde{J}_{\mathcal{E}}$ is indeed a left adjoint to $J^*_{\mathcal{E}}$, i.e. that for any $\Sigma : \underline{\mathbb{N}} \to \mathcal{E}$ and any $\Sigma' = (\Sigma', \alpha') \in \operatorname{Ob} \mathit{InnEndof}(\mathcal{E})$ we have $\operatorname{Hom}_{\mathit{InnEndof}(\mathcal{E})}(\tilde{\Sigma}, \Sigma') \cong \operatorname{Hom}_{\mathcal{E}^{\underline{\mathbb{N}}}}(\Sigma, J^*_{\mathcal{E}}\Sigma')$, where $\tilde{\Sigma} = (\tilde{\Sigma}, \alpha) := \tilde{J}_{\mathcal{E}}\Sigma$ as before.

So we have to show that natural transformations of inner endofunctors $\tilde{\gamma} : \tilde{\Sigma} \to \Sigma'$, $\tilde{\gamma}_X : \tilde{\Sigma}(X) \to \Sigma'(X)$, are in one-to-one correspondence with natural transformations $\gamma : \Sigma \to J^*_{\mathcal{E}}\Sigma'$, $\gamma_n : \Sigma(n) \to (J^*_{\mathcal{E}}\Sigma')(n) = \Sigma'(\mathbf{n}_{\mathcal{E}})$. Since $\tilde{\Sigma}(X) = \operatorname{Coker}(H_1(X) \rightrightarrows H_0(X))$, we see that giving a $\tilde{\gamma}_X : \tilde{\Sigma}(X) \to \Sigma'(X)$ is equivalent to giving a family of morphisms $\tilde{\gamma}^{(n)}_X : \Sigma(n) \times X^n \to \Sigma'(X)$, such that $\tilde{\gamma}^{(n)}_X \circ (\Sigma(\varphi) \times \mathrm{id}_{X^n}) = \tilde{\gamma}^{(m)}_X \circ (\mathrm{id}_{\Sigma(m)} \times X^{\varphi})$ for any $\varphi : \mathbf{m} \to \mathbf{n}$ (cf. **4.2.2**). Let's denote by $\theta^n_X : X^n \times \Sigma(n) \to \Sigma'(X)$ the same morphism as $\tilde{\gamma}^{(n)}_X$ with interchanged arguments. Since $X^n = \mathbf{Hom}_{\mathcal{E}}(\mathbf{n}_{\mathcal{E}}, X)$, we have some evaluation morphisms $\mathrm{ev}_{\mathbf{n},X} : X^n \times \mathbf{n}_{\mathcal{E}} \to X$. We know that $\tilde{\Sigma}(\mathbf{n}_{\mathcal{E}}) \cong \Sigma(n)$, and $\tilde{\gamma}$ is supposed to be compatible with plain inner structures; this proves the commutativity of the following diagram:

$$\begin{tikzcd}
X^n \times \Sigma(n) \arrow[r, "\alpha_{X^n,\mathbf{n}_\mathcal{E}}"] \arrow[d, "\mathrm{id}_{X^n}\times\gamma_n"] \arrow[rrd, dashed, "\theta_X^n"] & \tilde\Sigma(X^n \times \mathbf{n}_\mathcal{E}) \arrow[r, "\tilde\Sigma(\mathrm{ev}_{\mathbf{n},X})"] \arrow[d] & \tilde\Sigma(X) \arrow[d, "\tilde\gamma_X"] \\
X^n \times \Sigma'(\mathbf{n}_\mathcal{E}) \arrow[r, "\alpha'_{X^n,\mathbf{n}_\mathcal{E}}"] & \Sigma'(X^n \times \mathbf{n}_\mathcal{E}) \arrow[r, "\Sigma'(\mathrm{ev}_{\mathbf{n},X})"'] & \Sigma'(X)
\end{tikzcd} \tag{4.9.8.1}$$

Notice that the upper row defines the canonical morphism $X^n \times \Sigma(n) \to \tilde\Sigma(X)$, given by $X^n \times \Sigma(n) \to H_0(X) \to \tilde\Sigma(X)$, hence the diagonal arrow has to be equal to θ_X^n. This shows that the collection of the θ_X^n, hence also $\tilde\gamma_X$, is completely determined by the collection of γ_n, i.e. by γ. Conversely, given any $\gamma : \Sigma \to J_\mathcal{E}^*\Sigma'$, we can construct the morphisms $\theta_X^n : X^n \times \Sigma(n) \to \Sigma'(X)$ from the above diagram; they are easily seen to satisfy necessary compatibility conditions, so they define some $\tilde\gamma_X : \tilde\Sigma(X) \to \Sigma'(X)$, clearly functorial in X.

4.9.9. (Algebraic inner endofunctors and filtered inner inductive limits.) In the case $\mathcal{E} = \mathit{Sets}$ we had an equivalent description of the category of algebraic endofunctors $\mathit{Endof}_{alg}(\mathit{Sets}) \subset \mathit{Endof}(\mathit{Sets})$: it was the full subcategory, consisting of those endofunctors, which commute with filtered inductive limits. In the topos case all algebraic endofunctors are easily seen to commute with arbitrary filtered inductive limits; however, this property is insufficient to be an algebraic endofunctor over a topos $\mathcal{E}$, since in general an arbitrary object X of $\mathcal{E}$ cannot be represented as a filtered inductive limit of finite constant objects $\mathbf{n}_\mathcal{E}$.

It is possible to save the situation by considering the category of algebraic inner endofunctors $\mathit{InnEndof}_{alg}(\mathcal{E}) \subset \mathit{InnEndof}(\mathcal{E})$: these inner endofunctors can be characterized by their property to commute with filtered *inner* (or *local*) inductive limits, i.e. inductive limits along filtered inner index categories $\mathcal{I}$ in $\mathcal{E}$ (recall that this means that $\mathcal{I}$ has an object of objects $\mathcal{I}_0$ and an object of morphisms $\mathcal{I}_1$ instead of the sets of objects and morphisms, together with some morphisms $i : \mathcal{I}_0 \to \mathcal{I}_1$, $s, t : \mathcal{I}_1 \to \mathcal{I}_0$, $m : \mathcal{I}_1 \times_{t,s} \mathcal{I}_1 \to \mathcal{I}_1$, subject to certain conditions which need not be written here). For example, any $X \in \mathrm{Ob}\,\mathcal{E}$ can be written as an inner filtered inductive limit along a certain inner category $\mathcal{I}_X$, with $\mathcal{I}_{X,0} = \bigsqcup_{n\geq 0} X^n$ and $\mathcal{I}_{X,1} = \bigsqcup_{\varphi:\mathbf{m}\to\mathbf{n}} X^n$. If we write down explicitly the requirement for an inner endofunctor $\Sigma' = (\Sigma', \alpha')$ to commute with this particular inner inductive limit, we obtain $\Sigma' \cong \tilde J_\mathcal{E} J_\mathcal{E}^*\Sigma'$, and one can check that any functor of form $\tilde J_\mathcal{E}\Sigma$ commutes with filtered inner inductive limits, essentially in the same way as in **4.1.4**.

Such an approach has its advantages. For example, it shows immediately that *the composite of two algebraic inner endofunctors is again algebraic.* However, we don't want to follow this path, since it would require to explain in more detail the theory of inner categories and limits; instead, we prove the above statement about composites of algebraic endofunctors directly:

Proposition 4.9.10 *The composite of two algebraic inner endofunctors is again algebraic, i.e.* $InnEndof_{alg}(\mathcal{E})$ *is a full* $\otimes$*-subcategory of* $InnEndof(\mathcal{E})$, *and* $Endof_{alg}(\mathcal{E})$ *is a* $\otimes$*-subcategory of* $Endof(\mathcal{E})$. *Therefore, we have a canonical AU* $\otimes$*-structure on* $\mathcal{E}^{\underline{\mathbb{N}}} \cong InnEndof_{alg}(\mathcal{E})$, *and a canonical* $\oslash$*-action of this category on* $\mathcal{E}$. *Moreover,* $InnEndof_{alg}(\mathcal{E})$ *is stable under arbitrary inductive and finite projective limits of* $InnEndof(\mathcal{E})$, *and* $\tilde{J}_{\mathcal{E}}$ *commutes with these types of limits.*

Proof. First of all, notice that $InnEndof(\mathcal{E})$ admits arbitrary projective and inductive limits, which can be computed componentwise. Indeed, to define a plain inner structure $\alpha_{X,Y} : X \times (\varprojlim \Sigma_\iota)(Y) \to (\varprojlim \Sigma_\iota)(X \times Y)$ we simply consider the morphism with components $X \times \varprojlim \Sigma_\iota(Y) \to X \times \Sigma_\iota(Y) \to \Sigma_\iota(X \times Y)$, coming from the plain inner structures on Σ_ι. The case of inductive limits is similar, once we observe that the functor $S \mapsto X \times S$ commutes with arbitrary inductive limits, $\mathcal{E}$ being cartesian closed.

Now notice that $\mathcal{E}^{\underline{\mathbb{N}}}$ also admits arbitrary projective and inductive limits, which can be also computed componentwise, hence $J_{\mathcal{E}}^* : InnEndof(\mathcal{E}) \to \mathcal{E}^{\underline{\mathbb{N}}}$ commutes with all limits, and $\tilde{J}_{\mathcal{E}}$ commutes with arbitrary inductive limits, having a right adjoint $J_{\mathcal{E}}^*$. We claim that, similarly to what we had in **4.1.3**, $\tilde{J}_{\mathcal{E}}$ *commutes with finite projective limits as well*, hence $InnEndof_{alg}(\mathcal{E})$ *is stable under arbitrary inductive and finite projective limits of* $InnEndof(\mathcal{E})$. We have to show that $\widetilde{\varprojlim \Sigma_\iota} \to \varprojlim \tilde{\Sigma}_\iota$ is an isomorphism for any finite projective limit $\varprojlim \Sigma_\iota$ in $\mathcal{E}^{\underline{\mathbb{N}}}$. We can assume $\mathcal{E} = \tilde{\mathcal{S}}$ for some site $\mathcal{S}$; in this case our statement can be first shown for the extension $\hat{\Sigma}$ of Σ to the category of presheaves $\hat{\mathcal{S}}$, using **4.1.3** componentwise; then we extend this result to $\tilde{\Sigma} = a\hat{\Sigma}i$, using the left exactness of a.

Notice that $\tilde{J}_{\mathcal{E}}$ *transforms a constant functor* $\underline{\mathbb{N}} \to \mathcal{E}$ *with value* Z *into a constant endofunctor* $\mathcal{E} \to \mathcal{E}$ *with the same value*. Indeed, this is clear for $\mathcal{E} = Sets$, constant functors being algebraic, then the statement can be shown for $\hat{\mathcal{S}}$, applying the result for $Sets$ componentwise, and then we can extend it to the general case $\mathcal{E} = \tilde{\mathcal{S}}$ by sheafification.

Now let's show that the composite $\tilde{\Xi}\tilde{\Sigma}$ of two algebraic inner endofunctors is again algebraic. Indeed, for any $X \in \operatorname{Ob}\mathcal{E}$ we have $\tilde{\Xi}(X) = \operatorname{Coker}(H_1(X) \rightrightarrows H_0(X))$, hence $\tilde{\Xi}\tilde{\Sigma} = \operatorname{Coker}(H_1(\tilde{\Sigma}) \rightrightarrows H_0(\tilde{\Sigma}))$, where $H_0(\tilde{\Sigma}) = \bigsqcup_{n \geq 0} \Xi(n) \times \tilde{\Sigma}^n$ and $H_1(\tilde{\Sigma}) = \bigsqcup_{\varphi:\mathbf{m}\to\mathbf{n}} \Xi(m) \times \tilde{\Sigma}^n$ are computed in $InnEndof(\mathcal{E})$. This allows us to compute $\tilde{\Xi}\tilde{\Sigma}$ in $InnEndof(\mathcal{E})$ by means of finite products and some inductive limits, starting from algebraic inner endofunctor $\tilde{\Sigma}$ and constant inner endofunctors with values $\Xi(n)$, which are also algebraic; hence $\tilde{\Xi}\tilde{\Sigma}$ is algebraic as well.

4.9.11. Once we know that $\mathit{InnEndof}_{alg}(\mathcal{E}) \subset \mathit{InnEndof}(\mathcal{E})$ is stable under composition, arbitrary inductive, and finite projective limits, and that morphisms $\tilde{\Sigma}(X) \to Y$ admit a description similar to that given in **4.2.2** for any algebraic endofunctor Σ on $\mathcal{E}$ and any two objects X and Y of $\mathcal{E}$, we can extend all statements from **4.2** and **4.3** to the topos case. In particular, we see that an algebraic inner monad Σ over $\mathcal{E}$, i.e. an algebra in $\mathit{InnEndof}_{alg}(\mathcal{E})$, can be described in the same way as in **4.3.1** and **4.3.3**, so we obtain the same notion of an algebraic monad over $\mathcal{E}$ as in **4.9.1**, and the same notions of modules over such an algebraic monad. This proves the monadicity of $\Gamma_\Sigma : \mathcal{E}^\Sigma \to \mathcal{E}$, claimed in **4.9.4**.

4.9.12. Most statements and definitions given before in this chapter for algebraic monads over *Sets* generalize directly to the topos case, at least those of them which are intuitionistic, i.e. don't involve the logical law of excluded middle and the axiom of choice. However, we have to interpret these statements in the so-called *Kripke–Joyal semantics*, where existence is always understood as local existence, i.e. existence after restricting everything to a suitable cover. Consider for example **4.2.10**, which tells us that any element ξ of $\Sigma(X)$ can be written in form $t(\{x_1\}, \dots, \{x_n\})$ for some $n \geq 0$, $t \in \Sigma(n)$ and $x_i \in X$. In the topos case we can denote by $t(\{x_1\}, \dots, \{x_n\})$ the image of $(t, x_1, \dots, x_n) \in \Sigma(n)(S) \times X(S)^n$ in $\Sigma(X)(S)$, for any $S \in \operatorname{Ob}\mathcal{E}$ (or in $\operatorname{Ob}\mathcal{S}$, when $\mathcal{E} = \tilde{\mathcal{S}}$) and any $t \in \Sigma(n)(S)$ and $x_i \in X(S)$. However, the above statement has to be interpreted as follows: *for any $\xi \in (\Sigma(X))(S)$ we can find a cover $\{\varphi_\alpha : S_\alpha \to S\}$, some integers $n_\alpha \geq 0$, elements $t_\alpha \in (\Sigma(n_\alpha))(S_\alpha)$ and $x_{\alpha,i} \in X(S_\alpha)$, $1 \leq i \leq n_\alpha$, such that $\varphi_\alpha^*\xi = t_\alpha(\{x_{\alpha,1}\}, \dots, \{x_{\alpha,n_\alpha}\})$ in $(\Sigma(X))(S_\alpha)$ for any α.*

4.9.13. However, some statements over *Sets* have been proved before with the aid of the axiom of choice. We need to find other proofs of such statements, or to extend these results over *Sets* first to the categories of presheaves $\hat{\mathcal{S}}$, applying already known results componentwise, and then extend these statements to arbitrary $\mathcal{E} = \tilde{\mathcal{S}}$ by sheafification.

For example, we have seen that algebraic monads (even endofunctors) over *Sets* preserve epimorphisms and monomorphisms (cf. **4.3.10** and **4.6.4**); however, our proofs used the axiom of choice, so they cannot be generalized to the topos case directly. Nevertheless, we claim that *any algebraic endofunctor Σ over a topos $\mathcal{E}$ preserves monomorphisms and epimorphisms.*

The statement about monomorphisms is easily deduced from the known case by sheafification, since both $i : \tilde{\mathcal{S}} \to \hat{\mathcal{S}}$ and $a : \hat{\mathcal{S}} \to \tilde{\mathcal{S}}$ are left exact, and

$\tilde{\Sigma} = a\hat{\Sigma}i$. The statement about epimorphisms cannot be easily shown in this way, since i doesn't respect epimorphisms; however, it can be deduced from the fact that any finite product of epimorphisms in a topos is an epimorphism again, hence for any epimorphism $f : X \to Y$ the induced morphism $H_0(f) : H_0(X) = \bigsqcup_{n\geq 0} \Sigma(n) \times X^n \to H_0(Y)$ is again an epimorphism, and $H_0(Y) \to \Sigma(Y)$ is a (strict) epimorphism as well, hence $H_0(X) \to \Sigma(X) \to \Sigma(Y)$ is an epimorphism, hence $\Sigma(f) : \Sigma(X) \to \Sigma(Y)$ has to be an epimorphism as well.

Since all epimorphisms in $\mathcal{E}$ are strict, and any morphism in $\mathcal{E}$ decomposes uniquely into an epimorphism followed by a monomorphism, we are in position to generalize **4.6.4** to the topos case, proving that the image of any morphism of Σ-modules in $\mathcal{E}$ admits a Σ-structure. Most statements after **4.6.4** can be deduced from this fact in the same way as before. For example, arbitrary projective and inductive limits exist in $\mathcal{E}^\Sigma$.

4.9.14. Some statements that involve the law of excluded middle also need to be reconsidered. For example, an algebraic monad Σ over a topos $\mathcal{E}$ is said to be *a monad without constants* (resp. *with zero, with at most one constant*) iff $\Sigma(0)$ is isomorphic to the initial object $\varnothing_\mathcal{E}$ of $\mathcal{E}$ (resp. iff the canonical morphism $\Sigma(0) \to e_\mathcal{E}$ is an isomorphism, or a monomorphism for the last case). We see that $\Sigma(0)$ can be a non-trivial open object of $\mathcal{E}$ (i.e. a subobject of the final object $e_\mathcal{E}$, different from both $\varnothing_\mathcal{E}$ and $e_\mathcal{E}$); in this case Σ is a monad with at most one constant, but neither a monad without constants nor a monad with zero.

4.9.15. (Topos-specific properties.) Finally, there are some topos-specific properties, very similar to those discussed in SGA 4 IV. For example, any morphism of topoi $f : \mathcal{E}' \to \mathcal{E}$ induces functors $f^{*,\Sigma} : \mathcal{E}^\Sigma \to \mathcal{E}'^{f^*\Sigma}$ and $f_*^\Sigma : \mathcal{E}'^{f^*\Sigma} \to \mathcal{E}^\Sigma$ for any algebraic monad Σ over $\mathcal{E}$ (cf. **4.9.3**), which are easily seen to be adjoint. Since they coincide with f^* and f_* on the level of underlying objects, we usually omit the monad Σ in these notations.

In particular, we can apply this to localization morphisms $j_X : \mathcal{E}_{/X} \to \mathcal{E}$, for any object X of $\mathcal{E}$, and any algebraic monad Σ on $\mathcal{E}$. We usually write $\Sigma_{|X}$ or $\Sigma|X$ instead of $j_X^*\Sigma$; so we get a pair of adjoint functors $j_X^* = j_X^{*,\Sigma} : \mathcal{E}^\Sigma \to (\mathcal{E}_{/X})^{\Sigma_{|X}}$ and $j_{X,*} = (j_X)_*^\Sigma$ in the opposite direction, enjoying all the usual properties. Of course, we usually write $M_{|X}$ or $M|X$ instead of $j_X^{*,\Sigma}M$. Moreover, $j_X^{*,\Sigma}$ admits a left adjoint $j_{X,!}^\Sigma$, which has all the properties listed in SGA 4 IV 11.3.1; these properties can be shown in the same way as in SGA 4, and the existence of $j_{X,!}^\Sigma$ can be shown either by a reference to SGA 4 III 1.7, where the statement has been shown for arbitrary algebraic structures defined by finite projective limits, or directly: clearly, $j_{X,!}^\Sigma L_{\Sigma|X} = L_\Sigma j_{X,!}$, so we see how to define $j_{X,!}^\Sigma$ on free $\Sigma_{|X}$-modules; now by **3.3.20** an arbitrary $\Sigma_{|X}$-module N can be represented as a cokernel of a pair of morphisms p,

q between some free $\Sigma_{|X}$-modules $L_{\Sigma|X}(R)$ and $L_{\Sigma|X}(S)$; since $j_{X,!}$ has to commute with arbitrary inductive limits, $j^{\Sigma}_{X,!}N$ has to be equal to the cokernel of $j^{\Sigma}_{X,!}(p)$ and $j^{\Sigma}_{X,!}(q)$, and this cokernel is easily seen to have the universal property required from $j^{\Sigma}_{X,!}N$, hence $j^{\Sigma}_{X,!}$ is well-defined.

4.9.16. (Open and closed subtopoi.) Let $\mathcal{E}$ be a topos, Σ an algebraic monad over $\mathcal{E}$, $U \subset e_{\mathcal{E}}$ an open object of $\mathcal{E}$, $\mathcal{U}$ the open subtopos of $\mathcal{E}$ defined by U, and $\mathcal{F}$ the complementary closed subtopos of $\mathcal{E}$ (cf. SGA 4 IV 9.3). Let's denote the corresponding embeddings by $j : \mathcal{U} \to \mathcal{E}$ and $i : \mathcal{F} \to \mathcal{E}$; recall that $\mathcal{U} \cong \mathcal{E}_{/U}$, and $i_* : \mathcal{F} \to \mathcal{E}$ is a fully faithful functor, the essential image of which consists of all $X \in \operatorname{Ob}\mathcal{E}$, such that $j^*X = X_{|U}$ is isomorphic to the final object of $\mathcal{U} = \mathcal{E}_{/U}$. We obtain a left exact *gluing functor* $\rho := i^*j_* : \mathcal{U} \to \mathcal{F}$, and we know that $\mathcal{E}$ is canonically equivalent to the "comma category" $(\mathcal{F}, \mathcal{U}, \rho)$, the objects of which are triples (A, B, f), with $A \in \operatorname{Ob}\mathcal{F}$, $B \in \operatorname{Ob}\mathcal{U}$ and $f : A \to \rho B$ a morphism in $\mathcal{F}$. Recall that this equivalence $\Phi : \mathcal{E} \to (\mathcal{F}, \mathcal{U}, \rho)$ transforms $X \in \operatorname{Ob}\mathcal{E}$ into $(i^*X, j^*X, h(X) : i^*X \to \rho j^*X)$, where $h(X)$ is obtained by applying i^* to adjointness morphism $X \to j_*j^*X$ (cf. SGA 4 IV 9.5). All functors $j_!$, j^*, j_*, i^* and i_* admit descriptions in terms of this equivalence: $j_! : B \mapsto (\varnothing_{\mathcal{F}}, B, \varnothing_{\mathcal{F}} \to \rho B)$, $j^* : (A, B, f) \mapsto B$, $j_* : B \mapsto (\rho B, B, \mathrm{id}_{\rho B})$, $i^* : (A, B, f) \mapsto A$, and $i_* : A \mapsto (A, e_{\mathcal{U}}, A \to \rho e_{\mathcal{U}} = e_{\mathcal{F}})$.

Since all functors i^*, j^* and ρ are left exact, we can extend the above results to the category of modules over an algebraic monad Σ; we see that $\mathcal{E}^{\Sigma}$ is equivalent to the comma category $(\mathcal{F}^{i^*\Sigma}, \mathcal{U}^{j^*\Sigma}, \rho)$, similarly to SGA 4 IV 14. Moreover, *when Σ is an algebraic monad with zero, $i_* = i^{\Sigma}_*$ admits a right adjoint $i^! = i^{!,\Sigma} : \mathcal{E}^{\Sigma} \to \mathcal{F}^{i^*\Sigma}$, given by $i^! : (A, B, f) \mapsto \operatorname{Ker}(f : A \to \rho B)$.* Here, of course, $\operatorname{Ker}(f)$ means the kernel of the pair, consisting of f and of the zero morphism $0 : A \to \rho B$. We check that $i^!$ is a right adjoint to i_*, that $i_*i^!X \to X$ is a monomorphism, and that $i^!$ commutes with scalar restriction, essentially in the same way as in SGA 4 IV 14.5.

4.9.17. (Sheaves of homomorphisms.) Let Σ be an algebraic monad over a topos $\mathcal{E}$, and M, N be two Σ-modules. Then we can construct the "sheaf of local homomorphisms" $\mathbf{Hom}_{\Sigma}(M, N) \subset \mathbf{Hom}_{\mathcal{E}}(M, N)$ in two different ways. On one hand, we can repeat the reasoning of SGA 4 IV 12 and define $\mathbf{Hom}_{\Sigma}(M, N)$ by requiring $\operatorname{Hom}_{\mathcal{E}}(S, \mathbf{Hom}_{\Sigma}(M, N)) \cong \operatorname{Hom}_{\Sigma|S}(M_{|S}, N_{|S})$; this notion turns out to have almost all properties of SGA 4 IV 12, which can be proved essentially in the same way. On the other hand, Σ admits a canonical plain inner structure, so we might use **3.5.18** instead. *Both these definitions yield the same subobject of* $\mathbf{Hom}_{\mathcal{E}}(M, N)$. We don't give a detailed proof since we are not going to use this fact anyway; it can be shown, for example, by reduction to the case $\mathcal{E} = \hat{\mathcal{S}}$.

4.9.18. (Small families of generators.) Notice that $\mathcal{E}^\Sigma$ *admits a small family of generators.* Indeed, any topos admits a small family of generators, so let $\mathcal{G} \subset \operatorname{Ob}\mathcal{E}$ be such a family for $\mathcal{E}$. Then the corresponding free Σ-modules $\{\Sigma(S)\}_{S\in\mathcal{G}}$ are easily seen to constitute a small family of generators of $\mathcal{E}^\Sigma$.

4.9.19. (Algebraic bimodules.) Of course, we can consider algebraic bimodules (and left or right modules) over $\mathcal{E}$, i.e. bimodules in $\mathit{InnEndof}_{alg}(\mathcal{E})$. Most statements made in **4.7** generalize directly to the topos case. For example, we have matrix interpretations for all these notions, if we put $M(m,n;\Sigma) := \Sigma(m)^n$ as before; of course, now these are objects of $\mathcal{E}$, not just sets. However, (Σ,Λ)-bimodules cannot be interpreted as functors $\underline{\mathbb{N}}_\Lambda \to \mathcal{E}^\Sigma$ when Λ is non-constant, at least if we don't want to replace $\underline{\mathbb{N}}_\Lambda$ with an inner category in $\mathcal{E}$.

4.9.20. (Local END's.) For any $X \in \operatorname{Ob}\mathcal{E}$ we can construct an algebraic monad $\mathbf{END}(X)$ over $\mathcal{E}$, such that giving an action of some algebraic monad Σ on X is equivalent to giving a monad homomorphism $\rho : \Sigma \to \mathbf{END}(X)$. Indeed, we can put $(\mathbf{END}(X))(n) := \mathbf{Hom}_{\mathcal{E}}(X^n, X)$, similarly to **4.3.8**. We can also consider the algebraic monad $\operatorname{END}(X) := \Gamma_{\mathcal{E}}(\mathbf{END}(X))$ over *Sets*; then $(\operatorname{END}(X))(n) = \operatorname{Hom}_{\mathcal{E}}(X^n, X)$, and monad homomorphisms $\Sigma \to \operatorname{END}(X)$ are in one-to-one correspondence with actions of *constant* algebraic monads Σ on X. These notions generalize to the case of algebraic endofunctor X over $\mathcal{E}$, similarly to **4.7.3**.

5 Commutative monads

This chapter is dedicated to the notion of *commutativity* of algebraic monads, and to the properties of *generalized rings*, i.e. commutative algebraic monads, and categories of modules over them. We'll see that they have indeed properties very similar to classical commutative rings, so our terminology is justified. Moreover, sometimes we even consider arbitrary algebraic monads as *(non-commutative) generalized rings*; however, they can have properties very different from those of classical (non-commutative) rings.

We do some basic linear algebra over generalized rings; we construct a theory of exterior powers and determinants, at least for *alternating* generalized rings, and define and compute some K-groups. We are also able to construct a reasonable theory of traces, notwithstanding the absence of addition. The properties of these theories are illustrated by a series of important examples.

5.0. Notation. Given an algebraic monad homomorphism $\rho : \Sigma \to \Xi$, we denote by ${}^a\rho^* : \Sigma\text{-}Mod \to \Xi\text{-}Mod$ or even by ρ^* the base change functor (already denoted by ρ_* in **4.6.19**), and by ${}^a\rho_* : \Xi\text{-}Mod \to \Sigma\text{-}Mod$ or even ρ_* the scalar restriction functor (denoted before by ρ^*). This change of notation may seem slightly confusing, since ${}^a\rho^*$ is covariant in ρ, and ${}^a\rho_*$ is contravariant; the idea, of course, is that we think of pullback and direct image functors for the corresponding morphism of (generalized) schemes ${}^a\rho : \operatorname{Spec}\Xi \to \operatorname{Spec}\Sigma$. This new notation has its advantages: for example, ${}^a\rho^*$ is now a left adjoint to ${}^a\rho_*$.

Unless otherwise specified, all monads are supposed to be algebraic. We denote them by letters like Σ, Ξ, ..., and also by A, B, C, ..., especially when they are commutative; thus our notation reminds of classical rings. We also denote free A-module $L_A(S)$ by $A(S)$ or $A^{(S)}$; when $S = \mathbf{n} = \{1, 2, \ldots, n\}$, we write $A(n)$ or $A^{(n)}$, but never A^n; the latter notation actually means the product of n copies of $A = A^{(1)}$, which is usually not isomorphic to $A^{(n)}$, at least for a non-additive A.

We preserve the notation $|A|$ for $A(1)$, considered both as a monoid (with identity $\boldsymbol{e} \in |A|$), and an A-module. We denote by $\|A\|$ the (graded) set of operations $\bigsqcup_{n\geq 0} A(n)$.

5.1. (Definition of commutativity.)

Definition 5.1.1 *Let Σ be an algebraic monad.*

a) We say that two operations $t \in \Sigma(n)$ and $t' \in \Sigma(m)$ commute on a Σ-module X *if for any collection of elements $\{x_{ij}\}$, $1 \leq i \leq n$,*

$1 \le j \le m$ of X we have $t(x_{1.}, \ldots, x_{n.}) = t'(x_{.1}, \ldots, x_{.m})$, where we put $x_{i.} = t'(x_{i1}, \ldots, x_{im})$ and $x_{.j} = t(x_{1j}, \ldots, x_{nj})$.

b) *We say that two operations t, $t' \in \|\Sigma\|$* commute *if they commute on any Σ-module X; clearly, it suffices to require them to commute on the set of free generators of $\Sigma(\mathbf{n} \times \mathbf{m}) \cong \Sigma(mn)$.*

c) *For any subset $S \subset \|\Sigma\|$ we denote by $S' \subset \|\Sigma\|$ the* commutant *of Σ, i.e. the set of all operations of Σ, commuting with all operations from S. We'll show in a moment that S' is the underlying set of a certain submonad of Σ, which will be also denoted by S'. Finally, if $\Lambda \subset \Sigma$ is a submonad, we denote by Λ' the commutant of its underlying set $\|\Lambda\| \subset \|\Sigma\|$.*

d) *We say that $t \in \|\Sigma\|$ is a* central operation *of Σ if it commutes with all operations of Σ. All such operations form a submonad $\Sigma' \subset \Sigma$, called the* center *of Σ.*

e) *We say that Σ is* commutative, *if any two its operations commute, i.e. if Σ coincides with its center Σ'.*

5.1.2. Let's show that the commutant S' of any subset $S \subset \|\Sigma\|$ is indeed a submonad of Σ. Since $S' = \bigcap_{t \in S} \{t\}'$, we see that it is sufficient to show this for a one-element set $S = \{t\}$, with $t \in \Sigma(n)$. Given any two Σ-modules M and N, and any map of sets $f : M \to N$, we say that an operation $t' \in \Sigma(m)$ *commutes with* f, if $[t']_N \circ f^m = f \circ [t']_M : M^m \to N$. It follows immediately from the definitions that t' commutes with t iff it commutes with all maps $[t]_{\Sigma(k)} : \Sigma(k)^n \to \Sigma(k)$, $k \ge 0$; actually, $k = mn$ would suffice for $t' \in \Sigma(m)$. Thus we are reduced to proving the following statement:

5.1.3. *The operations of Σ, commuting with a given map $f : M \to N$ between two Σ-modules M and N, constitute a submonad $\Sigma_0 \subset \Sigma$. The map f is a Σ_0-homomorphism, and Σ_0 is the largest submonad of Σ with this property.* First of all, given any set X and any subset $Y \subset X$, we can consider the submonad $(Y : Y)$ of $\mathrm{END}(X)$, given by $(Y : Y)(n) = \{f : X^n \to X \mid f(Y^n) \subset Y\}$ (cf. **4.3.8**), i.e. $(Y : Y)$ consists of those maps $f : X^n \to X$, which stabilize Y. It is immediate that such maps are stable under composition, hence $(Y : Y)$ is indeed a submonad of $\mathrm{END}(X)$.

Now suppose that X is a Σ-module, i.e. we have a monad homomorphism $\rho : \Sigma \to \mathrm{END}(X)$. Consider the pullback $\Sigma_0 := \rho^{-1}(Y : Y)$; clearly, this is the submonad Σ_0 of Σ, consisting of all operations $t \in \Sigma(n)$ that stabilize Y (i.e. such that $[t]_X(Y^n) \subset Y$), hence Σ_0 is the largest submonad of Σ, such that Y is a Σ_0-submodule of X.

Finally, let's consider any map $f : M \to N$ of Σ-modules as above. Its graph $\Gamma_f \subset M \times N$ is a subset of the Σ-module $M \times N$, and we see that an operation $t \in \Sigma(n)$ commutes with f, i.e. $f \circ [t]_M = [t]_N \circ f^n$, iff t stabilizes the graph $\Gamma_f \subset M \times N$, i.e. $[t]_{M\times N}(\Gamma_f^n) \subset \Gamma_f$, and we have just seen that all such operations constitute a submonad $\Sigma_0 \subset \Sigma$.

5.1.4. Notice that our notion of commutant of sets $S \subset \|\Sigma\|$ enjoys the usual properties, e.g. $S \subset T$ implies $T' \subset S'$, and we have $S \subset S''$ and $S' = S'''$ for any S. We have seen that S' is always a submonad of Σ. This implies that for any subset $S \subset \|\Sigma\|$ its commutant S' coincides with the commutant $\langle S\rangle'$ of the algebraic submonad $\langle S\rangle \subset \Sigma$ generated by S (cf. **4.4.2**). Indeed, $S \subset \langle S\rangle$ (we deliberately confuse submonads of Σ with their underlying sets), hence $\langle S\rangle' \subset S'$. Conversely, $S \subset S''$ implies $\langle S\rangle \subset S''$, S'' being a submonad, hence $S' = S''' \subset \langle S\rangle'$. In particular, if S is a set of generators of Σ, then S' coincides with the center of Σ. Therefore, Σ is commutative iff $S' = \Sigma = \langle S\rangle$, i.e. iff $S \subset S'$, i.e. iff any two operations from S commute.

5.1.5. Given any two operations $t \in \Sigma(n)$, $t' \in \Sigma(m)$, we denote by $[t,t'] \in \Sigma(mn)^2$ the *commutator* of t and t', i.e. the relation between the free generators of $\Sigma(\mathbf{m} \times \mathbf{n}) \cong \Sigma(mn)$, described in **5.1.1**. In this way $[t,t']$ holds in Σ iff t and t' commute. Notice that the actual element $[t,t'] \in \Sigma(mn)^2$ depends on the choice of bijection $\mathbf{m} \times \mathbf{n} \cong \mathbf{mn}$, and on the order of t and t', but the compatible equivalence relation $\langle [t,t']\rangle$ generated by $[t,t']$ (cf. **4.4.9**) doesn't depend on these choices, since $\Sigma/\langle[t,t']\rangle$ is the largest strict quotient of Σ, such that the images of t and t' commute in this quotient. Similarly, for any two subsets S, $T \subset \|\Sigma\|$ we denote by $[S,T]$ the set of all relations $[s,t]$ for $s \in S$, $t \in T$; sometimes we denote the corresponding compatible equivalence relation $\langle [S,T]\rangle$ on Σ simply by $[S,T]$, especially when we form quotients like $\Sigma/[S,T]$. Clearly, this is the largest strict quotient of Σ, on which all operations from the image of S commute with those from the image of T. In particular, $\Sigma_{ab} := \Sigma/[\Sigma,\Sigma]$ is the largest commutative quotient of Σ, i.e. $\Sigma \mapsto \Sigma_{ab}$ is the left adjoint to the inclusion functor from the category of commutative algebraic monads into the category of all algebraic monads. Notice that for any two subsets S, $T \subset \|\Sigma\|$ we have $\big\langle [\langle S\rangle, \langle T\rangle]\big\rangle = \langle [S,T]\rangle$; this is shown by the same reasoning as in **5.1.4**.

5.1.6. (Category of generalized rings.) We denote by *GenR* the *category of generalized rings*, i.e. the full subcategory of the category $\mathit{Monads}_{alg}(\mathit{Sets})$ of algebraic monads, consisting of *commutative* algebraic monads. Clearly, *GenR* is stable under subobjects, strict quotients, and arbitrary projective limits of $\mathit{Monads}_{alg}(\mathit{Sets})$; in particular, arbitrary projective limits exist in *GenR*. As to inductive limits, their existence in *GenR* follows from their existence in $\mathit{Monads}_{alg}(\mathit{Sets})$ (cf. **4.5.19**), and from the existence of a left adjoint $\Sigma \to \Sigma_{ab}$ to the inclusion functor $GenR \to Monads_{alg}(Sets)$. Indeed, we have just to compute $\varinjlim_\alpha \Sigma_\alpha$ in the category of algebraic monads, and then take $(\varinjlim_\alpha \Sigma_\alpha)_{ab}$. For example, pushouts exist in $GenR$: given two homomorphisms of generalized rings $\rho_i : \Sigma \to \Sigma_i$, we construct their pushout, i.e. the "tensor product of Σ-algebras" $\Sigma_1 \otimes_\Sigma \Sigma_2$, by computing first the non-commutative tensor product $\Sigma_1 \boxtimes_\Sigma \Sigma_2$ (cf. **4.5.17**), and then taking quotient by the compatible equivalence relation, generated by all commutators between pairs of operations of this non-commutative tensor product. Since $\Sigma_1 \boxtimes_\Sigma \Sigma_2$ is generated by $\lambda_1(\Sigma_1) \cup \lambda_2(\Sigma_2)$, where $\lambda_i : \Sigma_i \to \Sigma_1 \boxtimes_\Sigma \Sigma_2$ is the canonical "inclusion" homomorphism, we see that this equivalence relation is generated by commutators between elements of this sort. Now the elements of $\lambda_1(\Sigma_1)$ will automatically commute between themselves in $\Sigma_1 \boxtimes_\Sigma \Sigma_2$, and similarly for $\lambda_2(\Sigma_2)$; hence $\Sigma_1 \otimes_\Sigma \Sigma_2 = (\Sigma_1 \boxtimes_\Sigma \Sigma_2)/[\lambda_1(\Sigma_1), \lambda_2(\Sigma_2)]$.

Notice that $GenR$ is stable under filtered inductive limits of the category of algebraic monads $Monads_{alg}(Sets)$, and such inductive limits can be computed componentwise: $(\varinjlim \Sigma_\alpha)(n) = \varinjlim \Sigma_\alpha(n)$.

5.1.7. (Commutativity for operations of lower arities.) Now we want to study the shape of commutativity relations between operations of lower arities, and their immediate consequences.

a) Commutativity between two constants c, $c' \in \Sigma(0)$ means simply that $c = c'$, i.e. *a commutative monad has at most one constant*, hence is either a monad without constants or a monad with zero. So we can denote all constants of commutative monads by the same symbol, say $0 \in \Sigma(0)$.

b) Commutativity between a constant $0 \in \Sigma(0)$ and a unary operation $u \in \Sigma(1)$ means that $u(0) = 0$, i.e. the zero constant of a commutative monad (if it exists) is necessarily fixed by all unary operations of Σ.

c) Similarly, commutativity between a constant $0 \in \Sigma(0)$ and an n-ary operation $t \in \Sigma(n)$ means $t(0, 0, \ldots, 0) = 0$, so the zero constant of a commutative monad has to be respected by operations of all arities.

d) Commutativity between two unary operations u, $u' \in |\Sigma| = \Sigma(1)$ means that $uu'\{1\} = u'u\{1\}$, i.e. $uu' = u'u$ inside the monoid $|\Sigma|$ (cf. **4.3.5** and **3.4.9**), so $|\Sigma|$ has to be a commutative monoid for any commutative algebraic monad Σ.

e) Commutativity between a unary operation u and an n-ary operation t means $t(u\{1\}, \ldots, u\{n\}) = ut(\{1\}, \ldots, \{n\})$. In particular, commutativity between unary u and binary $+$ actually means some sort of distributivity: $u(\{1\} + \{2\}) = u\{1\} + u\{2\}$.

f) Finally, the commutativity between two binary operations $+$ and $*$ means $(\{1\}+\{2\})*(\{3\}+\{4\}) = (\{1\}*\{3\})+(\{2\}*\{4\})$, i.e. $(x+y)*(z+t) = (x*z)+(y*t)$ for free variables x, y, z, t (or for any four elements of any Σ-

module X). Notice that any operation of arity ≤ 1 automatically commutes with itself, while this is not true for operations of higher arities: a binary operation $+$ commutes with itself iff $(x+y)+(z+t)=(x+z)+(y+t)$.

5.1.8. (Commutativity and addition.) Now we want to study the relationship between commutativity and additivity notions of **4.8**. Notice that these notions make sense only if we fix some constant $0 \in \Sigma(0)$; however, commutative monads have at most one constant, so we don't have to specify it explicitly.

a) Any two pseudoadditions $+$ and $*$ of a commutative monad Σ coincide. Indeed, we have $0+x=x=x+0$ and $0*x=x=x*0$ by definition of a pseudoaddition, and $(x+y)*(z+t)=(x*z)+(y*t)$ because of the commutativity between $+$ and $*$. We obtain $x+y=(x*0)+(0*y)=(x+0)*(0+y)=x*y$. A similar proof is valid for pseudoadditions t and t' of same arity $n \geq 3$: we just have to apply the commutativity relation between t and t' to the set of elements $x_{ij}=0$ for $i \neq j$, $x_{ii}=\{i\}$ in $\Sigma(n)$. This implies that any pseudoaddition in a commutative monad is unique, i.e. an addition, and it is automatically commutative and associative: we just apply uniqueness to $+$ and $\{2\}+\{1\}$ for the first property, and to $(\{1\}+\{2\})+\{3\}$ and $\{1\}+(\{2\}+\{3\})$ for the second; alternatively, we might put $z=0$ in $(x+y)+(z+t)=(x+z)+(y+t)$. Actually, all these statements are true for a central pseudoaddition $+$ in an arbitrary algebraic monad Σ.

b) Now we show that *any hyperadditive commutative monad Σ is in fact additive*; in other words, *if a commutative monad admits a pseudoaddition, it is necessarily additive.* Indeed, any hyperadditive monad Σ admits a pseudoaddition $+$ (cf. **4.8.5**), and commutativity implies that $+$ is an associative and commutative addition. Now let $t \in \Sigma(n)$ be an arbitrary operation of Σ, and let $\lambda=(\lambda_1,\ldots,\lambda_n)$ be the image of t under the comparison map $\pi_n : \Sigma(n) \to |\Sigma|^n$. Let us denote by $\tilde{+}$ the n-ary pseudoaddition (necessarily unique), constructed from $+$. We apply the commutativity relation between t and $\tilde{+}$ to the collection $x_{ij}=0$ if $i \neq j$, $x_{ii}=\{i\}$ of elements of $\Sigma(n)$. We have $x_{i.}=[\tilde{+}](0,\ldots,\{i\},\ldots,0)=0+\cdots+\{i\}+\cdots+0=\{i\}$, $x_{.j}=t(0,\ldots,\{j\},\ldots,0)=\lambda_j\{j\}$ by the definition of comparison map π_n, hence $t=t(\{1\},\ldots,\{n\})=t(x_{1.},\ldots,x_{n.})=[\tilde{+}](x_{.1},\ldots,x_{.n})=\lambda_1\{1\}+\cdots+\lambda_n\{n\}$, i.e. t is uniquely determined by its image under π_n, hence π_n is bijective and Σ is additive. Notice that the same proof actually shows that *if an algebraic monad admits a central pseudoaddition, it is automatically additive.*

c) We know that any additive monad Σ is of form Σ_R for a uniquely determined semiring R; in fact, $R=|\Sigma|$ (cf. **4.8.6**). We claim that *an additive monad Σ is commutative iff the semiring $|\Sigma|$ is commutative.* Indeed, the necessity of this condition has been shown in **5.1.7**,d); its sufficience follows from the fact that Σ is generated by $0^{[0]}$, $+^{[2]}$ and all its unary operations $|\Sigma|$ (cf. **4.8.6**), and the relations between these generators imply all necessary commutativity relations between them, with the only exception of commutativity among unary operations (cf. **5.1.7** and **4.8.6**). Therefore, the commutativity of $|\Sigma|$ implies that of Σ.

d) Similarly, classical commutative rings R correspond to commutative additive monads Σ that admit a *symmetry* $[-]$, i.e. a unary operation $- \in \Sigma(1)$, such that $\boldsymbol{e} + (-\boldsymbol{e}) = 0$ (this automatically implies $-^2 = \boldsymbol{e}$; cf. **4.8.7**). Notice that if these properties hold in Σ (i.e. Σ admits a pseudoaddition and a symmetry), and if we have any homomorphism of commutative monads $\rho : \Sigma \to \Xi$, they hold in Ξ as well, $\rho(+)$ and $\rho(-)$ being a pseudoaddition and a symmetry in Ξ, hence if Σ is given by a classical ring, the same is true for Ξ. Of course, we obtain similar results for semirings as well.

e) In general a commutative algebraic monad need not be additive. For example, $\mathbb{Z}_\infty \subset \mathbb{R}$, as well as $\mathbb{F}_\varnothing \subset \mathbb{F}_1 \subset \mathbb{F}_{\pm 1} \subset \mathbb{Z}_{(\infty)} \subset \mathbb{Q}$ are examples of hypoadditive commutative monads that are not additive (they are commutative, being submonads of commutative monad $\mathbb{R}$). In fact, a commutative algebraic monad needn't be even hypoadditive, as shown by $\mathbb{F}_\infty$ (cf. **4.8.13**; notice that $\mathbb{F}_\infty$ is commutative, being a strict quotient of $\mathbb{Z}_\infty$).

5.1.9. (Algebras over a generalized ring.) Let Λ be a generalized ring, i.e. a commutative algebraic monad. A *(commutative)* Λ*-algebra* Σ is by definition a homomorphism of generalized rings $\rho : \Lambda \to \Sigma$. We denote the category of (commutative) Λ-algebras by Λ-*CommAlg*. Clearly, arbitrary projective and inductive limits exist in this category; for example, coproducts of Λ-algebras are given by pushouts of *GenR*, i.e. the "tensor products" $\Sigma_1 \otimes_\Lambda \Sigma_2$ (cf. **5.1.6**).

Similarly, we define a *(not necessarily commutative)* Λ*-algebra* Σ to be a *central* homomorphism of algebraic monads $\rho : \Lambda \to \Sigma$, i.e. we require $\rho(\Lambda)$ to lie in the center of Σ. Clearly, if Λ is not commutative, any such ρ factorizes through Λ_{ab}, so it doesn't really make much sense to allow Λ to be non-commutative. Therefore, we shall consider algebras only over commutative monads, i.e. generalized rings Λ. We denote the category of (all) Λ-algebras by Λ-*Alg*. Notice that this category has arbitrary projective and inductive limits: all projective and filtered inductive limits are computed in the usual way (i.e. componentwise), $\mathrm{id} : \Lambda \to \Lambda$ is the initial object, and the pushouts are given by the non-commutative tensor products $\Sigma_1 \boxtimes_\Sigma \Sigma_2$ of **4.5.17**. We can also consider the commutative tensor products $\Sigma_1 \otimes_\Sigma \Sigma_2 := (\Sigma_1 \boxtimes_\Sigma \Sigma_2)/[\Sigma_1, \Sigma_2]$; they also lie in Λ-*Alg*, being strict quotients of corresponding non-commutative tensor products, and they have a universal property with respect to all Λ-algebras Ξ that fit into a commutative square with given

$\Sigma \to \Sigma_1$ and $\Sigma \to \Sigma_2$, and such that the images of Σ_1 and Σ_2 in Ξ commute.

5.1.10. (Algebras over an additive generalized ring.) If Λ is additive, i.e. given by a commutative semiring $|\Lambda|$, then any Λ-algebra Σ is additive as well: indeed, the structural homomorphism $\rho : \Lambda \to \Sigma$ has to be central, hence $\rho(+)$ is a central pseudoaddition of Σ, and we have seen in **5.1.8**,b) that this implies additivity of Σ, so Σ is given by a semiring $|\Sigma|$ and a central homomorphism $|\rho| : |\Lambda| \to |\Sigma|$. Similarly, if Λ is additive and admits a symmetry $-$, i.e. if Λ corresponds to a classical commutative ring $|\Lambda|$, then any Λ-algebra Σ admits a central pseudoaddition and a symmetry as well, hence it is given by a classical ring $|\Sigma|$ and a central homomorphism $|\rho| : |\Lambda| \to |\Sigma|$, i.e. a classical $|\Lambda|$-algebra. This explains our terminology.

Here is another result of the same kind: if Λ is a generalized ring with zero, then any Λ-algebra Σ is also a monad with zero. Indeed, Σ has at least one constant, namely, $\rho(0)$; and this constant is central, so any other constant must be equal to it (cf. **5.1.7**,a)).

5.1.11. (Scalar restriction and base change.) If $\sigma : \Lambda \to \Lambda'$ is a homomorphism of generalized rings, we can define the *scalar restriction* of a Λ'-algebra $\rho' : \Lambda' \to \Sigma'$ by considering $\rho' \circ \sigma : \Lambda \to \Sigma'$. We denote this Λ-algebra by ${}^a\sigma_*(\Sigma')$, $\sigma_*\Sigma'$ or ${}_\sigma\Sigma'$. This scalar restriction functor has an obvious left adjoint — the *base change functor* σ^*, given by the commutative tensor product: $\sigma^*\Sigma := \Lambda' \otimes_\Lambda \Sigma$. These considerations are valid both for the categories of all (non-commutative) algebras and for the categories of commutative algebras.

5.1.12. (Central elements.) We see directly from **5.1.1** that $t \in \Sigma(n)$ is a central element iff the maps $[t]_X : X^n \to X$ are Σ-homomorphisms for all Σ-modules X (or just for all $X = \Sigma(m)$). Another description: the center of Σ is the largest submonad $\Sigma_0 \subset \Sigma$, such that for all $t \in \Sigma(n)$ and all Σ-modules X (or just for all $X = \Sigma(m)$) the maps $[t]_X : X^n \to X$ are Σ_0-homomorphisms. In particular, Σ is commutative iff all maps $[t]_X : X^n \to X$ are Σ-homomorphisms, and $\rho : \Lambda \to \Sigma$ is central iff $[t]_X : X^n \to X$ is a Λ-homomorphism for any $t \in \Sigma(n)$ and any Σ-module X (again, $X = \Sigma(m)$ suffice).

We can generalize this a bit: the actions of two submonads $\Sigma_1, \Sigma_2 \subset \Xi$ commute on a Ξ-module X (i.e. any two operations $t \in \|\Sigma_1\|$, $t' \in \|\Sigma_2\|$ commute on X) iff for any $t \in \Sigma_1(n)$ the map $[t]_X : X^n \to X$ is a Σ_2-homomorphism. If this is true for any X (or for any $X = \Xi(m)$), then Σ_1 and Σ_2 commute inside Ξ.

We can apply this to $\Xi = \Sigma_1 \boxtimes_\Lambda \Sigma_2$, where Σ_1 and Σ_2 are Λ-algebras. We see that a Ξ-module X is just a set X with a Σ_1 and a Σ_2-structure, restricting to same Λ-structure (this actually follows from the universal property of Ξ, applied to $\Xi \to \mathrm{END}(X)$), and a $\Sigma_1 \otimes_\Lambda \Sigma_2$-module X is a set X with two *commuting* Σ_1 and Σ_2-structures, restricting to the same Λ-structure, i.e. we require Σ_1 to act by Σ_2-homomorphisms on X, or equivalently, Σ_2 to act by Σ_1-homomorphisms.

5.1.13. (Generators and relations.) Let's fix a generalized ring Λ. Then for any graded set $S = \bigsqcup_{n\geq 0} S_n$ we can construct the free (non-commutative) Λ-algebra $\Lambda\{S\}$ (resp. the free commutative Λ-algebra $\Lambda[S]$), enjoying the usual universal property with respect to non-commutative (resp. commutative) Λ-algebras Σ and graded maps $S \to \|\Sigma\|$. Indeed, we just have to put $\Lambda\{S\} := \Lambda\langle S \,|\, [S,\Lambda]\rangle = \Lambda\langle S\rangle/\langle[S,\Lambda]\rangle$ (cf. **4.5.11**), resp. $\Lambda[S] := \Lambda\langle S \,|\, [S,\Lambda],[S,S]\rangle$. If all operations from S are unary, then $\Lambda\{S\}$ is "the ring of polynomials in non-commuting variables from S" (that are nevertheless assumed to commute with constants from Λ), and $\Lambda[S]$ is the ring of polynomials in commuting variables from S.

Suppose that we are also given a system of "equations" or "relations" $E \subset \|\Lambda\{S\}\|^2$ (resp. $E \subset \|\Lambda[S]\|^2$); since $\Lambda\langle S\rangle \to \Lambda\{S\}$ (resp. $\cdots \to \Lambda[S]$) is surjective, we can assume that E comes from a system of relations $\tilde{E} \subset \|\Lambda\langle S\rangle\|^2$, which will be usually denoted by the same letter E. Then we can construct a (non-commutative) algebra $\Lambda\{S|E\} = \Lambda\{S\}/\langle E\rangle = \Lambda\langle S \,|\, \tilde{E}, [\Lambda,S]\rangle$, resp. a commutative algebra $\Lambda[S|E] = \Lambda[S]/\langle E\rangle = \Lambda\langle S \,|\, \tilde{E}, [\Lambda,S], [S,S]\rangle = \Lambda\{S \,|\, E, [S,S]\}$, having the usual universal property. When a non-commutative (resp. commutative) Λ-algebra Σ is represented in form $\Lambda\{S|E\}$ (resp. $\Lambda[S|E]$), we say that (S,E) *is a presentation of the non-commutative (resp. commutative)* Λ*-algebra* Σ.

5.1.14. (Finitely generated and pre-unary algebras.) We say that a subset $S \subset \|\Sigma\|$ *generates the non-commutative (resp. commutative)* Λ*-algebra* Σ if the induced homomorphism of Λ-algebras $\Lambda\{S\} \to \Sigma$ (resp. $\Lambda[S] \to \Sigma$) is surjective, i.e. a strict epimorphism. Since $\Lambda\langle S\rangle \to \Lambda\{S\} \to \Lambda[S]$ are also strict epimorphisms, we see that this condition is equivalent to the surjectivity of $\Lambda\langle S\rangle \to \Sigma$, i.e. to S being a set of generators of Σ over Λ in the sense of **4.4.4**. In particular, S is a set of generators of a commutative Λ-algebra Σ iff it generates Σ as a non-commutative algebra. This means that we can safely transfer all terminology from **4.4.4** and **4.4.5**; we obtain the notions of finitely generated Λ-algebras (or Λ-algebras of finite type), pre-unary Λ-algebras, and Λ-algebras, generated in arity $\leq n$, both in the commutative and non-commutative case.

5.1.15. (Finitely presented and unary algebras.) We can go further on and define the finitely presented and unary Λ-algebras to be those (commutative or non-commutative) Λ-algebras that admit a finite presentation (S,E) (resp. admit a unary presentation (S,E), i.e. a presentation with all operations from S and all relations from E unary). Relation $\Lambda[S|E] = \Lambda\{S \,|\, E,$

$[S,S]\rangle$ shows that a commutative Λ-algebra is finitely presented iff it is finitely presented as a non-commutative algebra, and similarly for unarity (note that $[u,u']$ is a unary relation when both u and u' are unary). However, these notions of finite presentation and unarity do not coincide with those of **4.5.15**; henceforth *finite presentation and unarity of algebras over generalized rings will be understood as just defined.* Nevertheless, they have essentially the same properties as listed before in **4.5.18**, when they are properly understood. For example, in the commutative case we have to replace the NC-tensor product $\boxtimes$ with its commutative counterpart $\otimes$, and in the non-commutative case we compute base change of a Λ-algebra Σ only with respect to homomorphisms of generalized rings $\Lambda \to \Lambda'$; such base change functors preserve finite generation and presentation, unarity and pre-unarity, due to the usual property $\Lambda' \otimes_\Lambda \Lambda\{S|E\} = \Lambda'\{S|E\}$ and its commutative version. Another consequence is that if a Λ'-algebra Σ' is finitely presented (resp. unary) over Λ, and Λ' is finitely generated (resp. pre-unary) over Λ, then Σ' is finitely presented (resp. unary) over Λ'.

5.1.16. (Examples.) Presentations of Λ-algebras (commutative or not) defined in **5.1.13** tend to be shorter than those of **4.5.14**, since some commutativity relations need not be written explicitly. Let's consider some examples.

a) We have $\mathbb{F}_1 = \mathbb{F}_\varnothing[0^{[0]}] = \mathbb{F}_\varnothing\{0^{[0]}\}$ and $\mathbb{F}_{\pm 1} = \mathbb{F}_1[-^{[1]} \,|\, -^2 = \boldsymbol{e}] = \mathbb{F}_1\{-^{[1]} \,|\, -^2 = \boldsymbol{e}\}$, and in general $\mathbb{F}_{1^n} = \mathbb{F}_1[\zeta_n^{[1]} \,|\, \zeta_n^n = \boldsymbol{e}]$ (notice that we don't have to require $\zeta_n 0 = 0$ explicitly, since it is a consequence of $[\zeta_n, 0]$).

b) For any n, $m \geq 1$ we have a canonical embedding $\mathbb{F}_{1^n} \to \mathbb{F}_{1^{mn}}$, given by $\zeta_n \mapsto \zeta_{mn}^m$; taking the (filtered) inductive limit along all such maps we obtain a new generalized ring $\mathbb{F}_{1^\infty} = \varinjlim \mathbb{F}_{1^n}$. Clearly, $\mathbb{F}_{1^\infty} = \mathbb{F}_1[\zeta_1, \zeta_2, \ldots \,|\, \zeta_1 = \boldsymbol{e}, \zeta_n = \zeta_{nm}^m]$.

c) Similarly, for any n, $m \geq 1$ we have a canonical surjection $\mathbb{F}_{1^{nm}} \to \mathbb{F}_{1^n}$, given by $\zeta_{nm} \mapsto \zeta_n$. We can compute the corresponding projective limit $\mathbb{F}_{1^{-\infty}} = \varprojlim \mathbb{F}_{1^n}$, and we have a special unary operation $\zeta \in \mathbb{F}_{1^{-\infty}}(1)$, defined by the projective system of the ζ_n. However, ζ doesn't generate this generalized ring over $\mathbb{F}_1$, and in fact $\mathbb{F}_1[\zeta] \subset \mathbb{F}_{1^{-\infty}}$ is isomorphic to the algebra of polynomials $\mathbb{F}_1[T^{[1]}]$.

d) Now we can write down shorter presentations for $\mathbb{Z}_{\geq 0}$ and $\mathbb{Z}$: we have $\mathbb{Z}_{\geq 0} = \mathbb{F}_1[+^{[2]} \,|\, 0 + \boldsymbol{e} = \boldsymbol{e} = \boldsymbol{e} + 0] = \mathbb{F}_\varnothing[0^{[0]}, +^{[2]} \,|\, 0 + \boldsymbol{e} = \boldsymbol{e} = \boldsymbol{e} + 0]$, and $\mathbb{Z} = \mathbb{Z}_{\geq 0}[-^{[1]} \,|\, (-\boldsymbol{e}) + \boldsymbol{e} = 0]$.

e) We also have $\mathbb{F}_\infty = \mathbb{F}_{\pm 1}\bigl[*^{[2]} \,|\, \boldsymbol{e} * (-\boldsymbol{e}) = 0,\ \boldsymbol{e} * \boldsymbol{e} = \boldsymbol{e},\ x * y = y * x,\ (x * y) * z = x * (y * z)\bigr]$; the remaining relations of **4.8.14** follow from commutativity.

5.1.17. (Some tensor products.) Since $\Lambda[S|E] \otimes_\Lambda \Lambda[S'|E'] = \Lambda[S, S'|E, E']$, we can use our results on commutative algebras and their generators to compute some tensor products, i.e. some pushouts in *GenR*. For example, $\mathbb{F}_1 = \mathbb{F}_\varnothing[0^{[0]}]$, hence $\mathbb{F}_1 \otimes_{\mathbb{F}_\varnothing} \mathbb{F}_1 \cong \mathbb{F}_\varnothing[0^{[0]}, 0'^{[0]}] \cong \mathbb{F}_\varnothing[0^{[0]}] = \mathbb{F}_1$, since the commutativity relation $[0, 0']$ implies $0 = 0'$ by **5.1.7**,a). Since the coproduct $\mathbb{F}_1 \otimes \mathbb{F}_1$ is isomorphic to $\mathbb{F}_1$ itself, we see that $\mathbb{F}_\varnothing \to \mathbb{F}_1$ is an epimorphism in *GenR*, but of course not an epimorphism in the category of all algebraic monads, since $\mathbb{F}_1 \boxtimes_{\mathbb{F}_\varnothing} \mathbb{F}_1 = \mathbb{F}_\varnothing\langle 0^{[0]}, 0'^{[0]}\rangle$, and this is definitely not isomorphic to $\mathbb{F}_1$ (in fact, the category of modules over this NC-tensor product is the category of sets with two marked points).

5.1.18. (NC-epimorphisms.) We see that a homomorphism $\rho : \Lambda \to \Sigma$ in *GenR* can be an epimorphism in *GenR* without being an epimorphism in the category of all algebraic monads. To distinguish between these two situations we say that a homomorphism of generalized rings $\rho : \Lambda \to \Sigma$ is an *NC-epimorphism* if it is an epimorphism in the category of all algebraic monads. Of course, any NC-epimorphism is automatically an epimorphism (in *GenR*). Applying the definition to homomorphisms $\Sigma \to \mathrm{END}(X)$ we see that ρ is an NC-epimorphism iff any Λ-module X admits at most one compatible Σ-structure (this is the behavior one expects from localization morphisms like $\Lambda \to \Lambda[S^{-1}]$, and in the next chapter we'll see that this expectation turns out to be well-founded). To show the sufficience of this condition we observe that the category of $\Sigma \boxtimes_\Lambda \Sigma$-modules is the category of sets, equipped with two different Σ-structures, restricting to the same Λ-structure; if any two such Σ-structures necessarily coincide, we have $(\Sigma \boxtimes_\Lambda \Sigma)$-*Mod* $\cong \Sigma$-*Mod*, hence $\Sigma \boxtimes_\Lambda \Sigma \cong \Sigma$ (any monad is completely determined by its category of modules, together with the forgetful functor).

Notice that while epimorphisms in *GenR* and in the category of algebraic monads are different, monomorphisms and strict epimorphisms coincide, so we can freely use these notions in both contexts, as well as notions of submonads and strict quotients, without any risk of confusion.

5.1.19. (Semirings and $\mathbb{Z}_{\geq 0}$-algebras.) We have seen in **5.1.10** that any $\mathbb{Z}_{\geq 0}$-algebra Σ admits a central pseudoaddition and a central constant, hence it is additive. Conversely, if Σ is an additive algebraic monad, it has exactly one constant 0, so this constant is automatically central, and its addition $+$ is also automatically central, a statement easily checked by applying the comparison map π_{2n} to both sides of the commutativity relation $[+, t]$ for an arbitrary $t \in \Sigma(n)$. We know that $\mathbb{Z}_{\geq 0} = \mathbb{F}_\varnothing[0^{[0]}, +^{[2]} \,|\, 0 + \boldsymbol{e} = \boldsymbol{e} = \boldsymbol{e} + 0]$, so we obtain a unique central homomorphism $\mathbb{Z}_{\geq 0} \to \Sigma$, i.e. any additive algebraic monad Σ admits a unique $\mathbb{Z}_{\geq 0}$-algebra structure. On the other hand, the list of generators and relations given in **4.8.6** shows that any such Σ is generated as a (non-commutative) $\mathbb{Z}_{\geq 0}$-algebra by the set of its unary operations $|\Sigma|$, subject to unary relations of form $\lambda' + \lambda'' = \lambda$ and $\lambda'\lambda'' = \lambda$;

all other relations of *loc.cit.* (associativity, commutativity and distributivity) are either consequences of similar relations in $\mathbb{Z}_{\geq 0}$, or consequences of implied commutativity relations between $\mathbb{Z}_{\geq 0}$ and the generators from $|\Sigma|$ (cf. **5.1.7**). Therefore, any such Σ is a unary $\mathbb{Z}_{\geq 0}$-algebra. We have just proved the following statement:

Proposition. *The following categories are canonically isomorphic: a) the category of additive algebraic monads; b) the category of (non-commutative) $\mathbb{Z}_{\geq 0}$-algebras; c) the category of (non-commutative) unary $\mathbb{Z}_{\geq 0}$-algebras.*

5.1.20. (Rings and $\mathbb{Z}$-algebras.) Of course, the above equivalence of categories establishes an equivalence between the category of (unary) commutative $\mathbb{Z}_{\geq 0}$-algebras and the category of additive generalized rings. Another consequence: *If Λ is an additive generalized ring, any Λ-algebra Σ is automatically additive and unary (over Λ).* Indeed, $\mathbb{Z}_{\geq 0} \to \Lambda \to \Sigma$ defines a $\mathbb{Z}_{\geq 0}$-algebra structure on Σ, hence Σ is additive and unary over $\mathbb{Z}_{\geq 0}$, hence it is unary over Λ as well, Λ being pre-unary over $\mathbb{Z}_{\geq 0}$ (cf. **5.1.15**). This is applicable to $\mathbb{Z} = \mathbb{Z}_{\geq 0}[-^{[1]} \,|\, (-\boldsymbol{e}) + \boldsymbol{e} = 0]$: we see that all $\mathbb{Z}$-algebras are additive and unary (over $\mathbb{Z}$), and that the category of $\mathbb{Z}$-algebras consists of those $\mathbb{Z}_{\geq 0}$-algebras, i.e. additive monads, that admit a symmetry $[-]$; clearly, this is just the category of classical semirings with symmetry, i.e. of classical rings. Similarly, any algebra Σ over a commutative $\mathbb{Z}$-algebra Λ, corresponding to a commutative classical ring $|\Lambda|$, is automatically additive and unary over Λ, hence given by classical ring $|\Sigma|$ and a central homomorphism $|\Lambda| \to| \Sigma|$, i.e. a $|\Lambda|$-algebra in the classical sense. This means that when we work over a classical base ring Λ, all arising algebras turn out to be unary, additive and classical, so we don't obtain anything new over such Λ. This also explains why we cannot observe the importance of unarity in the classical case, since in this case it turns out to be always fulfilled.

5.1.21. (Some examples: affine group schemes.) Consider, for example, the functor $M_n : \Sigma \mapsto M(n,n;\Sigma) = \Sigma(n)^n$ on the category of commutative $\mathbb{F}_\varnothing$-algebras, i.e. generalized rings. It is easily seen to be representable by a certain commutative $\mathbb{F}_\varnothing$-algebra A_n, namely, $A_n := \mathbb{F}_\varnothing[T_1^{[n]}, \ldots, T_n^{[n]}]$. Of course, A_n is not unary over $\mathbb{F}_\varnothing$ for $n \geq 2$ (indeed, if A_2 would be unary, its universal property would enable us to express any central binary operation in terms of some unary operations, hence any generalized ring generated in arity ≤ 2 would be also generated in arity ≤ 1; however, this is not true for $\mathbb{Z}$). But if we extend the base to $\mathbb{Z}$, we obtain a commutative $\mathbb{Z}$-algebra $A_{n,\mathbb{Z}} = \mathbb{Z} \otimes A_n = \mathbb{Z} \otimes_{\mathbb{F}_\varnothing} A_n$, representing the restriction of the same functor to the category of commutative $\mathbb{Z}$-algebras; of course, this algebra is additive and unary, and in fact it equals $\mathbb{Z}[T_{11}, T_{12}, \ldots, T_{nn}]$; so a non-unary algebra becomes unary over $\mathbb{Z}$.

Another example is given by $GL_n : \Sigma \mapsto GL_n(\Sigma) \subset M_n(\Sigma)$. It is representable by $A'_n = \mathbb{F}_\varnothing\bigl[T_1^{[n]}, \ldots, T_n^{[n]}, U_1^{[n]}, \ldots, U_n^{[n]} \bigm| U_i(T_1, \ldots, T_n) = \{i\}_n = T_i(U_1, \ldots, U_n)\bigr]$. Again, A'_n is not unary over $\mathbb{F}_\varnothing$ for $n \geq 2$, but it becomes unary over $\mathbb{Z}$; of course, $A'_{n,\mathbb{Z}}$ is the usual Hopf algebra of $GL_{n,\mathbb{Z}}$.

Proceeding in this way one can show that the only unary affine group schemes defined over $\mathbb{F}_\varnothing$ are the (split) diagonal groups $D_{\mathbb{F}_\varnothing}(A)$, parametrized by abelian groups A, e.g. the split tori $\mathbb{G}_m^n \cong D(\mathbb{Z}^n)$.

5.1.22. (Tensor square of $\mathbb{Z}$.) We have seen in **5.1.20** that for any generalized ring Σ there is at most one homomorphism $\mathbb{Z} \to \Sigma$. By definition this is equivalent to saying that $\mathbb{F}_\varnothing \to \mathbb{Z}$ is an epimorphism in *GenR*, or to $\mathbb{Z} \otimes_{\mathbb{F}_\varnothing} \mathbb{Z} \cong \mathbb{Z}$. This statement can be also shown directly starting from the presentation of $\mathbb{Z} \otimes \mathbb{Z}$ obtained from the presentation $\mathbb{Z} = \mathbb{F}_\varnothing[0^{[0]}, -^{[1]}, +^{[2]} \mid \boldsymbol{e} + 0 = \boldsymbol{e} = 0 + \boldsymbol{e}, (-\boldsymbol{e}) + \boldsymbol{e} = 0]$: we see that $\mathbb{Z} \otimes \mathbb{Z}$ is generated by two constants 0 and $0'$, two binary operations $+$ and $+'$, and two unary operations $-$ and $-'$, and we use **5.1.7** to show first $0 = 0'$, then $[+] = [+']$, and finally deduce $[-] = [-']$, hence $\mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}$. Yet another proof: use the well-known lemma which states that any two commuting group structures on the same set H coincide and are abelian (cf. e.g. SGA 3 II 3.10).

Of course, once we know that $\mathbb{F}_\varnothing \to \mathbb{Z}$ is an epimorphism, we deduce that $\mathbb{F}_1 \to \mathbb{Z}$ and $\mathbb{F}_{\pm 1} \to \mathbb{Z}$ are epimorphisms as well, and we can compose them with strict epimorphisms $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, thus obtaining statements like $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{F}_1} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/n\mathbb{Z}$. This is a phenomenon of commutativity: one easily checks that a countable set admits different $\mathbb{F}_p$-vector space structures, even if we fix in advance the zero and the symmetry, hence $\mathbb{F}_p \boxtimes_{\mathbb{F}_{\pm 1}} \mathbb{F}_p \ncong \mathbb{F}_p$.

Another important consequence is that for any two $\mathbb{Z}$-algebras Σ and Σ' (commutative or not) we have $\Sigma \otimes_{\mathbb{Z}} \Sigma' \cong \Sigma \otimes_{\mathbb{F}_\varnothing} \Sigma'$, so we can freely use the usual notation $\Sigma \otimes \Sigma'$ for such tensor products.

5.2. (Topos case.) Now we want to indicate a way to transfer all our previous results to the topos case. Everything here turns out to be quite similar to the classical case.

5.2.1. (Commutativity diagrams.) Notice that the commutativily of an algebraic monad Σ is equivalent to the commutativity of the following diagrams for all k, n, $m \geq 0$:

$$\begin{tikzcd} \Sigma(n) \times \Sigma(m) \times \Sigma(k)^{\mathbf{n \times m}} \arrow[r] \arrow[d] & \Sigma(n) \times \Sigma(k)^n \arrow[d, "\mu_k^{(n)}"] \\ \Sigma(m) \times \Sigma(k)^m \arrow[r, "\mu_k^{(m)}"] & \Sigma(k) \end{tikzcd} \tag{5.2.1.1}$$

Of course, the two remaining arrows are also given by products of $\mu_k^{(n)}$, $\mu_k^{(m)}$, so the commutativity of this diagram makes sense also if we are given an "algebraic monad" Σ over an arbitrary cartesian category $\mathcal{C}$ in the sense of **4.9.1**. We thus obtain the notion of a "commutative algebraic monad" or a "generalized ring" over such $\mathcal{C}$, and this commutativity property is stable under any left exact functors $h : \mathcal{C} \to \mathcal{C}'$, i.e. if Σ is commutative, the same is true for $h\Sigma$. We define the notion of a central homomorphism $\rho : \Lambda \to \Sigma$ by means of similar diagrams (comparing two morphisms from $\Lambda(n) \times \Sigma(m) \times \Sigma(k)^{\mathbf{m}\times\mathbf{n}}$ into $\Sigma(k)$), so we obtain the notion of a Λ-algebra (commutative or not) as well. These notions have essentially the same properties as before: we can either repeat the proofs or embed $\mathcal{C}$ into the category of presheaves $\hat{\mathcal{C}}$ as a full subcategory, and prove all statements for $\hat{\mathcal{C}}$ componentwise.

5.2.2. (Topos case.) Of course, we immediately obtain the notion of a commutative algebraic monad or a generalized ring in a topos $\mathcal{E}$, i.e. the notion of a sheaf of generalized rings on a site $\mathcal{S}$, if we consider $\mathcal{E} = \tilde{\mathcal{S}}$. Since the pullback, direct image and global section functors are left exact, they preserve commutativity of algebraic monads. Conversely, if Σ is an algebraic monad over $\mathcal{E}$, such that $\Sigma_{[U]} = \Gamma(U, \Sigma)$ is commutative for all U from a system of generating objects of $\mathcal{E}$, then Σ is commutative. If we are given a homomorphism $\rho : \Lambda \to \Lambda'$ of generalized rings in $\mathcal{E}$, we obtain a scalar restriction functor $\rho_* : \mathcal{E}^{\Lambda'} \to \mathcal{E}^{\Lambda}$; as usual, it admits a left adjoint, the base change functor ρ^*, denoted also by $\Lambda' \otimes_\Lambda -$; one can deduce its existence from the presheaf case by applying the "sheafification" functor $a : \hat{\mathcal{S}} \to \tilde{\mathcal{S}}$.

5.2.3. (Generalized ringed spaces and topoi.) This means that we can define a *generalized ringed topos* $\mathcal{E} = (\mathcal{E}, \Lambda)$ as a topos $\mathcal{E}$, together with a generalized ring Λ in $\mathcal{E}$ (the "structural sheaf" of $\mathcal{E}$). A *morphism* $f : (\mathcal{E}', \Lambda') \to (\mathcal{E}, \Lambda)$ is defined in the usual way: $f = (\varphi, \theta)$, with $\varphi : \mathcal{E}' \to \mathcal{E}$ a morphism of topoi, and $\theta : \Lambda \to \varphi_*\Lambda'$ a homomorphism of generalized rings. We obtain a pair of adjoint functors $f^* : \mathcal{E}^{\Lambda} \to \mathcal{E}'^{\Lambda'}$ and $f_* : \mathcal{E}'^{\Lambda'} \to \mathcal{E}^{\Lambda}$ in the usual way: f_* by composing $\varphi_*^{\Lambda'} : \mathcal{E}'^{\Lambda'} \to \mathcal{E}^{\varphi_*\Lambda'}$ with scalar restriction with respect to θ, and f^* by composing $\varphi^{*,\Lambda} : \mathcal{E}^{\Lambda} \to \mathcal{E}'^{\varphi^*\Lambda}$ with the base change functor $\Lambda' \otimes_{\varphi^*\Lambda} -$ with respect to $\theta^\sharp : \varphi^*\Lambda \to \Lambda'$. Sometimes we denote the underlying morphism of topoi φ by the same letter f; then we write f^{-1} for the pullback functor φ^* of sheaves of sets, and f^* for the functor just defined.

In this way we obtain the 2-category of generalized ringed topoi. Of course, we can consider topoi defined by some sites, thus obtaining the notions of generalized ringed sites and their morphisms. In particular, we can consider topoi defined by topological spaces; then we obtain the category of *generalized ringed spaces.*

We postpone until the next chapter the definitions of generalized locally ringed spaces and topoi, since they depend considerably on the choice of *theory of spectra* (or *localization theory*): $\mathcal{E} = (\mathcal{E}, A)$ is locally ringed iff for any $X \in \operatorname{Ob}\mathcal{E}$ the morphism of ringed topoi $(\mathcal{E}_{/X}, A_{/X}) \to (\mathit{Sets}, A(X))$ factorizes through $\operatorname{Spec} A(X)$.

5.2.4. (Commutativity in terms of plain inner structures.) Recall that an algebraic monad Σ over a topos $\mathcal{E}$ admits also a description in terms of certain plain inner monads (cf. **4.9.11**), so we get a monad $\Sigma : \mathcal{E} \to \mathcal{E}$ over $\mathcal{E}$, and a canonical plain inner structure α, $\alpha_{X,Y} : X \times \Sigma(Y) \to \Sigma(X \times Y)$. We can construct a diagonal inner structure ρ on endofunctor Σ, $\rho_{X,Y} : \Sigma(X) \times \Sigma(Y) \to \Sigma(X \times Y)$, and then ask whether this $\rho_{X,Y}$ is commutative (cf. **3.5.17**); one knows that in this case ρ is even compatible with the monad structure of Σ *(loc. cit.)* It turns out that *ρ is commutative iff Σ is commutative in the sense of* **5.2.1**. One checks this directly for $\mathcal{E} = \mathit{Sets}$, using the explicit description of the plain inner structure maps $\alpha_{X,Y} : X \times \Sigma(Y) \to \Sigma(X \times Y)$ and the construction of ρ from α, and then generalizes this result first to the categories of presheaves $\hat{\mathcal{S}}$, and then by sheafification to arbitrary $\mathcal{E} \cong \tilde{\mathcal{S}}$.

5.3. (Modules over generalized rings.) Now we want to study some properties of the category Λ-*Mod* of modules over a generalized ring Λ; sometimes we also need another generalized ring Λ' and a homomorphism $\rho : \Lambda \to \Lambda'$. Of course, all general properties obtained in **4.6** hold in the commutative case as well, e.g. Λ-*Mod* has arbitrary inductive and projective limits and so on. We want to study properties specific to the commutative case. The most important among them is the existence of a canonical ACU $\otimes$-structure on Λ-*Mod*, and of inner Homs with respect to this tensor product. This implies a theory of bilinear and multilinear forms, which admit a more direct description as well, and in particular of symmetric forms; we postpone the study of alternating forms, since it requires some additional structure on Λ. We can also use this $\otimes$-structure to consider the category of algebras (commutative or all) in Λ-*Mod*; this category turns out to be equivalent to the category of unary Λ-algebras.

5.3.1. (Σ-structure on $\operatorname{Hom}_\Sigma(N, P)$.) Recall that for any algebraic monad Σ, any Σ-module P and any set N we have a canonical Σ-structure on the set $\operatorname{Hom}(N, P) = \operatorname{Hom}_{\mathit{Sets}}(N, P) \cong P^N$, namely, the product structure; in other words, operations of Σ are applied to functions $N \to P$ pointwise (cf. **4.6.3**). Now suppose that N is also a Σ-module; then we have a subset $\operatorname{Hom}_\Sigma(N, P) \subset \operatorname{Hom}(N, P) \cong P^N$, in general not stable under all operations of Σ. We claim the following:

Proposition. *An algebraic monad Σ is commutative iff* $\operatorname{Hom}_\Sigma(N, P)$ *is a Σ-submodule of* $\operatorname{Hom}_{\mathit{Sets}}(N, P)$ *for any two Σ-modules N and P (or just* *for N and P free of finite rank, i.e. $\cong \Sigma(k)$). If this be the case, we denote by* $\mathbf{Hom}_\Sigma(N,P)$ *the set* $\mathrm{Hom}_\Sigma(N,P)$*, endowed with the Σ-structure induced from* $\mathrm{Hom}(N,P) \supset \mathrm{Hom}_\Sigma(N,P)$.

We deduce this statement from its more precise form:

Proposition 5.3.2 *An operation $t \in \Sigma(n)$ is central iff* $\mathrm{Hom}_\Sigma(N,P) \subset \mathrm{Hom}(N,P)$ *is stable under t for all Σ-modules N and P (or just for N and P free of finite rank). This condition is true for fixed t and P and variable N iff $t_P : P^n \to P$ is a Σ-homomorphism.*

Proof. Put $H := \mathrm{Hom}(N,P) \cong P^N$, $H' := \mathrm{Hom}_\Sigma(N,P) \subset H$. Then by definition $t_H : \mathrm{Hom}(N,P)^n \cong \mathrm{Hom}(N,P^n) \to \mathrm{Hom}(N,P)$ is given by composing maps $N \to P^n$ with $t_P : P^n \to P$. Now $H'^n \subset H^n$ is identified with $\mathrm{Hom}_\Sigma(N,P^n) \subset \mathrm{Hom}(N,P^n)$. Therefore, if $t_P : P^n \to P$ is a Σ-homomorphism, then t_H maps H'^n into $H' = \mathrm{Hom}_\Sigma(N,P)$, the composite of two Σ-homomorphisms being a Σ-homomorphism. Conversely, taking $N := P^n$ and considering the image under t_H of $\mathrm{id}_N \in \mathrm{Hom}_\Sigma(N,P^n) \cong H'^n$, equal to t_P, we see that the stability of $\mathrm{Hom}_\Sigma(P^n,P) \subset \mathrm{Hom}(P^n,P)$ under t implies that t_P is a Σ-homomorphism.

This proves the second statement of our proposition; notice that it would suffice to require the stability under t of $\mathrm{Hom}_\Sigma(N,P) \subset \mathrm{Hom}(N,P)$ not for all N, but only for all $N = \Sigma(k)$, $k \geq 0$: indeed, stability for such N's implies stability for arbitrary filtered inductive limits of these, in particular for all $\Sigma(S)$; now we can represent any N as a cokernel of a pair of morphisms $\Sigma(R) \rightrightarrows \Sigma(S)$ (cf. **3.3.20**), and a simple diagram chasing shows that the stability for $\Sigma(R)$ and $\Sigma(S)$ implies the stability for N.

Now the first statement follows from the second, in its stronger form given above, and from the following observation, already mentioned before on several occasions (cf. **5.1.12**): *t is central iff all $t_P : P^n \to P$ are Σ-homomorphisms iff all $t_{\Sigma(k)} : \Sigma(k)^n \to \Sigma(k)$ are Σ-homomorphisms.*

Definition 5.3.3 *(Bilinear forms.) Given three Λ-modules M, N and P over a generalized ring Λ and a map $\Phi : M \times N \to P$, we say that Φ* is Λ-bilinear *if for any $x \in M$ the map $s_\Phi(x) : N \to P$, $y \mapsto \Phi(x,y)$, is a Λ-homomorphism, and for any $y \in N$ the map $d_\Phi(y) : M \to P$, $x \mapsto \Phi(x,y)$ is a Λ-homomorphism as well. We denote by* $\mathrm{Bilin}_\Lambda(M,N;P)$ *the set of all Λ-bilinear maps $M \times N \to P$.*

5.3.4. Notice that $\mathrm{Bilin}_\Lambda(M,N;P) \subset \mathrm{Hom}(M \times N, P) \cong P^{M\times N}$, and the latter set is canonically isomorphic to $\mathrm{Hom}(M, \mathrm{Hom}(N,P))$, and this isomorphism is clearly Λ-linear (i.e. a Λ-homomorphism). We see that this bijection transforms a $\Phi \in \mathrm{Hom}(M \times N, P)$ into $s_\Phi \in \mathrm{Hom}(M, \mathrm{Hom}(N, P))$. Now the first condition in the definition of bilinearity actually requires $s_\Phi \in \mathrm{Hom}(M, \mathrm{Hom}_\Lambda(N, P)) \subset \mathrm{Hom}(M, \mathrm{Hom}(N, P))$; on the other hand, we have a canonical Λ-structure on $\mathrm{Hom}(N, P) \cong P^N$, and the second condition for the bilinearity of Φ is easily seen to be equivalent to $s_\Phi \in \mathrm{Hom}_\Lambda(M, \mathrm{Hom}(N, P))$. Now we know that $\mathrm{Hom}_\Lambda(N, P)$ is a Λ-submodule of $\mathrm{Hom}(N, P)$, so we get $\mathrm{Bilin}_\Lambda(M, N; P) \cong \mathrm{Hom}(M, \mathrm{Hom}_\Lambda(N, P)) \cap \mathrm{Hom}_\Lambda(M, \mathrm{Hom}(N, P)) = \mathrm{Hom}_\Lambda(M, \mathbf{Hom}_\Lambda(N, P))$, as expected.

The latter set admits a canonical Λ-structure, so we obtain a Λ-module $\mathbf{Bilin}_\Lambda(M, N; P)$, canonically isomorphic to $\mathbf{Hom}_\Lambda(M, \mathbf{Hom}_\Lambda(N, P))$. This Λ-structure is actually induced by that of $P^{M \times N} \cong \mathrm{Hom}(M \times N, P) \supset \mathrm{Bilin}_\Lambda(M, N; P)$.

5.3.5. (Tensor products.) We have just proved that $\mathbf{Hom}_\Lambda(N, P)$ represents $\mathrm{Bilin}_\Lambda(-, N; P)$; what about the representability of $\mathrm{Bilin}_\Lambda(M, N; -)$?

Proposition. *For any Λ-modules M and N the functor* $\mathrm{Bilin}_\Lambda(M, N; -)$ *is representable by a Λ-module $M \otimes_\Lambda N$, called the* tensor product of M and N, *and a Λ-bilinear map $\otimes : M \times N \to M \otimes N$.*

Proof. We construct $M \otimes_\Lambda N$ in the usual way. Consider first the free Λ-module $L := \Lambda(M \times N)$; let's denote its basis elements $\{(x, y)\}$, $x \in M$, $y \in N$, simply by $\{x, y\}$. Now consider the set of relations $E \subset L \times L$, given by the union of the following two families of relations:

$$t\big(\{x_1, y\}, \ldots, \{x_n, y\}\big) \equiv \{t(x_1, \ldots, x_n), y\}$$
$$\text{for any } t \in \Lambda(n),\ x_1,\ \ldots,\ x_n \in M \text{ and } y \in N; \quad (5.3.5.1)$$

$$t\big(\{x, y_1\}, \ldots, \{x, y_n\}\big) \equiv \{x, t(y_1, \ldots, y_n)\}$$
$$\text{for any } t \in \Lambda(n),\ x \in M, \text{ and } y_1,\ \ldots,\ y_n \in N. \quad (5.3.5.2)$$

We consider the compatible equivalence relation $\equiv$ on L, generated by this set of relations, and put $M \otimes_\Lambda N := L/{\equiv}$. We have a canonical map $M \times N \to \Lambda(M \times N) = L \to M \otimes_\Lambda N$, and its bilinearity and universal property follow immediately from the construction.

5.3.6. (Multilinear maps and closed ACU $\otimes$-structure.) Of course, we can define Λ-multilinear maps $M_1 \times \cdots \times M_n \to N$ in a similar fashion, by requiring them to become Λ-linear (i.e. Λ-homomorphisms) after we fix arbitrary values for all arguments but one. The set of multilinear maps from the product of fixed Λ-modules M_i into a variable Λ-module N is easily seen to be representable by a *multiple tensor product* $M_1 \otimes_\Lambda M_2 \otimes_\Lambda \cdots \otimes_\Lambda M_n$.

Clearly, $(M_1 \otimes_\Lambda M_2) \otimes_\Lambda M_3 \cong M_1 \otimes_\Lambda M_2 \otimes_\Lambda M_3 \cong M_1 \otimes_\Lambda (M_2 \otimes_\Lambda M_3)$, so we get associativity isomorphisms for this tensor product on Λ-*Mod*, easily seen to fit into the pentagon diagram, since all vertices of such a diagram are canonically isomorphic to a quadruple tensor product. We have a canonical bijection $\mathrm{Bilin}_\Lambda(M_1, M_2; N) \xrightarrow{\sim} \mathrm{Bilin}_\Lambda(M_2, M_1; N)$, hence canonical isomorphisms $M_1 \otimes_\Lambda M_2 \cong M_2 \otimes_\Lambda M_1$, fitting into the hexagon diagram.

We have inner Hom's as well: $\mathrm{Hom}_\Lambda(M \otimes_\Lambda N, P) \cong \mathrm{Bilin}_\Lambda(M, N; P) \cong \mathrm{Hom}_\Lambda(M, \mathbf{Hom}_\Lambda(N, P))$. The free Λ-module $|\Lambda| = \Lambda(1)$ is easily seen to be the unit for this tensor product: indeed, $\mathrm{Hom}_\Lambda(\Lambda(1), P) \cong \mathrm{Hom}_{\mathit{Sets}}(\mathbf{1}, P) \cong P$, and this bijection is easily seen to preserve the Λ-structures, so we have $\mathbf{Hom}_\Lambda(\Lambda(1), P) \cong P$ in Λ-*Mod*, and $\mathrm{Hom}_\Lambda(M \otimes_\Lambda \Lambda(1), P) \cong \mathrm{Hom}_\Lambda(M, \mathbf{Hom}_\Lambda(\Lambda(1), P)) \cong \mathrm{Hom}_\Lambda(M, P)$ functorially in P, hence $M \otimes_\Lambda \Lambda(1) \cong M$ by Yoneda.

We have shown that Λ-*Mod admits a closed ACU $\otimes$-structure.* When Λ is given by a classical ring $|\Lambda|$, this structure coincides with that given by classical multilinear algebra. Notice for this that it is sufficient to check bilinearity for a set of generators of Λ, e.g. $0 \in \Lambda(0)$, $[+] \in \Lambda(2)$ and $|\Lambda|$; this shows equivalence of our definition of bilinearity with the classical one, and isomorphicity of our tensor products and inner Homs to their classical counterparts follows. On the other hand, for $\Lambda = \mathbb{Z}_\infty$ our new definitions are compatible with those given in **2.6.1** and **2.6.3** for $\mathbb{Z}_\infty$-*Lat*, and in **2.10** and **2.10.1** for $\mathbb{Z}_\infty$-*Fl.Mod*, both full subcategories of $\mathbb{Z}_\infty$-*Mod*. This gives a motivation for defining tensor product in Λ-*Mod* the way we have just done it.

5.3.7. (Basic properties of tensor products.) We have some simple properties of our tensor product and inner Hom on Λ-*Mod*; most of them follow formally from the fact that Λ-*Mod* is a closed ACU $\otimes$-category, with arbitrary inductive and projective limits. For example, $M \otimes_\Lambda N$ commutes with arbitrary inductive limits in each variable (since it is commutative and has a right adjoint $\mathbf{Hom}_\Lambda$), and $\mathbf{Hom}_\Lambda(N, P)$ commutes with inductive limits in N and projective limits in P. We also have $\Lambda(S) \otimes_\Lambda M \cong M^{(S)}$ for any set S and any Λ-module M, where $M^{(S)}$ denotes the sum of S copies of M (apply $\mathrm{Hom}_\Lambda(-, N)$ to both sides), hence also $M^{(S)} \otimes_\Lambda N \cong (M \otimes_\Lambda N)^{(S)}$ (associativity), $M^{(S)} \otimes_\Lambda N^{(T)} \cong (M \otimes_\Lambda N)^{(S\times T)}$, $\Lambda(S) \otimes_\Lambda \Lambda(T) \cong \Lambda(S \times T)$, and in particular $\Lambda(m) \otimes_\Lambda \Lambda(n) \cong \Lambda(mn)$. Using adjointness between $\otimes_\Lambda$ and $\mathbf{Hom}_\Lambda$, we get $\mathbf{Hom}_\Lambda(N^{(S)}, P^T) \cong \mathbf{Hom}_\Lambda(N, P)^{S\times T}$; in particular, $\mathbf{Hom}_\Lambda(\Lambda(n), P) \cong P^n$.

Another easy consequence is that $\mathrm{Bilin}_\Lambda(\Lambda(m), \Lambda(n); P) \cong P^{mn}$, and in particular $\mathrm{Bilin}_\Lambda(\Lambda(m), \Lambda(n); |\Lambda|) \cong |\Lambda|^{mn}$, i.e. bilinear forms are given by $m \times n$-matrices (understood in the classical sense) with entries in $|\Lambda|$.

5.3.8. (Algebras in Λ-*Mod*.) Since Λ-*Mod* is an ACU $\otimes$-category, we can consider algebras inside this category, thus obtaining the category $Alg(\Lambda\text{-}Mod)$ of algebras in Λ-*Mod*, and its full subcategory $CommAlg(\Lambda\text{-}Mod)$, consisting of commutative algebras. Moreover, once we have an algebra A in Λ-*Mod*, we have the category of (left) A-modules $(\Lambda\text{-}Mod)^A$ (cf. **3.1.6** and **3.1.10**). For example, when $\Lambda = \mathbb{Z}_\infty$, we recover the $\mathbb{Z}_\infty$-algebras and modules over them, defined in **2.11**. On the other hand, if Λ corresponds to a classical ring $|\Lambda|$, we recover the classical notion of a $|\Lambda|$-algebra and of a module over such an algebra.

We are going to show the following: *The category of algebras in* Λ-*Mod is equivalent to the category of unary* Λ-*algebras (cf.* **5.1.9** *and* **5.1.15***), and similarly for commutative algebras. Moreover, if a unary* Λ-*algebra* A *corresponds to an algebra* $|A|$ *inside* Λ-*Mod under this equivalence, then* $(\Lambda\text{-}Mod)^{|A|}$ *is canonically equivalent (even isomorphic) to* A-*Mod, and this equivalence is compatible with the underlying set functors.*

Therefore, all $\mathbb{Z}_\infty$-algebras constructed before can be re-interpreted in terms of unary $\mathbb{Z}_\infty$-algebras, i.e. in terms of some algebraic monads.

5.3.9. Let's fix an algebra A in Λ-*Mod*. By definition, this means that we are given $\mu : A \otimes_\Lambda A \to A$ and $\varepsilon : |\Lambda| \to A$, satisfying the usual axioms. On the other hand, such Λ-homomorphisms μ are in one-to-one correspondence with Λ-bilinear maps $\tilde{\mu} : A \times A \to A$, and $\varepsilon : |\Lambda| \to A$ corresponds to an element $\tilde{\varepsilon} \in A$. We see that the algebra axioms can be now reformulated by requiring A to be both a Λ-module and a monoid (commutative, if we want A to be a commutative algebra), such that the monoid multiplication $A \times A \to A$ is Λ-bilinear. Similarly, an A-module M is just a Λ-module M with a Λ-bilinear monoid action $A \times M \to M$.

For example, if $\Lambda = \mathbb{Z}$ we recover classical rings and modules over them, and if $\Lambda = \mathbb{F}_\varnothing$ we recover monoids and sets with monoid actions.

5.3.10. (Some functors.) Now we construct some functors. First of all, we have a forgetful functor $\rho_* : (\Lambda\text{-}Mod)^A \to \Lambda\text{-}Mod$, $(M, \alpha) \mapsto M$. It admits a left adjoint ρ^*, given by $N \mapsto (A \otimes_\Lambda N, \mu_A \otimes \mathrm{id}_N)$ (this is true in any AU $\otimes$-category, cf. **3.1.14**). Moreover, existence of inner Homs allows us to construct a right adjoint $\rho^!$ to ρ_*: we put $\rho^! N := \mathbf{Hom}_\Lambda(A, N)$ for any $N \in \operatorname{Ob} \Lambda\text{-}Mod$, and define $\alpha_{\rho^! N} : A \otimes_\Lambda \mathbf{Hom}_\Lambda(A, N) \to \mathbf{Hom}_\Lambda(A, N)$ by requiring its adjoint $\alpha^\sharp_{\rho^! N} : A \otimes_\Lambda A \otimes_\Lambda \mathbf{Hom}_\Lambda(A, N) \to N$ to be the composite map of $\mu_A \otimes \mathrm{id}_{\mathbf{Hom}(A,N)}$ and the evaluation map $\mathrm{ev} : A \otimes_\Lambda \mathbf{Hom}_\Lambda(A, N) \to N$. In other words, an element $a \in A$ acts on $\mathbf{Hom}_\Lambda(A, N)$ by pre-composing maps $A \to N$ with the right multiplication $R_a : A \to A$. This shows that we've got a Λ-bilinear monoid action of A on $\rho^! N$, i.e. an A-module structure. Verification of $\operatorname{Hom}_A(M, \rho^! N) \cong \operatorname{Hom}_\Lambda(\rho_* M, N)$ can be done now essentially in the usual way; let us remark that $\rho_*\rho^! N = \mathbf{Hom}_\Lambda(A, N) \to N$ is given by the evaluation at the identity of A, and $M \to \rho^!\rho_* M = \mathbf{Hom}_\Lambda(A, M)$ is equal to $\bar{\alpha}^\flat$, where $\bar{\alpha} : M \otimes_\Lambda A \xrightarrow{\sim} A \otimes_\Lambda M \xrightarrow{\alpha} M$ is the A-action on M with permuted arguments.

Notice that arbitrary inductive and projective limits exist in $(\Lambda\text{-}Mod)^A$ and are essentially computed in Λ-*Mod*, scalar restriction ρ_* commutes with all these limits (and in particular it is exact), ρ^* commutes with inductive limits, and $\rho^!$ with projective limits.

5.3.11. (Forgetful functor and construction of an algebraic monad.) Now we can compose $\rho_* : (\Lambda\text{-}Mod)^A \to \Lambda\text{-}Mod$ with the forgetful functor $\Gamma_\Lambda : \Lambda\text{-}Mod \to Sets$; thus $\Gamma_A := \Gamma_\Lambda \circ \rho_* : (\Lambda\text{-}Mod)^A \to Sets$ is the "underlying set" functor, and it admits a left adjoint $L_A = \rho^* \circ L_\Gamma$, hence it defines a monad $\Sigma = \Lambda_A := \Gamma_A L_A$ over *Sets*, and a comparison functor $(\Lambda\text{-}Mod)^A \to \Sigma\text{-}Mod$.

We want to show that this functor is an equivalence (even an isomorphism) of categories, and that Λ_A is an algebraic monad (cf. **4.1.1**). The last statement is evident, since $\Lambda_A = \Gamma_\Lambda \rho_* \rho^* L_\Gamma$, and all functors involved commute with filtered inductive limits (a priori this is not clear only for ρ_*, but we have just seen that it admits a right adjoint $\rho^!$). Notice that this expression actually implies $\Lambda_A(S) = \Gamma_\Lambda(A \otimes_\Lambda \Lambda(S)) \cong \Gamma_\Lambda(A^{(S)})$, where $A^{(S)}$ denotes the direct sum in Λ-*Mod* of S copies of A. In particular, $|\Lambda_A| = A$ as a set.

5.3.12. (Monadicity of Γ_A.) So we are reduced to showing the monadicity of Γ_A. Consider for this the following unary Λ-algebra Ξ: Ξ is generated as a non-commutative Λ-algebra by unary generators $[a]$, $a \in A$, subject to unary relations $[e] = \boldsymbol{e}$, where $e \in A$ is the unity of A, $[a][a'] = [aa']$ for any a, $a' \in A$, and $\bigl[t(a_1, \ldots, a_n)\bigr] = t\bigl([a_1], \ldots, [a_n]\bigr)$ for any $t \in \Lambda(n)$, $a_i \in A$.

We see that a Ξ-module structure $\Xi \to \mathrm{END}(X)$ on a set X is the same thing as a Λ-module structure on X, plus a family of unary operations $a_X : X \to X$, one for each $a \in A$, commuting with all operations from Λ, i.e. all a_X have to be Λ-linear, and such that $(aa')_X = a_X \circ a'_X$, i.e. we get a monoid action $A \times X \to X$, linear in the second argument, and the remaining family of relations expresses its linearity with respect to the first argument.

Therefore, a Ξ-module structure is exactly a Λ-module structure plus an A-module structure, i.e. Ξ-*Mod* is canonically equivalent (and even isomorphic) to $(\Lambda\text{-}Mod)^A$, and this equivalence is compatible with the forgetful functors to *Sets*, hence Ξ has to coincide with the monad Λ_A constructed before, and we have thus shown $(\Lambda\text{-}Mod)^A \to \Lambda_A\text{-}Mod$ to be an isomorphism of categories.

Moreover, by construction we have a Λ-algebra structure on $\Lambda_A = \Xi$, i.e. a central homomorphism $\rho : \Lambda \to \Lambda_A$, and ρ_* obviously coincides with the functor $\rho_* : (\Lambda\text{-}\mathit{Mod})^A \to \Lambda\text{-}\mathit{Mod}$ considered before, so the left adjoints ρ^* also coincide. We have also proved that Λ_A is unary over Λ, and that ρ_* admits a right adjoint $\rho^!$. Another consequence: $|\Lambda_A| = A$.

5.3.13. (Unary envelopes of Λ-algebras.) Let Σ be an arbitrary Λ-algebra. Then $|\Sigma|$ is a Σ-module, hence it has a natural Λ-module structure as well by scalar restriction. On the other hand, $|\Sigma|$ has a monoid structure, given by the composition of unary operations of Σ, and we see that the multiplication $|\Sigma| \times |\Sigma| \to |\Sigma|$ is Λ-bilinear: Λ-linearity in the first argument is evident, while Λ-linearity in the second argument uses commutativity of all operations from the image of Λ with the unary operations of Σ, a consequence of the centrality of $\Lambda \to \Sigma$. Therefore, any such Σ defines an algebra $|\Sigma|$ in Λ-*Mod*, commutative for a commutative Σ.

Conversely, we have just seen that any algebra A in Λ-*Mod* defines a Λ-algebra Λ_A, such that $\Lambda_A\text{-}\mathit{Mod} \cong (\Lambda\text{-}\mathit{Mod})^A$, so we have a functor in the opposite direction. We claim that $A \mapsto \Lambda_A$ *is a left adjoint to* $\Sigma \mapsto |\Sigma|$. Indeed, this follows immediately from the description of Λ_A as the Λ-algebra, generated by unary operations $[a]$, $a \in A$, subject to certain unary relations, cf. **5.3.12**.

We also know that $|\Lambda_A| \cong A$, i.e. *the functor* $A \to \Lambda_A$ *is fully faithful.* We know that all Λ-algebras in its essential image are unary. Now we claim that *all unary algebras lie in the essential image of* $A \mapsto \Lambda_A$, hence *the category of unary* Λ*-algebras is equivalent to the category of algebras in* Λ*-Mod*, as announced in **5.3.8**. Indeed, we have to show that the homomorphism of Λ-algebras $\rho : \Lambda_{|\Sigma|} \to \Sigma$ is an isomorphism for a unary Σ; this follows from the fact that *if a homomorphism of unary* Λ*-algebras* $\rho : \Sigma' \to \Sigma$ *induces an isomorphism between their sets of unary operations, then* ρ *is itself an isomorphism.*

For a general Λ-algebra Σ the homomorphism $\Lambda_{|\Sigma|} \to \Sigma$ is not an isomorphism; we usually denote $\Lambda_{|\Sigma|}$ simply by $|\Sigma|$ and say that it is the *unary envelope* of Σ, since it has a universal property with respect to homomorphisms from unary Λ-algebras into Σ.

5.3.14. (Properties.) Notice that $A \mapsto \Lambda_A$ commutes with arbitrary inductive limits, and $\Sigma \mapsto |\Sigma|$ commutes with arbitrary projective and filtered inductive limits. However, the first functor does not commute in general with projective limits, e.g. products: all we can say is that $\Lambda_{A\times B}$ is the unary envelope of $\Lambda_A \times \Lambda_B$. For example, for $\Lambda = \mathbb{Z}_\infty$, and $A = B = |\mathbb{Z}_\infty|$, we see that $\mathbb{Z}_\infty \times \mathbb{Z}_\infty$ computed in the category of unary $\mathbb{Z}_\infty$-algebras (cf. **2.13.11**) differs from the same product computed in the category of all $\mathbb{Z}_\infty$-algebras. This explains why we've got $\operatorname{Spec}\mathbb{Z}_\infty \times \mathbb{Z}_\infty \neq \operatorname{Spec}\mathbb{Z}_\infty \sqcup \operatorname{Spec}\mathbb{Z}_\infty$ in **2.13.11**: this equality would be true if we had computed $\mathbb{Z}_\infty \times \mathbb{Z}_\infty$ in the larger category.

This also explains why unary $\mathbb{Z}_\infty$-algebras are insufficient for our purpose, so we need to enlarge our category.

Proposition 5.3.15 *Let Σ be a Λ-algebra, given by a central homomorphism $\rho : \Lambda \to \Sigma$. The following conditions are equivalent:*

a) *Σ is a unary Λ-algebra;*

b) *Σ is isomorphic to Λ_A for some algebra A in Λ-Mod;*

c) *Scalar restriction $\rho_* : \Sigma$-Mod $\to \Lambda$-Mod admits a right adjoint $\rho^!$;*

d) *ρ_* commutes with finite direct sums;*

d′) *ρ_* commutes with finite direct sums of free Σ-modules of finite rank.*

Proof. We have already proved *a*) ⇔ *b*), *b*) ⇒ *c*) has been shown in **5.3.10**, and *c*) ⇒ *d*) ⇒ *d′*) is trivial. Let's show *d′*) ⇒ *b*). Put $A := |\Sigma|$, and consider the canonical homomorphism $\Lambda_A \to \Sigma$. Now $\Lambda_A(n)$ can be identified with $A^{(n)}$, i.e. $(\rho_*\Sigma(1))^{(n)}$, and $\Sigma(n)$ with $\rho_*(\Sigma(1)^{(n)})$. Then $\Lambda_A(n) \to \Sigma(n)$ is identified with the canonical morphism $(\rho_*\Sigma(1))^{(n)} \to \rho_*(\Sigma(1)^{(n)})$, and *d′*) implies that all these maps are bijective, hence $\Lambda_A \cong \Sigma$, q.e.d.

5.3.16. (Base change and tensor product.) Let $\rho : \Lambda \to \Lambda'$ be a homomorphism of generalized rings. We claim that *the base change functor $\rho^* : \Lambda$-Mod $\to \Lambda'$-Mod is a $\otimes$-functor*, i.e. *we have canonical isomorphisms $\rho^*(M \otimes_\Lambda N) \xrightarrow{\sim} \rho^*M \otimes_{\Lambda'} \rho^*N$ for all Λ-modules M and N.*

First of all, for any three Λ'-modules M', N', P' we have a natural map $\mathrm{Bilin}_{\Lambda'}(M', N'; P') \to \mathrm{Bilin}_\Lambda(\rho_*M', \rho_*N'; \rho_*P')$, given by considering any Λ'-bilinear map $M' \times N' \to P'$ as a Λ-bilinear map with respect to Λ-module structures defined by scalar restriction. Put $M' := \rho^*M$, $N' := \rho^*N$; then $\mathrm{Bilin}_{\Lambda'}(\rho^*M, \rho^*N; P') \to \mathrm{Bilin}_\Lambda(\rho_*\rho^*M, \rho_*\rho^*N; \rho_*P') \to \mathrm{Bilin}_\Lambda(M, N; \rho_*P')$, where the second arrow is defined with the aid of canonical homomorphisms $M \to \rho_*\rho^*M$ and $N \to \rho_*\rho^*N$.

Using the universal property of tensor products we get natural maps $\mathrm{Hom}_{\Lambda'}(\rho^*M \otimes_{\Lambda'} \rho^*N, P') \cong \mathrm{Bilin}_{\Lambda'}(\rho^*M, \rho^*N; P') \to \mathrm{Bilin}_\Lambda(M, N; \rho_*P') \cong \mathrm{Hom}_\Lambda(M \otimes_\Lambda N, \rho_*P') \cong \mathrm{Hom}_{\Lambda'}(\rho^*(M \otimes_\Lambda N), P')$; by Yoneda the composite arrow is induced by a canonical Λ'-homomorphism $\rho^*(M \otimes_\Lambda N) \to \rho^*M \otimes_{\Lambda'} \rho^*N$. We want to show that it is an isomorphism, i.e. that for all M, N and P' the map $\mathrm{Bilin}_{\Lambda'}(\rho^*M, \rho^*N; P') \to \mathrm{Bilin}_\Lambda(M, N; \rho_*P')$ is a bijection.

Notice that the source of this map $\mathrm{Bilin}_{\Lambda'}(\rho^*M, \rho^*N; P')$ is isomorphic to $\mathrm{Hom}_{\Lambda'}(\rho^*M, \mathbf{Hom}_{\Lambda'}(\rho^*N, P')) \cong \mathrm{Hom}_\Lambda(M, \rho_*\,\mathbf{Hom}_{\Lambda'}(\rho^*N, P'))$, and the target is isomorphic to $\mathrm{Hom}_\Lambda(M, \mathbf{Hom}_\Lambda(N, \rho_*P'))$; by Yoneda our map is induced by some Λ-homomorphism $\rho_* \mathbf{Hom}_{\Lambda'}(\rho^* N, P') \to \mathbf{Hom}_\Lambda(N, \rho_* P')$, and all we have to do is to check that it is an isomorphism. Now this is trivial: taking global sections, i.e. looking at underlying sets, we get the adjointness map $\mathrm{Hom}_{\Lambda'}(\rho^* N, P') \to \mathrm{Hom}_\Lambda(N, \rho_* P')$, which is of course bijective.

One can construct in a similar way canonical homomorphisms, relating ρ^* with multiple tensor products; expressing multiple tensor products in terms of repeated binary tensor products we see that these homomorphisms are in fact isomorphisms, and their existence implies the compatibility of our isomorphisms $\rho^*(M \otimes_\Lambda N) \xrightarrow{\sim} \rho^* M \otimes_{\Lambda'} \rho^* N$ with the associativity and commutativity constraints, hence ρ^* is a $\otimes$-functor.

5.3.17. (Base change for unary algebras.) An immediate consequence is that $\rho^* A$ is an algebra in Λ'-*Mod* for any algebra A in Λ-*Mod*. Conversely, if A' is an algebra in Λ'-*Mod*, then $\rho_* A'$ admits a canonical algebra structure: this can be either seen directly from the description of algebras as monoids with bilinear multiplication, or deduced from the existence of canonical homomorphisms $\rho_* M' \otimes_\Lambda \rho_* N' \to \rho_*(M' \otimes_{\Lambda'} N')$.

We want to show that this base change is compatible with the general base change for Λ-algebras, i.e. that $\Lambda'_{\rho^* A}$ is canonically isomorphic to $\Lambda' \otimes_\Lambda \Lambda_A$; this would allow us to denote Λ_A by A and still be able to use notations like $\Lambda' \otimes_\Lambda A$ or $A_{(\Lambda')}$ without ambiguity. To prove this statement we observe that $\Sigma := \Lambda_A$ is a unary Λ-algebra, hence $\Sigma' := \Lambda' \otimes_\Lambda \Sigma$ is a unary Λ'-algebra, hence it is given by some algebra A' in Λ'-*Mod*, namely, $A' := |\Sigma'|$; all we have to show is $A' \cong \rho^* A$. This follows from the "base change theorem" (cf. **5.4.2** below), applied to Σ-module $|\Sigma|$: we see that doing base change to Σ' and then restricting scalars to Λ' yields the same result as first restricting scalars to Λ, and then applying base change ρ^*; now the first path yields $A' = |\Sigma'|$, and the second — $\rho^*|\Sigma| = \rho^* A$.

5.3.18. (Tensor algebras.) Given any Λ-module M, we can construct its *tensor algebra* $T_\Lambda(M)$. We construct $T_\Lambda(M) = T(M)$ first as an algebra in Λ-*Mod*, by putting $T^n(M) := M^{\otimes n}$ (*tensor powers* of M) and $T(M) := \bigoplus_{n \geq 0} T^n(M)$, and defining the multiplication $T(M) \otimes_\Lambda T(M) \to T(M)$ with the aid of canonical maps $T^n(M) \otimes_\Lambda T^m(M) \to T^{n+m}(M)$, using the fact that tensor products commute with infinite direct sums. One shows in the usual fashion that $M \to T_\Lambda(M)$ is universal among all Λ-homomorphisms from M into algebras in Λ-*Mod*.

We denote the unary Λ-algebra, defined by $T_\Lambda(M)$, by the same expression $T_\Lambda(M)$. Using the existence of unary envelopes one shows that $T_\Lambda(M)$ is universal among Λ-homomorphisms $M \to |\Sigma|$, with Σ a variable Λ-algebra.

Finally, if $\rho : \Lambda \to \Lambda'$ is a homomorphism of generalized rings, we have a canonical isomorphism $T_{\Lambda'}(\rho^* M) \cong \Lambda' \otimes_\Lambda T_\Lambda(M)$, obtained for example by comparing the universal properties of the two sides.

5.3.19. (Symmetric algebras.) Notice that the symmetric group $\mathfrak{S}_n$ acts on $T^n(M)$ by permuting the components. We can define the *symmetric powers* of M by putting $S^n_\Lambda(M) := T^n_\Lambda(M)/\mathfrak{S}_n$, i.e. we take the largest strict quotient of $T^n(M)$, on which $\mathfrak{S}_n$ acts trivially (in other words, the kernel of $T^n(M) \to S^n(M)$ is the compatible equivalence relation, generated by equations $\sigma(t) \equiv t$, $t \in T^n(M)$, $\sigma \in \mathfrak{S}_n$). These symmetric powers parametrize the spaces of symmetric multilinear maps $M^n \to P$, i.e. $\mathrm{Polylin}_\Lambda(M^n; P)^{\mathfrak{S}_n} \cong \mathrm{Hom}_\Lambda(S_\Lambda(M), P)$.

Now we can define canonical maps $S^n(M) \otimes S^m(M) \to S^{n+m}(M)$, induced by corresponding maps for tensor powers by passage to strict quotients, and construct the *symmetric algebra* $S^n_\Lambda(M) := \bigoplus_{n\geq 0} S^n_\Lambda(M)$, considered both as a commutative algebra inside Λ-*Mod*, and as a commutative unary Λ-algebra; clearly, $S_\Lambda(M)$ is a strict quotient of the tensor algebra $T_\Lambda(M)$.

We see that $S_\Lambda(M)$ has a universal property with respect to Λ-homomorphisms $M \to |\Sigma|$, where Σ runs through commutative Λ-algebras. This universal property implies stability of the construction under any base change $\rho : \Lambda \to \Lambda'$. In particular, $S^n_{\Lambda'}(\rho^* M) \cong \rho^*(S^n_\Lambda(M))$.

5.3.20. (Monoid and group algebras.) For any monoid G we can construct the corresponding *monoid algebra* $\Lambda[G]$ (called a *group algebra* if G is a group), as follows. We construct first the corresponding algebra in Λ-*Mod* by defining a multiplication on the free module $\Lambda(G) = \bigoplus_{g\in G} \Lambda\{g\}$ in the usual fashion, and then consider the corresponding unary Λ-algebra. It clearly has a universal property with respect to monoid homomorphisms $G \to |\Sigma|$, for variable Λ-algebra Σ (again, for a unitary Σ this is shown in the classical way, and in general case we replace Σ by its unary envelope). This gives another description of $\Lambda[G]$, namely, $\Lambda[G] = \Lambda\{[g], g \in G \,|\, [e] = \boldsymbol{e},\ [gh] = [g][h]\}$, and shows the compatibility with base change: $\Lambda' \otimes_\Lambda \Lambda[G] \cong \Lambda'[G]$. Clearly, $\Lambda[G]$ is commutative for a commutative G; considering homomorphisms $\Lambda[G] \to \mathrm{END}(X)$, we see that $\Lambda[G]$-modules are exactly Λ-modules, equipped with a Λ-linear action of G.

5.3.21. (Unary algebras over $\mathbb{F}_\varnothing$.) Recall that $\mathbb{F}_\varnothing$-*Mod* is just the category of sets, with direct sums (i.e. coproducts) given by disjoint unions, and tensor products equal to direct products (since any map $M \times N \to P$ is bilinear). This means that unary $\mathbb{F}_\varnothing$-algebras, i.e. algebras in $\mathbb{F}_\varnothing$-*Mod* $\cong$ *Sets*, are nothing else than *monoids* G (commutative algebras corresponding to commutative monoids). Clearly, in this case the unary $\mathbb{F}_\varnothing$-algebra defined by G coincides with the monoid algebra $\mathbb{F}_\varnothing[G]$, i.e. all unary algebras over $\mathbb{F}_\varnothing$ are monoid algebras. Of course, for any Λ we have $\Lambda[G] = \Lambda \otimes_{\mathbb{F}_\varnothing} \mathbb{F}_\varnothing[G] = \Lambda \otimes G$. For example, we have $(\mathbb{F}_\varnothing[G])(n) = \bigsqcup_{i\in\mathbf{n}} G = G \times \mathbf{n}$, with $(g, i) \in G \times \mathbf{n}$

corresponding to $g\{i\}$. Computing base change to Λ, we obtain $(\Lambda[G])(n) \cong \Lambda(G \times \mathbf{n})$.

5.3.22. (Unary polynomial algebras.) We can use the above results to describe the unary polynomial algebras $\Sigma = \Lambda[(T_i^{[1]})_{i\in I}]$. Indeed, an inspection of universal properties shows that it is isomorphic both to the symmetric algebra of the free Λ-module $\Lambda(I)$, hence it admits a natural grading, and to the monoid algebra $\Lambda[\mathbb{Z}_{\geq 0}^{(I)}]$ of the free commutative monoid $\mathbb{Z}_{\geq 0}^{(I)}$, generated by I. For example, the set of unary operations of $\mathbb{F}_\varnothing[T_1^{[1]}, \ldots, T_n^{[1]}]$ is canonically isomorphic to $\mathbb{Z}_{\geq 0}^n$, i.e. it is the set of monomials in the T_i; the set of m-ary operations is given by the product $\mathbb{Z}_{\geq 0}^n \times \mathbf{m}$, i.e. it consists of all expressions $T_1^{k_1} \cdots T_n^{k_n}\{i\}$, $k_j \geq 0$, $1 \leq i \leq m$.

Notice that the grading on the polynomial algebra, coming from its symmetric algebra description, defines an increasing filtration, so it makes sense to speak about the degree of a polynomial with respect to all or to some group of variables. However, in general we cannot extract from a polynomial its leading term, or any other coefficient, except the free term.

This isomorphism between polynomial algebras and symmetric algebras of free modules shows that symmetric powers of free modules are free, and (in the finite-dimensional case) of correct rank: $S_\Lambda^k(\Lambda(n))$ is free of rank $\binom{n+k-1}{k}$.

5.3.23. (Unary polynomial algebras in non-commuting variables.) Similarly, we see by comparing the universal properties that the free algebra $\Lambda\{S\}$, when all free generators from S are supposed to be unary, is canonically isomorphic to the monoid algebra of $W(S)$ (the free monoid over S, i.e. the set of monomials in non-commuting variables from S) over Λ. We deduce $(\Lambda\{S\})(n) \cong \Lambda(W(S) \times \mathbf{n})$, and in particular $|\Lambda\{S\}| \cong \Lambda(W(S))$.

5.3.24. (Commutativity and algebraic bimodules.) Finally, let us mention an application of commutativity to the category of algebraic (Λ, Λ)-bimodules. Namely, we have seen in **4.7.11** that any such bimodule M is given by a functor $\tilde{M} : \underline{\mathbb{N}}_\Lambda \to \Lambda\text{-}\mathit{Mod}$, where $\underline{\mathbb{N}}_\Lambda$ is the category of standard free Λ-modules $\Lambda(n)$ of finite rank. Therefore, for any n, $m \geq 0$ we have a map of sets $\mathrm{Hom}_\Lambda(\Lambda(n), \Lambda(m)) \to \mathrm{Hom}_\Lambda(\tilde{M}(n), \tilde{M}(n))$. When Λ is commutative, we obtain Λ-structures on these two sets, so it makes sense to consider the case when all these maps of sets are Λ-linear. When $\tilde{M}$ has this property, we say that M or $\tilde{M}$ is a *central* algebraic bimodule. This definition generalizes to the categories of algebraic (Σ, Ξ)-bimodules, with Σ and Ξ two Λ-algebras; then we speak about Λ-*central* algebraic bimodules. Of course, these notions admit some matrix descriptions as well.

5.3.25. (Comparison to [ShaiHaran].) Since $\Lambda(n) \otimes_\Lambda \Lambda(m) \cong \Lambda(nm)$ and $\Lambda(n) \oplus \Lambda(m) \cong \Lambda(n + m)$ for any generalized ring Λ, and we have defined the matrix sets $M(n,m;\Lambda)$ by $M(n,m;\Lambda) := \Lambda(n)^m \cong \mathrm{Hom}_\Lambda(\Lambda(m),\Lambda(n))$, functors $\oplus$ and $\otimes$ induce maps $\oplus : M(n,m;\Lambda)\times M(n',m';\Lambda) \to M(n+n',m+m';\Lambda)$ and $\otimes : M(n,m;\Lambda)\times M(n',m';\Lambda) \to M(nn',mm';\Lambda)$. We have the composition maps $\circ : M(n,m;\Lambda)\times M(m,k;\Lambda) \to M(n,k;\Lambda)$ as well, and when Λ is a (commutative) $\mathbb{F}_1$-algebra, we have also some maps $M(n,m;\mathbb{F}_1) \to M(n,m;\Lambda)$.

This collection of data, consisting of the sets $M(n,m;\Lambda)$, the $\circ$, $\otimes$ and $\oplus$-operations between them, and the maps $M(n,m;\mathbb{F}_1) \to M(n,m;\Lambda)$, is easily checked to satisfy all the conditions for an "$\mathbb{F}$-algebra" of [ShaiHaran] (up to a minor point — Shai Haran requires all $M(n,m;\mathbb{F}_1) \to M(n,m;\Lambda)$ to be injective; this excludes the trivial monad $\mathbf{1}$ and $\mathbf{1}_+ \subset \mathbf{1}$), and our (commutative) $\mathbb{F}_{\pm 1}$-algebras define "$\mathbb{F}_\pm$-algebras" in the sense of *loc. cit.* Notice that the approach of Shai Haran is more general, since it never requires $M(n,m;\Lambda) \cong M(n,1;\Lambda)^m$; actually, our category of commutative $\mathbb{F}_1$-algebras (almost) corresponds to the category of "$\mathbb{F}$-algebras", satisfying this additional condition.

Shai Haran's approach has the obvious advantage of being more general and more symmetric (one can "transpose" all matrices and define Λ^t by $M(n,m;\Lambda^t) := M(m,n;\Lambda)$ for any $\mathbb{F}$-algebra Λ). However, there are considerable disadvantages of such a generality as well: his approach is more categorial, while our approach is more algebraic and has a direct connection to algebraic systems. This enables us to construct free objects, and compute inductive limits (both in the categories of algebras and modules) quite explicitly, and transfer more statements from the classical case. At the same time our approach seems sufficiently general, at least for the present moment, to deal with objects arising from arithmetics and Arakelov geometry (e.g. the compactification of $\mathrm{Spec}\,\mathbb{Z}$). That's why we prefer not to replace our approach by the more general one.

5.4. (Flatness and unarity.) The notions of unarity and of flatness (defined below) seem to be in some sense "orthogonal", like properness and smoothness of morphisms with respect to étale cohomology.

Definition 5.4.1 *A homomorphism of generalized rings $\rho : \Sigma \to \Xi$ is* flat *(resp.* faithfully flat*), if the corresponding base change functor $\rho^* : \Sigma$-Mod $\to \Xi$-Mod is (left) exact (resp. left exact and faithful). A Λ-algebra Σ is said to be* flat *(resp.* faithfully flat*) if the corresponding central homomorphism $\Lambda \to \Sigma$ has this property.*

Clearly, ρ is faithfully flat iff it is flat, and ρ^* is conservative. If ρ is already known to be flat, the latter condition is equivalent to $\rho^* N \neq \rho^* M$ for any submodule $N \neq M$ of any Σ-module M.

We have the usual properties. For example, in the situation $\Sigma \to \Sigma' \to \Sigma''$, we have transitivity of flatness and of faithful flatness, and if Σ'' is flat (resp. faithfully flat) over Σ and faithfully flat over Σ', then Σ' is also flat (resp. faithfully flat) over Σ. Stability of flatness under base change, at least for algebras over a generalized ring, follows from the following statement:

Theorem 5.4.2 *("Base change theorem") Let $\sigma : \Lambda \to \Lambda'$ be a homomorphism of generalized rings, Σ be a Λ-algebra, given by a central homomorphism $\rho : \Lambda \to \Sigma$. Put $\Sigma' := \Lambda' \otimes_\Lambda \Sigma$, and denote by $\rho' : \Lambda' \to \Sigma'$ and $\tau : \Sigma \to \Sigma'$ the canonical homomorphisms, so as to obtain a cocartesian square:*

$$\begin{tikzcd} \Sigma' & \Lambda' \arrow[l, "\rho'"'] \\ \Sigma \arrow[u, "\tau"] & \Lambda \arrow[l, "\rho"] \arrow[u, "\sigma"'] \end{tikzcd} \tag{5.4.2.1}$$

Then we have a canonical natural transformation $\gamma : \sigma^*\rho_* \to \rho'_*\tau^*$ of functors Σ-Mod $\to \Lambda'$-Mod. When either a) ρ is unary, or b) σ is flat, this natural transformation $\gamma : \sigma^*\rho_* \to \rho'_*\tau^*$ is an isomorphism.

Proof. First of all, let us construct $\gamma : \sigma^*\rho_* \to \rho'_*\tau^*$. By adjointness it is sufficient to construct $\gamma^\flat : \rho_* \to \sigma_*\rho'_*\tau^* = \rho_*\tau_*\tau^*$, and to do this we just apply ρ_* to the adjointness natural transformation $\mathrm{Id} \to \tau_*\tau^*$.

We want to prove that γ is an isomorphism under some conditions. Let us fix some presentation $\Sigma = \Lambda\{S|E\}$; in case a) we choose a unary presentation, i.e. $S \subset |\Sigma|$ and $E \subset |\Lambda\{S\}|^2$. Now let's take some Σ-module M and put $\bar{M} := \rho_*M$; we have some additional structure on this Λ-module $\bar{M}$, namely, for each $s \in S$ we have a Λ-linear map $s_M : \bar{M}^{r(s)} \to \bar{M}$ (Λ-linearity follows from the centrality of $\Lambda \to \Sigma$), and appropriate composites of these maps satisfy the relations from E. Moreover, the category of Σ-modules is in fact equivalent to the category of Λ-modules, equipped with these additional Λ-linear maps, and since $\Sigma' = \Lambda' \otimes_\Lambda \Sigma = \Lambda'\{S|E\}$, we have a similar description of Σ'-modules in terms of Λ'-modules with some extra Λ'-linear maps.

We construct this additional structure on the Λ'-module $\sigma^*\bar{M}$ by putting $s_{\sigma^*M} := \sigma^*(s_M) \in \mathrm{Hom}_{\Lambda'}(\sigma^*(\bar{M}^{r(s)}), \sigma^*\bar{M})$. Notice that in case a) we have $r(s) = 1$, and in case b) σ^* commutes with finite products, so in both cases we have $\sigma^*(\bar{M}^{r(s)}) \cong (\sigma^*\bar{M})^{r(s)}$, hence s_{σ^*M} acts from $\sigma^*\bar{M}^{r(s)}$ to $\sigma^*\bar{M}$ as required.

We claim that these Λ'-homomorphisms s_{σ^*M} satisfy all the relations from E, thus defining a Σ'-module structure on $\sigma^*\bar{M}$; resulting Σ'-module will be provisionally denoted by $\tau^?M$. Indeed, this is clear in case b), since all relations from E equate some morphisms between products of several copies of a Λ- or a Λ'-module, and σ^* commutes with such products, σ being flat, hence the validity of E for $s_{\sigma^*M} = \sigma^*(s_M)$ follows from the validity of E for s_M; one might also say that we have a homomorphism of Λ-algebras $\mathrm{END}_\Lambda(\bar{M}) \to \mathrm{END}_{\Lambda'}(\sigma^*\bar{M})$. In case a) we have only unary relations, i.e. $E \subset |\Lambda\{S\}|^2$, and according to **5.3.23** any element of $|\Lambda\{S\}|$ can be written in form $t(u_1, \ldots, u_m)$, for some $t \in \Lambda(m)$, $m \geq 0$, and $u_i \in W(S)$. Let us denote by s the image in $|\Sigma|$ of such an element. All we have to show is that $s = t(u_1, \ldots, u_n)$ in Σ implies similar relation between s_{σ^*M} and u_{j,σ^*M}. Now $\sigma^* : \mathrm{End}_\Lambda(\bar{M}) \to \mathrm{End}_{\Lambda'}(\sigma^*\bar{M})$ is clearly a monoid homomorphism, so we can assume that S is closed under multiplication (in $|\Sigma|$), and that all $u_j \in S$. We know that $s_M = t(u_{1,M}, \ldots, u_{m,M})$ in $\mathbf{End}_\Lambda(\bar{M})$; now our statement follows from the Λ-linearity of $\mathbf{End}_\Lambda(\bar{M}) \to \mathbf{Hom}_\Lambda(\bar{M}, \sigma_*\sigma^*\bar{M}) \cong \sigma_*\,\mathbf{End}_{\Lambda'}(\sigma^*\bar{M})$, already shown in **5.3.16**.

Now let's show that $\tau^?M$ has the universal property required from τ^*M. Indeed, giving a Σ-homomorphism $f : M \to \tau_*N$ is equivalent to giving a Λ-homomorphism $\bar{f} : \bar{M} \to \sigma_*\bar{N}$, where $\bar{N} := \rho'_*N$, such that for all generators $s \in S$ the following diagram commutes:

$$\begin{tikzcd} \bar{M}^{r(s)} \arrow[r, "\bar{f}^{r(s)}"] \arrow[d, "s_M"] & \sigma_*\bar{N}^{r(s)} \arrow[d, "\sigma_*(s_N)"] \\ \bar{M} \arrow[r, "\bar{f}"] & \sigma_*\bar{N} \end{tikzcd} \tag{5.4.2.2}$$

Using adjointness between σ^* and σ_*, and the fact that either $r(s) = 1$ or σ is flat, we see that the commutativity of the above diagrams for $\bar{f}$ is equivalent to the commutativity of similar diagrams for $\bar{f}^\sharp : \sigma^*\bar{M} \to \bar{N}$:

$$\begin{tikzcd} \sigma^*\bar{M}^{r(s)} \arrow[r, "(\bar{f}^\sharp)^{r(s)}"] \arrow[d, "\sigma^*(s_M)"] & \bar{N}^{r(s)} \arrow[d, "s_N"] \\ \sigma^*\bar{M} \arrow[r, "\bar{f}^\sharp"] & \bar{N} \end{tikzcd} \tag{5.4.2.3}$$

On the other hand, giving such a Λ'-homomorphism $\bar{f}^\sharp : \sigma^*\bar{M} \to \bar{N}$ is equivalent to giving a Σ'-homomorphism $f^\sharp : \tau^?M \to N$, and we get the required universal property for $\tau^?M$.

We see that $\tau^*M \cong \tau^?M$, hence $\rho'_*\tau^*M \cong \rho'_*\tau^?M = \sigma^*\bar{M} = \sigma^*\rho_*M$, i.e. $\gamma_M : \sigma^*\rho_*M \to \rho'_*\tau^*M$ is an isomorphism, q.e.d.

5.4.3. Notice that the conclusion of the theorem is not true without any additional assumptions about σ or ρ. Indeed, let's take $\Lambda = \mathbb{F}_\varnothing$, $\Sigma = \Lambda' = \mathbb{Z}$; then $\Sigma' = \mathbb{Z} \otimes_{\mathbb{F}_\varnothing} \mathbb{Z} \cong \mathbb{Z}$ by **5.1.22**, and consider $M := \Sigma(1) = \mathbb{Z}$. Then

$\sigma^*\rho_*\mathbb{Z}$ equals $\mathbb{Z}[\mathbb{Z}]$, the monoid algebra over $\mathbb{Z}$ of the multiplicative monoid of $\mathbb{Z}$, i.e. the ring of Dirichlet polynomials $\sum_{n\in\mathbb{Z}} c_n \cdot n^\omega$. On the other hand, both τ and ρ' are isomorphisms, hence $\rho'_*\tau^*\mathbb{Z} = \mathbb{Z}$, and $\gamma_\mathbb{Z} : \mathbb{Z}[\mathbb{Z}] \to \mathbb{Z}$ is the map $\sum_n c_n n^\omega \mapsto \sum_n nc_n$. Clearly, it is not an isomorphism. This example also shows that $\mathbb{Z}$ is neither unary nor flat over $\mathbb{F}_\varnothing$, $\mathbb{F}_1$ and $\mathbb{F}_{\pm 1}$.

5.4.4. An important consequence is that *if* Σ *is flat over* Λ*, then* $\Sigma' = \Lambda' \otimes_\Lambda \Sigma$ *is flat over* Λ', i.e. flatness of algebras is stable under base change. Indeed, in the notations of **5.4.2** we have $\rho'_*\tau^* \cong \sigma^*\rho_*$; since ρ'_*, ρ_* and σ^* are left exact, and ρ'_* is in addition conservative, we see that τ^* is also left exact, hence τ is flat. One shows similarly that if in addition σ^* is conservative, then the same is true for τ^*, i.e. *faithful flatness is also stable under base change.*

Definition 5.4.5 *We say that a* Λ*-module* M *is flat (resp. faithfully flat) if the functor* $M \otimes_\Lambda - : \Lambda\text{-}Mod \to \Lambda\text{-}Mod$ *is (left) exact (resp. left exact and faithful).*

This definition is motivated by the following easy observation: a unary Λ-algebra A is flat (resp. faithfully flat) iff $|A|$ is flat (resp. faithfully flat) as a Λ-module.

Clearly, $|\Lambda| = \Lambda(1)$ is always faithfully flat, and if Λ is a monad with zero, then the zero module $\Lambda(0)$ is flat. However, in general we cannot say that all free Λ-modules are flat, since the direct sum of two monomorphisms is not necessarily a monomorphism. Actually, flatness of free modules seems to be deeply related to the existence of injectives in Λ-*Mod*.

5.4.6. (Projection formula.) Notice that the projection formula $\rho_*(M' \otimes_{\Lambda'} \rho^*N) \cong \rho_*M \otimes_\Lambda N$, for M' a Λ'-module, N a Λ-module, and $\rho : \Lambda \to \Lambda'$ a homomorphism of generalized rings, in general does *not* hold unless ρ is unary. Indeed, the validity of projection formula for $M' := \Lambda'(1)$, $N := \Lambda(n)$ means $\rho_*\Lambda'(n) \cong \Lambda'(1) \otimes_\Lambda \Lambda(n) \cong \Lambda'(1)^{\oplus n}$, hence the unarity of ρ by **5.3.15**,d'). Conversely, if ρ is unary, we can prove the projection formula first for $N = \Lambda(1)$ (trivial), then for $N = \Lambda(S) = \Lambda(1)^{(S)}$ (using the fact that ρ_* commutes in this case with direct sums), and then for arbitrary N, writing it as a cokernel of two morphisms between free modules.

5.4.7. (Products of flat algebras.) We have seen that in general we don't have a lot of flat modules and algebras in Λ-*Mod*: all we can say is that $\Lambda(1)$ is flat, and that $\Lambda(0)$ is flat when Λ admits a zero. On the other hand, we can construct more (non-unary) Λ-algebras: we know that Λ is flat over itself, and *when* Λ *is a monad with zero, the product* $\Sigma_1 \times \Sigma_2$ *of two flat* Λ*-algebras is itself a flat* Λ*-algebra.* This follows immediately from the following fact:

Theorem 5.4.8 *If Σ_1 and Σ_2 are two algebraic monads with zero, then $(\Sigma_1 \times \Sigma_2)$-Mod is canonically equivalent to Σ_1-Mod $\times$ Σ_2-Mod.*

Proof. Put $\Sigma := \Sigma_1 \times \Sigma_2$; by definition $\Sigma(n) = \Sigma_1(n) \times \Sigma_2(n)$, so any operation $t \in \Sigma(n)$ can be written in form $(t^{(1)}, t^{(2)})$, with $t^{(i)} \in \Sigma_i(n)$. Now we can construct a functor $F : \Sigma_1\text{-}Mod \times \Sigma_2\text{-}Mod \to \Sigma\text{-}Mod$: given any Σ_1-module M_1 and Σ_2-module M_2, we put $F(M_1, M_2) := M_1 \times M_2$, with the Σ-structure $\alpha^{(k)} : \Sigma(k) \times (M_1 \times M_2)^k \to M_1 \times M_2$ defined componentwise by means of the maps $\Sigma_i(k) \times M_i^k \to M_i$.

Let's construct a functor G in the opposite direction. Denote for this the only constant of Σ by 0, so we have $0 = (0^{(1)}, 0^{(2)})$, where $0^{(i)}$ is the zero of Σ_i. We also have $\boldsymbol{e} = (\boldsymbol{e}^{(1)}, \boldsymbol{e}^{(2)})$ for the identity of Σ, and we consider the elements $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ in $|\Sigma|$, given by $\boldsymbol{e}_1 = (\boldsymbol{e}^{(1)}, 0^{(2)})$ and $\boldsymbol{e}_2 = (0^{(1)}, \boldsymbol{e}^{(2)})$. Clearly, $|\Sigma| = |\Sigma_1| \times |\Sigma_2|$ as a monoid, hence $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are central idempotents in $|\Sigma|$, such that $\boldsymbol{e}_1\boldsymbol{e}_2 = \boldsymbol{e}_2\boldsymbol{e}_1 = 0$.

Now for an arbitrary Σ-module M we put $M_i := \boldsymbol{e}_i M \subset M$. Clearly, M_i is a Σ-submodule of M, $\boldsymbol{e}_i$ being central in Σ; M_i consists of all elements x of M, such that $\boldsymbol{e}_i x = x$. We define a Σ_i-action on M_i in the natural way ($t \in \Sigma_1(n)$ acts on M_1 by means of $[(t, 0^{(2)})]_{M_1} : M_1^n \to M_1$), and put $G(M) := (M_1, M_2)$.

We have to construct some isomorphisms $GF(M_1, M_2) \cong (M_1, M_2)$ and $M \xrightarrow{\sim} FG(M)$, defining an adjunction between G and F. First of them is obvious: we observe that $\boldsymbol{e}_1(M_1 \times M_2) = M_1 \times 0 \cong M_1$, and similarly for the second component of $GF(M_1, M_2)$. The Σ-homomorphism $\theta_M : M \to \boldsymbol{e}_1 M \times \boldsymbol{e}_2 M$ is also easy to construct: put $\theta_M(z) := (\boldsymbol{e}_1 z, \boldsymbol{e}_2 z)$. One checks immediately that these natural transformations do define an adjunction between G and F, and we have only to check that θ_M is an isomorphism.

Consider for this the element $\varphi := (\{1\}_{\mathbf{2}}^{(1)}, \{2\}_{\mathbf{2}}^{(2)}) \in \Sigma(2)$. It is clearly central in Σ, both its components being central, hence it defines a Σ-homomorphism $\varphi_M : M \times M \to M$ for any Σ-module M. Put $M_1 := \boldsymbol{e}_1 M$, $M_2 := \boldsymbol{e}_2 M$, and denote by φ'_M the composition of the embedding $M_1 \times M_2 \to M \times M$ with φ_M. We claim that φ'_M is an inverse for θ_M.

a) $\theta_M \varphi'_M = \mathrm{id}_{M_1 \times M_2}$. Let's take $x \in M_1$, $y \in M_2$; then $x = \boldsymbol{e}_1 x$, $y = \boldsymbol{e}_2 y$, and we want to show $\boldsymbol{e}_1\varphi(x, y) = x$, $\boldsymbol{e}_2\varphi(x, y) = y$. Clearly, it suffices to check $\boldsymbol{e}_1\varphi(\boldsymbol{e}_1\{1\}, \{2\}) = \boldsymbol{e}_1\{1\}$; we check this componentwise: $(\boldsymbol{e}_1\varphi(\boldsymbol{e}_1\{1\}, \{2\}))^{(1)} = \boldsymbol{e}_1^{(1)}\varphi^{(1)}(\boldsymbol{e}_1^{(1)}\{1\}, \{2\}) = \boldsymbol{e}^{(1)}\boldsymbol{e}^{(1)}\{1\} = \{1\} = \boldsymbol{e}_1^{(1)}\{1\}$, and $(\boldsymbol{e}_1\varphi(\boldsymbol{e}_1\{1\}, \{2\}))^{(2)} = \boldsymbol{e}_1^{(2)}\varphi^{(2)}(\boldsymbol{e}_1^{(2)}\{1\}, \{2\}) = 0^{(2)}\{2\} = 0^{(2)} = \boldsymbol{e}_1^{(2)}\{1\}$.

b) $\varphi'_M\theta_M = \mathrm{id}_M$. We have to check that $\varphi(\boldsymbol{e}_1 z, \boldsymbol{e}_2 z) = z$ for any $z \in M$, i.e. that $\varphi(\boldsymbol{e}_1, \boldsymbol{e}_2) = \boldsymbol{e}$ in $|\Sigma|$. We check this componentwise: $(\varphi(\boldsymbol{e}_1, \boldsymbol{e}_2))^{(1)} = \varphi^{(1)}(\boldsymbol{e}_1^{(1)}, \boldsymbol{e}_2^{(1)}) = \boldsymbol{e}_1^{(1)} = \boldsymbol{e}^{(1)}$, and similarly for the other component. So φ'_M is indeed an inverse to θ_M, hence F and G are adjoint equivalences, q.e.d.

5.4.9. One might try to generalize the above statement as follows. Suppose $\sigma_i : \Sigma_i \to \Sigma_0$, $i = 1, 2$, are two strict epimorphisms. Put $\Sigma := \Sigma_1 \times_{\Sigma_0} \Sigma_2$, and denote by ρ_i the canonical projections $\Sigma \to \Sigma_i$. Then ρ_1^* and ρ_2^* induce a functor G from Σ-*Mod* into the category of triples (M_1, M_2, φ), where $M_i \in \Sigma_i$-*Mod*, and $\varphi : \sigma_1^* M_1 \xrightarrow{\sim} \sigma_2^* M_2$ is a Σ_0-isomorphism. One can construct a functor F in the opposite direction by mapping a triple as above into the subset $M \subset M_1 \times M_2$, consisting of all pairs $(x, y) \in M_1 \times M_2$, such that $\varphi\gamma_1(x) = \gamma_2(y)$, where $\gamma_i : M_i \to \sigma_{i,*}\sigma_i^* M_i$ is the canonical map. It is easy to check that M is indeed a Σ-module (with the action of $\Sigma \subset \Sigma_1 \times \Sigma_2$ defined componentwise), and that G and F are adjoint. It seems plausible that they are always equivalences; in this case we might be able to prove that Λ-flatness of Σ_0, Σ_1 and Σ_2 implies that of Σ.

In any case similar considerations explain why in **5.4.8** we need to suppose that Σ_1 and Σ_2 admit some constants, i.e. that both $\Sigma_i \to \mathbf{1}$ are surjective. Without this assumption the theorem is false, as illustrated by the case $\Sigma_1 = \Sigma_2 = \mathbf{1}_+ \subset \mathbf{1}$: then $(\varnothing, \mathbf{1})$ and $(\mathbf{1}, \varnothing)$ are not isomorphic in Σ_1-*Mod*$\times\Sigma_2$-*Mod*, but their images under G are both equal to $\varnothing$, hence isomorphic.

5.4.10. (Flatness of algebraic bimodules.) Recall that any algebraic (Λ, Λ)-bimodule M induces a functor $\tilde{M} = M \otimes_\Lambda - : \Lambda$-*Mod* $\to \Lambda$-*Mod* (cf. **4.7.15**), such that $M \otimes_\Lambda \Lambda(n) \cong M(n)$. It is natural to say that M *is flat* if *this functor* $M \otimes_\Lambda -$ *is exact.* One defines similarly flatness for algebraic (Σ, Λ)-bimodules: in this case we require exactness of a functor Λ-*Mod* $\to \Sigma$-*Mod*. Clearly, a Λ-algebra Σ is flat iff it is flat as a (Σ, Λ)-bimodule.

5.5. (Alternating monads and exterior powers.) Now we would like to study some simple property (called *alternativity*) of generalized rings that enables us to construct a reasonable theory of exterior powers and determinants.

To do this we fix some commutative $\mathbb{F}_{\pm 1}$-algebra K; if we need another such algebra, we denote it by K'. Since $\mathbb{F}_{\pm 1} = \mathbb{F}_\varnothing[0^{[0]}, -^{[1]} \mid -^2 = \boldsymbol{e}]$, we see that this is equivalent to fixing a generalized ring K with zero 0 and a unary operation $-$, such that $-(-\boldsymbol{e}) = \boldsymbol{e}$.

5.5.1. (Bilinear forms and matrices.) Let Λ be a generalized ring. We know that Λ-bilinear maps $\Phi : \Lambda(m) \times \Lambda(n) \to X$ are in one-to-one correspondence with collections (Φ_{ij}) of elements of X, indexed by $(i, j) \in \mathbf{m} \times \mathbf{n}$, since $\mathrm{Bilin}_\Lambda(\Lambda(m), \Lambda(n); X) \cong \mathrm{Hom}_\Lambda(\Lambda(m) \otimes_\Lambda \Lambda(n), X) \cong \mathrm{Hom}_\Lambda(\Lambda(mn), X) \cong X^{mn} = X^{\mathbf{m}\times\mathbf{n}}$. Clearly, this correspondence is given by evaluating Φ on base elements: $\Phi_{ij} = \Phi(\{i\}, \{j\})$. Therefore, such bilinear maps are parametrized in the usual manner by (classical) $m \times n$-matrices with entries in X.

Given any elements $t \in \Lambda(m)$ and $t' \in \Lambda(n)$, we can construct a new element $t \otimes t' \in \Lambda(m) \otimes_\Lambda \Lambda(n) \cong \Lambda(\mathbf{m} \times \mathbf{n})$. For any Λ-module X the map $[t \otimes t']_X : X^{\mathbf{m}\times\mathbf{n}} \to X$ is computed in the same way as in **5.1.1**, first applying

t' to the rows of a matrix (x_{ij}), and then t to the elements of X thus obtained, or the other way around: $[t \otimes t']((x_{ij})) = [t' \otimes t]((x_{ji}))$. Clearly, if a bilinear map Φ is given by a matrix (Φ_{ij}) as above, $[t \otimes t']_X((\Phi_{ij})) = \Phi(t, t')$

Definition 5.5.2 *Let K be a commutative $\mathbb{F}_{\pm 1}$-algebra.*

a) *A K-bilinear map $\Phi : M \times M \to X$ is* skew-symmetric *if $\Phi(x, y) = -\Phi(y, x)$ for any x, $y \in M$.*

b) *A K-bilinear map $\Phi : M \times M \to X$ is* alternating *if it is skew-symmetric, and $\Phi(x, x) = 0$ for any $x \in M$.*

c) *A K-bilinear map $\Phi : K(n) \times K(n) \to X$ is said to be* pseudo-alternating *if it is given by an* alternating matrix *$(\Phi_{ij})_{1 \leq i,j \leq n}$, $\Phi_{ij} := \Phi(\{i\}, \{j\})$, i.e. if $\Phi_{ii} = 0$ and $\Phi_{ij} = -\Phi_{ji}$ for all $1 \leq i, j \leq n$.*

5.5.3. Clearly, any alternating K-bilinear map $\Phi : K(n) \times K(n) \to X$ is pseudo-alternating; the converse is in general not true. However, it is easy to see that any pseudo-alternating map is skew-symmetric: indeed, we have $\Phi(t, t') = [t \otimes t']((\Phi_{ij})) = [t' \otimes t]((\Phi_{ji})) = [t' \otimes t]((-\Phi_{ij})) = -[t' \otimes t]((\Phi_{ij})) = -\Phi(t', t)$ (notice the way we used here the commutativity of t, t' and $-$ between themselves). We see that a skew-symmetric map $\Phi : K(n) \times K(n) \to X$ is pseudo-alternating iff $\Phi(\{i\}, \{i\}) = 0$ for all $1 \leq i \leq n$.

Observe that there is a universal pseudo-alternating map $\Phi : K(n) \times K(n) \to X$ for a fixed n and variable X. To obtain it we put $X := K(n(n-1)/2)$, order the pairs (i, j) with $1 \leq i < j \leq n$ in some way (e.g. lexicographically), and put $\Phi_{ij} := \{i, j\}$ (the basis element corresponding to (i, j)) if $i < j$, $\Phi_{ii} := 0$, and $\Phi_{ij} := -\{j, i\}$ if $i > j$. We usually denote this free K-module with this base by $\bigwedge_K^2 K(n)$, and the universal pseudo-alternating map by $\wedge : K(n) \times K(n) \to \bigwedge_K^2 K(n)$; we write $x \wedge y$ instead of $\wedge(x, y)$.

Definition 5.5.4 *Let K be a commutative $\mathbb{F}_{\pm 1}$-algebra.*

a) *A K-linear map $f : K(m) \to K(n)$ is said to be* alternating *if for any pseudo-alternating bilinear map $\Phi : K(n) \times K(n) \to X$ (or just for the universal pseudo-alternating bilinear map $\wedge$) the bilinear map $\Phi \circ (f \times f) : K(m) \times K(m) \to X$ is also pseudo-alternating.*

b) *An operation $t \in K(n)$ is said to be* alternating *if the corresponding map $t : K(1) \to K(n)$ is alternating, i.e. if for any pseudo-alternating map $\Phi : K(n) \times K(n) \to X$ (or just for the universal one) we have $\Phi(t, t) = 0$, or equivalently, if for any alternating $n \times n$-matrix (Φ_{ij})*

(or just for the universal one, given by $\Phi_{ij} = -\Phi_{ji} = \{i, j\}$ for $i < j$, $\Phi_{ii} = 0$) we have $[t \otimes t]((\Phi_{ij})) = 0$.

c) *For any operation $t \in K(n)$ we denote by $\wedge t$ the corresponding* alternativity relation *for t, i.e. the relation of arity $n(n-1)/2$, obtained by equating to zero the value of $[t \otimes t]$ on the universal alternating $n \times n$-matrix with entries in $K(n(n-1)/2)$. Clearly, $\wedge t$ is fulfilled in K iff t is alternating.*

d) *We say that a commutative $\mathbb{F}_{\pm 1}$-algebra is* alternating *iff all its operations are alternating.*

e) *For any $n \geq 0$ we denote by $K^{alt}(n)$ the subset of $K(n)$, consisting of all alternating n-arity operations. We'll see in a moment that this collection of sets defines a submonad $K^{alt} \subset K$, clearly the largest alternating submonad of K.*

f) *We denote by K_{alt} or $K/_{alt}$ the quotient of K with respect to all alternativity relations $\wedge t$, for all $t \in \|K\|$. This is the largest alternating strict quotient of K.*

5.5.5. Notice that if K' is a submonad of K, and $t \in K'(n)$, then alternativity of t with respect to K is equivalent to its alternativity with respect to K', since both conditions require the same element $\wedge t \in K'(n(n-1)/2) \subset K(n(n-1)/2)$ to be equal to zero.

Let $f : K(m) \to K(n)$ be a K-linear map. Since $\mathrm{Hom}_K(K(m), K(n)) \cong K(n)^m$, f is given by a collection $(f_1, \ldots, f_m) \in K(n)^m$ of n-ary operations. Now let $\Phi : K(n) \times K(n) \to X$ be pseudo-alternating, and put $\Psi := \Phi \circ (f \times f)$. Clearly, Ψ is skew-symmetric, so it is pseudo-alternating iff all $\Psi_{ii} = \Phi(f(\{i\}), f(\{i\})) = \Phi(f_i, f_i)$ are equal to zero, hence *f is alternating iff all its components f_i are alternating.* It is evident that the composite of two alternating maps, say, $t : K(1) \to K(m)$ and $(t_1, \ldots, t_m) : K(m) \to K(n)$, is alternating again; we deduce that $[t]_{K(n)}(t_1, \ldots, t_m)$ is alternating whenever $t \in K(m)$ and all $t_i \in K(n)$ are alternating.

Hence $K^{alt} \subset K$ is closed under composition, and obviously contains all projections $\{i\}_{\mathbf{n}}$; therefore, K^{alt} is indeed an algebraic submonad of K, and $K^{alt}(n) = K(n)$ for $n \leq 1$, all operations of arity ≤ 1 being automatically alternating. Therefore, our operations $0 \in K(0)$ and $- \in K(1)$, such that $-^2 = \boldsymbol{e}$, lie in K^{alt}, hence K^{alt} is a commutative $\mathbb{F}_{\pm 1}$-algebra, clearly alternating.

An immediate consequence is this: *if $S \subset \|K\|$ generates K, then K is alternating iff all operations from S are alternating.* Since all unary operations are automatically alternating, we see that *any pre-unary algebra over an alternating $\mathbb{F}_{\pm 1}$-algebra is itself alternating*, and in particular *any commutative unary $\mathbb{F}_{\pm 1}$-algebra is alternating.*

5.5.6. (Alternativity and binary operations.) Clearly, all constants and unary operations are alternating. Let's consider the case of a binary operation $*$. We see that $\wedge *$ is the unary relation $(0 * \boldsymbol{e}) * (-\boldsymbol{e} * 0) = 0$. For an addition $+$ this is equivalent to $\boldsymbol{e} + (-\boldsymbol{e}) = 0$, i.e. to $-$ being a symmetry for $+$. Since any classical commutative ring is generated over $\mathbb{F}_{\pm 1}$ by its unary operations and its addition, we see that any classical commutative ring is alternating.

5.5.7. (Exterior square of a module.) For any commutative $\mathbb{F}_{\pm 1}$-algebra K and any K-module M we can construct a universal alternating map $\wedge : M \times M \to \bigwedge_K^2 M$, simply by taking the strict quotient of $M \times M$ modulo relations $x \otimes y \equiv -y \otimes x$ and $x \otimes x \equiv 0$. This K-module $\bigwedge_K^2 M$ is called the *exterior square of M*.

When $M = K(n)$ is free and K is alternating, all pseudo-alternating bilinear maps $\Phi : K(n) \times K(n) \to X$ are automatically alternating, condition $\Phi(t,t) = 0$ for any pseudo-alternating Φ being actually equivalent to the alternativity of $t \in K(n)$. Therefore, in this case the universal alternating map on $K(n) \times K(n)$ coincides with the universal pseudo-alternating map of **5.5.3**, hence $\bigwedge_K^2 K(n) \cong K(n(n-1)/2)$, and $\{i\} \wedge \{j\}$, $1 \leq i < j \leq n$ form a base of $\bigwedge_K^2 K(n)$, as one would expect. Notice that in general this is not true when K is not alternating.

5.5.8. (Exterior square of a direct sum.) We want to show that over an alternating K we have

$$\bigwedge\nolimits_K^2 (M_1 \oplus M_2) \cong \bigwedge\nolimits_K^2 M_1 \oplus (M_1 \otimes_K M_2) \oplus \bigwedge\nolimits_K^2 M_2 \tag{5.5.8.1}$$

First of all, consider bilinear maps $\Phi : M \times M' \to X$, where $M = M_1 \oplus M_2$ and $M' = M_1' \oplus M_2'$. Since $\mathrm{Bilin}_K(M, M'; X) \cong \mathrm{Hom}_K(M \otimes_K M', X) \cong \mathrm{Hom}_K(\bigoplus_{i,j} M_i \otimes_K M_j', X) \cong \prod_{i,j} \mathrm{Bilin}_K(M_i, M_j'; X)$, we see that giving such a Φ is equivalent to giving four bilinear maps $\Phi_{ij} : M_i \times M_j' \to X$; of course, these Φ_{ij} are just restrictions of Φ to corresponding components of direct sums.

Now consider the case $M = M'$, $M_i = M_i'$, and $\Phi : M \times M \to X$ skew-symmetric. Then $\Phi_1 := \Phi_{11} : M_1 \times M_1 \to X$ and $\Phi_2 := \Phi_{22}$ are clearly also skew-symmetric, and Φ_{21} is completely determined by Φ_{12} since $\Phi_{21}(y,x) = -\Phi_{12}(x,y)$. Therefore, a skew-symmetric map Φ on M gives rise to two skew-symmetric maps Φ_i on M_i and a bilinear map $\Phi_{12} : M_1 \times M_2 \to$

X. Conversely, any such collection $(\Phi_1, \Phi_2, \Phi_{12})$ allows us to reconstruct a bilinear map Φ (we recover Φ_{21} from Φ_{12} as above), clearly skew-symmetric.

We would like to obtain a similar description of alternating maps on M; this would prove (5.5.8.1), since both sides would represent the same functor. Clearly, if Φ is alternating, then both Φ_1 and Φ_2 are alternating. Let's prove the converse, i.e. let's prove $\Phi(z, z) = 0$ for a $z \in M = M_1 \oplus M_2$, assuming both Φ_1 and Φ_2 to be alternating.

We know that z can be written in form $z = t(x_1, \ldots, x_n, y_1, \ldots, y_m)$, for some n, $m \geq 0$, $t \in K(n+m)$, $x_i \in M_1$, and $y_j \in M_2$ (cf. **4.6.15**). Consider the map $f : K(n+m) \to M$, defined by these elements x_i and y_j. Clearly, $\Psi := \Phi \circ (f \times f)$ is pseudo-alternating: indeed, it is obviously skew-symmetric, and the diagonal elements of its matrix are equal either to some $\Phi(x_i, x_i) = \Phi_1(x_i, x_i)$ or to some $\Phi(y_j, y_j) = \Phi_2(y_j, y_j)$; in both cases we get zero, since both Φ_1 and Φ_2 have been assumed to be alternating. Now the alternativity of K means that any pseudo-alternating form is alternating; in particular, Ψ is alternating, hence $\Phi(z, z) = \Phi(f(t), f(t)) = \Psi(t, t) = 0$ as required.

Definition 5.5.9 *We say that a multilinear map $\Phi : M^n \to X$ is* skew-symmetric, *if it becomes a skew-symmetric bilinear map after an arbitrary choice of all arguments but two, or equivalently, if*

$$\Phi(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) = -\Phi(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n) \text{ for any } i < j \tag{5.5.9.1}$$

Similarly, we say that Φ is alternating *iff it is skew-symmetric, and it vanishes whenever any two of its arguments coincide:*

$$\Phi(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) = 0 \text{ if } x_i = x_j,\ i < j \tag{5.5.9.2}$$

Finally, when $M = K(m)$, we say that Φ is pseudo-alternating *iff it is skew-symmetric, and vanishes on collections (x_j) of base elements of $K(m)$, such that $x_i = x_j$ for some $i \neq j$.*

5.5.10. Clearly, (5.5.9.1) for all $j = i+1$ suffices for Φ to be skew-symmetric, since it implies

$$\Phi(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = \operatorname{sgn}(\sigma) \cdot \Phi(x_1, \ldots, x_n) \text{ for any } \sigma \in \mathfrak{S}_n \tag{5.5.10.1}$$

In order to show this one has just to decompose a permutation $\sigma \in \mathfrak{S}_n$ into a product of elementary transpositions. Similarly, it would suffice to consider $j = i+1$ in (5.5.9.1) and $i = 1$, $j = 2$ in (5.5.9.2) to establish that Φ is alternating.

Finally, one deduces from the bilinear map case that *if K is alternating, any pseudo-alternating multilinear map $\Phi : K(m)^n \to X$ is automatically alternating.*

5.5.11. (Exterior powers of a module.) For any K-module M and any $n \geq 0$ we can construct a universal alternating map $\wedge : M^n \to \bigwedge_K^n M$ by computing the strict quotient of $M^{\otimes n}$ modulo relations coming from (5.5.9.1) and (5.5.9.2). This K-module $\bigwedge_K^n M$ is called the *n-th exterior power of M.*

Notice that $\bigwedge_K^m M \otimes_K \bigwedge_K^n N$ has a universal property among all multilinear maps $M^m \times N^n \to X$, alternating with respect to each group of variables. Putting $N := M$, $X := \bigwedge_K^{n+m} M$, we obtain a canonical homomorphism $\bigwedge_K^m M \otimes_K \bigwedge_K^n M \to \bigwedge_K^{n+m} M$. This yields a graded algebra structure on $\bigwedge_K M := \bigoplus_{n\geq 0} \bigwedge_K^n M$. This algebra is called *the exterior algebra of M.* Clearly, it is a strict quotient of the tensor algebra of M; it has a universal property with respect to K-linear homomorphisms $f : M \to A$ from M into algebras A in K-*Mod*, such that $f(x)f(y) = -f(y)f(x)$ and $f(x)^2 = 0$ for any x, $y \in M$. In other words, we require $M \times M \xrightarrow{f\times f} A \times A \xrightarrow{\mu} A$ to be alternating.

5.5.12. (Exterior algebra of a direct sum.) Notice that $A := \bigwedge_K^n M = A^+ \oplus A^-$, $A^+ := \bigoplus_{n \text{ even}} \bigwedge_K^n M$, $A^- := \bigoplus_{n \text{ odd}} \bigwedge_K^n M$, is a supercommutative algebra in K-*Mod*, i.e. $xy = (-1)^{\deg(x)\deg(y)} yx$ for any x, $y \in A^{\pm}$. We define the tensor product $A \otimes_K B$ of two supercommutative algebras by the usual rule $(x \otimes y)(x' \otimes y') := (-1)^{\deg(x')\deg(y)}(xx' \otimes yy')$, $\deg(x \otimes y) := \deg(x) + \deg(y)$. One checks in the usual way that the resulting superalgebra is supercommutative, and that $A \otimes_K B$ is the coproduct in the category of supercommutative superalgebras in K-*Mod*.

It is easy to see that the exterior algebra $\bigwedge_K M$ together with the canonical map $M \to \bigwedge_K M$ is universal among all pairs (A, φ), where A is a supercommutative algebra, and $\varphi : M \to A^-$ is a K-linear map from M into the odd part of A, such that $\varphi(x)^2 = 0$ for any $x \in M$; this is equivalent to requiring $M \times M \to A^- \times A^- \to A^+$ to be alternating.

Now suppose that K is alternating. Let M_1 and M_2 be two K-modules, $A_i := \bigwedge_K M_i$, $f_i : M_i \to A_i$ the canonical embedding. Then we have a commutative superalgebra $A := A_1 \otimes_K A_2$, and a canonical K-linear map $f : M := M_1 \oplus M_2 \to A$, such that the restriction of f to M_i equals $M_i \xrightarrow{f_i} A_i \to A$. One checks immediately that f maps M into the odd part of A, hence $M \times M \xrightarrow{f\times f} A \times A \to A$ is skew-symmetric; moreover, the restriction of this bilinear map to $M_i \times M_i$ is alternating, hence it is alternating as well (cf. **5.5.8**; we use alternativity of K here). Now we see immediately that (A, f) has the universal property required from $\bigwedge_K M$, so we have proved

$$\bigwedge\nolimits_K (M_1 \oplus M_2) \cong \left(\bigwedge\nolimits_K M_1\right) \otimes_K \left(\bigwedge\nolimits_K M_2\right) \tag{5.5.12.1}$$

over an alternating monad K. Taking individual graded components we obtain (under the same assumption)

$$\bigwedge\nolimits_K^n(M_1 \oplus M_2) \cong \bigoplus_{p+q=n} \left(\bigwedge\nolimits_K^p M_1\right) \otimes_K \left(\bigwedge\nolimits_K^q M_2\right) \tag{5.5.12.2}$$

5.5.13. (Exterior algebra of a free module.) Notice that $\bigwedge_K K(1) = K(1) \oplus K(1)$. When K is alternating, one deduces from (5.5.12.1) and (5.5.12.2) that

$$\bigwedge\nolimits_K^r(M \oplus K(1)) \cong \bigwedge\nolimits_K^r M \oplus \bigwedge\nolimits_K^{r-1} M \tag{5.5.13.1}$$

$$\bigwedge\nolimits_K(M \oplus K(1)) \cong \bigwedge\nolimits_K M \otimes_K (K(1) \oplus K(1)) \tag{5.5.13.2}$$

From this one shows by induction that $\bigwedge_K K(n)$ is a free K-module of rank 2^n. Its basis elements e_I are parametrized by subsets $I \subset \mathbf{n}$ in the usual way: $I = \{i_1, i_2, \ldots, i_r\}$, $1 \leq i_1 < i_2 < \cdots < i_r \leq n$ corresponds to $e_I := \{i_1\} \wedge \{i_2\} \wedge \cdots \wedge \{i_r\}$. Considering individual graded pieces of the exterior algebra we see that the exterior power $\bigwedge_K^r K(n)$ is a free K-module of rank $\binom{n}{r}$, and its basis elements are parametrized by r-element subsets $I \subset \mathbf{n}$. One can extend these results to the exterior powers of $K(S)$, S any linearly ordered set, by observing that exterior powers and algebras commute with filtered inductive limits.

5.5.14. (Exterior algebra and pullbacks.) Notice that the exterior algebra $\bigwedge_K M$, considered as a unary K-algebra, is universal among all pairs (Σ, f), where Σ is a K-algebra, and $f : M \to |\Sigma|$ is a K-linear map, such that $f(x)^2 = 0$ for all $x \in M$. Indeed, the universal property among all unary Σ has been already discussed, and in the general case we can replace Σ by its unary envelope.

Another easy observation: when K is alternating, a skew-symmetric map $\Phi : M \times M \to X$ is alternating iff $\Phi(x, x) = 0$ for all x from a system of generators of M. Combining this with our previous remark we see that (over an alternating K) the exterior algebra $\bigwedge_K M$ is universal among all K-linear maps $f : M \to |\Sigma|$, Σ a K-algebra, such that $f(x)^2 = 0$ for all $x \in S$, where S is any fixed system of generators of M.

Now let $\rho : K \to K'$ be a homomorphism of alternating $\mathbb{F}_{\pm 1}$-algebras, M be a K-module, and S any system of generators of M (e.g. M itself). Then $\bigwedge_{K'} \rho^* M$ is universal among all K'-linear maps $f : \rho^* M \to |\Sigma'|$, with Σ' a K'-algebra, such that $f(\bar{s})^2 = 0$ for all $s \in S$, where $\bar{s}$ denotes the image of $s \in S \subset M$ under the canonical map $M \to \rho^* M$ (we use here the fact that the image of S in $\rho^* M$ is a system of generators). Clearly, such maps are in one-to-one correspondence with K-linear maps $f^\flat : M \to \rho_*|\Sigma| = |\rho_*\Sigma|$, such that $(f^\flat(s))^2 = 0$ for any $s \in S$. Such $f^\flat$ are in one-to-one correspondence with K-algebra homomorphisms $\bigwedge_K M \to \rho_*\Sigma$. Comparing the universal properties involved, we see that, for any ρ and M as above we have

$$\bigwedge_{K'} \rho^* M \cong \rho^*\bigl(\bigwedge_K M\bigr) = K' \otimes_K \bigwedge_K M \tag{5.5.14.1}$$

Considering individual graded pieces we obtain

$$\bigwedge_{K'}^r \rho^* M \cong \rho^*\bigl(\bigwedge_K^r M\bigr) \tag{5.5.14.2}$$

Notice that the alternativity of K' is important in this reasoning.

5.5.15. (Determinants.) Let K be an alternating monad (i.e. an alternating commutative $\mathbb{F}_{\pm 1}$-algebra), M be a free K-module of rank n, and $u \in \mathrm{End}_K(M)$. We have seen in **5.5.13** that $\bigwedge_K^n M$ is a free K-module of rank 1, hence the endomorphism $\wedge^n u$ of this module defines an element $\det(u) \in |K| \cong \mathrm{End}_K(\bigwedge_K^n M)$, such that $(\wedge^n u)(\alpha) = \det(u) \cdot \alpha$ for any $\alpha \in \bigwedge_K^n M$. Of course, we say that $\det(u)$ is the *determinant* of u; this definition can be extended to the case of any K-module M and any integer $n > 0$, such that $\bigwedge_K^n M$ is free of rank one. When $M = K(n)$, $\mathrm{End}_K(K(n)) \cong K(n)^n = M(n,n;K)$ is the set of $n \times n$-matrices over K, so we have a notion of determinant for such matrices as well.

Functoriality of exterior powers implies $\det(\mathrm{id}_M) = \boldsymbol{e}$ and $\det(v \circ u) = \det(v) \cdot \det(u)$, for any u, $v \in \mathrm{End}_K(M)$. Hence *if u is invertible,* $\det(u)$ *is also invertible, and* $\det(u^{-1}) = \det(u)^{-1}$. Unfortunately, the converse doesn't seem to be true in general; we'll discuss this point later in more detail.

It is also easy to check that $\det(u \oplus v) = \det(u)\det(v)$ for any $u \in M(n,n;K)$ and $v \in M(m,m;K)$, so we obtain indeed a reasonable theory of determinants, up to the point already mentioned above. Notice that these determinants are not so easy to compute as one might think. For example, if $A = (T, U) \in K(2)^2$ is a 2×2-matrix, we have

$$\det(A) = T(U(0,\boldsymbol{e}), U(-\boldsymbol{e},0)) = U(T(0,\boldsymbol{e}), T(-\boldsymbol{e},0)) \tag{5.5.15.1}$$

5.5.16. (Examples and constructions of alternating monads.) We see that we obtain a reasonable theory of exterior powers and determinants only over an alternating $\mathbb{F}_{\pm 1}$-algebra K. Therefore, we would like to know that there are sufficiently many of them.

a) First of all, any classical commutative ring is an alternating $\mathbb{F}_{\pm 1}$-algebra (cf. **5.5.6**). Thus $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are alternating.

b) Any $\mathbb{F}_{\pm 1}$-subalgebra K' of an alternating algebra K is also alternating, since the alternativity condition $\wedge t$ for an operation $t \in K'(n)$ in K' is equivalent to the corresponding condition in K (cf. **5.5.5**). Hence $\mathbb{Z}_\infty$, $\bar{\mathbb{Z}}_\infty$ and $\mathbb{Z}_{(\infty)}$ are alternating, being $\mathbb{F}_{\pm 1}$-subalgebras of $\mathbb{C}$.

c) Recall that an $\mathbb{F}_{\pm 1}$-algebra K is alternating iff it is generated by a system of alternating operations (cf. **5.5.5**), and that all operations of arity ≤ 1 are automatically alternating. Therefore, all $\mathbb{F}_{\pm 1}$-algebras (absolutely) generated in arity ≤ 1 are alternating, e.g. $\mathbb{F}_{\pm 1}$ itself, $\mathbb{F}_{1^n}$ for an even $n > 0$, and $\mathbb{F}_{1^\infty}$.

d) Notice that any projective limit of alternating $\mathbb{F}_{\pm 1}$-algebras is itself alternating. Indeed, if $K = \varprojlim K_\alpha$, then any $t \in K(n) = \varprojlim K_\alpha(n)$ is given by a compatible family of $t_\alpha \in K_\alpha(n)$, and the alternativity condition $\wedge t$ in $K(n(n-1)/2)^2$ is given by the compatible family of alternativity conditions $\wedge t_\alpha$, hence if all $\wedge t_\alpha$ hold, the same is true for $\wedge t$. In particular, $\mathbb{F}_{\pm 1^{-\infty}} = \varprojlim_{n \text{ odd}} \mathbb{F}_{1^{2n}}$ (cf. **5.1.16**,c) is alternating.

e) Any strict quotient of an alternating algebra is clearly alternating. Hence $\mathbb{F}_\infty$ is alternating, being a strict quotient of $\mathbb{Z}_\infty$. We might also check the alternativity condition $(0 * \boldsymbol{e}) * (-\boldsymbol{e} * 0) = 0$ for the binary generator $*$ of $\mathbb{F}_\infty$ directly, using $\mathbb{F}_\infty = \mathbb{F}_{\pm 1}[*^{[2]} \,|\, \boldsymbol{e} * (-\boldsymbol{e}) = 0,$ $\boldsymbol{e} * \boldsymbol{e} = \boldsymbol{e}$, $x * y = y * x$, $(x * y) * z = x * (y * z)]$.

f) Since an alternating operation remains alternating after an application of a homomorphism of commutative $\mathbb{F}_{\pm 1}$-algebras, we see that all sorts of inductive limits of alternating $\mathbb{F}_{\pm 1}$-algebras are alternating. For example, if K is a commutative $\mathbb{F}_{\pm 1}$-algebra, and K_1 and K_2 are alternating K-algebras, then $K_1 \otimes_K K_2$ is also alternating, being generated by the operations of K_1 and K_2.

g) Any (commutative) pre-unary algebra K' over an alternating $\mathbb{F}_{\pm 1}$-algebra K is alternating, since it is generated by the operations of K together with a family of unary, hence automatically alternating generators. For example, unary polynomial rings $\mathbb{F}_{\pm 1}[T^{[1]}, U^{[1]}, \ldots]$ and $\mathbb{Z}_\infty[T^{[1]}, U^{[1]}, \ldots]$ are alternating.

h) This is not true for binary polynomial rings: for example, $\mathbb{F}_{\pm 1}[T^{[2]}]$ is not alternating, i.e. $T(T(0, \boldsymbol{e}), T(-\boldsymbol{e}, 0)) \neq 0$ in this generalized ring (otherwise any binary operation would have been alternating!). We can consider the largest alternating quotients $\mathbb{F}_{\pm 1}[T_1^{[r_1]}, \ldots]/_{alt}$ of these polynomial rings instead, thus obtaining free alternating $\mathbb{F}_{\pm 1}$-algebras.

5.5.17. (Existence of $SL_{n,\mathbb{F}_{\pm 1}}$.) Notice that we can define an affine group scheme SL_n over $\mathbb{F}_{\pm 1}$, such that for any *alternating* $\mathbb{F}_{\pm 1}$-algebra K we have

$$SL_n(K) = \{g \in GL_n(K) \subset M(n, n; K) \,|\, \det(g) = \boldsymbol{e}\} \tag{5.5.17.1}$$

Of course, $SL_{n,\mathbb{Z}}$ coincides with the classical group scheme denoted in this way, since they both represent the same functor.

5.6. (Matrices with invertible determinant.) It would be nice to know that all matrices with invertible determinants are themselves invertible. However, we haven't managed to prove or disprove this statement so far. Therefore, we'll content ourselves with some partial results.

5.6.1. We consider the following properties of an alternating $\mathbb{F}_{\pm 1}$-algebra K:
(DET_r) Any $r \times r$-matrix $A \in M(r, r; K)$ with invertible determinant is invertible.
(DET'_r) For any $r \times r$-matrix $A \in M(r, r; K)$ one can find an integer $N \geq 1$ and a matrix $B \in M(r, r; K)$, such that $AB = BA = \det(A)^N$.
(DET^*_r) Same as above, but with N independent on the choice of A.
$(DET^?_\infty)$ means that the properties $(DET^?_r)$ are fulfilled for all $r \geq 0$; here the superscript ? can be replaced by $'$, $*$, or nothing at all.

Notice that these three properties trivially hold for $r \leq 1$ (with $N = 1$), so we might consider only $r \geq 2$. Also note that if (DET'_r) is satisfied for some $N \geq 1$, we can replace N by any $N' \geq N$: indeed, we just have to consider $B' := B \cdot \det(A)^{N'-N}$.

Clearly, $(DET^*_r) \Rightarrow (DET'_r) \Rightarrow (DET_r)$; hence (DET^*_∞) is the strongest among all conditions under consideration.

5.6.2. An important point is that $(DET^?_{r+1})$ implies $(DET^?_r)$, hence $(DET^?_r)$ implies all $(DET^?_s)$ with $0 \leq s \leq r$. Notice that this is true for $r = \infty$ as well.

So let's fix a matrix $A \in M(r, r; K)$ as required in $(DET^?_r)$, and put $A' := A \oplus \boldsymbol{e} \in M(r+1, r+1; K)$. Let's denote by $\lambda : K(r) \to K(r+1)$ and $\sigma : K(r+1) \to K(r)$ the natural embedding and projection (σ maps $\{r+1\}$ into 0). Then $\sigma\lambda = I_r = \mathrm{id}_{K(r)}$, and obviously $\sigma A' \lambda = A$ and $\det(A') = \det(A)$. Now let's apply $(DET^?_{r+1})$ to A'. We obtain a matrix $B' \in M(r+1, r+1; K)$, such that $A'B' = B'A' = \det(A')^N = \det(A)^N$ (with $N = 0$ for the (DET_r) property). Now put $B := \sigma B' \lambda \in M(r, r; K)$. Clearly, $AB = A\sigma B'\lambda = \sigma A' B' \lambda = \det(A)^N \cdot \sigma I_{r+1} \lambda = \det(A)^N \cdot I_r$, and similarly $BA = \sigma B' \lambda A = \sigma B' A' \lambda = \det(A)^N \cdot I_r$, i.e. $(DET^?_r)$ holds for A.

5.6.3. Notice that properties (DET^*_r) and (DET'_r) are stable under taking strict quotients, i.e. if $\rho : K \to \bar{K}$ is a strict epimorphism, and if one of these properties is valid for K, it is valid for K' as well, with the same value of $N = N(r)$ for the (DET^*_r) property. Indeed, the induced maps $\rho^r_r : M(r, r; K) = K(r)^r \to M(r, r; \bar{K})$ are surjective, so we can start with an arbitrary matrix $\bar{A} \in M(r, r; \bar{K})$, lift it arbitrarily to a matrix $A \in M(r, r; K)$, find a matrix $B \in M(r, r; K)$ that satisfies (DET'_r), and consider $\bar{B} := \rho^r_r(B) \in M(r, r; \bar{K})$. However, this reasoning is not applicable to the (DET_r) properties, since in general we have no means to lift a matrix with invertible determinant $\bar{A}$ to a matrix A over K without violating the invertibility of the determinant.

An immediate application is that if we show that $\mathbb{Z}_\infty$ satisfies (DET^*_∞) (the strongest of our properties), the same will be true for $\mathbb{F}_\infty$, and in particular any matrix over $\mathbb{F}_\infty$ with invertible determinant will be invertible. Actually, one can check (DET_r) for $\mathbb{F}_\infty$ directly, by observing that the matrices with non-zero, i.e. invertible determinant in $M(r, r; \mathbb{F}_\infty)$ are exactly the matrices from $GL_r(\mathbb{F}_{\pm 1}) \subset M(r, r; \mathbb{F}_\infty)$, i.e. the permutation matrices $(\pm\{\sigma_1\}, \ldots, \pm\{\sigma_r\})$, with $\sigma \in \mathfrak{S}_r$.

5.6.4. ((DET^*_∞) for $\mathbb{Z}_\infty$ and $\bar{\mathbb{Z}}_\infty$.) One can try to show directly that $\mathbb{Z}_\infty$ and $\bar{\mathbb{Z}}_\infty$ satisfy (DET^*_∞), as follows. A matrix $A \in M(r, r; \bar{\mathbb{Z}}_\infty)$ is actually a matrix $A = (a_{ij}) \in M(r, r; \mathbb{C}) = \mathbb{C}^{\mathbf{r}\times\mathbf{r}}$ with the L_1-norm $\|A\|_1 \leq 1$, i.e. with $\sum_i |a_{ij}| \leq 1$ for all j. We have to check that the matrix $B := A^* \cdot \Delta^{N-1}$ has L_1-norm ≤ 1 for some $N = N(r) > 0$, where A^* is the adjoint matrix of A and $\Delta := \det A$, i.e. that $\|A^*\|_1 \leq |\Delta|^{1-N}$ for all $A \in M(r, r; \mathbb{C})$ with $\|A\|_1 \leq 1$. We can remove the latter condition by rewriting our inequality:

$$\|A^*\|_1 \cdot |\det A|^{N_r - 1} \leq \|A\|_1^{rN_r - 1} \tag{5.6.4.1}$$

(The $\mathbb{Z}_\infty$ case then follows, since if this inequality holds for matrices with complex coefficients, it holds *a fortiori* for matrices with real coefficients.) This inequality can be checked directly; for example, for $r = 2$ we can take $N_r = 2$, thus obtaining the following inequality:

$$\bigl(|a_{11}| + |a_{21}|\bigr) \cdot \bigl|a_{11}a_{22} - a_{12}a_{21}\bigr| \leq \sup\bigl(|a_{11}| + |a_{12}|, |a_{21}| + |a_{22}|\bigr)^3 \tag{5.6.4.2}$$

We don't provide more details since we still hope to prove a more general result that would include unary polynomial rings over $\mathbb{Z}_\infty$ and $\bar{\mathbb{Z}}_\infty$ as well.

5.6.5. Another simple observation is that all classical rings satisfy (DET^*_r) with $N = 1$. Indeed, we might take $B = A^*$, the adjoint matrix of A; then $AA^* = A^*A = \det(A)$.

5.6.6. (Universal matrix.) Fix an integer $r \geq 0$ and consider the free alternating algebra $\Lambda := \mathbb{F}_{\pm 1}[T_1^{[r]}, \ldots, T_r^{[r]}]/_{alt}$ and the "universal $r \times r$-matrix" $A := (T_1, \ldots, T_r) \in \Lambda(r)^r = M(r, r; \Lambda)$. Clearly, (Λ, A) represents the functor $K \mapsto M(r, r; K)$ on the category of alternating $\mathbb{F}_{\pm 1}$-algebras. We see that if (DET'_r) is valid just for this matrix A over Λ, with some value of $N = N_r$, then the (DET'_r) and (DET^*_r), hence also (DET_r) properties are fulfilled for *all* alternating generalized rings, with the same value of N_r. Therefore, the validity of (DET'_r) for all alternating monads is equivalent to the validity of (DET^*_r) for all alternating monads.

5.6.7. (Universal matrix with invertible determinant.) In the above notations put $\Delta := \det A \in |\Lambda|$ and consider the localization $\tilde{\Lambda} := \Lambda[\Delta^{-1}] =$

$\Lambda[\bar{\Delta}^{[1]} \mid \Delta\bar{\Delta} = \boldsymbol{e}]$; this is an alternating $\mathbb{F}_{\pm 1}$-algebra, since it is unary over Λ. Clearly, the image $\tilde{A}$ of A in $M(r, r; \tilde{\Lambda})$ is the "universal $r \times r$-matrix with invertible determinant", i.e. $(\tilde{\Lambda}, \tilde{A})$ represents the functor $K \mapsto \{r \times r$-matrices over K with invertible determinant$\}$. Suppose that (DET_r) is valid just for this matrix $\tilde{A}$ over $\tilde{\Lambda}$, so that there is an inverse matrix $\tilde{B}$ to $\tilde{A}$. We'll see in the next chapter that the localization $\tilde{\Lambda} = \Lambda[\Delta^{-1}]$ admits the usual description, i.e. $\tilde{\Lambda}(n)$ consists of expressions t/Δ^k, $t \in \Lambda(n)$, $k \geq 0$, and $t/\Delta^k = t'/\Delta^{k'}$ iff $\Delta^{k'+m} \cdot t = \Delta^{k+m} \cdot t'$ for some $m \geq 0$. Using this fact we can write $\tilde{B} \in M(r, r; \tilde{\Lambda}) = \tilde{\Lambda}(r)^r$ in form B/Δ^m for some $B \in M(r, r; \Lambda)$. The equalities $(A/1) \cdot (B/\Delta^m) = I_r/1 = (B/\Delta^m) \cdot (A/1)$ imply the existence of some $N \geq m$, such that $A \cdot (\Delta^{N-m} B) = \Delta^N \cdot I_r = (\Delta^{N-m} B) \cdot A$ in $M(r, r; \Lambda)$. Therefore, in this case (DET'_r) is fulfilled for the universal matrix A with this value of $N = N_r$, hence (DET^*_r) is valid for all alternating monads with the same value of N_r by **5.6.6**. We see that (DET_r) *holds over all alternating monads iff it holds for* $\tilde{A}$ *over* $\tilde{\Lambda}$ *iff* (DET^*_r) *holds over all alternating monads*, and in this case *an universal value of* $N = N_r$ *can be chosen for all alternating monads.*

5.6.8. We see that proving or disproving the validity of (DET^*_r), (DET'_r) or (DET_r) for all alternating monads is equivalent to proving or disproving (DET'_r) for the universal matrix $A = (T_1, \ldots, T_r)$ over alternating generalized ring $\Lambda = \mathbb{F}_{\pm 1}[T_1^{[r]}, \ldots, T_r^{[r]}]/_{alt}$, so we'll concentrate our efforts on this case.

Let's introduce a technique that enables one to obtain lower bounds for the (universal) N_r, and to obtain good candidates for the matrix B of (DET'_r) for high values of N. We consider for this the canonical map $\Lambda \to \Lambda_{\mathbb{Z}} = \mathbb{Z} \otimes_{\mathbb{F}_{\pm 1}} \Lambda = \mathbb{Z}[T_1^{[r]}, \ldots, T_r^{[r]}] \cong \mathbb{Z}[T_{11}, T_{12}, \ldots, T_{1r}, T_{21}, \ldots, T_{rr}]$ (cf. **5.1.21**), where $(T_{i1}, \ldots, T_{ir})$ is the image of T_i under the comparison map $\pi_r : \Lambda(r) \to |\Lambda|^r$ (cf. **4.8.1**), e.g. $T_{i1} = T_i(\boldsymbol{e}, 0, \ldots, 0)$. Clearly, the map $\Lambda \to \Lambda_{\mathbb{Z}}$ maps $T_i \in \Lambda(r)$ into $T_{i1}\{1\} + \cdots + T_{ir}\{r\} \in \Lambda_{\mathbb{Z}}(r)$; we'll write this in the following form:

$$T_i = T_{i1}\{1\} + T_{i2}\{2\} + \cdots + T_{ir}\{r\} \quad (\text{over } \Lambda_{\mathbb{Z}}) \tag{5.6.8.1}$$

Let us denote by Λ' the image of $\Lambda \to \Lambda_{\mathbb{Z}}$, and by A', B' and Δ' the images under this map of the universal matrix $A = (T_1, \ldots, T_r)$, the (hypothetical) matrix B with property (DET_r) with respect to A and some integer $N = N_r \geq 1$, and the determinant $\Delta := \det(A)$.

Clearly, $A' = (T_{ji})$, so that $\Delta' = \det A' = \det(T_{ij})$ is the determinant of the universal matrix over $\mathbb{Z}$. If we have a matrix B as in (DET_r), we have $A'B' = (\Delta')^N = B'A'$ over $\Lambda' \subset \Lambda_{\mathbb{Z}} \subset \mathbb{Q}(T_{11}, T_{12}, \ldots, T_{rr})$. Since the determinant of A' is invertible in this field, A' is itself invertible, hence $B' = (\Delta')^N \cdot (A')^{-1} = (\Delta')^{N-1} \cdot (A')^*$ is completely determined by $N > 0$.

On the other hand, the components $B'_j = (b'_{1j}, b'_{2j}, \ldots, b'_{rj}) \in |\Lambda_{\mathbb{Z}}|^r =$

$\Lambda_{\mathbb{Z}}(r)$ of the matrix B' have to lie in $\Lambda'(r)$. Since $\Lambda' \subset \Lambda_{\mathbb{Z}}$ is generated by the images of the T_j in $\Lambda_{\mathbb{Z}}$, we see that $\Lambda'(r) \subset \Lambda_{\mathbb{Z}}^r$ consists of expressions that can be obtained by applying the following rules finitely many times:

- 0 and $\pm\{i\}$, for $1 \leq i \leq r$, belong to $\Lambda'(r)$;
- If $f_1, \ldots, f_r$ lie in $\Lambda'(r)$, the same is true for $T_{i1}f_1 + \cdots + T_{ir}f_r$, for any $1 \leq i \leq r$.

If (DET'_r) holds for A with some $N \geq 0$, then it holds for A' and the same N as well, hence the matrix $B' = (\Delta')^N \cdot (A')^{-1}$ lies in $\Lambda'(r)$. Clearly, for $r \geq 2$ this can happen only for $N > 0$, so that $B' = (\Delta')^{N-1} \cdot (A')^*$, i.e. $B = (b_{ij})$ with $b_{ij} = (-1)^{i+j}\Delta_{ji} \cdot \Delta'^{N-1}$, where Δ_{ji} is the corresponding principal minor of A'. Therefore, we must have $(b_{11}, b_{21}, \ldots, b_{r1}) \in \Lambda'(r)$ (and similarly for the other columns of B', but by symmetry it suffices to consider only the first column).

Therefore, if for some value of $N > 0$ we can show that $(b_{11}, \ldots, b_{r1})$ doesn't lie in $\Lambda'(r)$, then the value of universal N_r is at least $N + 1$. On the other hand, if this vector lies in $\Lambda'(r)$, then (DET'_r) is fulfilled for A' and this value of N. In this case we might consider different lifts B of B' to Λ and check whether (DET'_r) is fulfilled for any of them. Notice that if we would show the injectivity of $\Lambda \to \Lambda_{\mathbb{Z}}$, the latter step would be unnecessary.

5.6.9. Let's apply the above considerations to the first non-trivial case $r = 2$; the case $r \geq 3$ seems to be more complicated only from the technical point of view, so all the interesting issues can be seen already for $r = 2$.

In this case we write T and U instead of T_1 and T_2; so we have $\Lambda = \mathbb{F}_{\pm 1}[T^{[2]}, U^{[2]}]/_{alt}$, $A = (T, U)$, $\Lambda_{\mathbb{Z}} = \mathbb{Z}[T_1, T_2, U_1, U_2]$, where of course $T_1 = T(\boldsymbol{e}, 0)$, $T_2 = T(0, \boldsymbol{e})$ and so on. By (5.5.15.1) we obtain $\Delta = \det A = T(U(0, \boldsymbol{e}), U(-\boldsymbol{e}, 0)) = U(T(0, -\boldsymbol{e}), T(\boldsymbol{e}, 0))$; the image Δ' of Δ in $\Lambda_{\mathbb{Z}}$ is of course $T_1U_2 - T_2U_1$, and $A' = \left(\begin{smallmatrix} T_1 & U_1 \\ T_2 & U_2 \end{smallmatrix}\right)$, $A'^* = \left(\begin{smallmatrix} U_2 & -U_1 \\ -T_2 & T_1 \end{smallmatrix}\right)$.

We denote the components of B by (V, W); clearly, V and $W \in \Lambda(2)$ have to satisfy

$$T(V(\{1\}, \{2\}), W(\{1\}, \{2\})) = \Delta^N \cdot \{1\} \tag{5.6.9.1}$$

$$U(V(\{1\}, \{2\}), W(\{1\}, \{2\})) = \Delta^N \cdot \{2\} \tag{5.6.9.2}$$

$$V(T(\{1\}, \{2\}), U(\{1\}, \{2\})) = \Delta^N \cdot \{1\} \tag{5.6.9.3}$$

$$W(T(\{1\}, \{2\}), U(\{1\}, \{2\})) = \Delta^N \cdot \{2\} \tag{5.6.9.4}$$

We denote by B' the image of B in $\Lambda' \subset \Lambda_{\mathbb{Z}}$; its components $B' = (V', W')$ are given by $V' = U_2\Delta^{N-1}\{1\} - T_2\Delta^{N-1}\{2\}$ and $W' = -U_1\Delta^{N-1}\{1\} + T_1\Delta^{N-1}\{2\}$. We want to find those values of $N \geq 1$, for which V' and W' lie indeed in Λ', and find possible values of V and W for such values of N.

5.6.10. (Elements of Λ.) Notice that $\Lambda = \mathbb{F}_{\pm 1}[T^{[2]}, U^{[2]}]/_{alt}$ is generated by one constant 0, one unary operation $-$, and two binary operations T and U. Therefore, elements of $\Lambda(n)$ are formal expressions (terms, cf. **4.5.6**), usually written in prefix form, constructed with the aid of these operations from free variables $\{i\}_{\mathbf{n}}$, $1 \leq i \leq n$. Sometimes we omit braces around $\{i\}$, since positive integers have no other possible meaning in this context; thus we can write $T\ -1\,U\,2\,0$ instead of $T(-\{1\}, U(\{2\}, 0))$. Let us list all rules for the manipulation of these expressions, including implicit commutativity and alternativity relations; here x, y, z, $\ldots$ replace either free variables or arbitrary valid expressions:

1. $--x = x$, $-0 = 0$, $-Txy = T\ -x\ -y$, $-Uxy = U\ -x\ -y$;
2. $T00 = 0$, $U00 = 0$;
3. $T\,Txy\,Tzw = T\,Txz\,Tyw$; $\quad T\,Uxy\,Uzw = U\,Txz\,Tyw$;
   $U\,Uxy\,Uzw = U\,Uxz\,Uyw$;
4. $T\,T0x\,T-x0 = 0$; $\quad U\,U0x\,U-x0 = 0$.

The relation $--x = x$ is nothing else than the relation $-^2 = \boldsymbol{e}$ from the definition of $\mathbb{F}_{\pm 1}$, the last group of relations are just the alternativity relations for T and U, and the remaining relations are the commutativity relations between all pairs of generators.

Notice that the first set of relations allows us to rewrite any valid expression in such a way that unary $-$ is applied only to free variables. This means that we can represent valid expressions by means of binary trees, with interior nodes marked by T and U, and leaves marked by 0 or $\pm i$, $1 \leq i \leq n$ (of course, $+i$ or i stands for $\{i\}_{\mathbf{n}}$, and $-i$ for $-\{i\}_{\mathbf{n}}$).

When we need to define a new operation in $\Lambda(n)$, we usually write down either formulas like $Z\,x_1 \ldots x_n = \langle\text{expression in variables } \pm x_i\rangle$, or formulas like $Z = \langle\text{expression in } \pm i,\ 1 \leq i \leq n\rangle$. For example, Δ can be defined either by $\Delta\,z = T\ U\,0\,z\ U\ -z\,0$, or by $\Delta = T\ U\,0\,1\ U\ -1\,0$.

5.6.11. (Elements of Λ'.) Given two elements f and $f' \in \Lambda(n)$, we write $f \sim f'$ and say that f *and* f' *are weakly equivalent* if they have the same image $\bar{f}$ in $\Lambda'(n)$, or equivalenty, in $\Lambda_{\mathbb{Z}}(n)$. Then $\Lambda'(n)$ is in one-to-one correspondence with weak equivalence classes of elements of $\Lambda(n)$, and these elements can be represented by binary trees as described above. Of course, the image $\bar{f}$ of f in $\Lambda_{\mathbb{Z}}$ is computed by means of the following very simple rules:

- $0 \in \Lambda(n)$ is mapped into $0 = (0, \ldots, 0) \in \Lambda_{\mathbb{Z}}(n)$.

- $\pm i$, i.e. $\pm\{i\}_{\mathbf{n}}$ is mapped into $\pm\{i\} = (0, \ldots, \pm 1, \ldots, 0)$ in $\Lambda_{\mathbb{Z}}(n)$.
- If $f = T(f_1, f_2)$, i.e. if f is described by a tree with root marked by T, left subtree f_1, and right subtree f_2, then $\bar{f} = T_1 \cdot \bar{f}_1 + T_2 \cdot \bar{f}_2$.
- Similarly, $f = U(f_1, f_2)$ is mapped into $\bar{f} = U_1 \cdot \bar{f}_1 + U_2 \cdot \bar{f}_2$

We see that any leaf contributes some monomial into $\bar{f}$. More precisely, a leaf marked by zero contributes nothing, and a leaf marked by $\pm i$ contributes $\pm T_1^a T_2^b U_1^c U_2^d \{i\}$, where a is the amount of left branches, taken after vertices marked by T in the path from the root of the tree to the leaf under consideration, and integers b, c, $d \geq 0$ are defined similarly.

An immediate consequence is that *replacing f by a weakly equivalent tree if necessary, we can assume that there is no cancellation between contributions of separate leaves.* Indeed, suppose that $+T_1^a T_2^b U_1^c U_2^d \{i\}$ cancels with $-T_1^a T_2^b U_1^c U_2^d \{i\}$. These two terms are contributed by two leaves, one of them marked by i, and the other by $-i$. Then we can simply replace both labels by 0; since this operation doesn't change $\bar{f}$, but decreases the number of non-zero labels on leaves of f, this process will stop after a finite number of steps.

Using relations $T\,0\,0 = U\,0\,0 = 0$ if necessary, we see that *if $\bar{f}$ is homogeneous of degree d, f is weakly equivalent to a complete binary tree of depth d, i.e. to a binary tree, in which all leaves are at distance d from the root. We can also assume that there are no cancellations between contributions of different leaves of f.* Clearly, there are 2^d leaves in this case; each of these leaves can be labeled by 0 or $\pm i$. There are also $2^d - 1$ intermediate nodes as well, labeled by letters T and U.

5.6.12. (Lower bound for universal N_2.) Let's apply these consideration to obtain a lower bound for the universal N_2. So suppose that (DET_2') holds for the universal 2×2-matrix A for some $N > 0$. Then $V' = U_2 \Delta^{N-1}\{1\} - T_2 \Delta^{N-1}\{2\}$, $\Delta = T_1 U_2 - T_2 U_1$, is of form $\bar{V}$ for some binary tree V. Notice that V' is homogeneous of degree $2N - 1$. Using the above results we can replace V by a complete binary tree of depth $2N - 1$, such that there is no cancellation between contributions of individual leaves. Now let's consider individual monomials in V'. We see that all of them are of form $\pm T_1^a T_2^b U_1^c U_2^d \{i\}$ with $a + c = N - 1$, $b + d = N$, and that V' is the sum of 2^N such monomials (one checks that there is no cancellation between individual monomials in the expression for V' by substituting $T_1 := U_1 := U_2 := 1$, $T_2 := -1$). On the other hand, the only leaves in the complete binary tree of depth $2N - 1$ that might contribute such monomials are those that correspond to $(2N-1)$-digit binary numbers with $N - 1$ zeroes and N ones, if we agree to encode the path from the root to a leaf by means of a sequence of $2N-1$ binary digits, where 0 corresponds to taking the left branch, and 1 to the right branch. There are $\binom{2N-1}{N}$ such leaves, hence we can have at most $\binom{2N-1}{N}$ such monomials in $\bar{V}$. We have proved that the following inequality is necessary for the existance of V with $\bar{V} = V'$:

$$\binom{2N-1}{N} \geq 2^N \tag{5.6.12.1}$$

One checks immediately that this condition is equivalent to $N \geq 3$: indeed, for $N = 1$ we have $\binom{1}{1} = 1 < 2^1$, for $N = 2$ we have $\binom{3}{2} = 3 < 2^2$, but for $N = 3$ we get $\binom{5}{3} = 10 > 2^3$, and the inequality holds for $N > 3$ as well. We conclude that *universal* N_2 *is at least* 3. Notice that for $\mathbb{Z}_\infty$ and $\bar{\mathbb{Z}}_\infty$ we might take $N_2 = 2$, and for any classical ring we might even take $N_2 = 1$.

5.6.13. (Case $r = 2$, $N = 3$.) Consider the case $r = 2$, $N = 3$. Then $V' = (T_1U_2 - T_2U_1)^2(U_2\{1\} - T_2\{2\}) = T_1^2U_2^3\{1\} - 2T_1T_2U_1U_2^2\{1\} + T_2^2U_1^2U_2\{1\} - T_1^2T_2U_2^2\{2\} + 2T_1T_2^2U_1U_2\{2\} - T_3^2U_1^2\{2\}$. We want to find a tree $V \in \Lambda(2)$, such that $\bar{V} = V'$. We know that if such a V exists, it can be replaced by a weakly equivalent complete binary tree of depth 5, without cancellation of contributions of different leaves. So in order to find such a V we can draw a complete binary tree of depth 5, mark by a "?" those leaves that are encoded by a binary number with 3 ones and 2 zeroes (there will be $\binom{5}{3} = 10$ such leaves), mark the remaining $32 - 10 = 22$ leaves by 0, and then try to replace eight question marks with labels $\pm i$, $i = 1, 2$ (each label must occur exactly twice) and two remaining question marks by zeroes, and fill in the intermediate nodes (there are 31 of them) by letters T and U, so as to have the sum of contributions of leaves equal to V'.

It is easy to find at least one such configuration; using rules $T00 = U00 = 0$ we can replace subtrees with no non-zero leaves by a zero, thus obtaining a shorter expression for a V with $\bar{V} = V'$. Here is one possible value for such V:

$$V = TTU0U0U01U0UU0{-}1U{-}10TTU0U0{-}2UU020UUT0{-}2T20UU100 \tag{5.6.13.1}$$

Of course, we can obtain a W with $\bar{W} = W'$ by replacing Txy with Uyx and Uxy with Tyx everywhere in the expression for V, as well as $\pm\{i\}$ with $\pm\{3-i\}$; in other words, we reflect the tree for V with respect to the vertical line and interchange $T \leftrightarrow U$ in the inner nodes and $\pm 1 \leftrightarrow \pm 2$ in the leaves.

We can even find all possible values of V, arising from a complete binary tree of depth 5; this is slightly more complicated than it seems since we might have some cancellation, i.e. we have to consider trees with more than eight non-zero leaf labels as well. However, this problem can be tackled with the aid of a computer.

Then we might take all values of V thus obtained, try to check whether $VTxyUxy = \Delta^3 x$ for these values of V, then obtain the list of possible values of W by reflecting the list of possible values of V as described above. These W would satisfy $WTxyUxy = \Delta^3 y$. Then we might consider all pairs (V, W) from the direct product of these two sets of possible values and check whether $TVxyWxy = \Delta^3 x$ and $UVxyWxy = \Delta^3 y$. If this is true for some pair (V, W), we can put $B := (V, W) \in M(2, 2; \Lambda)$ and conclude that (DET_2') holds for this matrix with $N = 3$, and that universal $N_2 = 3$. On the other hand, if these conditions would fail for all (V, W), we might consider larger values of N in a similar manner.

Unfortunately, this procedure seems to be quite complicated to implement, on a computer or without it, since we don't have a reasonable way of checking whether two terms define the same element of $\Lambda(r)$ or not. This is the reason why we haven't managed to prove or disprove "universal $N_2 = 3$" so far.

Another interesting remark: one might try to use another procedure to find all possible values of V. Namely, since we must have $VTxyUxy = \Delta^3 x$, we might start by listing all complete binary tree of depth 6 corresponding to $\Delta^3\{1\}$, such that their lower level consists only of expressions like $T12$, $T-1-2$, $U12$, $U-1-2$, $T00$ and $U00$, and replace these expressions by 1, -1, 2, -2, 0 and 0, respectively; this would give us all candidates for V, at least among complete binary trees of depth 5.

5.6.14. ((DET_2^*) for torsion-free alternating algebras.) Notice that we have just proved that (DET_2') is fulfilled for the matrix A' over Λ' for $N = 3$, since we have shown that V' and W' lie in $\Lambda'(2)$, and not just in $\Lambda_{\mathbb{Z}}(2)$. This result is sufficient by itself to make some interesting conclusions.

Namely, we say that an $\mathbb{F}_{\pm1}$-algebra K is *torsion-free* if the canonical homomorphism $K \mapsto K_{\mathbb{Z}} := \mathbb{Z} \otimes_{\mathbb{F}_{\pm1}} K$ is injective, i.e. a monomorphism. Clearly, an $\mathbb{F}_{\pm1}$-algebra K is torsion-free iff K can be embedded into a (classical) $\mathbb{Z}$-algebra. In particular, $\Lambda' \subset \Lambda_{\mathbb{Z}}$ is torsion-free, and any $\mathbb{F}_{\pm1}$-algebra K admits a largest torsion-free quotient, namely, the image of $K \to K_{\mathbb{Z}}$. Another observation: any commutative torsion-free $\mathbb{F}_{\pm1}$-algebra is alternating, being a $\mathbb{F}_{\pm1}$-subalgebra of a classical commutative $\mathbb{Z}$-algebra $K_{\mathbb{Z}}$.

Notice that any homomorphism $f : \Lambda = \mathbb{F}_{\pm1}[T^{[2]}, U^{[2]}]/_{alt} \to K$ with a torsion-free K factorizes through $\Lambda \to \Lambda'$; this follows from the commutativity of the following diagram together with the fact that Λ' is the image of

$\Lambda \to \Lambda_{\mathbb{Z}}$:

$$\begin{tikzcd} \Lambda \arrow[r] \arrow[d, "f"] & \Lambda_{\mathbb{Z}} \arrow[d, "f_{\mathbb{Z}}"] \\ K \arrow[r] & K_{\mathbb{Z}} \end{tikzcd} \tag{5.6.14.1}$$

This implies that Λ' and $A' \in M(2,2;\Lambda')$ represent the functor $K \mapsto M(2,2;K)$ on the category of *torsion-free* (alternating) $\mathbb{F}_{\pm 1}$-algebras. Since we have shown that (DET_2') holds for A' with $N = 3$, we can conclude that (DET_2^*) *is fulfilled for all (alternating) torsion-free* $\mathbb{F}_{\pm 1}$*-algebras with* $N = 3$.

5.6.15. (Torsion-free algebras over $\mathbb{Z}_\infty$ and $\bar{\mathbb{Z}}_\infty$.) Notice that $\mathbb{Z}_\infty$ and $\bar{\mathbb{Z}}_\infty$ are torsion-free $\mathbb{F}_{\pm 1}$-algebras, since they can be embedded into $\mathbb{C}$. Another simple statement is that *any unary polynomial algebra* A *over a torsion-free* $\mathbb{F}_{\pm 1}$*-algebra* K *is itself torsion-free.* Indeed, a filtered inductive limit argument reduces everything to the case of $A = K[T_1^{[1]}, \ldots, T_n^{[1]}]$ (with finite n), and then an induction argument shows that it is sufficient to treat the case $A = K[T^{[1]}]$. Then K is torsion-free, hence $K \to K_{\mathbb{Z}}$ is a monomorphism, i.e. all $K(n) \to K_{\mathbb{Z}}(n)$ are injective; taking filtered inductive limits we see that $K(S) \to K_{\mathbb{Z}}(S)$ is injective for infinite sets S as well. Now consider the map $A(n) \to A_{\mathbb{Z}}(n)$. We know that $A(n) \cong K(\mathbb{Z}_{\geq 0} \times \mathbf{n})$, and similarly $A_{\mathbb{Z}}(n) = (K_{\mathbb{Z}}[T])(n) \cong K_{\mathbb{Z}}(\mathbb{Z}_{\geq 0} \times \mathbf{n})$. Then the map $A(n) \to A_{\mathbb{Z}}(n)$ is identified with the injective map $K(\mathbb{Z}_{\geq 0} \times \mathbf{n}) \to K_{\mathbb{Z}}(\mathbb{Z}_{\geq 0} \times \mathbf{n})$, hence $A \to A_{\mathbb{Z}}$ is a monomorphism, i.e. A is torsion-free.

We see that (DET_2^*) holds for any unary polynomial algebra over $\mathbb{Z}_\infty$, with $N = 3$. Since this property is stable under strict quotients, we see that (DET_2^*) *holds with* $N = 3$ *for all pre-unary alternating* $\mathbb{Z}_\infty$*- and* $\bar{\mathbb{Z}}_\infty$*-algebras, as well as for all strict quotients of torsion-free* $\mathbb{Z}_\infty$*-algebras.* For example, (DET_2^*) and (DET_2) hold for $\mathbb{F}_\infty[T^{[1]}]$.

5.6.16. (Open questions.) Let us list some questions that have naturally arisen so far, and make some comments on each of them.

1. For which values of r the property (DET_r^*) holds over all alternating generalized rings? Or, equivalently, for which r's (DET_r') is fulfilled for the universal matrix A over $\Lambda_r = \mathbb{F}_{\pm 1}[T_1^{[r]}, \ldots, T_r^{[r]}]/_{alt}$? What are the corresponding (universal) values of N_r? Is the property (DET_∞^*) universally true?

2. For which values of r the property (DET_r^*) holds over all torsion-free $\mathbb{F}_{\pm 1}$-algebras? Or, equivalently, what are the values of r, for which (DET_r') is fulfilled for the image of A in Λ_r', the image of $\Lambda_r \to \Lambda_{r,\mathbb{Z}}$?

3. Is it true that *all* polynomial algebras over $\mathbb{Z}_\infty$ are torsion-free?

5.6.17. (Some remarks.) Notice that a positive answer for 1) would establish (DET^*_∞), the strongest of our properties, for all alternating monads, thus making all further questions irrelevant. That's why we actually don't believe that 1) is true, or even that (DET^*_2) is universally true: this would be too nice to be true.

On the other hand, we think that 2) holds for all values of r, and that our proof for $r = 2$, $N = 3$ given above can be extended to the general case, especially if we choose very large values of N. Notice that 2) would imply (DET^*_∞) for all torsion-free $\mathbb{F}_{\pm 1}$-, $\mathbb{Z}_\infty$- and $\bar{\mathbb{Z}}_\infty$-algebras and their strict quotients, e.g. pre-unary alternating algebras over these generalized rings. This is more than enough for the applications in Arakelov geometry.

If both 2) and 3) are true, then (DET^*_∞) would hold for *all* alternating algebras over $\mathbb{Z}_\infty$. This statement would solve all our problems, but it seems again too strong to be true. So we don't believe in 3).

5.6.18. (Lower bounds for universal $N = N_r$.) Notice that one can obtain lower bounds for universal $N = N_r$ in 2) and 1), essentially by the same reasoning as in the proof of (5.6.12.1). This yields the following inequality for $N = N_r$ of 2) and 1):

$$\frac{(Nr-1)!}{(N-1)! \cdot N!^{r-1}} \geq r!^N \tag{5.6.18.1}$$

One can check that this inequality actually means $N \geq 3$ for $2 \leq r \leq 3$, and $N \geq 2$ for $r \geq 4$, so it doesn't really give us a lot of information. In fact even the inequality $N_r \leq N_{r+1}$ of **5.6.2** gives us a better lower bound $N_r \geq N_2 \geq 3$ for $r \geq 2$. However, we think that proving 2) for $r \geq 3$ will be much simpler for values of N much larger than any of these lower bounds. In fact, one might hope to find a regular way of constructing trees V with $\bar{V} = V'$ (this is sufficient to prove 2) for given r and N) when N is very large.

5.7. (Complements.) We collect in this subsection some interesting observations that didn't fit naturally anywhere in this chapter. Some of them are quite important, but require more sophisticated techniques for their study than we have already developped (e.g. localization and homotopical algebra techniques).

5.7.1. (Tensor square of $\mathbb{Z}_\infty$ and $\mathbb{F}_\infty$.) We have seen in **5.1.22** that $\mathbb{Z} \otimes_{\mathbb{F}_1} \mathbb{Z} = \mathbb{Z} \otimes_{\mathbb{F}_{\pm 1}} \mathbb{Z} = \mathbb{Z}$, i.e. that $\mathbb{F}_1 \to \mathbb{Z}$ is an epimorphism in the category of generalized rings. What can be said about tensor squares of $\mathbb{Z}_\infty$, $\mathbb{Z}_{(\infty)}$ and $\mathbb{F}_\infty$?

5.7.2. (Tensor square of $\mathbb{F}_\infty$.) Recall that $\mathbb{F}_\infty = \mathbb{F}_{\pm 1}[*^{[2]} \,|\, \boldsymbol{e}*\boldsymbol{e} = \boldsymbol{e}, \boldsymbol{e}*(-\boldsymbol{e}) = 0, x*y = y*x, (x*y)*z = x*(y*z)]$ (cf. **5.1.16**,e). We can use this to compute $\Sigma := \mathbb{F}_\infty \otimes_{\mathbb{F}_{\pm 1}} \mathbb{F}_\infty$. We see that Σ is generated as a commutative $\mathbb{F}_{\pm 1}$-algebra by two operations, say $*^{[2]}$ and $\vee^{[2]}$, each of them satisfies the conditions listed above for $*$, i.e. is commutative, associative, idempotent, and $\boldsymbol{e} * (-\boldsymbol{e}) = \boldsymbol{e} \vee (-\boldsymbol{e}) = 0$. We have to impose one additional condition as well, namely, the commutativity between $*$ and $\vee$:

$$(x \vee y) * (z \vee w) = (x * z) \vee (y * w) \tag{5.7.2.1}$$

Putting here $z = y$, $w = x$ and applying relations $x * x = x = x \vee x$, $x * y = y * x$ and $x \vee y = y \vee x$, we obtain

$$x \vee y = (x \vee y) * (y \vee x) = (x * y) \vee (y * x) = x * y \tag{5.7.2.2}$$

In other words, $* = \vee$, hence $\mathbb{F}_\infty \otimes_{\mathbb{F}_{\pm 1}} \mathbb{F}_\infty = \mathbb{F}_\infty$. (This simple argument has been communicated to me by A. Smirnov.)

5.7.3. (Tensor square of $\mathbb{Z}_{(\infty)}$.) The above argument can be easily generalized to prove $\mathbb{Z}_{(\infty)} \otimes_{\mathbb{F}_{\pm 1}} \mathbb{Z}_{(\infty)} = \mathbb{Z}_{(\infty)}$. First of all, denote by $s_N \in \mathbb{Z}_{(\infty)}(N)$, $N \geq 1$, the N-th averaging operation:

$$s_N = \frac{1}{N}\{1\} + \frac{1}{N}\{2\} + \cdots + \frac{1}{N}\{N\} \tag{5.7.3.1}$$

One sees immediately that $\mathbb{Z}_{(\infty)}$ is generated over $\mathbb{F}_{\pm 1}$ by all such operations s_N (actually $s_{NN'}$ can be easily expressed in terms of s_N and $s_{N'}$, so $\{s_p\}_{p \in \mathbb{P}}$ would suffice). Indeed, any rational octahedral combination $t = \sum_{i=1}^n \lambda_i \{i\} = \sum_{i=1}^n (m_i/N) \cdot \{i\} \in \mathbb{Z}_{(\infty)}(n)$, where N, $m_i \in \mathbb{Z}$, $N > 0$, and necessarily $\sum_i |m_i| \leq N$, can be rewritten as s_N applied to the list of arguments containing $\pm\{i\} = \operatorname{sgn} m_i \cdot \{i\}$ exactly $|m_i|$ times, for all $1 \leq i \leq n$, and with the remaining $N - \sum_i |m_i|$ arguments put equal to zero:

$$t = \sum_{i=1}^n \frac{m_i}{N}\{i\} = s_N\bigl(\overbrace{\pm\{1\}, \ldots, \pm\{1\}}^{|m_1| \text{ times}}, \overbrace{\pm\{2\}, \ldots}^{|m_2| \text{ times}}, \ldots, 0, \ldots, 0\bigr) \tag{5.7.3.2}$$

Now $\Sigma := \mathbb{Z}_{(\infty)} \otimes_{\mathbb{F}_{\pm 1}} \mathbb{Z}_{(\infty)}$ is generated over $\mathbb{F}_{\pm 1}$ by two commuting sets of such averaging operations s_N, s'_N, $N \geq 1$, and we are reduced to showing $s_N = s'_N$. Notice that s_N is invariant under any permutation of arguments, and that $s_N(x, x, \ldots, x) = x$, and similarly for s'_N; applying commutativity relation of **5.1.1** to s_N, s'_N, and $N \times N$-matrix $x = (x_{ij})_{1 \leq i,j \leq N}$, where $x_{ij} = \{((i + j - 2) \bmod N) + 1\} \in \Sigma(N)$ (i.e. x is a Latin square: any row or column contains each $\{i\}$, $1 \leq i \leq N$, exactly once), we obtain $x_{.j} = s_N(\{j\}, \{j+1\} \ldots, \{j-1\}) = s_N(\{1\}, \ldots, \{N\}) = s_N$, whence $x' = s'_N(x_{.1}, \ldots, x_{.N}) = s'_N(s_N, \ldots, s_N) = s_N$; a similar argument shows $x_{i.} = s'_N$ for any i and $x'' = s_N(x_{1.}, \ldots, x_{N.}) = s'_N$, so the commutativity relation $x' = x''$ yields $s_N = s'_N$ as claimed.

5.7.4. (Tensor square of $\mathbb{Z}_\infty$, and tensor squares over $\mathbb{F}_1$.) Notice that the above argument doesn't imply $\mathbb{Z}_\infty \otimes_{\mathbb{F}_{\pm 1}} \mathbb{Z}_\infty = \mathbb{Z}_\infty$, since $\mathbb{Z}_\infty$ is not generated over $\mathbb{F}_{\pm 1}$ by the s_N, and we actually expect $\mathbb{Z}_\infty \otimes_{\mathbb{F}_{\pm 1}} \mathbb{Z}_\infty \neq \mathbb{Z}_\infty$.

It also leaves the question about tensor squares of $\mathbb{F}_\infty$ and $\mathbb{Z}_{(\infty)}$ over $\mathbb{F}_1$ open. Notice that $\mathbb{Z}_{(\infty)}$ is generated over $\mathbb{F}_1$ by the symmetry $-^{[1]}$ subject to $[-]^2 = \boldsymbol{e}$, and by the averaging operations s_N, and the previous argument still shows $s_N = s'_N$ inside $\Sigma := \mathbb{Z}_{(\infty)} \otimes_{\mathbb{F}_1} \mathbb{Z}_{(\infty)}$ since it never used the symmetry. Therefore, Σ is generated over $\mathbb{Z}_{(\infty)}$ by an additional symmetry $\ominus^{[1]}$, commuting with $[-]$ and all s_N, i.e. $\mathbb{Z}_{(\infty)} \otimes_{\mathbb{F}_1} \mathbb{Z}_{(\infty)} = \mathbb{Z}_{(\infty)}[\ominus^{[1]}]$. Notice, however, that apart from relation $\ominus^2 = \boldsymbol{e}$, this new symmetry satisfies relations like $s_2(x, \ominus x) = 0$ since they are fulfilled by $-$. Similarly, $\mathbb{F}_\infty \otimes_{\mathbb{F}_1} \mathbb{F}_\infty = \mathbb{F}_\infty[\ominus^{[1]}]$, subject to relations $\ominus^2 = \boldsymbol{e}$ and $x * \ominus x = 0$. One can check that there are $\mathbb{F}_\infty$-modules (hence also $\mathbb{Z}_{(\infty)}$-modules) M with an involution $\ominus$ different from $-$: one can consider for example $M = \{0, a, b, -a, -b\}$ with $*$ given by $x * x = x$, $x * y = 0$ for $y \neq x$, and put $\ominus a := b$, $\ominus b := a$. This implies $\mathbb{F}_\infty \otimes_{\mathbb{F}_1} \mathbb{F}_\infty \neq \mathbb{F}_\infty$ and $\mathbb{Z}_{(\infty)} \otimes_{\mathbb{F}_1} \mathbb{Z}_{(\infty)} \neq \mathbb{Z}_{(\infty)}$.

5.7.5. Now let's show that $\mathbb{Z} \otimes_{\mathbb{F}_{\pm 1}} \mathbb{Z}_\infty = \mathbb{R}$ and $\mathbb{Z} \otimes_{\mathbb{F}_{\pm 1}} \mathbb{Z}_{(\infty)} = \mathbb{Q}$. To do this we use the localization theory developed in the next chapter, and in particular relations $\mathbb{Z}_\infty[(1/2)^{-1}] \cong \mathbb{R}$ and $\mathbb{Z}_{(\infty)}[(1/2)^{-1}] \cong \mathbb{Q}$; they will be also checked in the next chapter (cf. **6.1.23**). Now notice that the operation $1/2$ of $\mathbb{Z}_{(\infty)}$ becomes invertible in $\mathbb{Z} \otimes_{\mathbb{F}_{\pm 1}} \mathbb{Z}_{(\infty)}$: in fact, it admits an inverse $2 := \boldsymbol{e} * \boldsymbol{e}$, where we have temporarily denoted the addition of $\mathbb{Z}$ by $*$ to prevent confusion with octahedral linear combinations $\lambda_1\{1\} + \cdots + \lambda_n\{n\}$ of $\mathbb{Z}_{(\infty)}$.

Let us check that $2 = \boldsymbol{e} * \boldsymbol{e}$ is indeed an inverse to $1/2$. We use for this the commutativity relation between $*$ and $s_2 := (1/2)\{1\} + (1/2)\{2\}$. We get $((1/2)x + (1/2)y) * ((1/2)z + (1/2)w) = (1/2)(x * z) + (1/2)(y * w)$. Putting $y = z = 0$, $x = w$, we obtain $2 \cdot (1/2)x = ((1/2)x) * ((1/2)x) = (1/2)x + (1/2)x = x$, i.e. $2 \cdot (1/2) = \boldsymbol{e}$ in $\mathbb{Z} \otimes_{\mathbb{F}_{\pm 1}} \mathbb{Z}_{(\infty)}$ as claimed.

Once we know that $1/2$ becomes invertible in the tensor product, the rest follows immediately from universal properties of tensor products and localizations: $\mathbb{Z} \otimes_{\mathbb{F}_{\pm 1}} \mathbb{Z}_\infty = (\mathbb{Z} \otimes_{\mathbb{F}_{\pm 1}} \mathbb{Z}_\infty)[(1/2)^{-1}] = \mathbb{Z} \otimes_{\mathbb{F}_{\pm 1}} \mathbb{Z}_\infty[(1/2)^{-1}] = \mathbb{Z} \otimes_{\mathbb{F}_{\pm 1}} \mathbb{R} = \mathbb{Z} \otimes_{\mathbb{F}_{\pm 1}} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}$, since $\mathbb{Z} \otimes_{\mathbb{F}_{\pm 1}} \mathbb{Z} = \mathbb{Z}$ by **5.1.22**. Formula $\mathbb{Z} \otimes_{\mathbb{F}_{\pm 1}} \mathbb{Z}_{(\infty)} = \mathbb{Q}$ is deduced similarly from $\mathbb{Z}_{(\infty)}[(1/2)^{-1}] = \mathbb{Q}$.

5.7.6. (Generalized fields.) We would like to have a reasonable definition of generalized fields. Clearly, a generalized field is a generalized ring K with some additional properties. Our definition should coincide with the usual one for classical rings (i.e. a classical ring is a generalized field iff it is a classical field), and it shouldn't be too restrictive, otherwise we wouldn't be able to obtain a reasonable definition of "residue fields" of points of generalized schemes, studied in the next chapter.

In particular, we would like $\mathbb{F}_\infty$ to be a generalized field, since it is the only reasonable candidate for the residue field of $\mathbb{Z}_\infty$. This rules out a potential definition of a generalized field as a generalized ring K, such that all K-modules are free. Indeed, the subset $M := \{0, \pm?\{1\}\pm?\{2\}\} \subset \mathbb{F}_\infty(2)$ is a 5-element $\mathbb{F}_\infty$-module, clearly not free, since $\operatorname{card}\mathbb{F}_\infty(n) = 3^n$ according to **4.8.13**.

So we have to consider some other characteristic property of classical fields. For example, they have no non-trivial quotients. This leads to the following definition:

Definition 5.7.7 *We say that a generalized ring K is* subfinal *or* subtrivial *if it is isomorphic to a submonad of the final monad* $\mathbf{1}$, *i.e. to* $\mathbf{1}_+$ *or* $\mathbf{1}$. *According to* **4.3.13**, *K is subtrivial iff* $\{1\}_\mathbf{2} = \{2\}_\mathbf{2}$.

We say that a generalized ring K is a generalized field *if it is not subtrivial, but all its strict quotients different from K are subtrivial.*

When K is a generalized ring with zero, K is (sub)trivial iff $\boldsymbol{e} = 0$ in $|K|$. Therefore, if K is a generalized field with zero, we have $\boldsymbol{e} \neq 0$, i.e. $|K| = K(1)$ consists of at least two elements. This implies that the canonical homomorphism $K \to \mathrm{END}(|K|)$ is non-trivial, since 0 and $\boldsymbol{e}$ act on $|K|$ in different way, hence it has to be a monomorphism (otherwise the image of K in $\mathrm{END}(|K|)$ would be a non-trivial strict quotient of K), i.e. *if K is a generalized field with zero, $|K|$ is an exact K-module.* More precisely, $|K|$ is an exact K-module whenever it consists of at least two elements; and $K(2)$ is an exact K-module in all cases.

Another easy observation — *$|K|$ doesn't admit any non-trivial strict quotient A in K-Mod.* Indeed, if A is such a strict quotient, it is a quotient-monoid of $|K|$ as well, hence we get an algebra structure on A, and the corresponding unary algebra K_A over K will be a non-trivial strict quotient of K.

Conversely, *if K is a generalized ring with zero, such that $|K|$ is non-trivial, exact, and doesn't admit any non-trivial strict quotients in K-Mod, then K is a generalized field.* Indeed, let $f : K \to K'$ be a strict epimorphism. Then $|K'|$ is a strict quotient of $|K|$ in K-*Mod*. If $|K'| = 0$, $\boldsymbol{e} = 0$ in $|K'|$, and K' is itself trivial. On the other hand, if $|K'| \neq 0$, then necessarily $|K'| = |K|$, and the composite map $K \to K' \to \mathrm{END}(|K'|) = \mathrm{END}(|K|)$ is a monomorphism, $|K|$ being an exact K-module, hence $f : K \to K'$ is a monomorphism as well, hence $K' = K$.

An immediate consequence is that *a classical ring is a generalized field iff it is a field in the usual sense*, and that $\mathbb{F}_\infty$ *is a generalized field.*

5.7.8. (Examples of generalized fields.)

- We have seen that all classical fields, e.g. $\mathbb{F}_p$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$, are generalized fields.

- We have just seen that $\mathbb{F}_\infty$ is a generalized field. This example shows that in general there can be modules over a generalized field that are not free.

- It is easy to check with the aid of the criterion given above that $\mathbb{F}_\varnothing$ and $\mathbb{F}_1$ are generalized fields, while $\mathbb{F}_{\pm 1}$ and $\mathbb{F}_{1^n}$ for $n > 1$ are not, since they admit $\mathbb{F}_1$ as a strict quotient. In fact, $\mathbb{F}_{1^n}$ is something like a local artinian ring with residue field $\mathbb{F}_1$.

Notice that $|\mathbb{F}_{1^n}|$ is an exact $\mathbb{F}_{1^n}$-module, and all non-zero elements of $|\mathbb{F}_{1^n}|$ are invertible, but this is insufficient to conclude that $\mathbb{F}_{1^n}$ is a generalized field.

5.7.9. (Further properties of generalized fields.) Applying Zorn lemma to the set of compatible equivalence relations $\equiv$ on a generalized ring K, such that $\{1\}_2 \not\equiv \{2\}_2$, we see that *any non-subtrivial generalized ring K admits a strict quotient that is a generalized field.* Another interesting property is obtained by inspecting the image of $|K| \xrightarrow{s} |K|$, $s \in |K|$, and the image of $K \to K[s^{-1}]$ as well: we see that *if K is a generalized field with zero, and $s \in |K|$, $s \neq 0$, then the maps $[s]_{K(n)} : K(n) \to K(n)$ are injective for all $n \geq 0$.* We don't know whether all non-zero elements of $|K|$ are invertible or not.

5.7.10. (Elementary K-theory.) We can construct $K_0(\Sigma)$ for any generalized ring Σ by considering the quotient of the free abelian group generated by projective Σ-modules of finite type modulo relations $[M \oplus N] - [M] - [N]$. Clearly, the pullback functor defines a map $\rho^* : K_0(\Sigma) \to K_0(\Sigma')$ for any $\rho : \Sigma' \to \Sigma$.

We can try to construct $K_1(\Sigma)$ in the usual way, by considering the free abelian group generated by couples $[P, \varphi]$, P projective of finite type, $\varphi \in \mathrm{Aut}_\Sigma(P)$, modulo certain relations. However, some of these relations have to involve something like "short exact sequences"; a similar problem arises when we try to compute K_0 of larger categories of Σ-modules.

Probably the best way to deal with these problems is to construct a reasonable (triangulated) category of perfect complexes over Σ. For now we'll content ourselves by indicating that a bicartesian square

$$\begin{tikzcd} M \arrow[r] \arrow[d] & N \arrow[d] \\ M' \arrow[r] & N' \end{tikzcd} \tag{5.7.10.1}$$

is very much like a short exact sequence when we work over a classical ring, since in this case such a bicartesian square is essentially just a short exact sequence $0 \to M \to M' \oplus N \to N' \to 0$, hence it yields a relation $[M] - [M'] + [N] - [N'] = 0$ in K_0. We can try to construct something similar in the general case, but it seems that we have to impose some additional conditions on such squares (e.g. require all horizontal arrows to be constant perfect cofibrations in the sense of Chapter **10**) to obtain a reasonable theory.

We'll return to the algebraic K-theory for generalized rings later in Chapter **10**, where we'll apply a modification of Waldhausen construction to define all algebraic K-groups of a generalized ring Σ.

5.7.11. (Theory of traces.) We can construct an "abstract theory of characteristic polynomials" by considering the group $\tilde{K}_1(\Sigma)$, generated by pairs $[P, \varphi]$ with $\varphi \in \mathrm{End}_\Sigma(P)$ (now φ need not be invertible!) modulo relations similar to that of K_0. We can also construct an "abstract theory of traces" $K_1^+(\Sigma)$ by considering the $(\Sigma \otimes \mathbb{Z})$-module generated by couples $[P, \varphi]$ as above, subject to an additional family of relations, namely,

$$[P, t(\varphi_1, \ldots, \varphi_n)] = t\big([P, \varphi_1], \ldots, [P, \varphi_n]\big) \text{ for } t \in \Sigma(n).$$

5.7.12. (A simpler construction of traces.) However, we can construct a simpler theory of traces over a generalized ring Σ. Namely, we can put $\mathrm{tr}\,\varphi := \mathrm{tr}\,\rho^*(\varphi)$, where $\varphi : \Sigma \to \mathbb{Z} \otimes \Sigma$ is the canonical map (the tensor product is taken here over $\mathbb{F}_\varnothing$, $\mathbb{F}_1$ or $\mathbb{F}_{\pm 1}$ in the most appropriate way). This gives us a theory of traces of matrices over Σ with values in $\Sigma \otimes \mathbb{Z}$, and this theory is quite sensible: for example, matrices over $\mathbb{Z}_\infty$ have traces in $\mathbb{Z}_\infty \otimes_{\mathbb{F}_{\pm 1}} \mathbb{Z} = \mathbb{R}$, and matrices over $\mathbb{F}_{1^n}$ – in $\mathbb{Z}[\sqrt[n]{1}]$. However, $\mathbb{F}_\infty \otimes_{\mathbb{F}_{\pm 1}} \mathbb{Z} = 0$, so we don't obtain any theory of traces over $\mathbb{F}_\infty$ in this way.

5.7.13. (Valuation rings.) Recall that in **5.1.3** we have defined a submonad $(N : N) \subset \mathrm{END}(M)$ for any set M and its subset N; $(N : N)(n)$ consists of all maps $f : M^n \to M$, such that $f(N^n) \subset N$. When M is a Σ-module, we get a canonical homomorphism $\rho : \Sigma \to \mathrm{END}(M)$. Let us denote the preimage of $(N : N)$ under ρ by Σ_N; clearly, Σ_N is the largest algebraic submonad of Σ, such that N is a Σ_N-module.

Now suppose that K is a classical field, and $|\cdot|_v$ is a valuation on K (archimedean or not). Let's apply the above construction to $\Sigma = K$, $M = K$

(considered as a module over itself) and $N = N_v := \{x \in K : |x|_v \leq 1\}$. We obtain an algebraic submonad $\mathcal{O}_v := K_{N_v} \subset K$; we say that $\mathcal{O}_v$ *is the valuation ring of* $|\cdot|_v$. Clearly, $\mathcal{O}_v$ is a classical ring iff $|\cdot|_v$ is non-archimedian, and in this case it coincides with the classical valuation ring of $|\cdot|_v$. In any case, the underlying set $|\mathcal{O}_v|$ of $\mathcal{O}_v$ is easily seen to be equal to N_v, and $|\cdot|_v$ is completely determined by N_v, hence also by $\mathcal{O}_v$, up to equivalence. It is immediate that $\mathcal{O}_v$ is a hypoadditive alternating $\mathbb{F}_{\pm 1}$-subalgebra of K.

Notice that the valuation rings of archimedian valuations on $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Q}$ are nothing else than $\mathbb{Z}_\infty$, $\bar{\mathbb{Z}}_\infty$ and $\mathbb{Z}_{(\infty)}$. In general, it follows from the definitions that $\mathcal{O}_v(n)$ consists of all n-tuples $\lambda = (\lambda_1, \ldots, \lambda_n) \in K^n$, such that $|\lambda_1 x_1 + \cdots + \lambda_n x_n|_v \leq 1$ whenever all $|x_i|_v \leq 1$. If $|\cdot|_v$ is non-archimedian, this condition is equivalent to "all $|\lambda_i|_v \leq 1$", i.e. $\lambda \in N_v^n$. In the archimedian case one checks that this condition is equivalent to $|\lambda_1|_v + \cdots + |\lambda_n|_v \leq 1$ (the verification is based on choosing rational numbers y_i with $|\lambda_i|_v(1-\varepsilon) \leq y_i \leq |\lambda_i|_v$ and considering $x_i := y_i/\lambda_i$). Clearly, this is a generalization of the "octahedral combinations" used to construct $\mathbb{Z}_\infty$, $\bar{\mathbb{Z}}_\infty$ and $\mathbb{Z}_{(\infty)}$.

It would be nice to have an intrinsic definition of valuation rings inside a classical field K, and to transfer this definition to the case of generalized fields.

6 Localization, spectra and schemes

Now we are going to develop a reasonable theory of spectra for generalized rings. More precisely, we develop *two* such theories, as two special cases of a general construction of theories of spectra. We construct the *unary* or *prime spectrum* $\operatorname{Spec}^u A$ or $\operatorname{Spec}^p A$ of a generalized ring A. It admits a description in terms of prime ideals very similar to the classical case, and its basic properties are essentially the same. However, this spectrum fails to have some more sophisticated properties, e.g. finitely generated projective A-modules do not necessarily define locally free quasicoherent sheaves, and the image of a flat finitely presented morphism of such spectra is not necessarily open.

We deal with these problems by defining a more sophisticated theory of spectra, namely, the *total spectra* $\operatorname{Spec}^t A$ or simply $\operatorname{Spec} A$ of generalized rings. We have a comparison map $\operatorname{Spec}^t A \to \operatorname{Spec}^p A$, that turns out to be an isomorphism of ringed spaces for a classical A, but not in general.

Of course, once we construct the (total) spectrum of a generalized ring as a generalized (locally) ringed space or topos, we can immediately define the category of generalized schemes. We want to study their basic properties and discuss some examples, e.g. the projective spaces $\mathbb{P}^n_{\mathbb{F}_1}$ over $\mathbb{F}_1$.

6.1. (Unary localization.) The main definitions can be transferred verbatim from the classical case, since they depend (almost) only on the multiplicative structure of $|A|$. However, we must be careful while dealing with the extra structures.

Definition 6.1.1 *a) We say that a subset S of a monoid M is a* multiplicative system *if it is a submonoid of M, i.e. if it is closed under multiplication and contains the identity of M.*

b) For any subset $S \subset M$ we denote by $\langle S\rangle$ the multiplicative system (i.e. the submonoid) generated by S. If $S = \{f\}$, we write S_f instead of $\langle\{f\}\rangle = \{1, f, f^2, \ldots, f^n, \ldots\}$.

c) We say that a multiplicative system $S \subset M$ is saturated *if $ax \in S$, $a \in M$ implies $x \in S$, i.e. if S contains all divisors of any its element. The smallest saturated multiplicative system $\tilde{S}$ containing a given multiplicative system S (i.e. the set of all divisors of all elements of S) is called the* saturation *of S.*

Definition 6.1.2 *a) Given a generalized ring A and any subset $S \subset |A|$, we denote by $(A[S^{-1}], i_A^S)$ the universal (i.e. initial) object in the category of couples (B, ρ), where B is a generalized ring, and $\rho : A \to B$ a homomorphism, such that all elements of $\rho_1(S) \subset |B|$ are invertible in $|B|$ with respect to the* *canonical commutative monoid structure on this set. We say that $A[S^{-1}]$ is the* localization of A with respect to S. *When S is a multiplicative system, we write $S^{-1}A$ instead of $A[S^{-1}]$ as well, and we write A_f or $A[f^{-1}]$ instead of $S_f^{-1}A$ or $A[\{f\}^{-1}]$.*

b) Given a generalized ring A, a subset $S \subset |A|$, and an A-module M, we denote by $(M[S^{-1}], i_M^S)$ the universal (initial) object in the category of couples (N, f), with $f : M \to N$ an A-module homomorphism, such that all elements $s \in S$ act bijectively on N, i.e. such that all elements of S become invertible after the application of the canonical map $A \to \mathrm{END}(N)$. We say that $M[S^{-1}]$ is the localization of M with respect to S; *when S is a multiplicative system, we write $S^{-1}M$ instead of S, and we write M_f or $M[f^{-1}]$ instead of $S_f^{-1}M$.*

6.1.3. (Reduction to the multiplicative system case.) Let us denote by $\mathrm{Inv}(B)$ the set of invertible elements of monoid $|B|$, where B is any algebraic monad. Notice that this is a multiplicative system in $|B|$, hence its preimage $\rho^{-1}(\mathrm{Inv}(B)) \subset |A|$ is a multiplicative system as well. We conclude that for any subset $S \subset |A|$ the condition $\rho(S) \subset \mathrm{Inv}(B)$, i.e. $S \subset \rho^{-1}(\mathrm{Inv}(B))$, is equivalent to $\rho(\langle S\rangle) \subset \mathrm{Inv}(B)$. This means that $A[S^{-1}]$ and $A[\langle S\rangle^{-1}] = \langle S\rangle^{-1}A$ solve the same universal problem, hence are isomorphic (when they are representable). In other words, it suffices to show the existence of localizations with respect to multiplicative subsets $S \subset |A|$.

Applying the above observation to $B := \mathrm{END}_A(N)$ (this is a non-commutative A-algebra, but we didn't use commutativity of B so far), we see that $S \subset |A|$ acts by bijections on an A-module N iff $\langle S\rangle$ acts by bijections on N, hence $M[S^{-1}] = \langle S\rangle^{-1}N$, so we can restrict ourselves to the case when S is a multiplicative system in the module case as well.

6.1.4. (Construction of $S^{-1}M$.) Let $S \subset |A|$ be a multiplicative system and M be an A-module. We construct the localization $S^{-1}M$ in the usual way. Namely, we consider the equivalence relation $\sim$ on the set $M \times S$, given by $(x, s) \sim (y, t)$ iff there is a $u \in S$, such that $ut \cdot x = us \cdot y$, and put $S^{-1}M := (M \times S)/\sim$. One checks in the usual way that $\sim$ is indeed an equivalence relation on $M \times S$; we denote by x/s the class of (x, s) in $S^{-1}M$ as usual.

We have constructed a set $S^{-1}M$ so far. Notice that this construction used only the monoid structure of $|A|$ and the action of $|A|$ on $|M|$. Now consider small category $\mathcal{S}$, defined as follows: $\mathrm{Ob}\,\mathcal{S} = S$, i.e. the objects $[s]$ of $\mathcal{S}$ are in one-to-one correspondence with elements $s \in S$, and $\mathrm{Hom}_{\mathcal{S}}(s, t) := \{u \in S \mid su = t\}$; the composition of morphisms is given by the multiplication of S.

We can use our A-module M to define a functor $\tilde{M} : \mathcal{S} \to A\text{-}Mod$ by putting $\tilde{M}([s]) := M$ for all $s \in S$, and $\tilde{M}(u) := u_M : M \to M$ for any morphism $u : [s] \to [t]$ in $\mathcal{S}$ (notice that u_M is an A-endomorphism of M because of the commutativity of A). Now it is immediate that $S^{-1}M = \varinjlim_{\mathcal{S}} \tilde{M}$ in the category of sets. However, $\mathcal{S}$ is obviously a filtered category, and filtered inductive limits in A-*Mod* and *Sets* coincide (cf. **4.6.16**); hence we obtain a natural A-module structure on the set $S^{-1}M$ by putting $S^{-1}M := \varinjlim_{\mathcal{S}} \tilde{M}$ in A-*Mod*.

This A-module structure on $S^{-1}M$ can be written down explicitly: for any operation $a \in A(n)$ and any elements $x_i/s \in S^{-1}M$ with common denominator $s \in S$ we have

$$[a]_{S^{-1}M}(x_1/s, \ldots, x_n/s) = \big([a]_M(x_1, \ldots, x_n)\big)/s \tag{6.1.4.1}$$

In particular, an element $t \in S$ acts on $S^{-1}M$ by mapping x/s into tx/s; clearly, $x/s \mapsto x/st$ is the inverse map, i.e. S acts bijectively on $S^{-1}M$. We have a canonical A-linear map $i_M^S : M \to S^{-1}M$, $x \mapsto x/1$ as well; it is essentially given by the embedding of $M_{[1]}$ into $\varinjlim_{\mathcal{S}} M_{[s]}$.

We have to show that for any A-module N, such that S acts bijectively on N, we have $\mathrm{Hom}_A(S^{-1}M, N) \cong \mathrm{Hom}_A(M, N)$. This is quite clear: $\mathrm{Hom}_A(S^{-1}M, N) = \mathrm{Hom}_A(\varinjlim_{\mathcal{S}} M_{[s]}, N) \cong \varprojlim_{\mathcal{S}} \mathrm{Hom}(M_{[s]}, N)$ consists of families $(f_s)_{s\in S}$, $f_s : M \to N$, such that $f_{st} = f_s \circ [t]_M$ for any s, $t \in S$; this is equivalent to $f_{st} = [t]_N \circ f_s$, and in particular $f_s = [s]_N^{-1} \circ f_1$, i.e. such families are in one-to-one correspondence with A-homomorphisms $f_1 : M \to N$, q.e.d.

6.1.5. (Existence and unarity of $A[S^{-1}]$.) Notice that for any subset $S \subset |A|$ the localization $A[S^{-1}]$ can be constructed by considering the commutative A-algebra, generated by new unary operations s^{-1}, $s \in S$, subject to unary relations $s \cdot s^{-1} = \boldsymbol{e}$:

$$A[S^{-1}] = A[(s^{-1})_{s\in S} \,|\, s \cdot s^{-1} = \boldsymbol{e}] \tag{6.1.5.1}$$

This implies that $A[S^{-1}]$ *is a unary A-algebra*. Furthermore, we can define $A[S^{-1}]$ by a similar formula without imposing any implicit commutativity relations between the new generators and operations of A:

$$A[S^{-1}] = A\langle (s^{-1})_{s\in S} \,|\, s \cdot s^{-1} = \boldsymbol{e} = s^{-1} \cdot s\rangle \tag{6.1.5.2}$$

Indeed, we just have to check that the algebraic monad B defined by the RHS is commutative, i.e. that the generators s^{-1} commute between themselves and the operations of A. The first statement is clear, since $s^{-1}t^{-1} = (ts)^{-1} = (st)^{-1} = t^{-1}s^{-1}$ in $|B|$. Let's show that s^{-1} commutes with all operations of A. According to **5.1.12**, this is equivalent to $[s^{-1}]_{B(n)} : B(n) \to B(n)$

being an A-homomorphism for all $n \geq 0$. But $[s^{-1}]_{B(n)}$ is the inverse of the automorphism $[s]_{B(n)}$ of $B(n)$, and the latter is an A-homomorphism, since s commutes with all elements of A.

An immediate consequence of (6.1.5.2) is that $i_A^S : A \to A[S^{-1}]$ *is universal among all algebraic monad homomorphisms* $\rho : A \to B$*, such that* $\rho(S) \subset \mathrm{Inv}(B)$*,* i.e. $A[S^{-1}]$ is the "non-commutative localization" of A with respect to S as well.

Another immediate consequence is that $i_A^S : A \to A[S^{-1}]$ is an NC-epimorphism (cf. **5.1.18**), i.e. any monad homomorphism $A \to B$ extends to at most one monad homomorphism $A[S^{-1}] \to B$, or equivalently, any A-module admits at most one $A[S^{-1}]$-module structure.

6.1.6. ($A[S^{-1}]$-modules.) We see that algebraic monad homomorphisms $A[S^{-1}] \to B$ are in one-to-one correspondence with algebraic monad homomorphisms $\rho : A \to B$, such that all elements of $\rho(S) \subset |B|$ are invertible. Applying this to $B = \mathrm{END}(N)$ we see that $A[S^{-1}]$-*Mod is isomorphic to the full subcategory of* A-*Mod, consisting of those* A*-modules* N*, on which all elements* $s \in S$ *act by bijections.* Clearly, this isomorphism of categories is induced by the scalar restriction functor $(i_A^S)_* : A[S^{-1}]\text{-}\mathit{Mod} \to A\text{-}\mathit{Mod}$; in particular, *the scalar restriction functor* $(i_A^S)^*$ *is fully faithful.* Notice that it is also exact, and admits both a left and a right adjoint, since $A[S^{-1}]$ is unary over A (cf. **5.3.15**).

In this way we can identify $(i_A^S)_*$ with the natural embedding of the full subcategory of A-*Mod*, consisting of A-modules, on which S acts bijectively, into A-*Mod* itself. The left adjoint to this functor is clearly the functor $M \mapsto M[S^{-1}]$ of **6.1.2**, i.e. $(i_A^S)^* M = A[S^{-1}] \otimes_A M$ *is canonically isomorphic to* $M[S^{-1}]$*.* Another immediate consequence is that *any* $M[S^{-1}]$ *admits a canonical* $A[S^{-1}]$*-structure.*

6.1.7. (Description of $S^{-1}A$.) Now we can combine our results to obtain an explicit description of $S^{-1}A$ when $S \subset |A|$ is a multiplicative system. Indeed, $(S^{-1}A)(n) = (i_A^S)^*(A(n)) = S^{-1}A(n)$, hence according to **6.1.4** the set $(S^{-1}A)(n)$ of n-ary operations of $S^{-1}A$ consists of fractions a/s, where $a \in A(n)$, $s \in S$, and $a/s = b/t$ iff $tu \cdot a = su \cdot b$ for some $u \in S$.

We have already described the A-module structure on each $S^{-1}M$, hence also on $S^{-1}A(m)$; using the fact that all $s \in S$ act on these modules by bijections, we obtain

$$[a/s]_{S^{-1}M}(x_1/t, \ldots, x_n/t) = \big([a]_M(x_1, \ldots, x_n)\big)/st$$
$$\text{for any } a \in A(n),\ x_i \in M,\ s,\ t \in S \quad (6.1.7.1)$$

Putting here $M = A(m)$ we obtain the composition maps for the operations of $S^{-1}A$. Of course, when we want to apply a/s to some elements x_i/t_i

of $S^{-1}M$ with different denominators, we have to reduce them to the common denominator, e.g. $t = t_1 t_2 \cdots t_n$. For example, for a binary operation $*^{[2]}$ we get

$$x_1/t_1 * x_2/t_2 = (t_2 x_1 * t_1 x_2)/t_1 t_2 \tag{6.1.7.2}$$

When $* = +$, we obtain the usual formula for the sum of two fractions.

6.1.8. (Flatness of localizations.) Let's prove that $A[S^{-1}]$ *is a flat A-algebra for any $S \subset |A|$*, i.e. that $(i_A^S)^* : A\text{-}\mathit{Mod} \to A[S^{-1}]\text{-}\mathit{Mod}$ is (left) exact. Since $(i_A^S)_*$ is exact and conservative, it suffices to prove that $(i_A^S)_*(i_A^S)^* : A\text{-}\mathit{Mod} \to A\text{-}\mathit{Mod}$ is exact. But this functor is isomorphic to $M \mapsto M[S^{-1}]$; replacing S by the multiplicative system generated by S, we can assume that S is a multiplicative system, and then $M[S^{-1}] = S^{-1}M = \varinjlim_{\mathcal{S}} M_{[s]}$, $M_{[s]} := M$, where $\mathcal{S}$ is the filtered category constructed in **6.1.4**. Since filtered inductive limits commute with finite projective limits in *Sets*, hence also in A-*Mod*, we see that $M \mapsto S^{-1}M$ commutes with finite projective and arbitrary inductive limits, and in particular is exact.

6.1.9. (Classical localizations.) Suppose that A is a classical commutative ring, i.e. a commutative $\mathbb{Z}$-algebra. Then any A-algebra is a $\mathbb{Z}$-algebra, hence a classical ring as well, so the universal property of $A[S^{-1}]$ implies that $A[S^{-1}]$ coincides with classical A-algebra denoted in the same way, i.e. our "generalized" unary localizations coincide with classical localizations when computed over classical rings. The same is true for the localizations of A-modules: if M is a module over a classical ring A, then $S^{-1}M$ of **6.1.2** coincides with the classical A-module denoted in this way.

In this way our unary localization theory is completely compatible with the classical one.

6.1.10. (Invertible elements of localizations.) Let $S \subset |A|$ be a multiplicative system, and $\tilde{S}$ its saturation. We claim that the invertible elements of $|S^{-1}A|$ are exactly the elements of the form a/s, with $a \in \tilde{S}$ and $s \in S$, and that $(i_A^S)^{-1}(\mathrm{Inv}(S^{-1}A)) = \tilde{S}$. Indeed, the monoid $|S^{-1}A|$ is actually the localization of monoid $|A|$ with respect to S, i.e. everything depends here only on the multiplicative structures, and we can repeat the classical proof verbatim.

6.1.11. (Submodules of localizations.) Let us fix a generalized ring A, an A-module M, and a multiplicative system $S \subset |A|$. Any A-submodule $N \subset M$ defines an $S^{-1}A$-submodule $S^{-1}N \subset S^{-1}M$, since the localization functor is exact. We have an increasing map in the opposite direction as well: it transforms $N' \subset S^{-1}M$ into $(i_M^S)^{-1}(N') \subset M$. We say that an A-submodule N of M is *saturated* (with respect to S) if $sx \in N$ with $s \in S$, $x \in M$

implies $x \in N$. For any A-submodule $N \subset M$ we define its *saturation* $\tilde{N}$ to be the minimal saturated submodule of M containing N.

It is immediate that for any $N' \subset S^{-1}M$ the pullback $N := (i_M^S)^{-1}(N')$ is a saturated submodule of M, and in this case $S^{-1}N = N'$. Conversely, for a saturated $N \subset M$ we have $(i_M^S)^{-1}(S^{-1}N) = N$, and for a general N the LHS is exactly the saturation of N; it consists of all elements $x \in M$, such that $sx \in N$ for some $s \in S$. In this way *there is an increasing bijection between saturated A-submodules of M and all $S^{-1}A$-submodules of $S^{-1}M$*. Of course, this bijection and its inverse are given by $N \mapsto S^{-1}N$ and $N' \mapsto (i_M^S)^{-1}(N')$. In this respect the situation is exactly the same as in the classical case.

In particular, the above applies to the *ideals* of A, i.e. A-submodules of $M := |A|$. We see that the ideals of $S^{-1}A$ are in one-to-one correspondence with *saturated* ideals $\mathfrak{a} \subset |A|$.

6.1.12. (Strict quotients of $S^{-1}M$.) Now let's study the strict quotients of $S^{-1}M$. In the classical case we work in abelian categories, so the strict quotients of a module are in one-to-one correspondence with their kernels, and we don't obtain anything new compared to the situation just discussed. However, the general case is slightly more complicated. In any case any strict quotient $M \twoheadrightarrow P$ has a kernel $K := M \times_P M \subset M \times M$; it is an A-submodule of $M \times M$, and a compatible equivalence relation on M as well. Actually the compatibility condition means exactly that K is a submodule of $M \times M$, so we have just to require K to be an equivalence relation. Conversely, any such K defines a strict quotient M/K of M.

Notice that $S^{-1}K \subset S^{-1}M \times S^{-1}M$ is the kernel of $S^{-1}M \to S^{-1}P$ as well because of the exactness of localizations. Conversely, if we start with a strict quotient P' of the $S^{-1}A$-module $S^{-1}M$, its kernel $K' \subset S^{-1}M \times S^{-1}M$ is both an $S^{-1}A$-submodule and an equivalence relation on $S^{-1}M$, hence $K := (i_{M\times M}^S)^{-1}(K') \subset M \times M$ is a (saturated) A-submodule of $M \times M$ and an equivalence relation on M; this yields a strict quotient M/K of M. Another description: K is the kernel and M/K is the image of the composite map $M \to S^{-1}M \to P'$. The results of **6.1.11** imply $K' = S^{-1}K$, hence $P' = S^{-1}(M/K) = S^{-1}M/K' = P'$.

We have just shown that *there is a canonical increasing bijection between strict quotients of the $S^{-1}A$-module $S^{-1}M$ and the strict quotients of M with saturated kernels in $M \times M$.*

6.1.13. (Localization and base change.) Let A be a generalized ring, B a commutative A-algebra, $\rho : A \to B$ the structural homomorphism, $S \subset |A|$ a multiplicative set, M an A-module, and N a B-module.

First of all, notice that $\rho(S)^{-1}N$ is computed by means of the same filtered inductive limit as $S^{-1}(\rho_*N)$, so we get a canonical bijection $\rho(S)^{-1}N \cong$

$S^{-1}(\rho_* N)$, easily seen to be an $S^{-1}A$-module isomorphism. That's why we usually denote both sides by $S^{-1}N$.

Another easy observation: (6.1.5.1) implies $B[\rho(S)^{-1}] \cong B \otimes_A A[S^{-1}]$, i.e. $\rho(S)^{-1}B = B \otimes_A S^{-1}A$ when S is a multiplicative set. Since $S^{-1}M \cong S^{-1}A \otimes_A M$ for any A-module M, we get

$$\rho(S)^{-1}(B \otimes_A M) \cong \rho(S)^{-1}B \otimes_{S^{-1}A} S^{-1}M \tag{6.1.13.1}$$

Usually we write $S^{-1}B$ instead of $\rho(S)^{-1}B$. Notice that we can deduce the above isomorphism $\rho(S)^{-1}N \cong S^{-1}(\rho_* N)$ from the "base change theorem" **5.4.2** as well, using the flatness of $S^{-1}A$ over A.

Final remark: if A is a K-algebra and K' is another K-algebra, then $S^{-1}(A \otimes_K K') \cong (S^{-1}A) \otimes_K K'$, so we can write simply $S^{-1}A \otimes_K K'$.

6.1.14. (Strict quotients of $S^{-1}A$.) Let A and S be as above. Consider a strict quotient $\varphi : A \to A'$ of A. We say that A' *is saturated (with respect to S)* if $A'(n)$ is a saturated strict quotient of $A(n)$, i.e. the kernel $R(n) := A(n) \times_{A'(n)} A(n)$ is a saturated A-submodule of $A(n) \times A(n)$, for all $n \geq 0$.

Given any strict quotient A' of A as above, we can construct a strict quotient $S^{-1}A'$ of $S^{-1}A$; since $(S^{-1}A')(n) = S^{-1}(A'(n))$ as an $S^{-1}A$-module, this construction coincides with that of **6.1.12** on the level of individual strict quotients $A(n) \to A'(n)$. Conversely, we can start with an arbitrary strict quotient A'' of $S^{-1}A$ and construct a *saturated* strict quotient A' of A, namely, the image of homomorphism $A \to S^{-1}A \to A''$. On the level of individual components $A''(n)$ this construction coincides again with that of **6.1.12**. Therefore, *the two maps given above define a canonical increasing bijection between saturated strict quotients of A and all strict quotients of $S^{-1}A$.*

6.1.15. (Application to generalized fields.) The above statement immediately implies that *any localization $S^{-1}K$ of a generalized field K (cf.* **5.7.7***) is either subtrivial or a generalized field; in the latter case $K \to S^{-1}K$ is a monomorphism.*

Definition 6.1.16 *(Regular elements.) We say that an element $f \in |A|$ is* regular *with respect to an A-module M, or M-*regular*, if the map $f_M : M \to M$ is injective, or equivalently, if $M \to M_f$ is injective. We say that f is A-*regular*, or simply* regular*, if it is $A(n)$-regular for all $n \geq 0$, i.e. all maps $f_{A(n)} : A(n) \to A(n)$ are injective, or equivalently, if $A \to A_f$ is a monomorphism.*

Clearly, M-regular elements constitute a saturated multiplicative system $S_M \subset |A|$, hence any element of a multiplicative system generated by M-regular elements is itself M-regular. It is easy to see that $i_M^S : M \to M[S^{-1}]$ is injective iff S if M-regular, i.e. iff $S \subset S_M$.

Similarly, the set of all regular elements is a saturated multiplicative system $T \subset |A|$, and $A \to A[S^{-1}]$ is injective iff $S \subset T$. Therefore, $T^{-1}A$ is the largest localization of A, such that $A \to T^{-1}A$ is injective. We say that $T^{-1}A$ is *the total ring of fractions* of A.

When all non-zero elements of $|A|$ are regular (i.e. all elements of $|A|$, when A is a monad without zero), we say that A is a *domain*. For example, any generalized field K is a domain, its total fraction ring $T^{-1}K$ is a generalized field as well, $K \to T^{-1}K$ is a monomorphism, and any non-zero element of $|T^{-1}K|$ is invertible. In other words, *any generalized field can be embedded into a generalized field, in which all non-zero unary operations become invertible*. Clearly, $K \to T^{-1}K$ has a universal property among all such embeddings.

6.1.17. (Tensor products of localizations.) Let S and T be two subsets of A. Formulas (6.1.5.1) and (6.1.5.2) imply

$$A[S^{-1}] \otimes_A A[T^{-1}] \cong A[(S \cup T)^{-1}] \tag{6.1.17.1}$$
$$A[S^{-1}] \boxtimes_A A[T^{-1}] \cong A[(S \cup T)^{-1}] \tag{6.1.17.2}$$

In particular, the category of A-algebras of form $S^{-1}A$ is stable under coproducts and pushouts.

6.1.18. (Morphisms between localizations.) Since all $A \to S^{-1}A$ are epimorphisms (and even NC-epimorphisms), there is at most one A-algebra homomorphism between two such localizations, i.e. the category of all localizations of A is actually a pre-ordered set. The universal property of $A[S^{-1}]$ shows that $A \to T^{-1}A$ factorizes through $A \to A[S^{-1}]$ iff all elements of S become invertible in $T^{-1}A$, i.e. iff S is contained in $\tilde{T} = (i_A^T)^{-1}(\mathrm{Inv}(T^{-1}A))$, the saturation on T. When S is a multiplicative system, this condition is also equivalent to $\tilde{S} \subset \tilde{T}$; since $\tilde{S}^{-1}A \cong S^{-1}A$, we see that *the category of localizations of A is equivalent to the category defined by the set of saturated multiplicative systems in $|A|$, ordered by inclusion.*

In particular, $A \to A_f$ factorizes through $A \to A_g$ iff $\tilde{S}_g \subset \tilde{S}_f$ iff $g \in \tilde{S}_f$ iff g divides f^n for some $n \geq 0$.

6.1.19. (Filtered inductive limits of localizations.) If S_α is a collection of multiplicative subsets of $|A|$, filtered by inclusion, then $S := \bigcup_\alpha S_\alpha$ is a multiplicative system as well, and $\varinjlim_\alpha S_\alpha^{-1}A \xrightarrow{\sim} S^{-1}A$. This statement can be generalized to the case when we are given a filtered inductive system of generalized rings A_α as well, S_α is a multiplicative system in $|A_\alpha|$, and the transition morphisms $f_{\alpha\beta} : A_\alpha \to A_\beta$ satisfy $f_{\alpha\beta}(S_\alpha) \subset S_\beta$: we put $A := \varinjlim A_\alpha$, $S := \varinjlim S_\alpha \subset |A|$, and then $\varinjlim S_\alpha^{-1}A_\alpha \xrightarrow{\sim} S^{-1}A$. We can also take an inductive system of A_α-modules M_α, put $M := \varinjlim M_\alpha$, and obtain a canonical isomorphism of $S^{-1}A$-modules $\varinjlim S_\alpha^{-1}M_\alpha \xrightarrow{\sim} S^{-1}M$.

Notice that any multiplicative system $S \subset |A|$ is a filtered inductive limit of its finitely generated multiplicative subsystems $\langle f_1, \ldots, f_n\rangle$, $f_i \in S$. On the other hand, clearly $M[f_1^{-1}, \ldots, f_n^{-1}] \cong M[f^{-1}]$, where $f := f_1 \cdots f_n \in S$, for any A-module M. We conclude that

$$S^{-1}A \cong \varinjlim_{f\in S} A_f \quad \text{and} \quad S^{-1}M \cong \varinjlim_{f\in S} M_f \tag{6.1.19.1}$$

The inductive limits are computed here with respect to the following preorder on S: $f \prec g$ iff $\tilde{S}_f \subset \tilde{S}_g$ iff f divides some power of g. This condition guarantees the existence of maps $A_f \to A_g$ and $M_f \to M_g$ whenever $f \prec g$.

6.1.20. (Finitely presented localizations.) Notice that $A[S^{-1}]$ is a finitely generated A-algebra iff we can find a finite subset $S_0 \subset S$, such that $A[S_0^{-1}] \to A[S^{-1}]$ is surjective (cf. (6.1.5.1)). Similarly, $A[S^{-1}]$ is a finitely presented A-algebra iff there is a finite subset $S_0 \subset S$, such that $A[S_0^{-1}] \xrightarrow{\sim} A[S^{-1}]$. In this case we can replace S by S_0, and even by the product f of all elements of S_0, i.e. *finitely presented localizations of A are exactly the localizations of form A_f, $f \in |A|$.*

6.1.21. (Localization, tensor products, and **Hom**.) If $S \subset |A|$ is a multiplicative system and M, N two A-modules, we have canonical isomorphisms

$$S^{-1}(M \otimes_A N) \cong S^{-1}M \otimes_{S^{-1}A} S^{-1}N \cong S^{-1}M \otimes_A S^{-1}N \cong S^{-1}M \otimes_A N \tag{6.1.21.1}$$

In fact, the existence of the second and the third of these isomorphisms is due to the fact that S acts by bijections on $S^{-1}M \otimes_A S^{-1}N$ and $S^{-1}M \otimes_A N$, hence these two modules admit a natural $S^{-1}A$-module structure. Now the first isomorphism is given by the isomorphism of **5.3.16**, valid for any A-algebra A', when we put $A' := S^{-1}A$:

$$(M \otimes_A N)_{(A')} \cong M_{(A')} \otimes_{A'} N_{(A')} \tag{6.1.21.2}$$

Similarly, for any two A-modules M and N we have a canonical $S^{-1}A$-module homomorphism

$$S^{-1} \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N) \tag{6.1.21.3}$$

We claim that it is injective for a finitely generated M, and an isomorphism for a finitely presented M. In fact, we have a more general statement, valid for any *flat* A-algebra A' (hence also for $A' = S^{-1}A$), concerning the following canonical A'-module homomorphism:

$$\operatorname{Hom}_A(M, N)_{(A')} \xrightarrow{\gamma} \operatorname{Hom}_{A'}(M_{(A')}, N_{(A')}) \tag{6.1.21.4}$$

To show the injectivity (resp. bijectivity) of γ for a finitely generated (resp. presented) M, we first show that it is an isomorphism for $M = A(n)$: in this case the statement reduces to $(N^n)_{(A')} \cong (N_{(A')})^n$, an immediate consequence of the flatness of A'. Then we choose a strict epimorphism $A(n) \to M$ (resp. a presentation $A(m) \rightrightarrows A(n) \to M$), and the general result follows from the special case just considered by a standard diagram chasing.

6.1.22. (Localizations and alternativity.) Since $S^{-1}A$ is a unary A-algebra, $S^{-1}A$ is an alternating $\mathbb{F}_{\pm 1}$-algebra whenever A is one (cf. **5.5.16**,g). Moreover, any matrix $Z \in M(n, n; S^{-1}A) = (S^{-1}A)(n)^n \cong S^{-1}(A(n)^n)$ can be written in form Z_0/s, with $Z_0 \in M(n, n; A)$ and $s \in S$, and $Z_0/s = Z_0'/s'$ iff $ts' \cdot Z_0 = ts \cdot Z_0'$ for some $t \in S$. This immediately implies that *the properties* (DET_r') *and* (DET_r^*) *of* **5.6** *hold for* $S^{-1}A$ *whenever they hold for* A. We cannot say the same about properties (DET_r).

6.1.23. (Example.) We know that localizations of classical rings coincide with classical localizations, so we don't obtain anything new in the classical case. Let's show that $\mathbb{Z}_\infty[f^{-1}] \cong \mathbb{R}$, $\bar{\mathbb{Z}}_\infty[f^{-1}] \cong \mathbb{C}$, and $\mathbb{Z}_{(\infty)}[f^{-1}] \cong \mathbb{Q}$, for any $f \in |\mathbb{Z}_\infty|$ (resp…), such that $0 < |f| < 1$. Let's treat the first case; the two remaining cases are proved similarly.

First of all, $\mathbb{Z}_\infty \to \mathbb{R}$ is injective (i.e. a monomorphism), hence the same is true for $\mathbb{Z}_\infty[f^{-1}] \to \mathbb{R}[f^{-1}]$ (this is a general fact: *if* $A \to B$ *is a monomorphism, then* $S^{-1}A \to S^{-1}B$ *is also a monomorphism*). Since $f \neq 0$, $\mathbb{R}[f^{-1}] \cong \mathbb{R}$, and we can identify $B := \mathbb{Z}_\infty[f^{-1}]$ with a submonad of $\mathbb{R}$: $\mathbb{Z}_\infty \subset B = \mathbb{Z}_\infty[f^{-1}] \subset \mathbb{R}$. We want to show that $B = \mathbb{R}$, i.e. that any $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}(n) = \mathbb{R}^n$ lies in $B(n)$. Since $|f| < 1$, we can find an integer $n \geq 0$, such that $|\lambda_1| + \cdots + |\lambda_n| \leq f^{-n}$. Put $\mu := f^n\lambda$; then $|\mu_1| + \cdots + |\mu_n| \leq 1$, hence μ lies in $\mathbb{Z}_\infty(n) \subset B(n)$. On the other hand, $f^{-1} = 1/f \in B(1)$, hence $\lambda = \mu/f^n$ lies in $B(n)$ as well.

6.1.24. (Axiomatic description of localizations. Pseudolocalizations.) Consider the following set of properties of a commutative A-algebra B, with the structural morphism $\rho : A \to B$:

1) ρ is an NC-epimorphism, i.e. any A-module admits at most one compatible B-module structure;

2) $\rho_* : B\text{-}Mod \to A\text{-}Mod$ is fully faithful;

3) B is a flat A-algebra, i.e. ρ^* is exact;

4) B is a unary A-algebra;

5) B is finitely presented over A.

6) ρ_* is a $\otimes$-functor, i.e. $\rho_* M \otimes_A \rho_* N \to \rho_*(M \otimes_B N)$ is an isomorphism for any two B-modules M and N.

Notice that 2) $\Rightarrow$ 1), that 1) implies that ρ is an epimorphism, and that any (unary) localization $S^{-1}A$ satisfies 1)–4) and 6), and A_f satisfies all of the above conditions. However, the unarity of $S^{-1}A$ hasn't been used in most of the previous considerations. Therefore, we might define the *pseudolocalizations of* A to be the A-algebras B that satisfy 1)–3), and *open pseudolocalizations of* A by requiring 5) as well. Most properties of localizations should be also true for pseudolocalizations. However, one can check that 6) is a formal consequence of 2) and 4), but not of 2) alone (consider $\rho : \mathbb{Z}_\infty \to \mathbb{F}_\infty$, $M = N = \mathbb{F}_\infty(2)$ for this), so we cannot expect 6) to hold for all pseudolocalizations.

6.1.25. (Properties of pseudolocalizations.) Clearly, the set of pseudolocalization morphisms is closed under composition and base change. For the latter property we use corollary **5.4.4** of the "base change theorem" **5.4.2** to show that flatness is preserved by any base change; to check stability of 2) under some base change $f : A \to A'$ we observe that a $B' := A' \otimes_A B$-module structure on an A'-module M is the same thing as a B-module structure on $f_* M$ (automatically commuting with the action of A' on M: for any $t \in A'(n)$ the map $[t]_M : M^n \to M$ is an A-module homomorphism, hence also a B-module homomorphism, ρ_* being left exact and fully faithful; notice that the flatness of ρ is not needed here), hence B'-*Mod* can be identified with a full subcategory of A'-*Mod*, the preimage of B-*Mod* $\subset$ A-*Mod* under f_*.

All pseudolocalizations of A constitute a full subcategory of the category of commutative A-algebras; property 1) implies that this subcategory is a preordered set, and it is stable under coproducts and pushouts. The same is true for the subcategory of open pseudolocalizations.

6.1.26. Questions. Is it true that all unary pseudolocalizations are of form $S^{-1}A$? If not, is this true for a classical A? Are there any non-unary pseudolocalizations of generalized rings? Of alternating monads with (DET_∞^*) property?

Answers. No, no, yes, unknown.

6.1.27. (An example.) Notice that generalized rings without zero often have non-unary pseudolocalizations. For example, if A is a classical ring, and $S \subset |A|$ is a multiplicative system, the homomorphism of generalized rings $\mathrm{Aff}_A \to \mathrm{Aff}_{S^{-1}A}$ induced by $A \to S^{-1}A$ (cf. **3.4.12**,g) is easily seen to be a pseudolocalization. On the other hand, $|\,\mathrm{Aff}_A\,| = |\,\mathrm{Aff}_{S^{-1}A}\,| = \{\boldsymbol{e}\}$, so $\mathrm{Aff}_{S^{-1}A}$ is not unary over Aff_A when $S^{-1}A \neq A$. If $A \to S^{-1}A$ is not surjective, $\mathrm{Aff}_{S^{-1}A}$ cannot be even pre-unary over Aff_A.

Notice that there is only the trivial multiplicative system in $|\operatorname{Aff}_A| = \{\boldsymbol{e}\}$, so we don't have any non-trivial localizations of Aff_A.

6.1.28. (Localization with respect to matrices.) We can try to generalize our definition of unary localization by formally inverting families of square matrices $Z_i \in M(r_i, r_i; A) = A(r_i)^{r_i}$. Let's consider the case of one square matrix $Z \in M(r, r; A)$; the general case can be deduced from this one by (transfinite) induction.

So let's fix $Z = (Z_1, \ldots, Z_r) \in M(r, r; A)$ and define

$$A\langle Z^{-1}\rangle := A\langle Z_1^-, \ldots, Z_r^- \mid Z_i^-(Z_1, \ldots, Z_r) = \{i\}_r = Z_i(Z_1^-, \ldots, Z_r^-), \text{all } i\rangle \tag{6.1.28.1}$$

$$A[Z^{-1}] := A[Z_1^-, \ldots, Z_r^- \mid Z_i^-(Z_1, \ldots, Z_r) = \{i\}_r = Z_i(Z_1^-, \ldots, Z_r^-), \text{all } i] \tag{6.1.28.2}$$

Clearly, $A \to A\langle Z^{-1}\rangle$ is universal among all monad homomorphisms $\rho : A \to B$, such that $\rho(Z)$ is invertible over B, and $A \to A[Z^{-1}]$ has the same universal property among all commutative A-algebras.

We want to show that $B := A\langle Z^{-1}\rangle$ is commutative; this would imply $A[Z^{-1}] \cong A\langle Z^{-1}\rangle$, similarly to what we had in **6.1.5**. So we have to check that the generators Z_i^- commute among themselves and the operations of A. For any B-module N we denote by $[Z]_N : N^r \to N^r$ the map with components $[Z_j]_N : N^r \to N$, and define $[Z^-]_N$ similarly. Clearly, all Z_j^- commute with all operations from A iff all $[Z_j^-]_N$ are A-linear (for all N or just for all $N = B(n)$) iff $[Z^-]_N : N^r \to N^r$ is A-linear for such B-modules N. Now observe that $[Z^-]_N = [Z]_N^{-1}$, and $[Z]_N = [Z]_{\rho_* N}$ is A-linear for any N because of the commutativity of A.

It remains to check that Z_j^- commute among themselves. This means equality of two maps $N^{\mathbf{r}\times\mathbf{r}} \to N$ for all B-modules N and all $1 \leq i, j \leq r$. We can use the same trick as above and combine together these maps for all values of i and j into two maps $[Z^- \otimes Z^-]'_N, [Z^- \otimes Z^-]''_N : N^{\mathbf{r}\times\mathbf{r}} \to N^{\mathbf{r}\times\mathbf{r}}$; we have to check that these two maps coincide for all B-modules N. Notice that these two maps can be described as follows. The first of them takes a matrix $(z_{ij}) \in N^{\mathbf{r}\times\mathbf{r}}$, applies $[Z^-]_N : N^r \to N^r$ to each of its rows, and then applies $[Z^-]_N$ again to each of the columns, and the second map applies $[Z^-]_N$ first to columns, and then to rows. Hence $[Z^- \otimes Z^-]'_N = ([Z \otimes Z]''_N)^{-1}$ and $[Z^- \otimes Z^-]''_N = ([Z \otimes Z]'_N)^{-1}$, so it remains to check $[Z \otimes Z]''_N = [Z \otimes Z]'_N$. We can replace here N by $\rho_* N$, since Z is defined over A, and the resulting equality is true for any A-module M (and in particular for $M = \rho_* N$) because of the commutativity of A. One can also check that $A[Z^{-1}]$ *is alternating whenever A is alternating* essentially in the same way we have just verified the commutativity between the Z_j^-.

An immediate consequence of $A\langle Z^{-1}\rangle = A[Z^{-1}]$ is that giving a monad homomorphism $A[Z^{-1}] \to B$ is equivalent to giving a monad homomorphism $\rho : A \to B$, such that $\rho(Z)$ becomes invertible over B. Applying this to $B := \mathrm{END}(M)$, we see that $A[Z^{-1}]$-modules are exactly the A-modules M, such that $[Z]_M : M^r \to M^r$ is bijective. In particular, the scalar restriction functor $(i_A^Z)_*$ is fully faithful, and $i_A^Z : A \to A[Z^{-1}]$ is an NC-epimorphism.

In this way we see that $A[Z^{-1}]$ has the properties 1), 2) and 5) of **6.1.24**; if we localize with respect to an arbitrary set of matrices, we still obtain properties 1) and 2). However, we'll see in a moment that the property 3) (flatness) doesn't hold in general, so we don't obtain pseudolocalizations in this way.

6.1.29. (Example of a non-flat matrix localization.) Put $A := \mathbb{Z}_{\geq 0}$, $Z := \left(\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right) = (+, \{2\}) \in M(2,2;A) = A(2)^2 \cong \mathbb{Z}_{\geq 0}^{2\times 2}$. Then A-*Mod* is the category of commutative monoids, and $[Z]_M : M^2 \to M^2$ is the map $(x,y) \mapsto (x+y, y)$. We see that $[Z]_M$ is bijective iff M is an abelian group, hence $A[Z^{-1}]$-*Mod* $=$ $\mathbb{Z}$-*Mod*, hence $A[Z^{-1}] = \mathbb{Z}_{\geq 0}[\left(\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right)^{-1}] = \mathbb{Z}$, i.e. $\mathbb{Z}$ is a matrix localization of $\mathbb{Z}_{\geq 0}$.

We claim that $\mathbb{Z}$ is not flat over $\mathbb{Z}_{\geq 0}$. Indeed, the base change functor $\mathbb{Z}\otimes_{\mathbb{Z}_{\geq 0}}$ transforms any commutative monoid M into its symmetrization $\tilde{M}$, and it is quite easy to construct a commutative monoid M and a submonoid $N \subset M$, such that $\tilde{N} \to \tilde{M}$ is not injective. For example, put $M := \mathbb{Z}\times\mathbb{Z}_{\geq 0}\times\{\pm 1\}/\sim$, where the equivalence relation $\sim$ is given by $(x, n, +1) \sim (x, n, -1)$ for any $x \in \mathbb{Z}$ and $n > 0$, and $N := \mathbb{Z}\times 0\times\{\pm 1\} \subset M$. Then $\tilde{N} = \mathbb{Z}\times\{\pm 1\}$ and $\tilde{M} = \mathbb{Z}\times\mathbb{Z}$, and the map $\tilde{N} \to \tilde{M}$ is given by $(x, \pm 1) \to (x, 0)$, hence it is not injective.

6.1.30. (Matrix localization and (DET)-properties.) When A is a classical ring, a square matrix $Z \in M(r,r;A)$ is invertible in an A-algebra B iff $\det(Z)$ is invertible. More generally, suppose that A is an alternating $\mathbb{F}_{\pm 1}$-algebra, and $Z \in M(r,r;A)$ is a matrix, such that the property (DET'_r) of **5.6** is fulfilled for Z (e.g. A has the (DET'_∞)-property). Suppose that $A[Z^{-1}]$ is also alternating (something that we haven't checked in detail in **6.1.28**). Then $\det(Z)$ is invertible in $A[Z^{-1}]$, hence $A \to A[Z^{-1}]$ factorizes through $A[\det(Z)^{-1}]$. On the other hand, (DET'_r) means that we have a matrix $Z^* \in M(r,r;A)$ and an integer $N > 0$, such that $ZZ^* = Z^*Z = \det(Z)^N$; this implies that $Z^*/\det(Z)^N$ is an inverse to Z over $A[\det(Z)^{-1}]$, hence we have a homomorphism in the opposite direction $A[Z^{-1}] \to A[\det(Z)^{-1}]$. Since $A \to A[Z^{-1}]$ and $A \to A[\det(Z)^{-1}]$ are epimorphic, we conclude that under the above assumptions $A[Z^{-1}] \cong A[\det(Z)^{-1}]$, i.e. *matrix localization over alternating generalized rings with* (DET'_∞)*-property shouldn't give anything new compared to unary localizations.*

6.2. (Prime spectrum of a generalized ring.)

6.2.1. (Ideals in generalized rings.) Recall that an *ideal* $\mathfrak{a}$ in a generalized ring A is by definition an A-submodule of $|A|$. For any subset $S \subset |A|$ we denote by (S) the ideal generated by S, i.e. the smallest ideal in A containing S; it is equal to the image of $A(S) \to |A|$ (cf. **4.6.9**). When we have two subsets S, $T \subset |A|$, we usually write (S, T) instead of $(S \cup T)$, and similarly for larger finite families of subsets and/or elements of $|A|$. In particular, for any two ideals $\mathfrak{a}$ and $\mathfrak{b}$ we can construct their upper bound $(\mathfrak{a}, \mathfrak{b})$, denoted also by $\mathfrak{a} + \mathfrak{b}$, i.e. the smallest ideal containing both $\mathfrak{a}$ and $\mathfrak{b}$; clearly, $\mathfrak{a} + \mathfrak{b}$ coincides with the image of $\mathfrak{a} \oplus \mathfrak{b} \to |A|$. On the other hand, the intersection of any family of ideals and the union of any filtering family of ideals is an ideal again; in particular, we have lower bounds $\mathfrak{a} \cap \mathfrak{b}$ in the lattice of ideals of $|A|$ as well. Notice that the *unit ideal* $(1) = |A|$, and the *initial ideal* $\varnothing_A$ (denoted also by (0) or 0 and called the *zero ideal* when A is a generalized ring with zero) are the largest and the smallest elements of the ideal lattice of A.

For any two ideals $\mathfrak{a}$ and $\mathfrak{b}$ we can construct their *product* $\mathfrak{ab}$, i.e. the ideal generated by all products xy, with $x \in \mathfrak{a}$, $y \in \mathfrak{b}$. Another description: $\mathfrak{ab}$ is the image of $\mathfrak{a} \otimes_A \mathfrak{b} \to |A|$. Clearly, this multiplication of ideals is associative and commutative, with $(1) \cdot \mathfrak{a} = \mathfrak{a}$ and $\varnothing_A \cdot \mathfrak{a} = \varnothing_A$. We can extend our definition and define $\mathfrak{a}_1 \cdots \mathfrak{a}_n N \subset M$ (e.g. as the image of $\mathfrak{a}_1 \otimes_A \cdots \otimes_A \mathfrak{a}_n \otimes_A N \to M$), for any ideals $\mathfrak{a}_i \subset |A|$ and any A-submodule $N \subset M$.

Finally, if B is an A-algebra, and $\mathfrak{a}$ an ideal of A, we denote by $B\mathfrak{a}$ the ideal of B generated by the image of $\mathfrak{a}$, equal to the image of $B \otimes_A \mathfrak{a} \to B \otimes_A |A| = |B|$. Of course, all of the above constructions coincide with the usual ones over a classical A.

6.2.2. (Prime ideals.) We say that $\mathfrak{p} \subset |A|$ is a *prime ideal* if it an ideal, and its complement $S_{\mathfrak{p}} := |A| - \mathfrak{p}$ is a multiplicative system, i.e. $\boldsymbol{e} \notin \mathfrak{p}$, and $xy \in \mathfrak{p}$ implies that either x or y is in $\mathfrak{p}$. The localization $S_{\mathfrak{p}}^{-1}A$ is usually denoted by $A_{\mathfrak{p}}$, and similarly for the localizations of an A-algebra B or a B-module N.

6.2.3. (Maximal ideals.) We say that $\mathfrak{m} \subset |A|$ is a *maximal ideal* if it is a maximal element in the ordered set of all ideals of $|A|$ distinct from (1), i.e. if $\mathfrak{m} \subset \mathfrak{a}$ implies $\mathfrak{a} = \mathfrak{m}$ or $\mathfrak{a} = (1)$. Notice that *any maximal ideal is prime*: indeed, by definition $1 \notin \mathfrak{m}$, and if x, $y \notin \mathfrak{m}$, we have $(\mathfrak{m}, x) = (\mathfrak{m}, y) = (1)$, hence $(\mathfrak{m}, xy) \supset (\mathfrak{m}^2, x\mathfrak{m}, y\mathfrak{m}, xy) = (\mathfrak{m}, x)(\mathfrak{m}, y) = (1)$, and this is possible only if $xy \notin \mathfrak{m}$. Since any filtered union of ideals $\neq (1)$ is again an ideal $\neq (1)$, an application of Zorn lemma shows that *any ideal* $\mathfrak{a} \neq (1)$ *is contained in some maximal ideal*; more generally, any submodule N of a finitely generated A-module M is either equal to M or contained in a maximal submodule of M.

In particular, *any non-trivial generalized ring ($\mathbf{1}_+$ included) contains at least one maximal (hence also prime) ideal.*

Applying this to principal ideals (a) we see that *the union of all ideals $\neq (1)$ of A coincides with the union of all maximal ideals, equal to the complement of the set* $\operatorname{Inv}(A)$ *of invertible elements of A.*

6.2.4. (Generalized local rings.) It is natural to say that A is *local* if it has exactly one maximal ideal $\mathfrak{m} = \mathfrak{m}_A$. This is equivalent to saying that $\mathfrak{m} \subset |A|$ is an ideal, such that all elements of $|A| - \mathfrak{m}$ are invertible. Furthermore, a homomorphism of generalized local rings $f : A \to B$ is *local* if $f^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$.

6.2.5. (Prime spectrum.) The *prime spectrum* $\operatorname{Spec} A = \operatorname{Spec}^p A$ of a generalized ring A is simply the set of its prime ideals. Clearly, $\operatorname{Spec} A = \varnothing$ iff $A = \mathbf{1}$. For any subset $M \subset |A|$ we put $V(M) := \{\mathfrak{p} \in \operatorname{Spec} A \,|\, M \subset \mathfrak{p}\}$. We obtain immediately the usual formulas $V(M) = V(\mathfrak{a})$ if $\mathfrak{a} = (M)$, $V(\varnothing) = \operatorname{Spec} A$, $V(1) = \varnothing$, $V(\bigcup_\alpha M_\alpha) = \bigcap_\alpha V(M_\alpha)$, and finally $V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$, that show that all these $V(M)$ are the closed subsets for a certain topology on $\operatorname{Spec}^p A$, called the *spectral* or *Zariski topology* on $\operatorname{Spec}^p A$.

We write $V(f)$ instead of $V(\{f\})$ for any $f \in |A|$. Clearly, $V(M) = \bigcup_{f\in M} V(f)$ for any M, hence the *principal open subsets* $D(f) := \{\mathfrak{p} \,|\, f \notin \mathfrak{p}\}$ constitute a base of the Zariski topology on $\operatorname{Spec}^p A$. We have the usual formula $D(fg) = D(f) \cap D(g)$.

For any homomorphism of generalized rings $\varphi : A \to B$ and any prime ideal $\mathfrak{q} \subset |B|$ its pullback ${}^a\varphi(\mathfrak{q}) := |\varphi|^{-1}(\mathfrak{q})$ is a prime ideal in A, so we get a map ${}^a\varphi : \operatorname{Spec}^p B \to \operatorname{Spec}^p A$. This map is continuous because of the usual formula $({}^a\varphi)^{-1}(V(M)) = V(\varphi(M))$. In particular, $({}^a\varphi)^{-1}(D(f)) = D(\varphi(f))$ for any $f \in |A|$.

6.2.6. ($\operatorname{Spec}^p S^{-1}A$.) Let S be a multiplicative system in A. We know that $\mathfrak{a} \mapsto S^{-1}\mathfrak{a}$ induces a bijection between S-saturated ideals $\mathfrak{a} \subset |A|$ and all ideals of $S^{-1}A$ (cf. **6.1.11**), and a saturated $\mathfrak{a}$ coincides with the pullback of $S^{-1}\mathfrak{a}$. Clearly, the pullback of a prime ideal of $S^{-1}A$ is a saturated prime ideal $\mathfrak{p} \subset |A|$, and it is immediate that a prime ideal $\mathfrak{p}$ in A is S-saturated iff $\mathfrak{p} \cap S = \varnothing$. This implies that ${}^a i_S^A : \operatorname{Spec}^p S^{-1}A \to \operatorname{Spec}^p A$ *induces a bijection between* $\operatorname{Spec}^p S^{-1}A$ *and the set of prime ideals* $\mathfrak{p} \subset |A|$ *that do not intersect S.* Moreover, this bijection is actually a *homeomorphism* if we equip this set of prime ideals with the topology induced by that of $\operatorname{Spec}^p A$. Indeed, $D(f/s)$, $f \in |A|$, $s \in S$, constitute a base of the topology of $\operatorname{Spec}^p S^{-1}A$; but obviously $D(f/s) = D(f/1) = D(f) \cap \operatorname{Spec}^p S^{-1}A$. Henceforth we shall usually identify $\operatorname{Spec}^p S^{-1}A$ with this subspace of $\operatorname{Spec}^p A$.

In particular, $\operatorname{Spec}^p A_f = \operatorname{Spec}^p A[f^{-1}]$ is homeomorphic to the principal open subset $D(f) \subset \operatorname{Spec}^p A$.

6.2.7. (An application.) We can use the above constructions and definitions to show that *if A is a generalized ring with zero, and $f \in |A|$ is not nilpotent, then $f \notin \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Spec}^p A$.* Therefore, the intersection of all prime ideals of a generalized ring with zero coincides with the set of nilpotent elements. On the other hand, if A is a generalized ring without zero, then the initial ideal $\varnothing_A$ is prime, hence the intersection of all prime ideals is empty.

6.2.8. (Open covers and quasicompactness.) Let f_α be a family of elements of $|A|$, and let us denote by $\mathfrak{a}$ the ideal generated by this family. Then $\operatorname{Spec}^p A - \bigcup_\alpha D(f_\alpha) = \bigcap_\alpha V(f_\alpha) = V(\mathfrak{a})$, hence $\operatorname{Spec}^p A = \bigcup_\alpha D(f_\alpha)$ iff $V(\mathfrak{a}) = \varnothing$ iff $\mathfrak{a} = (1)$, since any non-unit ideal is contained in some maximal ideal. In other words, *principal open subsets $D(f_\alpha)$ constitute a cover of* $\operatorname{Spec}^p A$ *iff the ideal generated by* (f_α) *is equal to the unit ideal* (1). This means that $1 = t(f_{\alpha_1}, \ldots, f_{\alpha_n})$ for some $n \geq 0$, $t \in A(n)$ and α_i. In this case the finite set of elements f_{α_i} already generates the unit ideal, i.e. $\operatorname{Spec}^p A = \bigcup_{i=1}^n D(f_{\alpha_i})$. Using the fact that the principal open subsets are a base of topology of $\operatorname{Spec}^p A$, we see that $\operatorname{Spec}^p A$ *is quasicompact.* Of course, this is also true for any $D(g) \cong \operatorname{Spec}^p A_g$.

6.2.9. (Open covers of $D(g)$.) Notice that $D(g) = \operatorname{Spec}^p A$ iff $V(g) = \varnothing$ iff g is invertible in A. Therefore, $D(f) \subset D(g)$ iff $D(g) \cap \operatorname{Spec}^p A_f = \operatorname{Spec}^p A_f$ iff $D(g/1) = \operatorname{Spec}^p A_f$ iff g is invertible in A_f iff g divides some power of f iff $\tilde{S}_g \subset \tilde{S}_f$ iff $A \to A_f$ factorizes through $A \to A_g$.

When $D(g) \subset \bigcup_\alpha D(f_\alpha)$? This condition is equivalent to $\bigcup_\alpha D(f_\alpha/1) = \operatorname{Spec}^p A_g$, and this is equivalent to $\mathfrak{a}_g = (f_\alpha/1) \subset |A_g|$ being equal to $|A_g|$. However, $\mathfrak{a}_g = A_g \cdot \mathfrak{a}$, where $\mathfrak{a} = (f_\alpha) \subset |A|$ is the ideal of A generated by f_α. Clearly, $\mathfrak{a}_g = (1)$ iff the saturation of $\mathfrak{a}$ with respect to $S_g = \{1, g, g^2, \ldots\}$ is equal to the unit ideal iff $S_g \cap \mathfrak{a} \neq \varnothing$ iff $g^n \in \mathfrak{a} = (f_\alpha)$ iff $g \in \mathfrak{r}(\mathfrak{a}) := \{g \mid g^n \in \mathfrak{a}$ for some $n > 0\}$.

6.2.10. (The radical of an ideal.) For any ideal $\mathfrak{a} \subset |A|$ we denote by $\mathfrak{r}(\mathfrak{a})$ the *radical* of $\mathfrak{a}$, i.e. the set of all elements $g \in |A|$, such that $g^n \in \mathfrak{a}$ for some $n > 0$. We claim that $\mathfrak{r}(\mathfrak{a})$ *is always an ideal, namely, the intersection of all prime ideals containing* $\mathfrak{a}$. Clearly, $\mathfrak{r}(\mathfrak{a})$ is contained in this intersection, since $g^n \in \mathfrak{a} \subset \mathfrak{p}$ implies $g \in \mathfrak{p}$. Conversely, suppose that $g \notin \mathfrak{r}(\mathfrak{a})$, i.e. $S_g \cap \mathfrak{a} = \varnothing$. Then the S_g-saturation of $\mathfrak{a}$ doesn't contain $\boldsymbol{e}$, hence $\mathfrak{a}_g := S_g^{-1}\mathfrak{a} \neq (1)$, hence this ideal is contained in some maximal ideal $\mathfrak{m}$ of A_g, and the preimage $\mathfrak{p}$ of $\mathfrak{m}$ in A is a prime ideal of A that contains $\mathfrak{a}$, but does not contain g.

6.2.11. Another simple statement is that for any $N > 0$ a collection of elements $f_i \in |A|$ generates the unit ideal iff the same is true for (f_i^N). Indeed, we know that f_i generate the unit ideal iff $\bigcup_i D(f_i) = \operatorname{Spec}^p A$, and similarly for f_i^N, and our statement follows from $D(f_i) = D(f_i^N)$.

6.2.12. One can check that the irreducible closed subsets of $\operatorname{Spec}^{\mathcal{P}} A$ are exactly the subsets of the form $V(\mathfrak{p})$, with $\mathfrak{p}$ any prime ideal. Clearly, these closed subsets admit a unique generic point, namely, $\mathfrak{p}$ itself. This shows that $\operatorname{Spec}^{\mathcal{P}} A$ is a sober topological space, and in particular a Kolmogorov space.

6.2.13. (Examples.) Let's consider some examples of prime spectra. Examples b) and e) show that we shouldn't be too optimistic about the theory of prime spectra in the generalized context.

a) Of course, for a classical commutative ring A we recover its prime spectrum $\operatorname{Spec} A$ in the usual sense.

b) On the other hand, $\operatorname{Spec}^{\mathcal{P}} \operatorname{Aff}_A$ is a one-element set for any classical ring A, since $|\operatorname{Aff}_A| = \{\boldsymbol{e}\}$, and the only prime ideal in Aff_A is the initial ideal $\varnothing$. This shows that the theory of prime spectra is not very nice for such generalized rings: one would rather expect $\operatorname{Spec}^{\mathcal{P}} \operatorname{Aff}_A = \operatorname{Spec}^{\mathcal{P}} A$ as a topological space (but with a different structure sheaf).

c) Similarly, $\operatorname{Spec}^{\mathcal{P}} \mathbb{F}_\varnothing$, $\operatorname{Spec}^{\mathcal{P}} \mathbb{F}_1$, $\operatorname{Spec}^{\mathcal{P}} \mathbb{F}_{\pm 1}$ and $\operatorname{Spec}^{\mathcal{P}} \mathbb{F}_{1^n}$ are one-element sets, since the initial ideal is the only prime ideal in any of these generalized rings.

d) We know already that $\operatorname{Spec}^{\mathcal{P}} \mathbb{Z}_\infty = \{0, \mathfrak{m}_\infty\}$ is a two-point set with a generic and a closed point, hence it is similar to the spectrum of a discrete valuation ring.

e) The only prime ideals in $\operatorname{Spec}^{\mathcal{P}} \mathbb{F}_1[T]$ are (0) and (T), and a similar result holds for all $\operatorname{Spec}^{\mathcal{P}} \mathbb{F}_{1^n}[T]$, i.e. these spectra also look like spectra of DVRs. Notice that the canonical strict epimorphism $\varphi : \mathbb{F}_1[T] \to \mathbb{F}_{1^n} = \mathbb{F}_1[\zeta \mid \zeta^n = \boldsymbol{e}]$, $T \mapsto \zeta$, induces a map ${}^a\varphi : \operatorname{Spec}^{\mathcal{P}} \mathbb{F}_{1^n} \to \operatorname{Spec}^{\mathcal{P}} \mathbb{F}_1[T]$ that is *not* closed: indeed, its image consists of the generic point of $\operatorname{Spec}^{\mathcal{P}} \mathbb{F}_1[T]$.

6.3. (Localization theories.) Now we want to define the *localization theories*, which will be sometimes also called *theories of spectra*, giving rise to (different) spectra of generalized rings, and study in some detail two of them.

6.3.1. (Categories of pseudolocalizations.) Recall that a *pseudolocalization* of a generalized ring A is a commutative A-algebra B, or equivalently, a homomorphism of generalized rings $\rho : A \to B$, such that (cf. **6.1.24**): 1) the scalar restriction functor $\rho_* : B\text{-}\mathit{Mod} \to A\text{-}\mathit{Mod}$ is fully faithful; this condition implies that ρ is an NC-epimorphism, hence an epimorphism as well; 2) B is a flat A-algebra, i.e. the base change functor ρ^* is exact. If in addition

B is finitely presented over A, we say that it is an *open pseudolocalization* of A.

Let's fix some A and consider the category $\bar{\mathcal{P}}_A$ of all pseudolocalizations of A (considered as a full subcategory of A-*Alg*), and its full subcategory $\mathcal{P}_A$, consisting of all open pseudolocalizations of A.

Recall that for any homomorphism of generalized rings $f : A \to A'$ the base change functor $f^* := A' \otimes_A -$ transforms a pseudolocalization $\rho : A \to B$ of A into a pseudolocalization $\rho' : A' \to B'$ (cf. **6.1.25**); we have actually shown that B'-*Mod* $\subset A'$-*Mod* can be identified with the preimage of B-*Mod* $\subset A$-*Mod* under the scalar restriction functor $f_* : A'$-*Mod* $\to A$-*Mod*, and that any compatible B-module structure on an A'-module N automatically commutes with the A'-module structure, hence $B' = A' \otimes_A B \cong A' \boxtimes_A B$, i.e. B'/A' is the non-commutative pullback of B/A as well.

We can combine the above observations to define a fibered category $p : \bar{\mathcal{P}} \to$ *GenR*, with the fiber over A equal to $\bar{\mathcal{P}}_A$, and the pullback functors $f^* : \bar{\mathcal{P}}_A \to \bar{\mathcal{P}}_{A'}$ defined for any $f : A \to A'$ in the natural way. We have a fibered subcategory of open localizations $\mathcal{P} \subset \bar{\mathcal{P}}$ as well.

6.3.2. (Finite inductive limits of pseudolocalizations.) Recall that both $\mathcal{P}_A$ and $\bar{\mathcal{P}}_A$ are essentially given by preordered sets, i.e. there is at most one morphism between two pseudolocalizations of A. Moreover, all morphisms in these categories are epimorphisms and even NC-epimorphisms (i.e. they are epimorphic in the category of algebraic monads as well). Another simple observation: both these categories are closed under coproducts, since pseudolocalization morphisms are stable under base change and composition. This implies immediately that both $\mathcal{P}_A$ and $\bar{\mathcal{P}}_A$ are closed under finite inductive limits.

6.3.3. (Filtered inductive limits.) We claim that $\bar{\mathcal{P}}_A$ *is stable under filtered inductive limits.* Together with our previous results on finite inductive limits this implies the stability of pseudolocalizations under arbitrary inductive limits.

Now suppose that $B = \varinjlim B_\alpha$, where all $A \to B_\alpha$ are pseudolocalizations. Put $\mathcal{A} := A$-*Mod*, $\mathcal{B}_\alpha := B_\alpha$-*Mod*, identified with a strictly full subcategory of $\mathcal{A}$. Notice that the image of the scalar restriction functor B-*Mod* $\to \mathcal{A}$ is contained in each $\mathcal{B}_\alpha$, hence it induces a functor B-*Mod* $\to \mathcal{B} := \bigcap_\alpha \mathcal{B}_\alpha$. We claim that it is an equivalence; this would imply that $A \to B$ is a pseudolocalization, flatness of arbitrary inductive limits of flat algebras being evident. So let M be an object of $\mathcal{B}$, i.e. an A-module that admits a (necessarily unique) compatible B_α-module structure for each α. This means that $A \to \mathrm{END}(M)$ (uniquely) factorizes through each B_α, hence also through $B = \varinjlim B_\alpha$, i.e. M admits a unique B-module structure, compatible with its

A-module structure. This proves the essential surjectivity of $B\text{-}Mod \to \mathcal{B}$. Now it remains to check that $\mathrm{Hom}_B(N, N') = \mathrm{Hom}_A(N, N')$ for any two B-modules N and N'. But this is clear: $\mathrm{Hom}_B(N, N') = \bigcap_\alpha \mathrm{Hom}_{B_\alpha}(N, N')$, and $\mathrm{Hom}_{B_\alpha}(N, N') = \mathrm{Hom}_A(N, N')$ for all α.

Question. We know that any localization $S^{-1}A$ is a filtered inductive limit of open (i.e. finitely presented) localizations A_f. Is it true that any pseudolocalization is an inductive (hence also filtered inductive) limit of open pseudolocalizations?

6.3.4. (Localization theories.) A *localization theory* $\mathcal{T}$ (or a *theory of spectra*) of generalized rings (resp. of K-algebras, where K is a fixed generalized ring) is a choice of strictly full subcategories $\mathcal{T}_A \subset \mathcal{P}_A$ for each generalized ring A (resp. each K-algebra A), such that the following conditions are satisfied:

0) All isomorphisms $(A \xrightarrow{\sim} A')$ belong to $\mathcal{T}_A$.

1) $\mathcal{T}_A$ is stable under base change, i.e. if $(A \to B)$ belongs to $\mathcal{T}_A$, then $(A' \to A' \otimes_A B)$ belongs to $\mathcal{T}_{A'}$ for any $f : A \to A'$. In other words, $\mathcal{T}$ is a full fibered subcategory of $\mathcal{P}$.

2) $\mathcal{T}$ is closed under composition, i.e. if $(A \to B)$ belongs to $\mathcal{T}_A$, and $(B \to C)$ belongs to $\mathcal{T}_B$, then $(A \to C)$ belongs to $\mathcal{T}_A$.

3) All open unary localizations $(A \to A_f)$, $f \in |A|$, belong to $\mathcal{T}_A$.

We denote by $\hat{\mathcal{T}}_A$ the full subcategory of $\bar{\mathcal{P}}_A$, consisting of filtering inductive limits of pseudolocalizations from $\mathcal{T}_A$. This gives rise to a full fibered subcategory $\hat{\mathcal{T}} \subset \bar{\mathcal{P}}$.

Clearly, $\mathcal{P}$ is the maximal ("finest") possible localization theory, while $\mathcal{L}$ (consisting of all open unary localizations $A \to A_f$) is the minimal ("coarsest") one.

6.3.5. (Basic properties.)

4) $\mathcal{T}_A$ is closed under coproducts, hence also under finite inductive limits of $\mathcal{P}_A$. (Notice that such inductive limits in $\mathcal{P}_A$ might be different from their values when computed in $A\text{-}Alg$.)

5) $\hat{\mathcal{T}}_A$ is closed under arbitrary inductive limits. It contains $\hat{\mathcal{L}}_A$, the category of all unary localizations $A \to S^{-1}A$.

6) If $(A \to A')$ and $(A \to A'')$ both belong to $\mathcal{T}_A$, and $f : A' \to A''$ is an A-algebra homomorphism, then $(A' \xrightarrow{f} A'')$ belongs to $\mathcal{T}_{A'}$.

7) For any homomorphism of generalized rings $f : A \to B$ there is a pseudolocalization $A \to \bar{B}$ in $\hat{\mathcal{T}}_A$, such that f factorizes through a pseudolocalization $A \to A'$ in $\hat{\mathcal{T}}_A$ iff $A' \leq \bar{B}$ (i.e. iff there is an A-algebra homomorphism $A' \to \bar{B}$). In other words, the inclusion functor $\hat{\mathcal{T}}_A \to A\text{-}Alg$ admits a right adjoint $B \mapsto \bar{B}$.

8) Arbitrary projective limits exist in $\hat{\mathcal{T}}_A$ (not necessarily coinciding with those of $A\text{-}Alg$).

Proof. Let's prove these properties. First of all, 1) and 2) imply 4): indeed, if $A \to A'$ and $A \to A''$ both belong to $\mathcal{T}_A$, then $A' \to A' \otimes_A A'' \cong A' \boxtimes_A A''$ belongs to $\mathcal{T}'_A$ by 1), hence $A \to A' \to A' \otimes_A A''$ belongs to $\mathcal{T}_A$ by 2). Now 5) is an immediate consequence of 4), and 6) follows from the fact that all pseudolocalizations are epimorphisms, hence $f : A' \to A''$ can be identified with the pushout of $A \to A''$ with respect to $A \to A'$, and we just have to apply 1). Next, to show 7) we simply put $\bar{B}$ to be the inductive limit of all pseudolocalizations $A \to A'$, such that $A \to B$ factorizes through this pseudolocalization, and use 5) to show that $A \to \bar{B}$ lies indeed in $\hat{\mathcal{T}}_A$. Finally, 8) is a formal consequence of 7) and of existence of arbitrary projective limits in $A\text{-}Alg$: we first compute the required projective limit B in $A\text{-}Alg$, and then apply $B \mapsto \bar{B}$ to it.

6.3.6. (Spectra.) Let $\mathcal{T}^? \subset \mathcal{P}$ be any localization theory, A a generalized ring (or a K-algebra, if $\mathcal{T}^?$ is defined only for K-algebras). We construct the *(strong) $\mathcal{T}^?$-spectrum of* A, denoted by $\operatorname{Spec}_s^{\mathcal{T}^?} A$, $\operatorname{Spec}_s^? A$, or even $\operatorname{Spec}^? A$, as follows.

a) We consider the opposite category $\mathcal{S}_A^? := (\mathcal{T}_A^?)^{op}$ to $\mathcal{T}_A^?$. Clearly, finite projective limits exist in $\mathcal{S}_A^?$, and all morphisms of $\mathcal{S}_A^?$ are monomorphisms.

b) For any A-module M we define a presheaf of sets $\tilde{M}$ on $\mathcal{S}_A^?$, i.e. a functor $\tilde{M} : (\mathcal{S}_A^?)^{op} = \mathcal{T}_A^? \to Sets$, by putting $\tilde{M}(A') := A' \otimes_A M$ (for now we consider only the underlying set of $A' \otimes_A M$). The functor $M \mapsto \tilde{M}$ is exact, all pseudolocalizations A' being flat over A.

c) We consider on $\mathcal{S}_A^?$ the finest topology, for which all $\tilde{M}$ become sheaves. The results of SGA 4 tell us that $(U_\alpha \to U)$ is a covering in $\mathcal{S}_A^?$ iff

$$\tilde{M}(U) \longrightarrow \prod_\alpha \tilde{M}(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} \tilde{M}(U_\alpha \cap U_\beta) \tag{6.3.6.1}$$

is left exact for all A-modules M.

d) We denote by $\operatorname{Spec}^? A$ the topos defined by the site $\mathcal{S}_A^?$ thus constructed, i.e. $\operatorname{Spec}^? A$ is the category of sheaves of sets on $\mathcal{S}_A^?$.

e) By construction all $\tilde{M}$ are sheaves on $\operatorname{Spec}^? A$. In particular, this is true for all $\widetilde{A(n)} : A' \mapsto A'(n)$. Thus the collection of $(\widetilde{A(n)})$ defines a sheaf of generalized rings $\tilde{A}$, characterized by $\tilde{A}(A') = A'$ (cf. **4.9**), hence $\mathrm{Spec}^? A := (\mathrm{Spec}^? A, \tilde{A})$ is a generalized ringed topos (cf. **5.2.3**). We usually denote by $\mathscr{O}_{\mathrm{Spec}^? A}$ the sheaf of generalized rings $\tilde{A}$ just constructed, and say that it is *the structural sheaf of* $\mathrm{Spec}^? A$. Since any $\tilde{M}(A')$ has a natural A'-module structure, we see that *all* $\tilde{M}$ *are* $\mathscr{O}_{\mathrm{Spec}^? A}$*-modules.*

f) Similarly, any (generalized) A-algebra B defines a $\mathscr{O}_{\mathrm{Spec}^?_A}$-algebra $\tilde{B}$, with individual components given by $\tilde{B}(n) := \widetilde{B(n)}$. Clearly, the value of $\tilde{B}$ on a localization A' equals $A' \otimes_A B$ (we use here the affine base change theorem **5.4.2** again). Moreover, any B-module N can be treated as an A-module, thus definiting a sheaf of $\mathscr{O}_{\mathrm{Spec}^?_A}$-modules $\tilde{N}$, and it is easy to check that $\tilde{N}$ is in fact a sheaf of $\tilde{B}$-modules.

6.3.7. (Set-theoretical issues.) Notice that if A is $\mathcal{U}$-small (this is what we've tacitly assumed everywhere in this text), then the set of finitely presented A-algebras is also $\mathcal{U}$-small, since any such algebra admits a description in terms of a finite list of generators (with appropriate arities) and relations. In particular, the set of all open pseudolocalizations of A is $\mathcal{U}$-small, hence the same holds for $\mathcal{T}_A^?$ and $\mathcal{S}_A^?$, i.e. the site $\mathcal{S}_A^?$ is $\mathcal{U}$-small, so we can safely consider corresponding topos $\mathrm{Spec}^? A$.

6.3.8. (Topology of $\mathcal{S}_A^?$.) The attentive reader may have noticed that SGA 4 actually gives a stronger requirement for a family $(U_\alpha \to U)$ to be a covering with respect to the finest topology on $\mathcal{S}_A^?$, for which all $\tilde{M}$ become sheaves. Namely, a family as above is a covering iff for any $V \in \mathrm{Ob}\, \mathcal{S}_A^?$ and any $M \in \mathrm{Ob}\, A\text{-}\mathit{Mod}$ the following sequence of sets is left exact:

$$\tilde{M}(V \cap U) \longrightarrow \prod_\alpha \tilde{M}(V \cap U_\alpha) \rightrightarrows \prod_{\alpha,\beta} \tilde{M}(V \cap U_\alpha \cap U_\beta) \qquad (6.3.8.1)$$

However, in our case (6.3.8.1) is fulfilled for all V and M iff (6.3.6.1) is fulfilled for all M, so the description of covers in $\mathcal{S}_A^?$ given above in **6.3.6**,c) turns out to be accurate.

Indeed, let's deduce (6.3.8.1) from (6.3.6.1). An object $V \in \mathrm{Ob}\, \mathcal{S}_A^?$ is given by some pseudolocalization $A \xrightarrow{\rho} A'$ of $\mathcal{T}_A^?$. Put $N := \rho_* \rho^* M$. According to **5.4.2**, ρ_* commutes with any *flat* base change. This is also true for ρ^* for trivial reasons, and the definition of $\tilde{M}$ now implies $\tilde{M}(V \cap U) \cong \tilde{N}(U)$ for any $U \in \mathrm{Ob}\, \mathcal{S}_A^?$. Therefore, (6.3.6.1) for N implies (6.3.8.1) for M and V.

6.3.9. We claim that *the topology of* $\mathcal{S}_A^?$ *is subcanonical*, i.e. all representable presheaves $\mathrm{Hom}(-, V)$ on $\mathcal{S}_A^?$ are actually sheaves. This is easily reduced to proving the following: *if* $(U_\alpha \to U)$ *is a covering, and all* $U_\alpha \subset V$*, then* $U \subset V$*.*

Proof. Let's denote $\mathscr{O}_{\mathrm{Spec}^? A}$ simply by $\mathscr{O}$. Then for any $V \in \mathrm{Ob}\,\mathcal{S}_A^?$ the generalized ring of sections $\mathscr{O}(V)$ is nothing else than the pseudolocalization of A corresponding to V, with the A-algebra structure given by the restriction map $A = \mathscr{O}(e) \to \mathscr{O}(V)$, where e is the final object of $\mathcal{S}_A^?$ (corresponding to the trivial pseudolocalization $A \to A$). Recall that $\mathscr{O}$ is a sheaf of generalized rings on $\mathcal{S}_A^?$; this implies the exactness of the following diagram of generalized rings (or A-algebras):

$$\mathscr{O}(U) \longrightarrow \prod_\alpha \mathscr{O}(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} \mathscr{O}(U_\alpha \cap U_\beta) \tag{6.3.9.1}$$

Now $U_\alpha \subset V$ means the existence of a (necessarily unique) A-algebra homomorphism $f_\alpha : \mathscr{O}(V) \to \mathscr{O}(U_\alpha)$, and uniqueness shows that the two maps $\mathscr{O}(V) \to \mathscr{O}(U_\alpha \cap U_\beta)$ obtained from f_α and f_β coincide. Left exactness of the above diagram now implies the existence of an A-algebra homomorphism $\mathscr{O}(V) \to \mathscr{O}(U)$, i.e. $U \subset V$, q.e.d.

6.3.10. (Functoriality.) Let $f : A \to B$ be a homomorphism of generalized rings (or K-algebras, if $\mathcal{T}^?$ is defined over K-*Alg*). We want to construct a morphism of ringed topoi ${}^a f : \mathrm{Spec}^? B \to \mathrm{Spec}^? A$, enjoying all the usual properties. First of all, f induces a base change functor $f^* : \mathcal{T}_A^? \to \mathcal{T}_B^?$, that can be considered as a functor $f^{-1} : \mathcal{S}_A^? \to \mathcal{S}_B^?$ between opposite categories. This functor f^{-1} is left exact by construction; we have to check that it is a morphism of sites, i.e. that $(f^{-1}U_\alpha \to f^{-1}U)$ is a cover in $\mathcal{S}_B^?$ whenever $(U_\alpha \to U)$ is a cover in $\mathcal{S}_A^?$.

Fix a cover $(U_\alpha \to U)$ and let's prove that $(f^{-1}U_\alpha \to f^{-1}U)$ is a cover as well. By definition, this means left exactness of the counterpart of (6.3.6.1) for $\tilde{N}$, N any B-module, and the family $(f^{-1}U_\alpha \to f^{-1}U)$. Consider $M := f_*N$, the scalar restriction of N. We claim that $\tilde{M}(U) \cong \tilde{N}(f^{-1}U)$ for any $U \in \mathrm{Ob}\,\mathcal{S}_A^?$. This would show that (6.3.6.1) for $\tilde{N}$ and $(f^{-1}U_\alpha \to f^{-1}U)$ is equivalent to (6.3.6.1) for $\tilde{M}$ and $(U_\alpha \to U)$, valid by definition of the topology of $\mathcal{S}_A^?$.

So let's prove $\tilde{M}(U) \cong \tilde{N}(f^{-1}U)$. Notice that U corresponds to some pseudolocalization $A \to A'$. Then $f^{-1}U$ corresponds to $B \to B' := A' \otimes_A B$, and $\tilde{M}(U) = A' \otimes_A M = A' \otimes_A f_*N$, $\tilde{N}(f^{-1}U) = B' \otimes_B N$. Since A' is flat over A, our statement now follows from the "affine base change theorem" **5.4.2**.

Notice that we have a natural homomorphism $\theta : \mathscr{O}_{\mathrm{Spec}^? A} \to ({}^a f)_* \mathscr{O}_{\mathrm{Spec}^? B}$ of structural sheaves as well, defined by the canonical maps $\theta_U : \mathscr{O}_{\mathrm{Spec}^? A}(U) = A' \to \mathscr{O}_{\mathrm{Spec}^? B}(f^{-1}U) = B \otimes_A A'$. Therefore, ${}^a f = ({}^a f, \theta) : \mathrm{Spec}^? B \to \mathrm{Spec}^? A$ can be considered as a morphism of generalized ringed topoi.

6.3.11. (Spectra of pseudolocalizations.) Let $f : A \to A'$ be a pseudolocalization of $\mathcal{T}_A^?$. It corresponds to some object U in $\mathcal{S}_A^?$. Notice that U is an *open object* of $\mathcal{S}^?_A$, i.e. a subobject of the final object e_U. Therefore, we can consider the open subtopos $\operatorname{Spec}^? A_{/U}$ of $\operatorname{Spec}^? A$, defined by site $\mathcal{S}^?_A/U$. Notice that $\mathcal{S}^?_A/U$ is canonically equivalent to $\mathcal{S}^?_{A'}$ (cf. property 5) of **6.3.5**). We claim that this equivalence is compatible with the natural topologies on these sites. Indeed, the topology of $\mathcal{S}^?_A/U$ is easily seen to be the finest topology, for which all $\tilde{M}|_U$, $M \in \operatorname{Ob} A\text{-}\mathit{Mod}$, become sheaves. On the other hand, the topology of $\mathcal{S}^?_{A'}$ is the finest topology, for which all $\tilde{N}$, $N \in \operatorname{Ob} A'\text{-}\mathit{Mod}$, become sheaves. Now it is sufficient to check that these two families of sheaves of sets correspond to each other under equivalence $\mathcal{S}^?_A/U \cong \mathcal{S}^?_{A'}$. This is immediate: $\tilde{M}|_U$ is isomorphic to $\widetilde{f^*M}$; conversely, if we start with some $N \in \operatorname{Ob} A'\text{-}\mathit{Mod}$, then $\tilde{N}$ is isomorphic to $\widetilde{f_*N}|_U$.

Clearly, $\mathscr{O}_{\operatorname{Spec}^? A}|_U$ is canonically isomorphic to $\mathscr{O}_{\operatorname{Spec}^? A'}$, if we identify $\operatorname{Spec}^? A'$ with the open subtopos of $\operatorname{Spec}^? A$ defined by U. In other words, $\operatorname{Spec}^? A'$ is equivalent to the open ringed subtopos $\operatorname{Spec}^? A_{/U}$ of $\operatorname{Spec}^? A$. The morphism of ringed topoi $\operatorname{Spec}^? A' \xrightarrow{\sim} \operatorname{Spec}^? A_{/U} \to \operatorname{Spec}^? A$ is easily seen to be isomorphic to ${}^af$, i.e. *if $f : A \to A'$ is a pseudolocalization of $\mathcal{T}^?_A$, ${}^af : \operatorname{Spec}^? A' \to \operatorname{Spec}^? A$ is an open embedding of generalized ringed topoi.*

6.3.12. (Quasicoherent sheaves.) Fix $X = (X, \mathscr{O}_X) := \operatorname{Spec}^? A$ and consider the global sections functor $\Gamma = \Gamma_X := \Gamma(X, -)$, given by $\Gamma(X, \mathscr{F}) := \mathscr{F}(e)$, where e is the final object of $\operatorname{Spec}^? A$ or of $\mathcal{S}^?_A$. Since $\Gamma_X(\mathscr{O}_X) = A$, Γ_X induces a functor from the category of (sheaves of) $\mathscr{O}_X$-modules into the category of A-modules: $\Gamma_X : \mathscr{O}_X\text{-}\mathit{Mod} \to A\text{-}\mathit{Mod}$. We have a functor $\Delta = \Delta_X$ in the opposite direction as well, given by $\Delta_X(M) := \tilde{M}$.

It is easy to see that Δ_X is a left adjoint to Γ_X, i.e.

$$\operatorname{Hom}_{\mathscr{O}_X}(\tilde{M}, \mathscr{F}) \cong \operatorname{Hom}_A(M, \Gamma(X, \mathscr{F})) \tag{6.3.12.1}$$

Indeed, this follows immediately from the universal property of $\tilde{M}(U) = \mathscr{O}_X(U) \otimes_A M$.

Next, it is immediate that $\Gamma(X, \tilde{M}) = M$, i.e. $\Gamma_X \circ \Delta_X \cong \mathrm{Id}$; this means that Δ_X is fully faithful, hence Δ_X *induces an equivalence between A-Mod and the essential image of Δ_X, and the quasiinverse equivalence is given by the restriction of Γ_X to this essential image.*

We say that an $\mathscr{O}_X$-module $\mathscr{F}$ is *quasicoherent* if it lies in the essential image of Δ_X, i.e. if it is isomorphic to some $\tilde{M}$. We have just shown that *the category of quasicoherent sheaves of* $\operatorname{Spec}^? A$ *is equivalent to A-Mod.* On the other hand, Δ_X is exact and commutes with arbitrary inductive limits, hence *the category of quasicoherent sheaves is closed under finite projective and arbitrary inductive limits of $\mathscr{O}_X$-Mod.*

6.3.13. (Quasicoherent sheaves and functoriality.) Let $f : A \to B$ be a homomorphism of generalized rings, ${}^af : Y := \operatorname{Spec}^? B \to X := \operatorname{Spec}^? A$

be the associated morphism of spectra. Notice that we have actually proved in **6.3.10** that $({}^af)_*\circ\Delta_Y \cong \Delta_X\circ f_*$, i.e. $({}^af)_*(\tilde{N}) \cong \widetilde{f_*N}$ for any B-module N. In particular, *the direct image functor* $({}^af)_*$ *transforms quasicoherent sheaves on* $\operatorname{Spec}^? B$ *into quasicoherent sheaves on* $\operatorname{Spec}^? A$.

On the other hand, computing the left adjoint to $\Gamma_X\circ({}^af)_* = f_*\circ\Gamma_Y$ we obtain $({}^af)^*\circ\Delta_X \cong \Delta_Y\circ f^*$, i.e. $({}^af)^*\tilde{M} \cong \widetilde{f^*M}$ for any A-module M. In particular, *the pullback functor* $({}^af)^*$ *transforms quasicoherent* $\mathscr{O}_X$*-modules into quasicoherent* $\mathscr{O}_Y$*-modules.*

6.3.14. (Flat topology.) We can generalize the above constructions as follows. We can consider category $\mathcal{S}_A^{fl}$, opposite to the category of flat A-algebras, and define there a topology by requiring all presheaves of sets $\tilde{M} : B \to B\otimes_A M$ to be sheaves. The description of covers given in **6.3.6**,c) is still valid, since the reasoning of **6.3.8** is still applicable. (Problem: not all fibered products exist in $\mathcal{S}_A^{fl}$.)

One can check that $\mathcal{S}_{fl}^A$ corresponds to the small flat site of $\operatorname{Spec} A$ (with respect to the (fpqc) topology) when A is a classical commutative ring. That's why we say that $\mathcal{S}_A^{fl}$ *is the small flat site of* A, and the corresponding topos $(\operatorname{Spec} A)_{fl} = \operatorname{Spec}^{fl} A := \tilde{\mathcal{S}}_A^{fl}$ *is the small flat topos of* A.

If we choose some localization theory $\mathcal{T}^?$ over A, we get a natural inclusion functor $\mathcal{S}_A^? \to \mathcal{S}_A^{fl}$, easily seen to be left exact and to transform covers into covers (more precisely, $(U_\alpha \to U)$ is a cover in $\mathcal{S}_A^?$ iff it is a cover in $\mathcal{S}_A^{fl}$), i.e. a morphism of sites. It induces a morphism of topoi $(\operatorname{Spec} A)_{fl} \to \operatorname{Spec}^? A$. We have a natural sheaf of generalized rings $\mathscr{O}$ or $\mathscr{O}_X$ on $(\operatorname{Spec} A)_{fl}$, that transforms any object of $\mathcal{S}_A^{fl}$, given by some flat A-algebra B, into B itself, and all $\tilde{M}$ are actually sheaves of $\mathscr{O}$-modules on $(\operatorname{Spec} A)_{fl}$.

6.3.15. (Comparison morphisms.) Given two localization theories $\mathcal{T}'$ and $\mathcal{T}''$, we say that $\mathcal{T}''$ *is a refinement of* $\mathcal{T}'$ if $\mathcal{T}' \subset \mathcal{T}''$, i.e. any open pseudolocalization of $\mathcal{T}'$ lies also in $\mathcal{T}''$. This defines a partial order between localization theories. Clearly, the theory $\mathcal{T}^u$ consisting only of unary localizations of form $A \to A_f$ is the smallest element with respect to this order, while the theory $\mathcal{P}$ consisting of all open pseudolocalizations is the largest element. The latter will be called the *total localization theory* $\mathcal{T}^t := \mathcal{P}$*, and corresponding spectra will be denoted by* $\operatorname{Spec}^t A$.

In any case, given two such theories $\mathcal{T}' \subset \mathcal{T}''$ as above, we obtain canonical inclusion functors $\mathcal{S}'_A \to \mathcal{S}''_A$; they are left exact and preserve covers, hence we have a morphism of sites and of topoi in the opposite direction $\operatorname{Spec}'' A \to \operatorname{Spec}' A$, easily seen to be a morphism of generalized ringed topoi, depending functorially on A. We say that $\operatorname{Spec}'' A \to \operatorname{Spec}' A$ *is a comparison morphism with respect to these two theories.*

6.3.16. (Unary localization spectra $\mathrm{Spec}^u A$.) Let us study spectra $\mathrm{Spec}^u A$ with respect to the smallest possible theory $\mathcal{T}^u$. Clearly, we have comparison morphisms $\mathrm{Spec}^? A \to \mathrm{Spec}^u A$ for any other theory $\mathcal{T}^?$; this explains the importance of $\mathrm{Spec}^u A$. First of all, the site $\mathcal{S}_A^u$ is the opposite category to $\mathcal{T}_A^u$, hence it consists (up to equivalence) of objects $D^u(f)$, $f \in |A|$, that correspond to unary localizations $A \to A_f$. We know that $\mathcal{S}_A^u$ is a preordered set, i.e. between any two objects $D^u(f)$ and $D^u(g)$ there is at most one morphism; when such a morphism exists, we write $D^u(f) \subset D^u(g)$. By definition $D^u(f) \subset D^u(g)$ iff $A \to A_f$ factorizes through $A \to A_g$ iff f divides some power of g iff $D(f) \subset D(g)$ in the prime spectrum $\mathrm{Spec}^p A$ (cf. **6.2.9**). Therefore, $\mathcal{S}_A^u$ can be identified with the ordered set of principal open subsets of the prime spectrum $\mathrm{Spec}^p A$ of **6.2.5**.

6.3.17. (Covers in $\mathrm{Spec}^u A$.) We claim that $(D^u(f_\alpha) \to D^u(g))$ *is a cover in* $\mathcal{S}_A^u$ *iff* $(D(f_\alpha) \to D(g))$ *is a cover in* $\mathrm{Spec}^p A$*, i.e. iff* $D(g) = \bigcup_\alpha D(f_\alpha)$. Since the principal open subsets form a base of topology of $\mathrm{Spec}^p A$, this would prove that $\mathrm{Spec}^u A$ *coincides with the topos defined by topological space* $\mathrm{Spec}^p A$. Moreover, this would imply that $\mathrm{Spec}^u A$ has enough points, and that the topological space associated to $\mathrm{Spec}^u A$ is canonically homeomorphic to $\mathrm{Spec}^p A$, the latter topological space being sober.

So let us show our statement about covers. Since replacing A with A_g replaces both $\mathrm{Spec}^u A$ and $\mathrm{Spec}^p A$ with corresponding principal open subsets or subtopoi (cf. **6.2.6** and **6.3.11**), we can assume that $A = A_g$, hence $D^u(g) = D^u(1) = e$ is the final object of $\mathcal{S}_A^u$, and $D(g) = D(1) = \mathrm{Spec}^p A$.

$\Rightarrow$) Suppose that $(D^u(f_\alpha) \to D^u(1))$ is a cover in $\mathcal{S}_A^u$, but $\bigcup_\alpha D(f_\alpha) \neq \mathrm{Spec}^p A$. According to **6.2.8**, this means that the ideal $\mathfrak{a} := (f_\alpha)$ is $\neq (1)$. Then $\mathfrak{a}_{f_\alpha} = A_{f_\alpha}$ for any α, and the diagrams (6.3.6.1) are left exact for both $\tilde{\mathfrak{a}}$ and $\tilde{A}$. Consider the following diagram with left exact rows:

$$\begin{tikzcd}
\tilde{\mathfrak{a}}(U) \arrow[r] \arrow[d] & \prod_\alpha \tilde{\mathfrak{a}}(U_\alpha) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\sim"] & \prod_{\alpha,\beta} \tilde{\mathfrak{a}}(U_\alpha \cap U_\beta) \arrow[d, "\sim"] \\
\tilde{A}(U) \arrow[r] & \prod_\alpha \tilde{A}(U_\alpha) \arrow[r, shift left] \arrow[r, shift right] & \prod_{\alpha,\beta} \tilde{A}(U_\alpha \cap U_\beta)
\end{tikzcd} \tag{6.3.17.1}$$

The vertical arrows are induced by natural embeddings; we know that the central and the right vertical arrows are isomorphisms, hence the same is true for the remaining vertical arrow, i.e. $\mathfrak{a} = (1)$. This is absurd.

$\Leftarrow$) Suppose that $\bigcup_\alpha D(f_\alpha) = \mathrm{Spec}^p A$. We have to show that (6.3.6.1) is exact for $(D(f_\alpha) \to D(1))$ and any A-module M. We know that $\mathrm{Spec}^p A$ is quasicompact, so we can find a finite subcover, consisting of some $D(f_i) := D(f_{\alpha_i})$, $1 \leq i \leq n$. It is easy to see that it suffices to check left exactness of

(6.3.6.1) for this finite subcover. This means left exactness of the following diagram:

$$M \longrightarrow \prod_i M_{f_i} \rightrightarrows \prod_{i,j} M_{f_if_j} \qquad (6.3.17.2)$$

provided the f_i generate the unit ideal in A (cf. **6.2.8**).

a) First of all, let us prove that if $x,\ y \in M$ are such that $x/1 = y/1$ in any M_{f_i}, then $x = y$. Indeed, $x/1 = y/1$ in M_{f_i} means that $xf_i^N = yf_i^N$ for some $N > 0$, and we can choose the same N for all values of i, our cover being finite. We know that the elements f_i^N still generate the unit ideal (since $\bigcup D(f_i^N) = \bigcup D(f_i) = \operatorname{Spec}^p A$), so we can find an $H \in A(n)$, such that $H(f_1^N, \ldots, f_n^N) = \boldsymbol{e}$. Then we obtain $x = \boldsymbol{e} \cdot x = H(f_1^N x, \ldots, f_n^N x) = H(f_1^N y, \ldots, f_n^N y) = y$.

b) Now suppose we are given a family of elements x_i/f_i^N in all M_{f_i}, such that $x_i/f_i^N = x_j/f_j^N$ in $M_{f_if_j}$ for any i and j. We use again the finiteness of our cover to choose a common value of N for all i. Let us prove that $x_i/f_i^N = x/1$ for a suitably chosen $x \in M$. We know that $x_i/f_i^N = x_j/f_j^N$ means $x_i f_i^K f_j^{N+K} = x_j f_i^{N+K} f_j^K$ for some integer $K \geq 0$, chosen simultaneously for all values of i and j. Replacing x_i with $x_i f_i^K$ and N with $N + K$, we can assume $K = 0$, i.e. $x_i f_j^N = x_j f_i^N$.

By the same reasoning as before we can find some $H \in A(n)$, such that $H(f_1^N, \ldots, f_n^N) = \boldsymbol{e}$. Put $x := H(x_1, \ldots, x_n)$. For any i we have $xf_i^N = H(x_1, \ldots, x_n) \cdot f_i^N = H(x_1 f_i^N, \ldots, x_n f_i^N) = H(f_1^N x_i, \ldots, f_n^N x_i) = H(f_1^N, \ldots, f_n^N) \cdot x_i = \boldsymbol{e} x_i = x_i$, hence $x_i/f_i^N = x/1$, q.e.d.

6.3.18. (Open pseudolocalizations of classical rings.) Let $f : A \to B$ be a homomorphism of classical commutative rings. We claim that f *is an open pseudolocalization iff* ${}^a f : \operatorname{Spec} B \to \operatorname{Spec} A$ *is an open embedding.* The "if" part is very easy to check, since ${}^a f$ turns out to be flat and finitely presented, and ${}^a f_*$ induces a fully faithful functor from the category of quasicoherent sheaves on $\operatorname{Spec} B$ into the category of quasicoherent sheaves on $\operatorname{Spec} A$. Conversely, suppose that f is an open pseudolocalization. This implies that B is flat and finitely presented over A, hence ${}^a f$ is an open map, and in particular $U := {}^a f(\operatorname{Spec} B)$ is a quasicompact open subset of $\operatorname{Spec} A$. We have to check that the induced map $\operatorname{Spec} B \to U$ is an isomorphism; this is equivalent to checking that ${}^a f_{D(s)} : {}^a f^{-1}(D(s)) \to D(s)$ is an isomorphism whenever $D(s) \subset U$, $s \in A$, i.e. that f becomes an isomorphism after any base change $A \to A' := A_s$ for any such $s \in A$. Indeed, put $B' := A' \otimes_A B = B_s$. Then $f' : A' \to B'$ is an open pseudolocalization again, but now ${}^a f'$ is both flat and surjective, hence faithfully flat, i.e. B' is a faithfully flat A'-algebra. On the other hand, we know that $B' \otimes_{A'} B' \cong B'$, since any pseudolocalization is an NC-epimorphism. In other words, $A' \to B'$ becomes an isomorphism after tensoring with B' over A', hence it is itself an isomorphism by faithful flatness, q.e.d.

6.3.19. (Comparison morphisms for spectra of classical rings.) One can easily check that a family of open pseudolocalizations $(B \to B_\alpha)$ defines a cover in the sense of **6.3.6**,c) iff $\operatorname{Spec} B$ is the union of its open subsets $\operatorname{Spec} B_\alpha$. An immediate consequence is that *for any localization theory* $\mathcal{T}^?$ *and any classical ring* A *the comparison morphism* $\operatorname{Spec}^? A \to \operatorname{Spec}^u A = \operatorname{Spec} A$ *is an equivalence of ringed topoi.* In other words, all localization theories yield the same result for classical rings; we might have defined $\operatorname{Spec} A$ to be equal to $\operatorname{Spec}^t A$ as well.

One can check essentially in the same way that whenever we have a localization theory $\mathcal{T}^?$ (say, for generalized K-algeras), such that for any K-algebra A and any finitely presented flat A-algebra B induced morphism $\operatorname{Spec}^? B \to \operatorname{Spec}^? A$ has open image (i.e. decomposes into a faithfully flat morphism followed by an open embedding), then the comparison morphisms $\operatorname{Spec}^t A \to \operatorname{Spec}^? A$ are equivalences, i.e. any such theory can be actually replaced by the total localization theory.

6.3.20. (Localizations vs. pseudolocalizations of classical rings.) Notice that an open pseudolocalization $A \to B$ of classical rings need not be an open localization $A \to A_s$, i.e. not all affine open subsets of $\operatorname{Spec} A$ are principal open subsets. Let us illustrate the subtlety of the notion of open pseudolocalizations by a seemingly simple question.

Question. Let $A := k[T_1, \ldots, T_n]$ be a polynomial ring over a field k. Is there a normal k-subalgebra $B \subset A$, distinct from A, such that $B \to A$ is a) an open localization; b) an open pseudolocalization?

While the answer to a) is obviously negative (all invertible elements of A lie in k, so we cannot have $A = B_s$ for $s \notin k$), the negative answer to b) would in fact imply the jacobian conjecture, so the second part of the question has to be very subtle.

6.3.21. (Points and quasicompactness of spectra.) One can check directly or by invoking general results of SGA 4 that once we know that all spectra for a certain theory $\mathcal{T}^?$ are quasicompact, they admit enough points (being coherent; cf. SGA 4), and in any case they are generated by open subobjects, hence they correspond to some sober topological spaces, uniquely determined up to a canonical homeomorphism. These sober topological spaces will be also denoted by $\operatorname{Spec}^? A$; they are generalized ringed topological spaces, of course.

The only problem here is to prove the quasicompactness. I don't know whether quasicompactness holds for all theories and all generalized rings, so we have to circumvent this problem in another way.

6.4. (Weak topology and quasicoherent sheaves.) The "strong" topology just defined on $\mathcal{S}_A^? = (\mathcal{T}_A^?)^{op}$ and corresponding "strong" spectra $\operatorname{Spec}^? A = \operatorname{Spec}_s^? A$, where A is a generalized ring and $\mathcal{T}^?$ is any localization theory, has a serious drawback. Namely, when $X := \operatorname{Spec}_A^?$ is not quasicompact (we know already that this can happen only if A is non-additive and $\mathcal{T}^? \neq \mathcal{T}^u$), then the property of an $\mathscr{O}_X$-module $\mathscr{F}$ to be quasicoherent (i.e. of the form $\tilde{M}$ for some A-module M) is not necessarily local. Therefore, even if we construct "strong" schemes by gluing together strong spectra (for some fixed theory $\mathcal{T}^?$) along open subsets, we won't obtain a reasonable notion of quasicoherent sheaves over such schemes.

6.4.0. (Terminology and notations.) In order to overcome this difficulty we consider two weaker topologies on the same category $\mathcal{S}_A^? = (\mathcal{T}_A^?)^{op}$, namely, the *weak* and the *finite* topology. Weak topology will be weaker than the strong topology considered so far, and the finite topology will be even weaker than the weak topology. Both these topologies have correct properties with respect to quasicoherent sheaves, and the finite topology has another advantage: resulting spectra turn out to be quasicompact, hence also coherent in the sense of SGA 4, hence they admit enough points, i.e. they are given by some sober topological spaces, so if we restrict our attention to finite spectra, we can work with generalized ringed topological spaces and forget about topoi.

Notation. We denote by $\mathcal{S}_{A,s}^?$, $\mathcal{S}_{A,w}^?$ or $\mathcal{S}_{A,f}^?$ the category $\mathcal{S}_A^?$, considered as a site by means of its strong, weak or finite topology, respectively. Corresponding topoi (or topological spaces, when these topoi have enough points) will be denoted by $\operatorname{Spec}_s^? A$, $\operatorname{Spec}_w^? A$ or $\operatorname{Spec}_f^? A$, respectively.

If A is additive (*additive* actually means here "additive with a compatible symmetry", i.e. "given by a classical ring"), or if $\mathcal{T}^?$ is the unary localization theory $\mathcal{T}^u$, then all of the above topologies will coincide (cf. **6.4.13**), so we'll still write $\mathcal{S}_A^?$ and $\operatorname{Spec}^? A$, or even $\operatorname{Spec} A$.

6.4.1. (Fibered category of quasicoherent $\mathscr{O}$-modules.) Let us fix a generalized ring A and a localization theory $\mathcal{T}^?$. Put $\mathcal{S}_A^? := (\mathcal{T}_A^?)^{op}$ as before. For any $U \in \operatorname{Ob} \mathcal{S}_A^?$, given by some open pseudolocalization $A \to A'$, denote by $\mathbf{QCOH}(U)$ the category of quasicoherent modules over U, or, equivalently, the category of A'-modules: $\mathbf{QCOH}(U) = A'$-*Mod*. Whenever $V \subset U$, we have a base change (scalar extension) functor $\mathbf{QCOH}(U) \to \mathbf{QCOH}(V)$, and these base change functors are transitive up to a canonical isomorphism. This means that the collection of categories $\{\mathbf{QCOH}(U)\}$ together with these base change functors defines a fibered category $\mathbf{QCOH} = \mathbf{QCOH}_A^? \to \mathcal{S}_A^?$ over category $\mathcal{S}_A^?$ (cf. **9.1**, where definitions of fibered categories and stacks are recalled, or [SGA1, VI] and [Giraud]).

6.4.2. (Descent families.) Let $\{U_\alpha \to U\}$ be any family of morphisms in category $\mathcal{S}^?_A$. Denote by $Desc(\{U_\alpha \to U\}, \mathbf{QCOH})$ the category of descent data in $\mathbf{QCOH}$ with respect to this family; such a descent datum is a family of modules $M_\alpha \in \mathbf{QCOH}(U_\alpha)$ together with isomorphisms $\theta_{\alpha\beta} : M_\alpha|_{U_{\alpha\beta}} \xrightarrow{\sim} M_\beta|_{U_{\alpha\beta}}$ satisfying the cocycle relation, where $U_{\alpha\beta} = U_\alpha \times_U U_\beta = U_\alpha \cap U_\beta$ in our situation, and the "restriction" $M|_V$ denotes the image of M under appropriate pullback, i.e. scalar extension functor.

We have a natural functor $\mathbf{QCOH}(U) \to Desc(\{U_\alpha \to U\}, \mathbf{QCOH})$. Recall that $\{U_\alpha \to U\}$ is said to be a *family of descent* (resp. *of efficient descent*) for $\mathbf{QCOH}$ if this functor is fully faithful (resp. an equivalence of categories). If we fix a topology on $\mathcal{S}^?_A$, then $\mathbf{QCOH}$ is said to be a *prestack* (resp. *stack*) if any cover for this topology is a family of descent (resp. of efficient descent) for $\mathbf{QCOH}$. Finally, recall that all *universal* families of descent (resp. of efficient descent) for a fibered category, $\mathbf{QCOH}$ in our case, constitute a topology on $\mathcal{S}^?_A$, called the descent (resp. efficient descent) topology for $\mathbf{QCOH}$ (cf. [Giraud]). This is the strongest topology on $\mathcal{S}^?_A$, for which $\mathbf{QCOH}$ is a prestack (resp. a stack).

6.4.3. (Structural presheaf of $\mathcal{S}^?_A$.) Notice that some of the statements made before about sheaves on $\mathcal{S}^?_A$ with respect to the strong topology are still valid as statements about presheaves, without any choice of topology. For example, we still have the *structural presheaf (of generalized rings)* $\mathscr{O}$ on $\mathcal{S}^?_A$, which maps $U \in \operatorname{Ob}\mathcal{S}^?_A$ given by some open pseudolocalization $A \to A'$ into A', and any A-module M defines a *quasicoherent presheaf of* $\mathscr{O}$*-modules* $\tilde{M}$ by $\tilde{M}(U) := M \otimes_A A'$. Furthermore, $\mathcal{S}^?_A/U$ can be identified with $\mathcal{S}^?_{A'}$, and $\tilde{M}_{/U}$ is identified then with $\widetilde{M|_U}$, where $M|_U = M \otimes_A A'$. Finally, $\Delta : M \mapsto \tilde{M}$ is still the left adjoint to global sections functor $\Gamma : \mathscr{F} \mapsto \mathscr{F}(e)$, i.e.

$$\operatorname{Hom}_{\mathscr{O}}(\tilde{M}, \mathscr{F}) \cong \operatorname{Hom}_A(M, \Gamma(\mathscr{F})) \quad \text{for any presheaf of } \mathscr{O}\text{-modules } \mathscr{F}. \tag{6.4.3.1}$$

Since $\Gamma\Delta = \mathrm{Id}_{A\text{-}Mod}$, we see that Δ is still fully faithful, i.e. $\operatorname{Hom}_{\mathscr{O}}(\tilde{M}, \tilde{N}) \cong \operatorname{Hom}_A(M, N)$ for any two A-modules M and N.

6.4.4. (Descent topology on $\mathcal{S}^?_A$.) We claim that *the descent topology on* $\mathcal{S}^?_A$ *with respect to* $\mathbf{QCOH}$ *coincides with the strong topology of* **6.3.6**. More precisely, *a family* $\{U_\alpha \to U\}$ *in* $\mathcal{S}^?_A$ *is a family of descent for* $\mathbf{QCOH}$ *iff it is a cover for the strong topology, i.e. iff it satisfies the condition of* **6.3.6***,c).* This means in particular that *any family of descent for* $\mathbf{QCOH}$ *is universal, i.e. remains such after any base change.*

(a) For any $U \in \operatorname{Ob}\mathcal{S}^?_A$ and any $M,\ N \in \operatorname{Ob}\mathbf{QCOH}(U) = A'\text{-}Mod$, where U is given by $A \to A'$, we define a presheaf of sets $\mathbf{Hom}_U(M, N)$ on $\mathcal{S}^?_A/U$ by mapping $V \to U$ into $\operatorname{Hom}_{\mathbf{QCOH}(V)}(M|_V, N|_V)$. It is a general fact that $\{U_\alpha \to U\}$ is a family of descent for **QCOH** iff all $\mathbf{Hom}_U(M,N)$ satisfy the sheaf condition for this cover. On the other hand, $(\mathbf{Hom}_U(M,N))(V) = \mathrm{Hom}_{\mathbf{QCOH}(V)}(M|_V, N|_V) = \mathrm{Hom}_{\mathscr{O}(V)}(M|_V, N|_V) = \mathrm{Hom}_{\mathscr{O}|_V}(\widetilde{M|_V}, \widetilde{N|_V}) = \mathrm{Hom}_{\mathscr{O}_V}(\tilde{M}|_V, \tilde{N}|_V)$, i.e. $\mathbf{Hom}_U(M,N)$ coincides with the usual local Hom $\mathbf{Hom}_{\mathscr{O}_U}(\tilde{M}, \tilde{N})$.

(b) Now consider the strong topology on $\mathcal{S}^?_A$. By definition any $\tilde{N}$ is a sheaf over $\mathcal{S}^?_A/U$, hence $\mathbf{Hom}_{\mathscr{O}_U}(\tilde{M}, \tilde{N})$ is a sheaf as well, hence any cover $\{U_\alpha \to U\}$ for the strong topology is a descent family for **QCOH**.

(c) Conversely, let $\{U_\alpha \to U\}$ be a descent family for **QCOH**. Then by (a) the presheaf $\mathbf{Hom}_{\mathscr{O}_U}(\tilde{A}|_U, \tilde{M}|_U)$ satisfies the sheaf condition for this family. On the other hand, this presheaf is canonically isomorphic to $\tilde{M}|_U$ since $\tilde{A} = \mathscr{O}$, i.e. $\tilde{M}$ satisfies the sheaf condition for $\{U_\alpha \to U\}$. By **6.3.6**,c) this means that this family is a cover for the strong topology, q.e.d.

Definition 6.4.5 *Let $\mathcal{T}^?$ be a localization theory and A be a generalized ring. Define $\mathcal{S}^?_A := (\mathcal{T}^?_A)^{op}$ and* **QCOH** $\to \mathcal{S}^?_A$ *as before. The* weak topology *on $\mathcal{S}^?_A$ is the efficient descent topology with respect to* **QCOH**. *Category $\mathcal{S}^?_A$, considered as a site with respect to this topology, will be denoted by $\mathcal{S}^?_{A,w}$. The* weak $\mathcal{T}^?$-spectrum *of A is the corresponding topos $\mathrm{Spec}^?_w A$, considered as a generalized ringed topos with structural sheaf $\mathscr{O}$. A sheaf of $\mathscr{O}$-modules $\mathscr{F}$ is said to be* quasicoherent *if it is isomorphic to $\tilde{M}$ for some A-module M.*

Since any universal family of efficient descent is a universal family of descent, the weak topology is indeed weaker than the strong topology of **6.3.6**. Therefore, any sheaf for the strong topology on $\mathcal{S}^?_A$ is a sheaf for the weak topology as well. This is applicable in particular to $\mathscr{O}$ and $\tilde{M}$, i.e. the structural presheaf and the quasicoherent presheaves are indeed sheaves for the weak topology.

6.4.6. (Restriction to open objects.) Of course, for any $U \in \mathrm{Ob}\,\mathcal{S}^?_A$ given by some open pseudolocalization $A \to A'$ the weak spectrum $\mathrm{Spec}^?_w A'$ can be identified (as a generalized ringed topos) with the restriction $\mathrm{Spec}^?_w A_{/U}$ of $\mathrm{Spec}^?_w A$ to its open object represented by U. This observation enables us to assume $U = e$, $A' = A$ while studying covers or families of morphisms $\{U_\alpha \to U\}$.

6.4.7. (Quasicoherence is a local property.) Notice that the quasicoherence is a local property with respect to the weak topology. In other words, if $\{U_\alpha \to e\}$ is a cover of the final object of $\mathcal{S}^?_A$ with respect to the weak topology, and an $\mathscr{O}$-module $\mathscr{F}$ is such that each $\mathscr{F}|_{U_\alpha}$ is quasicoherent, i.e. $\mathscr{F}|_{U_\alpha} \cong \tilde{M}_\alpha$ for some $\mathscr{O}(U_\alpha)$-module M_α, then $\mathscr{F}$ is itself quasicoherent. This statement actually follows immediately from the definition of weak topology:

$\{U_\alpha \to e\}$ is a family of efficient descent for **QCOH**, and the family of $\mathscr{O}(U_\alpha)$-modules M_α together with isomorphisms $\theta_{\alpha\beta} : M_\alpha|_{U_{\alpha\beta}} \xrightarrow{\sim} M_\beta|_{U_{\alpha\beta}}$ induced by $\tilde{M}_\alpha|_{U_{\alpha\beta}} \xrightarrow{\sim} \mathscr{F}|_{U_{\alpha\beta}} \xrightarrow{\sim} \tilde{M}_\beta|_{U_{\alpha\beta}}$ obviously is a descent datum for **QCOH** with respect to this cover. We deduce existence of an A-module M such that $M|_{U_\alpha} \cong M_\alpha$ in a manner compatible with all $\theta_{\alpha\beta}$. In other words, we have a compatible family of sheaf isomorphisms $\tilde{M}|_{U_\alpha} \cong \tilde{M}_\alpha \cong \mathscr{F}|_{U_\alpha}$, which can be glued into an isomorphism of $\mathscr{O}$-modules $\tilde{M} \cong \mathscr{F}$, i.e. $\mathscr{F}$ is quasicoherent, q.e.d.

6.4.8. (Pullbacks of efficient descent families and functoriality of weak spectra.) Let $f : A \to B$ be a homomorphism of generalized rings, thus defining a left exact pullback functor $f^{-1} : \mathcal{S}_A^? \to \mathcal{S}_B^?$ as in **6.3.10**. We claim that f^{-1} *preserves families of efficient descent for* **QCOH**. Since f^{-1} is left exact, the same is true for universal efficient descent families (at least if f is an open pseudolocalization from $\mathcal{T}_A^?$; however, this case already suffices to show **6.4.9**, which tells us that all efficient descent families are automatically universal), i.e. f^{-1} becomes a morphism of sites, thus defining a morphism of generalized ringed topoi ${}^a f : \operatorname{Spec}_w^? B \to \operatorname{Spec}_w^? A$ (the "generalized ringed" part follows from corresponding strong topology result **6.3.10** once we show that f^{-1} is a morphism of sites.)

(a) So let $\{U_\alpha \to U\}$ be an efficient descent family for **QCOH** in $\mathcal{S}_A^?$. By **6.4.6** we may assume $U = e$. Put $A_\alpha := \mathscr{O}(U_\alpha)$; then $f^{-1}U_\alpha$ is given by $B \to B_\alpha := B \otimes_A A_\alpha$, the pushout of $A \to A_\alpha$ with respect to $f : A \to B$. Let $(M_\alpha, \theta_{\alpha\beta})$ be a descent datum for **QCOH** with respect to $\{f^{-1}U_\alpha \to e\}$. Notice that M_α is a B_α-module, and $\theta_{\alpha\beta}$ is an isomorphism of $B_{\alpha\beta}$-modules, where of course $U_{\alpha\beta} := U_\alpha \cap U_\beta$, $A_{\alpha\beta} := \mathscr{O}(U_{\alpha\beta}) = A_\alpha \otimes_A A_\beta$, $B_{\alpha\beta} = B \otimes_A A_{\alpha\beta}$ and so on. By scalar restriction we can treat M_α as A_α-module and $\theta_{\alpha\beta}$ as an $A_{\alpha\beta}$-isomorphism, thus obtaining a descent datum for $\{U_\alpha \to e\}$. By assumption any such descent datum is efficient, so we obtain an A-module M together with A_α-isomorphisms $\gamma_\alpha : M_\alpha \xrightarrow{\sim} M|_{U_\alpha} = M \otimes_A A_\alpha$. All we have to check is that this A-module structure on M extends in a unique way to a B-module structure, such that all γ_α become B_α-isomorphisms. (Notice that we are implicitly using here the "affine base change theorem" **5.4.2**, which enables us to identify $M \otimes_A A_\alpha$ with $M \otimes_B B_\alpha$ since $A \to A_\alpha$ is flat.)

(b) Let $t \in B(n)$ be any operation of B. We want to show that the A-homomorphism $[t]_M : M^n \to M$ is uniquely determined by our data. Indeed, flatness of all $A \to A_\alpha$ implies $M^n|_{U_\alpha} = (M|_{U_\alpha})^n \cong M_\alpha^n$, and the requirement of all γ_α to be B_α-homomorphisms means $[t]_M|_{U_\alpha} = [t]_{M_\alpha}$. Therefore, the collection $\bigl\{[t]_{M_\alpha} : M_\alpha^n \to M_\alpha\bigr\}_\alpha$ defines a morphism from the descent datum for M^n into the descent datum for M; by descent we obtain a unique A-homomorphism $[t]_M : M^n \to M$ with required properties, and all relations between operations t, t', $\ldots$ of B will be automatically fulfilled for $[t]_M$, $[t']_M$, $\ldots$ just because they hold for all $[t]_{M_\alpha}$, $[t']_{M_\alpha}$, $\ldots$, and because of the uniqueness of $[t]_M$. Thus M admits a unique B-module structure with required properties, so our original descent datum is efficient, q.e.d.

6.4.9. (All efficient descent families are universal.) Applying the above result to an open pseudolocalization $A \to A'$ in $\mathcal{T}_A^?$, corresponding to some $V \in \operatorname{Ob} \mathcal{S}_A^?$, we see that *any efficient descent family for* **QCOH** *in* $\mathcal{S}_A^?$ *is universal.* Therefore, *covers for the weak* $\mathcal{T}^?$*-topology on* $\mathcal{S}_A^?$ *are exactly the efficient descent families for* **QCOH**.

6.4.10. (Finite descent families are efficient.) Clearly, any efficient descent family (for **QCOH**) is a descent family, i.e. any cover for the weak topology is a cover for the strong topology. Let us prove a partial converse: *any* **finite** *descent family is efficient*, i.e. *any finite cover for the strong topology is a cover for the weak topology.*

So let $\{U_i \to e\}_{1\leq i\leq n}$ be a finite cover of the final object e of $\mathcal{S}_A^?$ with respect to the strong topology. Put $U_{ij} := U_i \cap U_j$, $A_i := \mathscr{O}(U_i)$ and $A_{ij} := \mathscr{O}(U_{ij})$ as before, and consider any descent datum (M_i, θ_{ij}) with respect to this cover. For any i and j we have two maps $M_i \to M_{ij} = M_i \otimes_{A_i} A_{ij}$: one is the natural scalar extension map, while the other is obtained by composing another scalar extension map $M_i \to M_{ji}$ with $\theta_{ij}^{-1} : M_{ji} \xrightarrow{\sim} M_{ij}$. Now we can define an A-module M by means of the following left exact diagram:

$$M \longrightarrow \prod_i M_i \rightrightarrows \prod_{i,j} M_{ij} \tag{6.4.10.1}$$

Since this is a finite projective limit diagram, it remains such after any flat base change, e.g. with respect to $A \to A_k$, i.e. $M \otimes_A A_k$ is canonically isomorphic to $\operatorname{Ker}(\prod_i M_{ik} \rightrightarrows \prod_{i,j} M_{ijk})$. One easily sees that this kernel is canonically isomorphic to $M_k = M_{kk}$, just because now we are solving a descent problem with respect to the trivial cover $\{U_{ik} \to U_k\}_{1\leq i\leq n}$, hence $M \otimes_A A_k \cong M_k$ for all $1 \leq k \leq n$, i.e. M is the solution of our original descent problem, q.e.d.

This result leads us to the following definition:

Definition 6.4.11 *(Finite topology and spectra.) The* finite topology *on* $\mathcal{S}_A^?$ *is the topology generated by all finite covers with respect to the strong (or equivalently, the weak) topology on* $\mathcal{S}_A^?$*. In other words, a family* $\{U_\alpha \to U\}$ *is a cover for the finite topology iff it admits a finite refinement* $\{V_i \to U\}$*, which is an (efficient) descent family for* **QCOH**. *The corresponding site will be denoted by* $\mathcal{S}_{A,f}^?$*, and the corresponding generalized ringed topos by* $\operatorname{Spec}_f^? A$*; it will be called the* finite $\mathcal{T}^?$-spectrum *of* A.

6.4.12. (Properties of finite topology and spectra.) All the nice properties of weak topology and weak spectra can be almost immediately transferred to the finite topology case since finite topology is even weaker than the weak topology. In particular, all $\tilde{M}$ are sheaves of $\mathscr{O}$-modules on $\mathrm{Spec}_f^? A$, quasicoherence is a local property for the finite topology, and any $\varphi : A \to B$ induces a morphism of generalized ringed topoi ${}^a\varphi : \mathrm{Spec}_f^? B \to \mathrm{Spec}_f^? A$. Furthermore, all objects $U \in \mathrm{Ob}\,\mathcal{S}_A^?$ are quasicompact for the finite topology. Since $\mathcal{S}_A^?$ is closed under finite projective limits, this implies that $\mathrm{Spec}_f^? A$ is *coherent* in the sense of SGA 4, hence it admits enough points by the general results of *loc. cit.* On the other hand, $\mathrm{Spec}_f^? A$ is clearly generated by its open objects (actually $U \in \mathrm{Ob}\,\mathcal{S}_A^?$ would suffice), hence the topos $\mathrm{Spec}_f^? A$ is defined by some (uniquely determined) sober topological space, which will be usually denoted by $\mathrm{Spec}_f^? A$ as well. Therefore, the finite spectrum $\mathrm{Spec}_f^? A$ can be thought of as a generalized ringed topological space, not a topos, and this topological space will be automatically quasicompact.

6.4.13. (Comparison of strong, weak and finite spectra.) Clearly, we have morphisms of generalized ringed topoi $\mathrm{Spec}_s^? A \to \mathrm{Spec}_w^? A \to \mathrm{Spec}_f^? A$, induced by the identity functor on $\mathcal{S}_A^?$. In some cases these arrows are equivalences:

(a) Suppose that any object $U \in \mathrm{Ob}\,\mathcal{S}_A^?$ is quasicompact with respect to the weak topology. Then by definition weak and finite topologies on $\mathcal{S}_A^?$ coincide, hence $\mathrm{Spec}_w^? A = \mathrm{Spec}_f^? A$. In particular, these two topoi are given by some topological space.

(b) Similarly, if any object $U \in \mathrm{Ob}\,\mathcal{S}_A^?$ is quasicompact with respect to the strong topology, then strong, weak and finite topologies on $\mathcal{S}_A^?$ coincide, so we get $\mathrm{Spec}_s^? A = \mathrm{Spec}_w^? A = \mathrm{Spec}_f^? A$.

(c) This is applicable in particular for the unary localization theory $\mathcal{T}^u$ and any A, since $\mathrm{Spec}_s^u A$ can be identified with the prime spectrum $\mathrm{Spec}^p A$ (cf. **6.3.17**), which is quasicompact by **6.2.8**, hence $\mathrm{Spec}_f^u A = \mathrm{Spec}_w^u A = \mathrm{Spec}_s^u A = \mathrm{Spec}^p A$, i.e. we needn't distinguish the three variants of spectra for the unary localization theory.

(d) If A is additive, then the comparison morphism $\mathrm{Spec}_s^? A \to \mathrm{Spec}_s^u A = \mathrm{Spec}\,A$ is an isomorphism for any localization theory $\mathcal{T}^?$ (cf. **6.3.19**), hence $\mathrm{Spec}^? A$ is quasicompact. If $A \to A'$ is any open pseudolocalization from $\mathcal{T}_A^?$, corresponding to some $U \in \mathrm{Ob}\,\mathcal{S}_A^?$, then A' is still additive, hence U is quasicompact with respect to the strong topology by the above reasoning. Applying (b) we obtain $\mathrm{Spec}_f^? A = \mathrm{Spec}_w^? A = \mathrm{Spec}_s^? A = \mathrm{Spec}^u A = \mathrm{Spec}\,A$, i.e. *any construction of spectra yields the prime spectrum* $\mathrm{Spec}\,A$ *when applied to an additive (i.e. classical) commutative ring A.*

6.4.14. (Choice of localization theory and topology.) Whenever we want to concider spectra of generalized rings, and generalized schemes, obtained from such spectra by gluing, we have to fix a localization theory $\mathcal{T}^?$ and a topology (weak or finite; strong topology won't do for schemes since quasicoherence is not local for strong topology). We have just seen that these choices do not affect spectra of classical rings and classical schemes, so any choice would be compatible with the classical theory of Grothendieck schemes in this respect. Usually we do one of the extreme choices:

(a) Minimal choice: unary localization theory, and any topology. We obtain $\operatorname{Spec}^u A = \operatorname{Spec}^p A$, the prime spectrum of A in the sense of **6.2.5**.

(b) Maximal choice: total localization theory $\mathcal{T}^t$, consisting of all open pseudolocalizations, and the finite topology. We obtain reasonable spectra $\operatorname{Spec}^t_f A$, which coincide with $\operatorname{Spec} A$ in the classical case, but do not coincide with $\operatorname{Spec}^u A$ for, say, $A = \operatorname{Aff}_{\mathbb{Z}}$ (cf. **6.2.13**). In this way we still have quasicompactness and nice properties of quasicoherent sheaves, these spectra are still given by (generalized ringed) topological spaces, and we obtain more natural results at least for generalized rings like Aff_R.

The only obstruction to adopting (b) in all situations is the absence of a direct description of all open pseudolocalizations of a generalized ring A, or of points of $\operatorname{Spec}^t_f A$. For such reasons we may want to use unary spectra $\operatorname{Spec}^u A$ sometimes.

6.4.15. (Local rings.) Once we fix a localization theory $\mathcal{T}^?$ and a topology (usually we choose the finite topology), we say that a generalized ring A is $\mathcal{T}^?$*-local* if the global sections functor Γ on $\operatorname{Spec}^?_f A$ is a point (or a fiber functor) on this topos. Another description: any cover of the final object e in $\mathcal{S}^?_f$ contains the final object itself. For example, a generalized ring is $\mathcal{T}^u$-local iff it has a unique maximal ideal, i.e. is local in the sense of **6.2.4**. Similarly, a homomorphism $\varphi : A \to B$ of $\mathcal{T}^?$-local rings is *local* if ${}^a\varphi : \operatorname{Spec}^?_f B \to \operatorname{Spec}^?_f A$ maps the only closed point of $\operatorname{Spec}^?_f B$ (given by Γ_B) into the only closed point of $\operatorname{Spec}^?_f A$. This is again consistent with existing terminology.

6.4.16. (Generalized locally ringed spaces and topoi.) Now we can define generalized locally ringed spaces (with respect to some theory $\mathcal{T}^?$) by requiring the stalks of the structural sheaf of a generalized ringed space to be $\mathcal{T}^?$-local, and local morphisms of such spaces are then defined in the natural way. Furthermore, we might extend these definitions to topoi, and show the usual universal property of $\operatorname{Spec}^?_f A$ among all generalized locally ringed spaces (or topoi) with the global sections of the structural sheaf isomorphic to A.

6.5. (Generalized schemes.) Once we fix a localization theory $\mathcal{T}^?$ (say, of commutative K-algebras), and choose a topology (weak or finite; we'll usually choose finite) we can define a *scheme* for this theory, or a $\mathcal{T}^?$*-scheme*, to be a topological space or topos X, together with a structural sheaf $\mathscr{O}_X$ of (generalized) K-algebras, such that $(X, \mathscr{O}_X)$ is locally isomorphic to generalized ringed spaces or topoi of form $\operatorname{Spec}^? A$. In the topos case this means that we have a cover $(U_\alpha \to e)$ of the final object by open objects, such that each $(X_{/U_\alpha}, \mathscr{O}_X|_{U_\alpha})$ is equivalent to some $\operatorname{Spec}^? A$. Of course, once we know that all spectra for the chosen theory are quasicompact, we have enough points, so we can work with sober topological spaces, and forget about topoi.

This is applicable in particular if we choose finite topology on all our spectra. We are going to do this, so as to obtain a theory of schemes with almost all basic classical properties fulfilled. Furthermore, we fix some localization theory $\mathcal{T}^?$ and write $\operatorname{Spec} A$ instead of $\operatorname{Spec}^?_f A$.

6.5.1. (Affine schemes.) A *(generalized) affine scheme* X is just a generalized (locally) ringed space isomorphic to $\operatorname{Spec} A$ for some generalized ring A. An *open subscheme* U of a scheme X is an open subset $U \subset X$ with induced generalized ringed structure: $\mathscr{O}_U = \mathscr{O}_X|_U$. Notice that any generalized scheme can be covered by affine open subschemes, essentially by definition of a scheme, and that affine open subsets constitute a base of topology of any affine scheme $\operatorname{Spec} A$, essentially by our construction of $\operatorname{Spec} A = \widetilde{\mathcal{S}^?_{A,f}}$, hence of any scheme. This observation also implies that any open subscheme U of a scheme X is indeed a scheme.

These properties would hold even if we had chosen the weak topology on $\mathcal{S}^?_A$ instead of the finite one; however, the special property of the finite topology is that *any affine scheme is quasicompact.*

6.5.2. (Morphisms of generalized schemes.) A *morphism* of generalized schemes $f : X \to Y$ is by definition a *local* morphism of generalized locally ringed spaces or topoi, cf. **6.4.16**. Another description: for any (generalized) affine open subschemes $U \subset X$ and $V \subset Y$, such that $f(U) \subset V$, the restricted morphism $f_{U,V} : U \to V$ must be induced by a homomorphism of generalized rings $\mathscr{O}(V) \to \mathscr{O}(U)$. One can either show the equivalence of these two descriptions, or just adopt the second of them as a definition of scheme morphism. One can also show that it would suffice to require that the open subschemes U with the above property (for variable V) cover X (cf. **6.5.3** below). An immediate consequence is that *morphisms of affine schemes* $f : \operatorname{Spec} B \to \operatorname{Spec} A$ *are in one-to-one correspondence with generalized ring homomorphisms* $A \to B$. Covering an arbitrary scheme X by affine open subschemes $U_i = \operatorname{Spec} B_i$, and then each $U_i \cap U_j$ by affine open $U_{ij}^{(k)}$, we deduce that *scheme morphisms* $f : X \to \operatorname{Spec} A$ *are in one-to-one correspondence with homomorphisms* $A \to \Gamma(X, \mathscr{O}_X)$, exactly as in the classical situation.

6.5.3. (Localness of scheme morphisms.) The statement implicitly used above is the following: *if $f : \operatorname{Spec} B \to \operatorname{Spec} A$ is a generalized ringed space morphism, such that all $f|_{U_i} : U_i = \operatorname{Spec} B_i \to \operatorname{Spec} A$ are given by some homomorphisms $\varphi_i : A \to B_i$, where $\operatorname{Spec} B = \bigcup_i U_i$ is an affine open cover, then f is itself given by some homomorphism $\varphi : A \to B$.* Indeed, $U_{ij} = U_i \cap U_j \to U_i$ is obviously given by a homomorphism, namely, $B_i \to B_{ij} := B_i \otimes_B B_j$, essentially by definition of $\mathcal{S}^?_{B,f}$ (cf. also **6.4.6**), hence $f|_{U_{ij}}$ is given by $\varphi_{ij} : A \xrightarrow{\varphi_i} B_i \to B_{ij}$, equal to the map induced by $f|_{U_{ij}}$ on the global sections of structural sheaves. In other words, we get maps $\varphi_i : A \to B_i = \mathscr{O}_{\operatorname{Spec} B}(U_i)$, and the restrictions of φ_i and φ_j to $\mathscr{O}_{\operatorname{Spec} B}(U_{ij})$ coincide. Since $\mathscr{O}_{\operatorname{Spec} B}$ is a sheaf, the sheaf condition for cover $\{U_i\}$ implies existence and uniqueness of a generalized ring homomorphism $\varphi : A \to \mathscr{O}_{\operatorname{Spec} B}(e) = B$, such that $A \xrightarrow{\varphi} B \to B_i$ coincide with φ_i. This means ${}^a\varphi|_{U_i} = {}^a\varphi_i = f|_{U_i}$ for any i, hence ${}^a\varphi = f$, q.e.d.

6.5.4. (Fibered products of generalized schemes.) Previous statement immediately implies existence of fibered products of affine generalized schemes: $\operatorname{Spec} A \times_{\operatorname{Spec} C} \operatorname{Spec} B$ is given by usual formula $\operatorname{Spec}(A \otimes_C B)$, where this tensor product of (generalized commutative) C-algebras is understood in the sense of **5.1.6**. Once we know this, existence of fibered products $X \times_S Y$ of arbitrary generalized schemes is shown exactly in the same way as in EGA I, by gluing together fibered products of appropriate affine open $U \subset X$ and $V \subset Y$ over $W \subset S$. Most properties of fibered products of schemes can be generalized to our case. Notice, however, that the map $|X \times_S Y| \to |X| \times_{|S|} |Y|$, where $|Z|$ denotes the underlying set of a generalized scheme Z, is not necessarily surjective, as illustrated by $X = Y = \operatorname{Spec} \mathbb{Z}$, $S = \operatorname{Spec} \mathbb{F}_\varnothing$, $X \times_S X = \operatorname{Spec} \mathbb{Z} = X$ (cf. **5.1.22**), in contrast with the classical case.

6.5.5. (Final generalized scheme.) Another consequence of **6.5.2** is that the category of generalized schemes admits a final object, namely, $\operatorname{Spec} \mathbb{F}_\varnothing$, simply because $\mathbb{F}_\varnothing$ is initial in the category of generalized rings (cf. **3.4.12**,h). In this way any generalized scheme admits a unique $\mathbb{F}_\varnothing$-scheme structure, so we might say *$\mathbb{F}_\varnothing$-schemes* instead of *generalized schemes.*

6.5.6. (Generalized schemes with zero.) Notice that ${}^a\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ is a monomorphism in the category of generalized schemes whenever $\varphi : A \to B$ is an epimorphism of generalized rings. This is applicable in particular to $\mathbb{F}_\varnothing \to \mathbb{F}_1$ since $\mathbb{F}_1 \otimes_{\mathbb{F}_\varnothing} \mathbb{F}_1 = \mathbb{F}_1$ (recall that $\mathbb{F}_1 = \mathbb{F}_\varnothing[0^{[0]}]$, hence $\mathbb{F}_1 \otimes \mathbb{F}_1 = \mathbb{F}_\varnothing[0^{[0]}, \bar{0}^{[0]}]$, and any two commuting constants coincide). We see that $\operatorname{Spec} \mathbb{F}_1 \to \operatorname{Spec} \mathbb{F}_\varnothing$ is a monomorphism, i.e. *any generalized scheme X admits at most one $\mathbb{F}_1$-structure.* If this be the case, we say that X is an $\mathbb{F}_1$*-scheme*, or a *(generalized) scheme with zero.* Clearly, X is a generalized scheme with zero iff $\Gamma(X, \mathscr{O}_X)$ admits a constant, i.e. iff the set $\Gamma(X, \mathscr{O}_X(0))$ is non-empty.

Another consequence of the fact that $\operatorname{Spec} \mathbb{F}_1 \to \operatorname{Spec} \mathbb{F}_\varnothing$ is a monomorphism is that $X \times_{\mathbb{F}_1} Y = X \times_{\mathbb{F}_\varnothing} Y$ for any two $\mathbb{F}_1$-schemes X and Y, so we can write simply $X \times Y$.

6.5.7. (Additive generalized schemes.) Similarly, $\mathbb{F}_1 \to \mathbb{Z}$ and $\mathbb{F}_\varnothing \to \mathbb{Z}$ are epimorphisms of generalized rings since $\mathbb{Z} \otimes_{\mathbb{F}_\varnothing} \mathbb{Z} = \mathbb{Z}$ (cf. **5.1.22**). This means that $\operatorname{Spec} \mathbb{Z} \to \operatorname{Spec} \mathbb{F}_\varnothing$ is a monomorphism of generalized schemes, i.e. *any generalized scheme X admits at most one $\mathbb{Z}$-structure.* If this be the case, we say that X is a *$\mathbb{Z}$-scheme*, or an *additive (generalized) scheme.* If X is additive, we have a homomorphism $\mathbb{Z} \to \mathscr{O}_X(U)$ for any open $U \subset X$, hence all $\mathscr{O}_X(U)$ are additive generalized rings with symmetry $[-]$, i.e. are given by classical commutative rings, so $\mathscr{O}_X$ can be thought of as a sheaf of classical rings. Taking into account our previous results **5.1.10** and **6.4.13**, we see that *the category of additive generalized schemes is canonically equivalent to the category of Grothendieck schemes.* Notice that $X \times_{\mathbb{Z}} Y = X \times_{\mathbb{F}_\varnothing} Y$ for any additive schemes X and Y, so we can safely write $X \times Y$.

6.5.8. (Quasicoherent sheaves.) Let $X = (X, \mathscr{O}_X)$ be a generalized scheme. We say that an $\mathscr{O}_X$-module $\mathscr{F}$ is *quasicoherent* if for any affine open $U = \operatorname{Spec} A \subset X$ the restriction $\mathscr{F}|_U$ is isomorphic to $\mathscr{O}_U$-module $\tilde{M}$ for some A-module M, and then necessarily $M = \Gamma(U, \mathscr{F})$. Since quasicoherence is a local property with respect to weak and finite topologies (cf. **6.4.7**), we see that it would suffice to require this condition for U from some affine open cover of X. Once we know this, all usual properties of quasicoherent sheaves follow. In particular, quasicoherent sheaves on X constitute a full subcategory $\mathscr{O}_X$-*QCoh* in $\mathscr{O}_X$-*Mod*, stable under arbitrary inductive and projective limits, and for any scheme morphism $f : Y \to X$ the pullback functor $f^* : \mathscr{O}_X\text{-}\mathit{Mod} \to \mathscr{O}_Y\text{-}\mathit{Mod}$ preserves quasicoherence. When both X and Y are affine, the action of f^* on quasicoherent sheaves is given by scalar extension with respect to generalized ring homomorphism φ corresponding to f. Furthermore, in this situation (but not in general) f_* also preserves quasicoherence, and its action on quasicoherent sheaves is given by scalar restriction along φ (we apply the "affine base change theorem" **5.4.2** here; cf. **6.3.13**).

6.5.9. (Quasicoherent sheaves of finite type.) Recall that a sheaf of $\mathscr{O}_X$-modules $\mathscr{F}$ on a generalized ringed space $X = (X, \mathscr{O}_X)$ is said to be *(locally) finitely generated,* or *of finite type*, if one can find an open cover $\{U_\alpha\}$ and strict epimorphisms (i.e. surjections) $p_\alpha : \mathscr{O}_{U_\alpha}(n_\alpha) \twoheadrightarrow \mathscr{F}|_{U_\alpha}$, where $n_\alpha \geq 0$ are integers, and $\mathscr{O}_U(n)$ or $L_{\mathscr{O}_U}(n)$ denotes the free $\mathscr{O}_U$-module of rank n.

Now suppose that X is a generalized scheme, and that $\mathscr{F}$ is quasicoherent; since the above cover can be always chosen to consist of affine U_α, and functor

$\Delta : M \mapsto \tilde{M}$, $\mathscr{O}(U)$-*Mod* $\to$ $\mathscr{O}_U$-*Mod* is exact for any open affine $U \subset X$, we see that *an $\mathscr{O}_X$-module $\mathscr{F}$ is quasicoherent of finite type iff one can find an affine open cover $\{U_\alpha\}$, such that each $\mathscr{F}|_{U_\alpha}$ is isomorphic to $\tilde{M}_\alpha$ for some finitely generated $\mathscr{O}(U_\alpha)$-module M_α.* If we know already that $\mathscr{F}$ is quasicoherent, this is equivalent to requiring all $\Gamma(U_\alpha, \mathscr{F})$ to be finitely generated as $\mathscr{O}(U_\alpha)$-modules.

6.5.10. (Global sections of quasicoherent sheaves of finite type.) We claim that *if $\mathscr{F}$ is quasicoherent of finite type, then $\Gamma(U, \mathscr{F})$ is a finitely generated $\mathscr{O}(U)$-module for any affine open $U \subset X$.* We can assume $X = U = \operatorname{Spec} A$, $\mathscr{F} = \tilde{M}$ for some A-module M. Since X is quasicompact, we can find a finite affine open cover $X = \bigcup_{i=1}^n U_i$, $U_i = \operatorname{Spec} A_i$, such that $\mathscr{F}|_{U_i} = \tilde{M}_i$ for some finitely generated A_i-module M_i, clearly equal to $M \otimes_A A_i$. Now write M as the filtered inductive limit of all its finitely generated submodules: $M = \varinjlim_\alpha M_\alpha$, and put $M_{i\alpha} := M_\alpha \otimes_A A_i$. Then $M_i = \varinjlim_\alpha M_{i\alpha}$, and M_i is finitely generated, whence $M_{i\alpha} = M_i$ for all $\alpha \geq \alpha(i)$. Taking $\alpha \geq \alpha(i)$ simultaneously for $i = 1, \ldots, n$, we get $M_i = M_{i\alpha}$ for all i, hence $M = M_\alpha$ is finitely generated.

6.5.11. (Finitely presented sheaves.) Similarly, an $\mathscr{O}_X$-module $\mathscr{F}$ is said to be *(locally) finitely presented,* or *of finite presentation* if it becomes finitely presented on some open cover $\{U_\alpha\}$, i.e. if $\mathscr{F}|_{U_\alpha} = \operatorname{Coker}\bigl(p_\alpha, q_\alpha : \mathscr{O}_{U_\alpha}(m_\alpha) \rightrightarrows \mathscr{O}_{U_\alpha}(n_\alpha)\bigr)$ for suitable m_α, $n_\alpha \geq 0$ and morphisms p_α, q_α. Now if X is a scheme, such a cover can be always assumed to be affine. Clearly, each $\mathscr{F}|_{U_\alpha}$ is quasicoherent, being an inductive limit of quasicoherent sheaves, hence *any finitely presented $\mathscr{O}_X$-module $\mathscr{F}$ over a scheme X is quasicoherent.* Furthermore, we see that $M_\alpha = \Gamma(U_\alpha, \mathscr{F})$ is a finitely presented $\mathscr{O}_X(U_\alpha)$-module, being a cokernel of two maps between free modules of finite rank.

We claim that *if $\mathscr{F}$ is finitely presented, then $\Gamma(U, \mathscr{F})$ is a finitely presented $\mathscr{O}(U)$-module for any affine open $U \subset X$.* We can assume $X = U = \operatorname{Spec} A$ and $\mathscr{F} = \tilde{M}$. By **6.5.10** M is finitely generated, so we can find a surjection $\pi : A(n) \to M$. Put $R := A(n) \times_M A(n)$, and write $R = \varinjlim_\alpha R_\alpha$ where R_α are finitely generated submodules of R, and put $M_\alpha := A(n)/R_\alpha = \operatorname{Coker}(R_\alpha \rightrightarrows A(n))$. We get $M = \varinjlim_\alpha M_\alpha$, all M_α are finitely presented, and all transition morphisms are surjective. The proof is finished in the same way as in **6.5.10**: we know that $M_i := M \otimes_A A_i$ are finitely presented A_i-modules, so, in the notations of *loc.cit.* $M_{i\alpha} \to M_i$ is an isomorphism for $\alpha \geq \alpha(i)$; choosing $\alpha \geq \alpha(i)$ for all i, we see that $M = M_\alpha$ is finitely presented.

6.5.12. (Sheaves of $\mathscr{O}_X$-algebras.) Notice that we have two different notions of an $\mathscr{O}_X$-algebra $\mathscr{B}$, commutative or not, over a generalized scheme X:

(a) We can define such an algebra as a sheaf of generalized rings $\mathscr{B}$ over X together with a central homomorphism $\mathscr{O}_X \to \mathscr{B}$.

(b) We can define such an $\mathscr{B}$ as an algebra in $\mathscr{O}_X$-*Mod*, considered as a $\otimes$-category with respect to $\otimes_{\mathscr{O}_X}$.

Similarly to what we had in **5.1.9** and **5.3.8**, any $\mathscr{O}_X$-algebra in the sense of (b) defines a *unary* $\mathscr{O}_X$-algebra in the sense of (a), and the functor thus defined is an equivalence of categories, essentially for the same reasons as in **5.3.8**.

Furthermore, when $\mathscr{B}$ is a quasicoherent $\mathscr{O}_X$-algebra, then for any affine open $U \subset X$ we have $\mathscr{B} = \tilde{B}$ for some $\mathscr{O}_X(U)$-algebra B in the sense of **5.1.9**, where $\tilde{B}$ is understood as the sheaf of generalized rings over U defined by the family of sheaves of sets $\{\widetilde{B(n)}\}$, with the structural maps $\tilde{B}(\varphi)$ and $\mu_n^{(k)}$ induced by corresponding structural maps for A (it is important here that all structural maps involved are in fact $\mathscr{O}(U)$-homomorphisms, i.e. that $\mathscr{O}(U) \to B$ is at least central, even if we don't assume B to be commutative).

In this way we obtain a natural equivalence between quasicoherent $\mathscr{O}_X$-algebras $\mathscr{B}$ over an affine scheme $X = \operatorname{Spec} A$ and A-algebras B, given by $\mathscr{B} \mapsto \Gamma(X, \mathscr{B})$, $B \mapsto \tilde{B}$. This equivalence preserves unarity of algebras.

6.5.13. (Affine morphisms.) Given a generalized scheme S and a quasicoherent $\mathscr{O}_S$-algebra $\mathscr{B}$, we can construct a generalized scheme $X = \operatorname{Spec} \mathscr{B}$ over S by gluing together $\operatorname{Spec} \mathscr{B}(U)$ for all affine open $U \subset X$, in the same way as in EGA II 1.3.1. A scheme morphism $f : X \to S$ is said to be *affine* if it is isomorphic to a morphism $\operatorname{Spec} \mathscr{B} \to S$ of the above form; in this case obviously $\mathscr{B} = f_* \mathscr{O}_X$, where f_* is understood as a direct image functor for sheaves of generalized rings over X.

All general properties of affine morphisms from EGA II 1 can be now transferred to our situation. For example, $f : X \to S$ is affine iff for any affine open $U \subset S$ its pullback $f^{-1}(U)$ is also affine iff this condition holds for some open affine cover $\{U_\alpha\}$ of S.

6.5.14. (Unary affine morphisms.) We have a phenomenon specific to our situation: an affine morphism $f : X \to S$ is said to be *unary* if corresponding quasicoherent $\mathscr{O}_S$-algebra $f_* \mathscr{O}_X$ is unary. One can easily show that unary affine morphisms are stable under composition, their diagonals and graphs are also unary affine etc.

6.5.15. (Quasicompact morphisms.) We say that a generalized scheme morphism $f : X \to S$ is *quasicompact* if $f^{-1}(U)$ is quasicompact for any quasicompact open subset $U \subset S$, i.e. if it can be covered by finitely many affine open subsets of X. It is sufficient to require this for all open affine $U \subset S$, or just for open affine U_α from some cover of S.

6.5.16. (Quasiseparated morphisms.) We say that a morphism $f : X \to Y$ is *quasiseparated* if the diagonal $\Delta_{X/Y} : X \to X \times_Y X$ is quasicompact. We say that a generalized scheme X is *quasiseparated* if $X \to \operatorname{Spec} \mathbb{F}_\varnothing$ is quasiseparated. These notions have all their usual properties known from EGA IV 1. For example, X is quasiseparated iff it admits an affine open cover $\{U_\alpha\}$, such that all $U_\alpha \cap U_\beta$ are quasicompact. All affine schemes and morphisms are quasicompact and quasiseparated, and $\Delta_{X/Y}$ is quasiseparated for any $X \xrightarrow{f} Y$.

6.5.17. (Open embeddings.) We say that a generalized scheme morphism $f : Y \to X$ is an *open embedding* or an *open immersion* if it is isomorphic to an embedding morphism $U \to X$ for some open subset $U \subset X$ (clearly equal to $f(Y)$). Open embeddings are always monomorphisms, hence quasiseparated, but not necessarily quasicompact. An *open subscheme* $U \subset X$ is a subobject of X in the category of schemes defined by an open embedding. Most of their elementary properties from EGA I generalize immediately to our situation.

6.5.18. (Closed embeddings?) However, we don't seem to have a good notion of closed embeddings for generalized schemes. This is partly due to the fact that while the localization theory can be transferred to generalized rings almost literally, this is not true for ideal/quotient ring theory.

Consider the following example. Let A be a generalized ring with zero, and put $X := \operatorname{Spec} A$. One can consider some strict quotient B of A, e.g. by choosing two elements f, $g \in |A|$ and setting $B := A/\langle f = g\rangle$ (cf. **4.4.9**). Since $A \to B$ is a (strict) epimorphism, $i : Y := \operatorname{Spec} B \to X$ is a monomorphism of generalized schemes. In classical situation Y would have been a closed subscheme of X, "the locus where $f = g$", equal to $V(f - g)$.

However, in our situation i will be a closed embedding in the topological sense only if we divide by several equations of the form $f_k = 0$ (then Y will be homeomorphic to $V(\mathfrak{a})$, where $\mathfrak{a}$ is the ideal generated by (f_k), if we work with prime spectra). Since we don't have subtraction, we cannot replace equations $f_k = g_k$ with $f_k - g_k = 0$ as in the classical case, so in general $i : Y \to X$ can have a non-closed image, as illustrated by the following example.

6.5.19. (Diagonal of $\mathbb{A}^1_{\mathbb{F}_1}$.) Put $X := \mathbb{A}^1_{\mathbb{F}_1} = \operatorname{Spec} \mathbb{F}_1[T]$ and consider $\Delta_X : X \to X \times X$. Clearly, Δ_X is defined by strict epimorphism $\mathbb{F}_1[T] \otimes \mathbb{F}_1[T] = \mathbb{F}_1[T, T'] \to \mathbb{F}_1[T]$, mapping both T and T' into T. Now let's compute the corresponding prime spectra: $\operatorname{Spec} \mathbb{F}_1[T] = \{0, (T)\}$, $\operatorname{Spec} \mathbb{F}_1[T, T'] = \{0, (T), (T'), (T, T')\}$, and the image of Δ_X equals $\{0, (T, T')\}$, which is not closed in $\operatorname{Spec} \mathbb{F}_1[T, T']$, i.e. the image of Δ_X is not closed, at least if we use $\mathcal{T}^u$.

6.5.20. (Separated morphisms?) At this point we see that we don't have any reasonable notion of separated morphisms $f : X \to Y$: if we require $\Delta_{X/Y}$ to be a closed immersion (whatever that means), then even the affine line $\mathbb{A}^1_{\mathbb{F}_1}$ wouldn't be separated.

All we can do is to consider quasiseparated morphisms instead, whenever this is possible. It is not always enough: for example, a section $\sigma : S \to X$ of an affine morphism $f : X \to S$ won't be a closed immersion, just because $\Delta_{X/S}$ is not.

6.5.21. (Immersions and subschemes.) However, some related notions can be still saved. Namely, let us say that $f : Y \to X$ is an *immersion* if it is a monomorphism, locally (in Y and X) given by ${}^a\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$, where $\varphi : A \to B$ is a homomorphism of a generalized ring A into a finitely generated A-algebra B, such that the scalar restriction functor $\varphi_* : B\text{-}\mathit{Mod} \to A\text{-}\mathit{Mod}$ is fully faithful (hence φ is a NC-epimorphism, and in particular an epimorphism, cf. **6.3.1**), and retains this property after any base change. Examples of such homomorphisms are given by open pseudolocalizations and strict epimorphisms (i.e. surjective homomorphisms).

A *subscheme* $Y \subset X$ is any subobject of X in the category of schemes, such that $Y \to X$ is an immersion. Clearly, immersions are stable under composition and base change, open embeddings are immersions, and any diagonal morphism $\Delta_{X/S}$ is an immersion, hence the graph $\Gamma_f : X \to X \times_S Y$ of any morphism $f : X \to Y$ of (generalized) S-schemes is an immersion as well.

6.5.22. (Monomorphisms are not injective!) Notice that, in contrast to the classical case, a monomorphism of generalized schemes can be not injective on points, as illustrated by $\operatorname{Spec} \mathbb{Z} \to \operatorname{Spec} \mathbb{F}_\varnothing$. It is an interesting question whether an immersion can be non-injective.

6.5.23. ("Closed" immersions.) We say that a morphism of generalized schemes $j : Y \to X$ is a "closed" immersion if it is affine, and if the induced homomorphism of sheaves of generalized rings $\mathscr{O}_X \to f_*\mathscr{O}_Y$ is surjective. Any "closed" immersion is an immersion in the sense of **6.5.21**; however, it is not necessarily closed in the topological sense, whence the quotes. A "closed" immersion is locally of the form $j : \operatorname{Spec} B \to \operatorname{Spec} A$, where $\varphi : A \twoheadrightarrow B$ is a surjective homomorphism of generalized rings. If we work in $\mathcal{T}^u$ (i.e. with prime spectra), then such a j is injective on points (since $\varphi^{-1}(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p}')$ implies $\mathfrak{p} = \mathfrak{p}'$), and the topology on $j(\operatorname{Spec} B)$ coincides with that induced from $\operatorname{Spec} A$, since $j^{-1}(D(f)) = D(\varphi(f))$, open sets $D(\bar{f})$ constitute a base of topology of $\operatorname{Spec} B$, and any $\bar{f} \in |B|$ can be lifted to some $f \in |A|$, hence $D(\bar{f}) = j^{-1}(D(f))$.

These "closed" immersions are stable under composition and base change, and the diagonal of an affine morphism is a "closed" immersion. Therefore, it might make sense to define a separated (or rather "separated") morphism $f : X \to Y$ by requiring its diagonal $\Delta_{X/Y}$ to be a "closed" immersion. Notice that "closed" immersions are always affine, but not necessarily unary.

Of course, we define a "closed" subscheme $Y \subset X$ as a subobject in the category of generalized schemes, such that $Y \to X$ is a "closed" immersion.

6.5.24. (Disjoint union of affine schemes.) Let $X_1 = \operatorname{Spec} A_1$ and $X_2 = \operatorname{Spec} A_2$ be two affine generalized schemes with zero. We claim that $\operatorname{Spec}(A_1 \times A_2) \cong X_1 \sqcup X_2$; in particular, the disjoint union $X_1 \sqcup X_2$ is affine.

Indeed, according to **5.4.8**, $(A_1 \times A_2)$-*Mod* is equivalent to A_1-*Mod* $\times$ A_2-*Mod*; the inverse of this equivalence transforms (M_1, M_2) into $M_1 \times M_2$, with the action of $t = (t_1, t_2) \in (A_1 \times A_2)(n) = A_1(n) \times A_2(n)$ defined componentwise. Now it is immediate that A_1 equals $A[\boldsymbol{e}_1^{-1}]$, where $A := A_1 \times A_2$ and $\boldsymbol{e}_1 = (\boldsymbol{e}, 0) \in |A|$, and similarly for A_2, whence two open embeddings $\lambda_i : X_i = \operatorname{Spec} A_i \to X := \operatorname{Spec} A$. Next, $X_1 \cap X_2$ equals $\operatorname{Spec} A[(\boldsymbol{e}_1\boldsymbol{e}_2)^{-1}] = \operatorname{Spec} A[0^{-1}] = \varnothing$; applying **5.4.8** once more, we see that any quasicoherent sheaf $\widetilde{M}$ on X satisfies the sheaf property for family of open (pseudo)localizations $\{X_i \to X\}_{i=1,2}$, hence $X = X_1 \cup X_2$ by the definition of (strong) topology, cf. **6.3.6** and **6.4.10**, q.e.d.

6.5.25. (Unarity is not local in the source.) Notice that the unarity of affine schemes over a fixed base generalized ring C is not a local property, as illustrated by $C = \mathbb{Z}_\infty$, $X = \operatorname{Spec}(\mathbb{Z}_\infty \times \mathbb{Z}_\infty) = \operatorname{Spec} \mathbb{Z}_\infty \sqcup \operatorname{Spec} \mathbb{Z}_\infty$: we see that X is a union of unary affine C-schemes, but it is not unary itself.

6.5.26. (Morphisms of finite type and of finite presentation.) We say that a generalized scheme morphism $f : X \to Y$ is *locally of finite type* if one can cover Y by affine open subschemes V_i, and each $f^{-1}(V_i)$ by affine open U_{ij}, such that each $\mathscr{O}(U_{ij})$ is a finitely generated $\mathscr{O}(V_i)$-algebra in the sense of **5.1.14**. Similarly, f is *locally of finite presentation* if one can find similar covers with the property that each $\mathscr{O}(U_{ij})$ be a finitely presented $\mathscr{O}(V_i)$-algebra in the sense of **5.1.15**.

Furthermore, we say that f is *of finite type* if it is quasicompact and locally of finite type, and that f is *of finite presentation* if it is quasicompact, quasiseparated and locally of finite presentation.

These notions have all formal properties of EGA I 6 and EGA IV 1, modulo the following statement:

6.5.27. *If* $\operatorname{Spec} B \to \operatorname{Spec} A$ *is locally of finite type (resp. presentation), then* B *is a finitely generated (resp. finitely presented)* A*-algebra.* Unfortunately, we don't have a proof of this statement for $\mathcal{T}^? \neq \mathcal{T}^u$. However, for the unary localization theory $\mathcal{T}^u$ the proofs from EGA work with minimal modifications.

Let us show for example the statement about finite generation. Suppose that $s_1, \ldots, s_n \in |B|$ are such that all B_{s_i} are finitely generated A-algebras, and choose some finite lists of generators $f_{ij} \in B_{s_i}(r_{ij})$; we can write $f_{ij} = g_{ij}/s_i^{k_{ij}}$ for some $k_{ij} \geq 0$ and $g_{ij} \in B(r_{ij})$, choose a common value k for all k_{ij} and replace s_i by s_i^k, thus reducing to the case $k_{ij} = 1$. Next, since $\bigcup_{i=1}^n D(s_i) = \operatorname{Spec} B$, these elements s_i generate the unit ideal over B (cf. **6.3.17** and **6.2.8**), so we can find an operation $h \in B(n)$, such that $h(s_1, \ldots, s_n) = 1$ (cf. **4.6.9**). Now we put $B' := A[s_i^{[1]}, g_{ij}^{[r_{ij}]}, h^{[n]}] \subset B$ and show that $B' = B$ in the same way as in EGA I 6.5.

If all B_{s_i} are finitely presented over A, we can find a surjection $\rho : B' := A[T_1^{[r_1]}, \ldots, T_m^{[r_m]}] \to B$ from an algebra of generalized polynomials, consider its kernel $R := B' \times_B B'$, choose finite lists of equations $(f_{ij}/\tilde{s}_i^N, g_{ij}/\tilde{s}_i^N)$ generating compatible equivalence relation $R_{\tilde{s}_i}$ on $B_{\tilde{s}_i}$, the kernel of $B'_{\tilde{s}_i} \to B_{s_i}$, where $\tilde{s}_i \in B'$ are some lifts of s_i, and show that R is generated by equations $\{f_{ij} = g_{ij}\}$ as a compatible equivalence relation on generalized ring B', hence $B = B'/R$ is finitely presented over A.

6.5.28. (Vector bundles and projective modules.) A *vector bundle* $\mathscr{E}$ over a generalized scheme S is defined as a *locally free $\mathscr{O}_S$-module of finite type*, i.e. we require $\mathscr{E}|_{U_i} \cong \mathscr{O}_{U_i}(n_i)$ as an $\mathscr{O}_{U_i}$-module, for some open cover U_i of S and some integers $n_i \geq 0$. If all n_i can be chosen to have the same value n, we say that *$\mathscr{E}$ is a vector bundle of rank n*. A vector bundle is obviously a (locally) finitely presented quasicoherent $\mathscr{O}_S$-module.

Proposition. *If $\mathscr{E}$ is a vector bundle over S, then $\Gamma(U, \mathscr{E})$ is a finitely generated projective module over generalized ring $\Gamma(U, \mathscr{O}_S)$, for any affine open $U \subset S$.*

Proof. It suffices to consider the case $U = S = \operatorname{Spec} A$, $\mathscr{E} = \tilde{P}$ for some A-module P, finitely presented by **6.5.11**. We have to check that P is projective, i.e. that the functor $\operatorname{Hom}_A(P, -) : A\text{-}Mod \to A\text{-}Mod$ preserves strict epimorphisms (surjections of A-modules). Notice for this that $\widetilde{\operatorname{Hom}_A(P, M)} \cong \mathbf{Hom}_{\mathscr{O}_S}(\tilde{P}, \tilde{M})$ for any finitely presented A-module P. The proof proceeds as in the classical case of EGA I: fix a finite presentation of P, use exactness properties of functors involved to reduce to the case of a free $P = A(n)$ of finite rank n, and then we are reduced to proving $\widetilde{M^n} \cong \tilde{M}^n$, true by exactness of $\Delta : M \to \tilde{M}$. Applying this statement to our P and noticing that $f : M \to N$ is a strict epimorphism iff $\tilde{f} : \tilde{M} \to \tilde{N}$ is one, we see that projectivity of P is equivalent to $\mathbf{Hom}_{\mathscr{O}_S}(\mathscr{E}, \mathscr{F}) \to \mathbf{Hom}_{\mathscr{O}_S}(\mathscr{E}, \mathscr{G})$

being a strict epimorphism for any strict epimorphism $\varphi : \mathscr{F} \to \mathscr{G}$ of quasicoherent $\mathscr{O}_S$-modules. This is a local property, obviously fulfilled for any free $\mathscr{E}$ of finite rank, hence for any locally free $\mathscr{E}$ of finite type, q.e.d.

6.5.29. (Converse is not true.) Contrary to the classical case, $\tilde{P}$ is not necessarily a vector bundle over $\operatorname{Spec} A$ whenever P is a finitely generated projective A-module, at least if we use unary localization theory $\mathcal{T}^u$ to construct our spectra, as illustrated by the non-free projective $\mathbb{F}_\infty$-module P of **10.4.20**, since $\operatorname{Spec}^u \mathbb{F}_\infty$ is a one-point set.

6.5.30. (Line bundles and Picard group.) A *line bundle* $\mathscr{L}$ over S is defined as a vector bundle of rank 1, i.e. an $\mathscr{O}_S$-module, locally isomorphic to $|\mathscr{O}_S| = \mathscr{O}_S(1)$. Tensor product of two line bundles is obviously again a line bundle, and $\mathscr{L}^{\otimes -1} := \check{\mathscr{L}} := \mathbf{Hom}(\mathscr{L}, |\mathscr{O}_S|)$ is also a line bundle, such that the canonical map $\mathscr{L} \otimes_{\mathscr{O}_S} \check{\mathscr{L}} \to |\mathscr{O}_S|$ is an isomorphism.

This enables us to define the *Picard group* $\operatorname{Pic}(S)$ as the set of isomorphism classes of line bundles over S, with the group operation given by the tensor product. If we denote by $|\mathscr{O}_S|^\times$ the sheaf of invertible elements in $|\mathscr{O}_S|$, then $\operatorname{Pic}(S) = H^1(S, |\mathscr{O}_S|^\times)$, where the RHS is understood as the usual cohomology of a sheaf of abelian groups.

6.5.31. (Flat morphisms.) We say that a generalized scheme morphism $f : X \to S$ is *flat* if for any affine open $U \subset X$ and $V \subset S$, such that $U \subset f^{-1}(V)$, $\mathscr{O}(U)$ is a flat $\mathscr{O}(V)$-algebra. This is equivalent to the existence of an affine open cover $\{V_i\}$ of S and affine open covers $\{U_{ij}\}$ of $f^{-1}(V_i)$, such that each $\mathscr{O}(U_{ij})$ is a flat $\mathscr{O}(V_i)$-algebra.

In order to check equivalence of these two descriptions we have to show that an A-algebra B is flat whenever there is an affine cover $\{U_i \to X\}$, such that each B_i is flat over A, where we put $X := \operatorname{Spec} B$, $S := \operatorname{Spec} A$, $U_i := \operatorname{Spec} B_i$ and denote $\operatorname{Spec} B \to \operatorname{Spec} A$ by f. This is immediate: indeed, functors $f^*|_{U_i} : \mathscr{O}_S\text{-}\mathit{QCoh} \to \mathscr{O}_{U_i}\text{-}\mathit{QCoh}$ are exact, hence $f^* : \mathscr{O}_S\text{-}\mathit{QCoh} \to \mathscr{O}_X\text{-}\mathit{QCoh}$ is exact, i.e. the scalar extension functor $A\text{-}\mathit{Mod} \to B\text{-}\mathit{Mod}$ is exact, so B is indeed a flat A-algebra.

6.5.32. (Faithfully flat morphisms.) We say that $f : X \to S$ is *faithfully flat* if it is flat, and exact functor $f^* : \mathscr{O}_S\text{-}\mathit{Mod} \to \mathscr{O}_X\text{-}\mathit{Mod}$ is fully faithful. This is equivalent to requiring all $f_U^* : \mathscr{O}_U\text{-}\mathit{QCoh} \to \mathscr{O}_{f^{-1}(U)}\text{-}\mathit{QCoh}$ to be fully faithful, where $U \subset S$ is any affine open subscheme, and $f_U : f^{-1}(U) \to U$ is the restriction of f.

6.5.33. (Flat topology (fpqc).) Consider the category $\mathcal{S}$ of affine generalized schemes, or, equivalently, the opposite category of the category *GenR* of generalized rings. Introduce on $\mathcal{S}$ the topology generated by finite families of flat morphisms $\{\operatorname{Spec} A_i \to \operatorname{Spec} A\}$, which are (universal efficient) families of descent for **QCOH**. The resulting topology is the *flat topology (fpqc)*. Any scheme X represents a functor $\tilde{X} : \operatorname{Spec} A \mapsto X(A) = \operatorname{Hom}(\operatorname{Spec} A, X)$, which is easily seen to be a sheaf for the flat topology. Functor $h : X \mapsto \tilde{X}$ from generalized schemes into the flat topos $\tilde{\mathcal{S}}$ is fully faithful, so we can define the flat topology on the category of all generalized schemes simply by pulling back the canonical topology of $\tilde{\mathcal{S}}$ with respect to h. Of course, the topos defined by this larger site will be still $\tilde{\mathcal{S}}$.

One can describe this flat topology more explicitly. We don't want to do it here; let us remark that a large portion of faithfully flat descent theory (cf. e.g. SGA 1) can be generalized to our case with the aid of flat topology just defined.

6.5.34. (Étale coverings, étale and smooth morphisms.) Once we have a notion of flat topology, we can define for example *étale coverings* $X \to S$ as morphisms, locally for the flat topology (over S) isomorphic to "constant étale coverings" $S \times \mathbf{n} \to S$, where $S \times \mathbf{n} = S \sqcup \ldots \sqcup S$ denotes the disjoint union of n copies of S. After that we can define *étale morphisms* $f : X \to Y$ as morphisms that locally for the flat topology on Y, and after that — locally for the Zariski topology on X are of the form $X \xrightarrow{j} X' \xrightarrow{\pi} Y$, where j is an open embedding and π an étale covering. Finally, we can define *smooth morphisms* in a similar manner, where this time j is to be étale and $\pi : X' \to Y$ is to be of the form $Y \times \mathbb{A}^n \to Y$.

However, we are not going to study or use these definitions in this work; they've been mentioned just to demonstrate a possible way of extending some classical notions to generalized schemes.

6.5.35. (Proper and projective morphisms.) Unfortunately, the obvious definition of a proper morphism of generalized schemes involving universal closedness doesn't seem to be useful, because diagonals of affine morphisms are not proper in this sense. However, we have a reasonable notion of a projective morphism, so we are going to study projective generalized schemes and morphisms instead.

6.6. (Projective generalized schemes and morphisms.) Now we want to define projective generalized schemes and morphisms and related notions. There are three major differences from the classical case of EGA II. Firstly, we don't have reasonable noetherian conditions, so we have to use finitely presented morphisms and quasicoherent sheaves instead. Secondly, we have to deal with *non-unary* generalized graded rings and their projective spectra, together with various arising phenomena that have no counterparts in the classical case. Thirdly, we have to consider "closed" immersions in the sense of **6.5.23**, which are not really closed as maps of topological spaces. This also complicates things considerably.

6.6.1. (Graded algebras and modules in a tensor category.) Let Δ be a commutative monoid (e.g. $\mathbb{N}_0$ or $\mathbb{Z}$), $\mathscr{A}$ be an ACU $\otimes$-category, such that direct sums (i.e. coproducts) indexed by subsets of Δ exist in $\mathscr{A}$ and commute with $\otimes$. Suppose that $M \to M \oplus N$ is a monomorphism for any M, $N \in \operatorname{Ob}\mathscr{A}$ (condition automatically fulfilled if $\mathscr{A}$ has a zero object). Then we can define a *Δ-graded algebra S in $\mathscr{A}$* as an algebra $S = (S, \mu, \varepsilon)$ in $\mathscr{A}$ together with a direct sum decomposition $S = \bigoplus_{\alpha\in\Delta} S^\alpha$, such that $\varepsilon : \mathbf{1}_{\mathscr{A}} \to S$ factorizes through $S^0 \subset S$, and $S^\alpha \otimes S^\beta \to S \otimes S \xrightarrow{\mu} S$ factorizes through $S^{\alpha+\beta} \subset S$, for any α, $\beta \in \Delta$. Next, we can define a *graded (left) S-module M in $\mathscr{A}$* as an S-module $M = (M, \gamma)$ with a direct sum decomposition $M = \bigoplus_{\alpha\in\Delta} M^\alpha$, such that $S^\alpha \otimes M^\beta \to S \otimes M \xrightarrow{\gamma} M$ factorizes through $M^{\alpha+\beta} \subset M$ for all α and $\beta \in \Delta$.

6.6.2. (Alternative descriptions.) The above situation admits an alternative description. Namely, we can define a Δ-graded algebra S in $\mathscr{A}$ as a family $\{S^\alpha\}_{\alpha\in\Delta}$ of objects of $\mathscr{A}$ together with identity map $\varepsilon : \mathbf{1}_{\mathscr{A}} \to S^0$ and multiplication maps $\mu_{\alpha,\beta} : S^\alpha \otimes S^\beta \to S^{\alpha+\beta}$, satisfying natural associativity and unit relations. A graded S-module M can be described similarly, as a family $\{M^\beta\}_{\beta\in\Delta}$ together with action maps $\gamma_{\alpha,\beta} : S^\alpha \otimes M^\beta \to M^{\alpha+\beta}$, satisfying obvious relations.

This approach has its advantages. It yields definitions equivalent to those of **6.6.1** when Δ-indexed direct sums exist in $\mathscr{A}$ and commute with $\otimes$, and all $M \to M \oplus N$ are monic. However, these new definitions don't need any of these assumptions and are valid in any ACU $\otimes$-category.

Yet another description: if Δ-indexed direct sums exist in $\mathscr{A}$ and commute with $\otimes$, we can consider the category $\mathscr{A}^\Delta$ of Δ-indexed families of objects of $\mathscr{A}$, and introduce a $\otimes$-structure on it by putting $\{X^\alpha\} \otimes \{Y^\beta\} := \{\bigoplus_{\alpha+\beta=\gamma} X^\alpha \otimes Y^\beta\}_{\gamma\in\Delta}$. Then Δ-graded algebras and modules are nothing else than algebras and modules in $\mathscr{A}^\Delta$.

6.6.3. (Application: Δ-graded unary algebras and modules over a generalized ring.) Let us fix a generalized base ring C. Since C-*Mod* is an ACU $\otimes$-category with arbitrary direct sums commuting with the tensor product, we can apply any of the above constructions to define Δ-graded C-algebras and modules over them. Notice that the condition "$M \to M \oplus N$ injective" is fulfilled for all C-modules M and N whenever C admits a zero, so **6.6.1** is applicable at least over such C's. On the other hand, the alternative approach of **6.6.2** is applicable without any restrictions on C, and has another convenient property. Namely, let $C_0 \subset C$ be a generalized subring; we would like to be able to treat any graded C-algebra $S = \bigoplus S^\alpha$ as a graded C_0-

algebra via scalar restriction. However, if C is not unary over C_0, direct sums over C and C_0 differ, so the scalar restriction of S wouldn't be equal to the direct sum of scalar restrictions of its components S^α.

Nevertheless, if we adopt the "family approach" of **6.6.2**, this problem doesn't arise: family $\{S^\alpha\}_{\alpha\in\Delta}$ together with $1 \in S^0$ and C-bilinear maps $\mu_{\alpha,\beta} : S^\alpha \times S^\beta \to S^{\alpha+\beta}$ retain all their properties after scalar restriction, so we can work with Δ-graded algebras and modules without worrying too much about the choice of C.

6.6.4. (Non-unary Δ-graded C-algebras.) The above description is somewhat incomplete. Namely, according to **5.3.8**, algebras in C-*Mod* correspond to *unary* C-algebras, so we've just described only unary graded C-algebras. If we manage to describe non-unary graded C-algebras, then we can indeed replace C by $\mathbb{F}_\varnothing$ and forget about it altogether, thus studying Δ-graded generalized rings and graded modules over them.

6.6.5. (Graded algebraic monads and generalized rings: non-unary case.) A Δ-graded algebraic monad or generalized ring S can be defined as follows. We must have a collection of sets $\{S^\alpha(n)\}_{\alpha\in\Delta,n\geq 0}$, together with transition maps $S^\alpha(\varphi) : S^\alpha(n) \to S^\alpha(m)$ for each map $\varphi : \mathbf{n} \to \mathbf{m}$, such that $S^\alpha : \mathbf{n} \mapsto S^\alpha(n)$ becomes a functor $\underline{\mathbb{N}} \to \mathit{Sets}$, which can be uniquely extended to an algebraic endofunctor S^α on *Sets*, if we wish to. Informally, $S^\alpha(n)$ is just the degree α component $S(n)^\alpha$ of free S-module $S(n)$ of rank n.

Next, we must have an identity $\boldsymbol{e} \in S^0(1)$, or, equivalently, a functorial morphism $\varepsilon : \mathrm{Id}_{\mathit{Sets}} \to S^0$, and composition maps $\mu^{(k,\alpha)}_{n,\beta} : S^\alpha(k) \times S^\beta(n)^k \to S^{\alpha+\beta}(n)$, satisfying certain compatibility relations, which are just "graded versions" of those of **4.3.3**. If we want to study Δ-graded generalized rings, not just algebraic monads, we must impose "graded commutativity relations" on μ's, involving commutative diagrams similar to (5.2.1.1).

After this we can define a graded S-module M as a collection of sets $\{M^\beta\}_{\beta\in\Delta}$ together with action maps $\gamma^{(k,\alpha)}_\beta : S^\alpha(k) \times (M^\beta)^k \to M^{\alpha+\beta}$. In this way we obtain a reasonable category of graded S-modules, together with a monadic forgetful functor $\Gamma : S\text{-}\mathit{GrMod}_\Delta \to \mathit{Sets}_{/\Delta}$ into the category of Δ-graded sets. This yields an algebraic monad over $\mathit{Sets}_{/\Delta}$ in the sense of **4.5.20**, i.e. Δ-graded algebraic monads just defined are special cases of those considered in *loc.cit.*

6.6.6. (Underlying set of a graded R-module.) Notice that the underlying set $|M| = M$ of a graded R-module M is just the Δ-graded set $|M| := \bigsqcup_{\alpha\in\Delta} M^\alpha$. In other words, any element $x \in M$ is automatically homogeneous.

6.6.7. (Trivially graded generalized rings.) Any generalized ring C can be treated as a trivially Δ-graded generalized ring $\bar{C}$ (usually denoted also by C), given by $\bar{C}^0(n) := C(n)$, $\bar{C}^\alpha(n) := \varnothing$ for $\alpha \neq 0$, i.e. $\bar{C}^0 := C$, $\bar{C}^\alpha := \varnothing$ for $\alpha \neq 0$ in terms of corresponding algebraic endofunctors. Conversely, R^0 has a natural generalized ring structure for any Δ-graded generalized ring R, and one easily checks that these two functors $C \mapsto \bar{C}$ and $R \mapsto R^0$ are adjoint. In particular, a C-algebra structure on a Δ-graded generalized ring R is the same thing as a homomorphism $C \to R^0$, i.e. a C-algebra structure on R^0.

6.6.8. (Degree shift of graded modules.) Let M be a graded module over Δ-graded generalized ring S, and $\lambda \in \Delta$ be an arbitrary element. We denote by $M[\lambda]$, or sometimes by $M(\lambda)$, when no confusion can arise, the corresponding *degree shift* of M, given by $M[\lambda]^\alpha := M^{\lambda+\alpha}$, $\gamma^{(k,\alpha)}_{M[\lambda],\beta} := \gamma^{(k,\alpha)}_{M,\lambda+\beta}$. These operations have all their usual properties, e.g. $(M[\lambda])[\mu] = M[\lambda+\mu]$ and $M[0] = M$.

6.6.9. (Free graded modules.) We denote by $L_R(n)$ or simply by $R(n)$ the graded R-module defined by collection $\{R^\alpha(n)\}_{\alpha\in\Delta}$, with the R-action given by appropriate maps $\mu^{(k,\alpha)}_{n,\beta}$. We can extend this definition to all sets I by putting $L_R(I) := \varinjlim_{\mathbf{n}\xrightarrow{\varphi} I} L_R(n)$. Another description: $L_R : \mathit{Sets} \to R\text{-}\mathit{GrMod}_\Delta$ is left adjoint functor to $M \mapsto M^0$. Furthermore, we can always construct a left adjoint to $\Gamma_R : R\text{-}\mathit{GrMod}_\Delta \to \mathit{Sets}_{/\Delta}$; when Δ is a group, it can be defined by transforming a family of sets $\{I_\alpha\}_{\alpha\in\Delta}$ into $\bigoplus_\alpha L_R(I_\alpha)[-\alpha]$ (of course, arbitrary inductive and projective limits of graded R-modules exist). It makes sense to call such graded R-modules also *(graded) free*.

6.6.10. (Properties of graded rings and modules.) One can transfer to graded case almost all our general considerations done before in the non-graded case. For example, we can define graded algebras given by generators and relations (with the degree of each generator explicitly mentioned) by their universal property, and show that they exist, and use this result to prove existence of coproducts in the category of Δ-graded generalized rings, i.e. the "graded tensor product" $R' \otimes_R R''$ of two graded algebras R' and R'' over another graded generalized ring R.

Furthermore, the categories of graded modules over graded generalized rings also retain most of their nice properties, e.g. existence of inductive and projective limits, scalar restriction and scalar extension functors and so on. We can also define a *graded R-bilinear map* $\varphi : M \times N \to P$, where M, N, P are graded R-modules, by requiring all $s_\varphi(x) : N \to P[\deg x]$, $y \mapsto \varphi(x,y)$ to be graded R-homomorphisms for all $x \in M$, and similarly for $d_\varphi(y) : M \to P[\deg y]$, for any $y \in N$ (here $M[\deg x]$ denotes the degree shift of M by $\deg x$, cf. **6.6.8**). After this we can define and construct graded tensor products $M \otimes_R N$ and inner Homs $\mathbf{Hom}_R(M,N)$; one can check that $\mathbf{Hom}_R(M,N)^\alpha = \mathrm{Hom}_R(M, N[\alpha])$, i.e. *graded component of* $\mathbf{Hom}_R(M,N)$

of degree α consists of homorphisms $M \to N$ of degree α.

6.6.11. (Monoid of unary operations.) Let R be a Δ-graded generalized ring. Maps $\mu_{1,\beta}^{(1,\alpha)} : R^\alpha(1) \times R^\beta(1) \to R^{\alpha+\beta}(1)$ define on $|R| = |R(1)| := \bigcup_\alpha R^\alpha(1)$ a (commutative) monoid structure with identity $\boldsymbol{e} \in R^0(1)$, and the degree map $\deg : |R| \to \Delta$, equal by definition to α on $R^\alpha(1)$, is obviously a monoid homomorphism.

6.6.12. (Graded multiplicative systems. Localization.) A *(graded) multiplicative system* $S \subset |R|$ is simply a submonoid of $|R|$ with respect to monoid structure just discussed. Given any subset $S \subset |R|$, such that $\deg S \subset \Delta$ consists of invertible elements of Δ, we can consider the multiplicative system (i.e. monoid) $\langle S \rangle \subset |R|$, generated by S. Next, we can consider graded R-algebras $R \xrightarrow{\rho} R'$ with the property that all elements of $\rho(S) \subset |R'|$ become invertible in $|R'|$, and define the *localization* $R[S^{-1}]$ as the universal (initial) R-algebra with this property. Localizations always exist and can be defined by putting $R[S^{-1}] := R[s^{-1} | ss^{-1} = \boldsymbol{e}]_{s \in S}$, where $\{s^{-1}\}_{s \in S}$ are new unary generators with $\deg s^{-1} := -\deg s$.

Next, we can consider the full subcategory of R-$GrMod_\Delta$, consisting of those graded R-modules M, on which all $s \in S$ act bijectively, i.e. the map $[s]_M : M \to M$ (or equivalently, each of the maps $[s]_M^\alpha : M^\alpha \to M^{\alpha+\deg s}$) is bijective. One checks in the usual manner that this category is equivalent to $R[S^{-1}]$-$GrMod$ by means of the scalar restriction functor, hence the scalar extension $M[S^{-1}]$ of a graded R-module M to $R[S^{-1}]$ can be characterized by its universal property among all R-homomorphisms from M into R-modules N, on which S acts bijectively.

6.6.13. (Classical description of localizations.) One can always replace S by $\langle S \rangle$ in the above considerations, thus assuming S to be a (graded) multiplicative system. In this case we also write $S^{-1}R$ and $S^{-1}M$ instead of $R[S^{-1}]$ and $M[S^{-1}]$. Then we can describe these objects in the classical way (cf. **6.1.7**). For example, $S^{-1}M = M \times S/\sim$, where $(x, s) \sim (y, t)$ iff there is some $u \in S$, such that $utx = usy$. If we denote the class of (x, s) by x/s, then the grading on $S^{-1}M$ is given by $\deg(x/s) := \deg x - \deg s$ (recall that $\deg S \subset \Delta$ was supposed to consist of invertible elements of Δ). Since $(S^{-1}R)(n) = S^{-1}(R(n))$, $S^{-1}R(n)$ admits a similar description as $R(n) \times S/\sim$, with equivalence relation $\sim$ and grading given by the same formulas.

6.6.14. (Flatness of localizations.) Furthermore, graded localizations admit a filtered inductive limit description similar to that of **6.1.4**. Namely, let $S \subset |R|$ be a multiplicative system, such that $\deg S \subset \Delta$ consists of invertible elements, and let M be a graded R-module. Consider small category $\mathcal{S}$ of **6.1.4**, with $\mathrm{Ob}\,\mathcal{S} = S$, and morphisms given by $\mathrm{Hom}_{\mathcal{S}}([s], [s']) := \{t \in S :$

$ts = s'\}$. Define a functor $\bar{M} : \mathcal{S} \to R\text{-}\mathit{GrMod}$ by putting $\bar{M}_{[s]} := M[\deg s]$, with the transition morphisms $t : M_{[s]} \to M_{[s']}$ given by the action of $t \in S \subset |R|$ on M. Then $S^{-1}M = \varinjlim_{\mathcal{S}} M_{[s]}$, and since filtered inductive limits are left exact, $M \mapsto S^{-1}M$ is exact, i.e. $S^{-1}R$ is flat over R.

6.6.15. (Degree zero part of a localization.) For any $f \in |R|$ with $\deg f$ invertible in Δ we denote by $R_{(f)}$ the degree zero component $(R_f)^0$ of localization $R_f = R[f^{-1}]$, and similarly $M_{(f)} := (M_f)^0$ for any graded R-module M. Clearly $M_{(f)}$ is a $R_{(f)}$-module; it consists of fractions x/f^n, where $x \in M$, $n \geq 0$ and $\deg x = n \deg f$. We have $x/f^n = y/f^m$ iff $f^{m+N}x = f^{n+N}y$ for some $N \geq 0$. Components $R_{(f)}(n) = R(n)_{(f)}$ of generalized ring $R_{(f)}$ admit a similar description. Notice that $M_{(f)}$ depends only on components M^α with $\alpha = n \deg f$ for some $n \geq 0$, and similarly for $R_{(f)}$.

6.6.16. (Localization of $R_{(f)}$.) Fix any element $\bar{g} = g/f^n \in |R_{(f)}| = R_{(f)}(1)$, where necessarily $g \in |R|$ is of degree $n \deg f$. One sees immediately that $(R_{(f)})_{\bar{g}} = R_{(f)}[\bar{g}^{-1}]$ can be identified with degree zero part of $(R_f)_{\bar{g}} = R[f^{-1}, g^{-1}] = R_{fg}$, i.e. with $R_{(fg)}$. More generally, if f, $g \in |R|$ are such that $n \deg f = m \deg g$ for some integers n, $m \geq 0$, then $R_{(fg)}$ is canonically isomorphic both to $R_{(f)}[(g^m/f^n)^{-1}]$ and $R_{(g)}[(f^n/g^m)^{-1}]$.

6.6.17. (Cases $\Delta = \mathbb{N}_0$ and $\Delta = \mathbb{Z}$.) Henceforth all graded rings and modules we consider will be either $\mathbb{Z}$-graded or $\mathbb{N}_0$-graded. Furthermore, any $\mathbb{N}_0$-graded ring R will be treated as a *positively* $\mathbb{Z}$-graded algebra by putting $R_d(n) := \varnothing$ for $d < 0$. Any $\mathbb{N}_0$-graded module M over such a ring R can be also treated as a positively $\mathbb{Z}$-graded module by putting $M_d := \varnothing$ for $d < 0$. Notice that now we use lower indices to denote graded components, just to avoid confusion with direct power M^d (product of several copies of M).

6.6.18. (Projective spectrum of a positively graded generalized ring.) Let R be a positively graded generalized ring, considered here as a $\mathbb{Z}$-graded generalized ring. Consider monoid $|R| := \bigsqcup_{n\in\mathbb{Z}} R_n(1) = \bigsqcup_{n\geq 0} R_n(1)$, and its submonoid $|R|^+$ consisting of elements of strictly positive degree. For any $f \in |R|^+$ we denote affine generalized scheme $\operatorname{Spec} R_{(f)}$ by $D_+(f)$. If $g \in |R|^+$ is another such element, then $D_+(f)$ contains principal open subset $D_+(f)_g := \operatorname{Spec}(R_{(f)})[(g^{\deg f}/f^{\deg g})^{-1}]$, canonically isomorphic to $D_+(fg) = \operatorname{Spec} R_{(fg)}$ by **6.6.16**, and similarly for $D_+(g)$. Gluing together generalized ringed spaces $\{D_+(f)\}_{f\in|R|^+}$ along their isomorphic open subsets $D_+(f)_g \cong D_+(fg) \cong D_+(g)_f$ in the usual fashion, we obtain a generalized ringed space $\operatorname{Proj} R$, which admits an open cover by subspaces isomorphic to $D_+(f)$ (so they will be identified with $D_+(f) = \operatorname{Spec} R_{(f)}$), in such a way that $D_+(f) \cap D_+(g) = D_+(fg)$ inside $\operatorname{Proj} R$, with induced morphisms $D_+(fg) \to D_+(f)$ equal to those considered above.

Since $\{D_+(f)\}_{f\in|R|^+}$ constitute an affine open cover, $\operatorname{Proj} R$ is a generalized scheme (for the localization theory chosen). It will be called the *projective spectrum* of R.

6.6.19. (Quasicoherent sheaf defined by a graded module.) Let M be a graded module over a positively graded generalized ring R. We can construct a quasicoherent sheaf $\tilde{M}$ on $P := \operatorname{Proj} R$ as follows. Consider quasicoherent sheaves $\widetilde{M_{(f)}}$ on $D_+(f)$; canonical isomorphisms $(M_{(f)})_{g^{\deg f}/f^{\deg g}} \cong M_{(fg)}$ induce canonical isomorphisms $\widetilde{M_{(f)}}|_{D_+(f)\cap D_+(g)} \cong \widetilde{M_{(fg)}} \cong \widetilde{M_{(g)}}|_{D_+(f)\cap D_+(g)}$, so these quasicoherent sheaves can be patched together into a quasicoherent $\mathscr{O}_P$-module denoted by $\tilde{M}$.

This construction is obviously functorial, so we get a functor R-*GrMod* $\to$ $\mathscr{O}_{\operatorname{Proj} R}$-*QCoh*. This functor is left exact. It even commutes with arbitrary inductive limits if $\operatorname{Proj} R = \bigcup_{f\in R_1} D_+(f)$, individual functors $M \mapsto M_{(f)}$ having this property whenever $\deg f = 1$. Since the localization functor R-*GrMod* $\to R_f$-*GrMod*, $M \mapsto M_f$, is exact and commutes with arbitrary inductive limits, the only complicated point is to show that left exact functor R_f-*GrMod* $\to R_{(f)}$-*Mod*, $N \mapsto N_0$, commutes with inductive limits.

One checks for this that the graded homomorphism $R_{(f)}[T^{\pm 1}] \to R_f$ of $R_{(f)}$-algebras, induced by $T \mapsto f$, where T is a unary generator of degree one, is in fact an isomorphism, i.e. R_f is a *unary* graded $R_{(f)}$-algebra, hence the inductive limits in R_f-*GrMod* can be computed degreewise in $R_{(f)}$-*Mod*, hence $N \mapsto N_0$ commutes with arbitrary inductive limits.

Another proof: if $\operatorname{Proj} R = \bigcup_{f\in R_1} D_+(f)$, we can prove that Δ commutes with arbitrary inductive limits by constructing explicitly a right adjoint functor Γ_*, cf. **6.6.26**.

6.6.20. (Covers of $\operatorname{Proj} R$.) Let (f_α) be any family of elements of $|R|^+$. Denote by $\mathfrak{a} \subset |R|$ the graded ideal (i.e. R-submodule of $|R| = R(1)$) generated by f_α. Notice that $\mathfrak{a}$ is contained in $|R|^+$ since $|R|^+$ is itself a graded ideal in $|R|$, R being supposed to be positively graded. We claim that $D_+(g) \subset \bigcup_\alpha D_+(f_\alpha)$ *iff* $g^n \in \mathfrak{a}$ *for some* $n > 0$, *i.e. iff* g *belongs to* $\mathfrak{r}(\mathfrak{a})$, *the radical of* $\mathfrak{a}$. Since $\operatorname{Proj} R$ is by construction the union of all such $D_+(g)$, we conclude that $\bigcup_\alpha D_+(f_\alpha) = \operatorname{Proj} R$ *iff* $\mathfrak{r}(\mathfrak{a}) = |R|^+$.

Let us show our statement. Since $D_+(g) \cap D_+(f_\alpha)$ is identified with $D(\bar{f}_\alpha) \subset D_+(g) = \operatorname{Spec} R_{(g)}$, where $\bar{f}_\alpha = f_\alpha^{\deg g}/g^{\deg f_\alpha}$, we see that $D_+(g) \subset \bigcup_\alpha D_+(f_\alpha)$ iff $D(\bar{f}_\alpha)$ constitute a cover of $D_+(g) = \operatorname{Spec} R_{(g)}$. According to **6.3.17** and **6.2.8**, this is equivalent to saying that the ideal in $R_{(g)}$ generated by the $\bar{f}_\alpha$ contains the identity of $R_{(g)}$. (Notice that such a family constitutes a cover or not independently on the choice of localization theory, since this issue can be reduced to whether this is a family of efficient descent for **QCOH** or not, a condition independent on the choice of theory.) One checks, more or less in the classical fashion, that this ideal equals $\mathfrak{b}_{(g)}$, where $\mathfrak{b}$ denotes the graded ideal in $|R|$ generated by the $f_\alpha^{\deg g}$ (the key point here is that $\Delta : M \mapsto \tilde{M}$ always preserves strict epimorphisms, even when it is not known to be right exact); therefore, $1 \in \mathfrak{b}_{(g)}$ iff $g^n \in \mathfrak{b}$ for some $n > 0$ by construction of $\mathfrak{b}_{(g)}$, i.e. iff $g \in \mathfrak{r}(\mathfrak{b}) = \mathfrak{r}(\mathfrak{a})$, q.e.d.

6.6.21. (Radicals of graded ideals. Graded prime ideals.) The above reasoning used the following fact: *the radical of a graded ideal in $|R|$ is itself a graded ideal.* Indeed, we knew that $\mathfrak{b} \subset \mathfrak{a}$, so $\mathfrak{r}(\mathfrak{b}) \subset \mathfrak{r}(\mathfrak{a})$; on the other hand, all generators f_α of $\mathfrak{a}$ belong to $\mathfrak{r}(\mathfrak{b})$, so $\mathfrak{a} \subset \mathfrak{r}(\mathfrak{b})$ and $\mathfrak{r}(\mathfrak{a}) \subset \mathfrak{r}(\mathfrak{b})$. In order to conclude $\mathfrak{a} \subset \mathfrak{r}(\mathfrak{b})$ we need to know that $\mathfrak{r}(\mathfrak{b})$ is a graded ideal. This can be shown in several different ways. For example, we can introduce the notion of a *graded prime ideal* $\mathfrak{p} \subset |R|$, i.e. a graded ideal, such that $|R| - \mathfrak{p}$ is a submonoid of $|R|$, and show that $\mathfrak{r}(\mathfrak{b})$ coincides with the intersection of all graded prime ideals containing $\mathfrak{b}$, more or less in the same way as in **6.2.10**.

Furthermore, if we work in the unary localization theory $\mathcal{T}^u$, we can describe $\operatorname{Proj} R$ as the set of all graded prime ideals $\mathfrak{p} \not\supset |R|^+$, and $D_+(f)$ is identified then with the set of all $\mathfrak{p} \not\ni f$, exactly as in the classical theory of projective spectra of EGA II 2.

6.6.22. (Projective spaces.) Now we are in position to define the *(unary) projective space* $\mathbb{P}^n_C$ over a generalized ring C. By definition, this is nothing else than $\operatorname{Proj} C[T_0^{[1]}, T_1^{[1]}, \ldots, T_n^{[1]}]$, the projective spectrum of the unary polynomial algebra over C generated by $n+1$ indeterminates of degree one. Our previous results show that $D_+(T_i)$, $0 \le i \le n$, constitute an affine cover of $\mathbb{P}^n_C$, and that each of $D_+(T_i)$ is isomorphic to $\mathbb{A}^n_C = \operatorname{Spec}[T_1^{[1]}, \ldots, T_n^{[1]}]$, the unary affine space over C. For example, $\mathbb{P}^1_C$ is constructed by gluing together two copies of the affine line in the classical fashion. In particular, $\mathbb{P}^1_{\mathbb{F}_1}$ consists of three points (at least if we use unary localization theory): two closed points 0 and ∞, corresponding to graded prime ideals (T_0) and (T_1) in $\mathbb{F}_1[T_0, T_1]$, and one generic point ξ, corresponding to the zero ideal (0).

6.6.23. (Serre twists.) Suppose that $\operatorname{Proj} R$ is quasicompact, so that it can be covered by some $D_+(f_1)$, $\ldots$, $D_+(f_n)$ with $\deg f_i > 0$. We may even assume that all $\deg f_i$ are equal to some $N > 0$, by replacing f_i with appropriate powers, and then that all $\deg f_i = 1$, replacing R by $R^{[N]}$, given by $R_d^{[N]}(n) := R_{Nd}(n)$ (this is possible at least if R_0 admits a zero), similarly to the constructions of EGA II, 2.4.7.

One easily checks that $M_{(f)} \cong M[n]_{(f)}$ for any graded R-module M and any $n \in \mathbb{Z}$ whenever $\deg f = 1$. Therefore, if we assume that $D_+(f)$ with $\deg f = 1$ cover $\operatorname{Proj} R$, we see that $\widetilde{M[n]}$ is locally isomorphic to $\tilde{M}$.

Applying this to $M = |R| = R(1)$, we obtain *Serre line bundles* $\mathscr{O}[n] =$

$\mathscr{O}_{\operatorname{Proj} R}[n] := \widetilde{R(1)[n]}$, for all $n \in \mathbb{Z}$. They are indeed line bundles, since they are locally isomorphic to $\mathscr{O}$ by construction. They have their usual properties $\mathscr{O}[0] = \mathscr{O}$, $\mathscr{O}[n] \otimes_{\mathscr{O}} \mathscr{O}[m] \cong \mathscr{O}[n+m]$ and $\widetilde{M[n]} \cong \tilde{M} \otimes_{\mathscr{O}} \mathscr{O}[n]$. We define *Serre twists* $\mathscr{F}[n] := \mathscr{F} \otimes_{\mathscr{O}} \mathscr{O}[n]$ for any $\mathscr{O}_{\operatorname{Proj} R}$-module $\mathscr{F}$.

When no confusion can arise, we use classical notation $\mathscr{O}(n)$ and $\mathscr{F}(n)$ instead of $\mathscr{O}[n]$ and $\mathscr{F}[n]$. Notice, however, that in our context $\mathscr{O}(n)$ sometimes means the sheaf of n-ary operations of $\mathscr{O}$, i.e. the free $\mathscr{O}$-module $L_{\mathscr{O}}(n)$ of rank n.

6.6.24. (Functor Γ_*.) Now suppose that we are given a generalized scheme X and a line bundle $\mathscr{L}$ on X, i.e. a (necessarily quasicoherent) $\mathscr{O}_X$-module, locally isomorphic to $|\mathscr{O}_X|$. Denote by $\mathscr{L}^{\otimes n}$, $n \in \mathbb{Z}$, the tensor powers of $\mathscr{L}$. We can define a positively graded generalized ring $R = \Gamma_*(\mathscr{O}_X)$ by putting

$$R_d(n) := \Gamma(X, L_{\mathscr{O}_X}(n) \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes d}), \quad n, d \geq 0, \tag{6.6.24.1}$$

where $L_{\mathscr{O}_X}(n)$ denotes the free $\mathscr{O}_X$-module of rank n, i.e. the trivial vector bundle of rank n. We define the "multiplication maps" $R_d(k) \times R_e(n)^k \to R_{d+e}(n)$ by "twisting" the ($\mathscr{O}_X$-bilinear) multiplication maps $\mu^{(k)}_{\mathscr{O}_X,n} : \mathscr{O}_X(k) \times \mathscr{O}_X(n)^k \to \mathscr{O}_X(n)$ for $\mathscr{O}_X$ by appropriate powers of $\mathscr{L}$ (notice that tensoring with a line bundle is an exact functor) and taking global sections. (Commutativity of $\mathscr{O}_X$ is paramount here: we need to know that the multiplication maps we consider here are $\mathscr{O}_X$-bilinear maps of $\mathscr{O}_X$-modules.) All relations implied in our definition of a graded generalized ring are easily seen to hold; actually **6.6.5** was originally motivated exactly by this situation.

Next, for any $\mathscr{O}_X$-module $\mathscr{F}$ we can define a graded $\Gamma_*(\mathscr{O}_X)$-module $M := \Gamma_*(\mathscr{F})$ by putting

$$M_d := \Gamma(X, \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes d}) \tag{6.6.24.2}$$

Here we might use all $d \in \mathbb{Z}$, or restrict our attention only to $d \geq 0$, whatever we like most.

In this way we obtain a functor $\Gamma_* : \mathscr{O}_X\text{-}\mathit{QCoh} \to R\text{-}\mathit{GrMod}$, obviously left exact.

6.6.25. (Application to $X = \operatorname{Proj} R$, $\mathscr{L} = \mathscr{O}_X[1]$.) Now let R be a positively graded generalized ring, such that $X := \operatorname{Proj} R$ is covered by $D_+(f)$ with $f \in R_1(1)$. Then $\mathscr{L} := \mathscr{O}_X[1]$ is a line bundle by **6.6.23**, so we can apply the above constructions to this case. Notice that for any graded R-module M we obtain a natural map of $\mathbb{Z}$-graded sets $\xi_M : M \to \Gamma_*(\tilde{M})$. Its component of degree d, $\xi_{M,d} : M_d \to \Gamma_d(\tilde{M}) = \Gamma(X, \tilde{M} \otimes \mathscr{L}^{\otimes d}) = \Gamma(X, \widetilde{M[d]})$ maps an element $x \in M_d$ into global section $\xi(x) \in \Gamma(X, \widetilde{M[d]})$, such that $\xi(x)|_{D_+(f)} = x/1$ for any $f \in |R|^+$.

Applying this to free modules $M = R(n)$, we get maps of graded sets $R(n) \to (\Gamma_*(\mathscr{O}_X))(n)$, easily seen to combine together into a graded generalized ring homomorphism $R \to \Gamma_*(\mathscr{O}_X)$. Now if $\mathscr{F}$ is any $\mathscr{O}_X$-module, $\Gamma_*(\mathscr{F})$ can be considered as an R-module by means of scalar restriction via $R \to \Gamma_*(\mathscr{O}_X)$. The map of graded sets $\xi_M : M \to \Gamma_*(\tilde{M})$ is easily seen to be a homomorphism of graded R-modules, functorial in M, i.e. $\xi : \mathrm{Id}_{R\text{-}GrMod} \to \Gamma_* \circ \Delta$ is a natural transformation, where $\Delta : R\text{-}GrMod \to \mathscr{O}_X\text{-}QCoh$ denotes the functor $M \mapsto \tilde{M}$.

6.6.26. (Adjointness of Δ and Γ_*.) Let us show that in the above situation functors Δ and Γ_*, considered as functors between R-*GrMod* and $\mathscr{O}_X$-*QCoh* (or even $\mathscr{O}_X$-*Mod*), are adjoint, and that $\xi_M : M \to \Gamma_*(\tilde{M})$ is one of adjointness morphisms. We show this by constructing the other adjointness morphism $\eta_{\mathscr{F}} : \widetilde{\Gamma_*(\mathscr{F})} \to \mathscr{F}$, functorial in $\mathscr{F}$. It would suffice to define $\eta_{\mathscr{F}}|_{D_+(f)} : \widetilde{\Gamma_*(\mathscr{F})}|_{D_+(f)} \to \mathscr{F}|_{D_+(f)}$ for all $f \in |R|^+$. Since the source of this map is the quasicoherent sheaf on affine scheme $D_+(f) = \operatorname{Spec} R_{(f)}$ given by module $\Gamma_*(\mathscr{F})_{(f)}$, we just have to define $R_{(f)}$-homomorphism $\eta'_{\mathscr{F},f} : \Gamma_*(\mathscr{F})_{(f)} \to \Gamma(D_+(f), \mathscr{F})$. We do this by mapping an element x/f^n, where $x \in \Gamma_{n \deg f}(\mathscr{F}) = \Gamma(X, \mathscr{F}[n \deg f])$, into $x|_{D_+(f)} \otimes f^{-n}$, or equivalently, into the preimage of $x|_{D_+(f)}$ under the isomorphism $\mathrm{id}_{\mathscr{F}|_{D_+(f)}} \otimes f^n$, obtained by tensoring $\mathscr{F}$ with $f^n : \mathscr{O}_X|_{D_+(f)} \xrightarrow{\sim} \mathscr{O}_X[n \deg f]|_{D_+(f)}$.

Once η is constructed, the verification of required compatibility between ξ and η is done exactly in the same way as in EGA II 2.6.5, since related proofs of *loc.cit.* are "multiplicative", i.e. don't use addition at all.

The following lemma is a counterpart of EGA I 9.3.2:

Lemma 6.6.27 *Let X be quasicompact quasiseparated generalized scheme, $\mathscr{L}$ be a line bundle over X, $f \in \Gamma(X, \mathscr{L})$ be a global section, $X_f \subset X$ be the largest open subscheme of X, such that f is invertible on X_f. Let $\mathscr{F}$ be a quasicoherent sheaf on X. Then:*

(a) If $a, b \in \Gamma(X, \mathscr{F})$ are two global sections of $\mathscr{F}$, such that $a|_{X_f} = b|_{X_f}$, then $af^n = bf^n \in \Gamma(X, \mathscr{F} \otimes \mathscr{L}^{\otimes n})$ for some integer $n \geq 0$.

(b) For any section $a' \in \Gamma(X_f, \mathscr{F})$ one can find an integer $n \geq 0$ and a global section $a \in \Gamma(X, \mathscr{F})$, such that $a' f^n = a|_{X_f}$.

Proof. Proof goes essentially in the same way as in *loc.cit.* Since X is quasicompact and $\mathscr{L}$ locally trivial, we can find a finite affine open cover $U_i = \operatorname{Spec} A_i$ of X, $1 \leq i \leq n$, such that $\mathscr{L}|_{U_i} \cong \mathscr{O}_X|_{U_i}$. Let $f_i \in A_i = \Gamma(U_i, \mathscr{O}_X)$ be the image of $f|_{U_i}$ under this isomorphism; then $X_f \cap U_i = U_{i,f} = \operatorname{Spec} A_i[f_i^{-1}] = D(f_i) \subset \operatorname{Spec} A_i$.

(a) Since the cover U_i is finite, it is enough to show existence of $n \geq 0$, such that $af^n|_{U_i} = bf^n|_{U_i}$, separately for each i, i.e. we can replace X by U_i and assume $X = \operatorname{Spec} A$ affine, $\mathscr{L} = \mathscr{O}_X$ trivial, $f \in \Gamma(X, \mathscr{O}_X) = A$, and $a, b \in M := \Gamma(X, \mathscr{F})$ such that $a|_{D(f)} = b|_{D(f)}$, i.e. $a/1 = b/1$ in M_f. This implies existence of $n \geq 0$, such that $af^n = bf^n$, by definition of localizations.

(b) Reasoning as in (a), we can find an integer $n \geq 0$, same for all i if we want, such that $a'f^n|_{U_i \cap X_f}$ extends to a section a_i of $\mathscr{F}$ over U_i. However, $a_i|_{U_i \cap U_j}$ needn't coincide with $a_j|_{U_i \cap U_j}$; all we know is that their restrictions to $U_i \cap U_j \cap X_f$ are equal, so applying (a) to quasicompact generalized scheme $U_i \cap U_j$ we obtain $a_i f^m|_{U_i \cap U_j} = a_j f^m|_{U_i \cap U_j}$, and we can choose an $m \geq 0$ valid for all $1 \leq i, j \leq n$. Then $a_i f^m \in \Gamma(U_i, \mathscr{F})$ coincide on intersections of these open sets, so they can be glued to a global section $a \in \Gamma(X, \mathscr{F})$, and obviously $a|_{X_f} = a'f^{n+m}$, q.e.d.

Once we have the above lemma, we can prove the following:

Proposition 6.6.28 *Let R be a positively graded generalized ring, such that $P := \operatorname{Spec} R$ is union of finitely many $D_+(f)$ with $\deg f = 1$. Then $\eta_{\mathscr{F}} : \widetilde{\Gamma_*(\mathscr{F})} \to \mathscr{F}$ is an isomorphism for any quasicoherent $\mathscr{O}_P$-module $\mathscr{F}$, functor $\Gamma_* : \mathscr{O}_P$-QCoh $\to$ R-GrMod is fully faithful, and its left adjoint $\Delta : R$-GrMod $\to \mathscr{O}_P$-QCoh is a localization functor (in particular it is essentially surjective), i.e. $\mathscr{O}_P$-QCoh is equivalent to the localization of R-GrMod with respect to set of all homomorphisms in R-GrMod, which become isomorphisms after application of Δ, and this localization can be computed by means of left fraction calculus.*

Proof. The statement about $\eta_{\mathscr{F}}$ is deduced from **6.6.27** in the same way as in EGA II 2.7.5, and the remaining statements follow for general (category-theoretic) reasons, cf. [GZ].

6.6.29. (Δ preserves finite type and presentation.) Suppose that a positively graded ring R is such that $P := \operatorname{Proj} R = \bigcup_{f \in R_1} D_+(f)$, e.g. ideal $|R|^+$ is generated by R_1. Then $\widetilde{|R|[n]} = \mathscr{O}_P[n]$ is a locally free and in particular finitely presented $\mathscr{O}_P$-module. Therefore, if M is a (graded) free R-module of finite rank k, i.e. if $M = \bigoplus_{i=1}^k |R|[n_i]$, then $\tilde{M}$ is a vector bundle of rank k. Since $\Delta : M \mapsto \tilde{M}$ is right exact, we see that $\tilde{M}$ is a quasicoherent $\mathscr{O}_P$-module of finite type (resp. finite presentation) whenever M is a graded R-module of finite type (resp. finite presentation).

6.6.30. (Partial converse.) Let's suppose in addition that $P = \operatorname{Proj} R$ be quasicompact, i.e. we are under the conditions of **6.6.28**. Let $\mathscr{F}$ be any quasicoherent $\mathscr{O}_P$-module of finite type. According to **6.6.28**, we can identify

$\mathscr{F}$ with $\tilde{M}$ for a suitable graded R-module M, e.g. $M = \Gamma_*(\mathscr{F})$. Next, we can write $M = \varinjlim_\alpha M_\alpha$, filtered inductive limit of its graded submodules of finite type. Then $\mathscr{F} = \tilde{M} = \varinjlim_\alpha \tilde{M}_\alpha$, and using quasicompactness of P we find an index α, such that $\tilde{M}_\alpha = \tilde{M} = \mathscr{F}$ (cf. **6.5.10** for a similar reasoning), i.e. *any quasicoherent $\mathscr{O}_P$-module $\mathscr{F}$ of finite type is isomorphic to $\tilde{N}$ for some graded R-module N of finite type.* Reasoning further as in **6.5.11**, we obtain a similar description of finitely presented $\mathscr{O}_P$-modules: *any finitely presented $\mathscr{O}_P$-module $\mathscr{F}$ is isomorphic to $\tilde{N}$ for some finitely presented graded R-module N.*

Notice that we don't claim here $\Gamma_*(\mathscr{F})$ to be a finitely generated or presented graded R-module.

6.6.31. (Finitely presented $\mathscr{O}_P$-modules as a fraction category.) Let's keep the previous assumptions. Let $\mathcal{C}$ be the category of finitely presented graded R-modules, $\mathcal{D}$ be the category of finitely presented $\mathscr{O}_P$-modules, denote by $S \subset \operatorname{Ar}\mathcal{C}$ the set of morphisms in $\mathcal{C}$ that become isomorphisms after an application of $\Delta : \mathcal{C} \to \mathcal{D}$. By universal property of localizations of categories we obtain a functor $\bar{\Delta} : \mathcal{C}[S^{-1}] \to \mathcal{D}$, essentially surjective by **6.6.30**. We claim the following:

Proposition. *Subset $S \subset \operatorname{Ar}\mathcal{C}$ admits a left fraction calculus, so that morphisms $f' : X \to Y$ in $\mathcal{C}[S^{-1}]$ can be described as $s^{-1}f$ for some $f : X \to Z$ and $s : Y \to Z$, $s \in S$ (cf.* [GZ, I.2.3]*), and $\bar{\Delta} : \mathcal{C}[S^{-1}] \to \mathcal{D}$ is an equivalence of categories.*

Proof. (a) Conditions a)–d) of [GZ, I.2.2] immediately follow from the fact that $\mathcal{C}$ is stable under finite inductive limits of *R-GrMod*, and that $\Delta : \mathcal{C} \to \mathcal{D}$ is right exact, i.e. commutes with such limits. Therefore, S admits left fraction calculus.

(b) Functor $\bar{\Delta}$ is faithful, i.e. injective on morphisms. Indeed, suppose that $s^{-1}f$ and $t^{-1}g \in \operatorname{Hom}_{\mathcal{C}[S^{-1}]}(X, Y)$ are such that $\tilde{s}^{-1}\tilde{f} = \tilde{t}^{-1}\tilde{g}$ in $\mathcal{D}$. Constructing a pushout diagram on s and t and using obvious stability of S under pushouts, we get a diagram

$$\begin{tikzcd}
X \arrow[dr, "f"'] \arrow[drrr, "g"] & & Y \arrow[dl, "s"'] \arrow[dr, "t"] & \\
& Y' \arrow[dr, "t'"] & & Y'' \arrow[dl, "s'"'] \\
& & Z \arrow[r, dashed, "u"] & W
\end{tikzcd} \tag{6.6.31.1}$$

Put $h := t'f$, $h' := s'g$, $W := \mathrm{Coker}(h, h' : X \to Z)$. Clearly $\tilde{h} = \tilde{h}'$, hence $Z \xrightarrow{u} W$ belongs to S, Δ being exact, W is finitely presented, being a finite inductive limit of finitely presented graded R-modules, and obviously $ut'f = uh = uh' = us'g$ and $ut's = us't$; we conclude that $s^{-1}f = t^{-1}g$ in $\mathcal{C}[S^{-1}]$.

(c) Functor $\bar{\Delta}$ is surjective on morphisms, hence fully faithful by (b). Indeed, let $\varphi : \tilde{M} \to \tilde{N}$ be any $\mathscr{O}_P$-morphism, for any $M, N \in \mathrm{Ob}\,\mathcal{C}$. Put $N' := \Gamma_*(\tilde{N})$. By adjointness of Δ and Γ_* we get graded homomorphisms $f' = \varphi^\flat : M \to N'$ and $s' = \mathrm{id}_{\tilde{N}}^\flat : N \to N'$, such that $\tilde{s}'$ is an isomorphism by **6.6.28**, and $\varphi = (\tilde{s}')^{-1} \circ \tilde{f}'$. Now write N' as a filtered inductive limit of finitely presented modules N_α and use that $\mathrm{Hom}(M, -)$ and $\mathrm{Hom}(N, -)$ commute with filtered inductive limits, M and N being finitely presented. The conclusion is that both f' and s' factorize through some finitely presented N_α, and by taking an even larger α we can assume $i_\alpha : N_\alpha \to N'$ to belong to S. Then $s : N \to N_\alpha$ will also belong to S, since both i_α and $s' = i_\alpha s$ belong to S, and considering $f : M \to N_\alpha$, such that $f' = i_\alpha f$, we get $\varphi = \tilde{s}^{-1}\tilde{f} = \bar{\Delta}(s^{-1}f)$.

(d) Finally, $\bar{\Delta}$ is essentially surjective by **6.6.30**, hence an equivalence of categories, q.e.d.

6.6.32. (Almost isomorphisms of graded modules.) We say that a homomorphism (of degree zero) $f : M \to N$ of graded R-modules is *almost isomorphism* if $f_n : M_n \to N_n$ is a bijection for $n \gg 0$. Clearly, any almost isomorphism f induces an isomorphism $\tilde{f} : \tilde{M} \to \tilde{N}$ between corresponding quasicoherent sheaves. The converse seems to hold only for finitely presented M and N, and only under some additional restrictions, e.g. when R is generated as a graded R_0-algebra by a finite subset of R_1, and in particular if R is a finitely presented graded unary R_0-algebra. This includes the projective spaces $\mathbb{P}^n_{R_0} = \mathrm{Proj}\, R_0[T_0^{[1]}, \ldots, T_n^{[1]}]$. However, we don't want to provide more details here.

6.6.33. (Functoriality.) Any homomorphism of graded generalized rings $\varphi : R \to R'$ induces an affine morphism of generalized schemes $\mathrm{Proj}(\varphi) := \Phi : G(\varphi) \to \mathrm{Proj}\, R$, where $G(\varphi) \subset \mathrm{Proj}\, R'$ is the open subscheme of $\mathrm{Proj}\, R'$ equal to $\bigcup_{f \in |R|^+} D_+(\varphi(f))$, essentially in the same way as in EGA II 2.8, i.e. $\Phi^{-1}(D_+(f)) = D_+(\varphi(f))$, and the restriction of Φ to $D_+(\varphi(f))$ is induced by canonical homomorphism $R_{(f)} \to R'_{(\varphi(f))}$. We don't need to provide more details since the reasoning of *loc.cit.* is completely "multiplicative", i.e. doesn't use addition in any essential way.

6.6.34. (Base change.) Let $C \to C'$ be an extension of generalized rings, R be a positively graded C-algebra. Consider $R' := R \otimes_C C'$ (understood for example as a pushout in the category of graded generalized rings, where C and C' are considered with their trivial grading). Graded homomorphism $\varphi : R \to R'$ induces a morphism $\Phi : G(\varphi) \to \operatorname{Proj} R$, where $G(\varphi) = \bigcup_{f \in |R|^+} D_+(\varphi(f)) \subset \operatorname{Proj} R'$. We obtain a commutative diagram

$$\begin{tikzcd} G(\varphi) \arrow[r, "\Phi"] \arrow[d] & \operatorname{Proj} R \arrow[d] \\ \operatorname{Spec} C' \arrow[r] & \operatorname{Spec} C \end{tikzcd} \tag{6.6.34.1}$$

One checks, essentially in the same way as in EGA II, 2.8.9, that the above diagram is cartesian (one just checks $R'_{(f)} = R_{(f)} \otimes_C C'$ for this), and that in fact $G(\varphi) = \operatorname{Proj} R'$, at least if R is a unary graded C-algebra or if C' is flat over C (this follows from the fact that in any of these cases the graded R'-ideal $|R'|^+$ is generated by the image of $|R|^+$ in $|R'|$, a consequence of the graded version of the "affine base change" theorem **5.4.2**).

Therefore, construction of $\operatorname{Proj} R$ always commutes with *flat* base change, and if R is a unary graded C-algebra, then it commutes with arbitrary base change.

6.6.35. (Quasicoherent sheaves of graded algebras.) Now let S be any generalized scheme. One can define a *quasicoherent sheaf of graded $\mathscr{O}_S$-algebras* $\mathscr{R}$ by combining together **6.6.5** and **6.5.12** in the natural way. Any such $\mathscr{R}$ has the property that $\mathscr{R}|_U = \tilde{R}$ for any open affine $U = \operatorname{Spec} C \subset S$, where R is some graded C-algebra. Reasoning further as in EGA II, 3.1, we construct a generalized S-scheme $\operatorname{Proj} \mathscr{R}$ by gluing together $\operatorname{Proj} \Gamma(U, \mathscr{R})$ for all open affine $U \subset S$. Graded quasicoherent sheaves of $\mathscr{R}$-modules on S define quasicoherent sheaves on $\operatorname{Proj} \mathscr{R}$ by the same reasoning as in *loc.cit.*, and further properties like the relative version of **6.6.28** hold as well, at least if we assume S to be quasicompact and quasiseparated.

6.6.36. (Projective bundles.) An important application is given by *projective bundles.* Let S be a generalized scheme, $\mathscr{E}$ be a vector bundle over S (i.e. a locally free $\mathscr{O}_S$-module of finite type), or even any quasicoherent $\mathscr{O}_S$-module. Consider the symmetric algebra $\mathscr{R} := S_{\mathscr{O}_S}(\mathscr{E}) = \bigoplus_{n \geq 0} S^n_{\mathscr{O}_S}(\mathscr{E})$. It is a $\mathbb{Z}$-graded algebra in the $\otimes$-category of quasicoherent $\mathscr{O}_S$-modules, hence it can be considered as a quasicoherent sheaf of unary graded $\mathscr{O}_S$-algebras. Putting $\mathbb{P}(\mathscr{E}) := \operatorname{Proj}_{\mathscr{O}_S} \mathscr{R}$, we construct the *projective bundle over S corresponding to vector bundle $\mathscr{E}$.* If $\mathscr{E}$ is a vector bundle of rank $n+1$, then $\mathbb{P}(\mathscr{E})$ is locally (over S) isomorphic to $\mathbb{P}^n_S/S$.

Some of basic properties of projective bundles known from EGA II generalize to our case. For example, canonical homomorphism $\mathscr{E} \otimes_{\mathscr{O}_S} S_{\mathscr{O}_S}(\mathscr{E}) \to$

$S_{\mathscr{O}_S}(\mathscr{E})[1]$ induces a surjection $\pi : p^*\mathscr{E} \twoheadrightarrow \mathscr{O}[1]$ on $\mathbb{P}(\mathscr{E})$, where $p : \mathbb{P}(\mathscr{E}) \to S$ denotes the structural morphism.

6.6.37. (Sections of projective bundles.) Notice that sections $\Gamma_S(\mathbb{P}_S(\mathscr{E})/S)$ are in one-to-one correspondence with strict quotients $\mathscr{E} \twoheadrightarrow \mathscr{L}$, which are line bundles over S. Namely, given an S-section $\sigma : S \to \mathbb{P}_S(\mathscr{E})$, we obtain such a strict epimorphism $\varphi := \sigma^*(\pi) : \mathscr{E} = \sigma^* p^* \mathscr{E} \twoheadrightarrow \sigma^*\mathscr{O}[1] =: \mathscr{L}$ by applying σ^* to $\pi : p^*\mathscr{E} \twoheadrightarrow \mathscr{O}[1]$. Conversely, given $\varphi : \mathscr{E} \twoheadrightarrow \mathscr{L}$, we construct a section $\sigma := \operatorname{Proj} S_{\mathscr{O}_S}(\varphi) : S \cong \mathbb{P}_S(\mathscr{L}) \to \mathbb{P}_S(\mathscr{E})$. This morphism is defined on the whole of $\mathbb{P}_S(\mathscr{L}) = S$ since $S_{\mathscr{O}_S}(\varphi) : S_{\mathscr{O}_S}(\mathscr{E}) \to S_{\mathscr{O}_S}(\mathscr{L})$ is surjective. One checks that these two maps $\sigma \leftrightarrow \varphi$ are indeed inverse to each other, essentially as in EGA II 4.2.3. Noticing that the construction of $\mathbb{P}_S(\mathscr{E})$ commutes with any base change $S' \xrightarrow{f} S$, we see that the set of S'-valued points of S-scheme $\mathbb{P}_S(\mathscr{E})$ can be identified with the set of those strict quotients $\mathscr{L}$ of $\varphi^*\mathscr{E}$, which are line bundles on S', exactly as in the classical case.

Another remark: in general σ will be a "closed" immersion in the sense of **6.5.23**, for example because of surjectivity of $S_{\mathscr{O}_S}(\varphi)$, or because of $\mathbb{P}_S(\mathscr{E})$ being "separated" over S.

6.6.38. (Example: points of $\mathbb{P}^n$ over some generalized rings.) This is applicable in particular to $\mathbb{F}_\varnothing$-scheme $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{F}_\varnothing} := \operatorname{Proj} \mathbb{F}_\varnothing[T_0^{[1]}, \ldots, T_n^{[1]}] = \mathbb{P}(\mathbb{F}_\varnothing(n+1))$. We see that R-valued points of $\mathbb{P}^n$ correspond to strict quotients P of free R-module $L_R(n+1) = R(n+1)$, which are projective "of rank 1" (i.e. $\tilde{P}$ has to be a line bundle over $\operatorname{Spec} R$), for any generalized ring R. When $\operatorname{Pic}(R) = \operatorname{Pic}(\operatorname{Spec} R)$ is trivial (e.g. when R is $\mathcal{T}^?$-local), P has to be isomorphic to $|R|$, i.e. we have to consider the set of all surjective R-homomorphisms $\varphi : L_R(n+1) \twoheadrightarrow L_R(1) = |R|$, modulo multiplication by invertible elements of $\operatorname{Aut}_R(|R|) = |R|^\times$. By the universal property of free modules φ is completely determined by the set $(X_0, \ldots, X_n) \in |R|^n$ of its values $X_i := \varphi(\{i+1\})$ on the basis elements of $L_R(n+1)$, and the image of φ is exactly the ideal $\langle X_0, \ldots, X_n \rangle \subset |R|$. Therefore, *if* $\operatorname{Pic}(\operatorname{Spec} R) = 0$*, e.g. if R is $\mathcal{T}^?$-local, then* $\mathbb{P}^n(R)$ *consists of* $(n+1)$*-tuples* $(X_0, \ldots, X_n) \in |R|^{n+1}$ *of elements of* $|R|$*, generating the unit ideal in* $|R|$*, modulo multiplication by invertible elements from* $|R|^\times$ (cf. EGA II, 4.2.6). Of course, the class of $(X_0, \ldots, X_n)$ modulo this equivalence relation will be denoted by $(X_0 : \ldots : X_n)$.

For example, $\mathbb{P}^n(\mathbb{F}_1)$ consists of $2^{n+1} - 1$ points $(X_0 : X_1 : \ldots : X_n)$ with not all $X_i \in |\mathbb{F}_1| = \{0, 1\}$ equal to zero. In particular, $\mathbb{P}^1_{\mathbb{F}_1}/\operatorname{Spec}\mathbb{F}_1$ has three sections $\mathbb{P}^1(\mathbb{F}_1) = \{(0:1), (1:0), (1:1)\}$, each supported at a different point of $\mathbb{P}^1_{\mathbb{F}_1} = \operatorname{Proj} \mathbb{F}_1[X, Y] = \{0, \infty, \xi\} = \{(X), (Y), (0)\}$ (for the unary localization theory; cf. **6.6.22**). Notice that one of these sections is supported at the generic point ξ! Of course, this section is a "closed" immersion in the sense of **6.5.23**, but not closed as a map of topological spaces.

6.6.39. (Projective morphisms and ample line bundles.) One can transfer more results from EGA II to our case. For example, one can start from a line bundle $\mathscr{L}$ over a generalized scheme X (better supposed to be quasi-compact and quasicoherent), construct graded generalized ring $\Gamma_*(\mathscr{O}_X)$ by the procedure of **6.6.24**, and a generalized scheme morphism $r_{\mathscr{L}} : G(\mathscr{L}) \to \operatorname{Proj} \Gamma_*(\mathscr{O}_X)$, defined on an open subscheme $G(\mathscr{L}) \subset X$ essentially in the same way as in EGA II, 3.7. It is natural to say that $\mathscr{L}$ is *ample* if $G(\mathscr{L}) = X$ and $r_{\mathscr{L}}$ is an immersion in the sense of **6.5.21**.

We say that X is *projective* if it admits an ample $\mathscr{L}$, such that $r_{\mathscr{L}}$ is an isomorphism of generalized schemes, and that X is *quasi-projective* if it admits an ample $\mathscr{L}$, such that $r_{\mathscr{L}}$ is an open embedding. We also need to assume X to be finitely presented over some affine base $S = \operatorname{Spec} C$ (e.g. $\operatorname{Spec} \mathbb{F}_\varnothing$) in these definitions.

These definitions have relative versions, at least over a quasicompact quasiseparated base S. One can transfer some of the classical results to our situation, e.g. $\mathscr{O}[1]$ is S-ample on $\mathbb{P}_S(\mathscr{E})$.

6.6.40. (Projective schemes vs. "closed" subschemes of $\mathbb{P}^N$.) Notice that if $X = \operatorname{Proj} R$ is projective (say, over some $S = \operatorname{Spec} C$) in the above sense, we still cannot expect X to embed into $\mathbb{P}^N_C$, or even into an infinite-dimensional projective bundle $\mathbb{P}_C(\mathscr{E})$, where $\mathscr{E} := |R|_1$, simply because we cannot expect the canonical graded C-algebra homomorphism $S_C(|R|_1) \to R$ to be surjective, even after replacing R by some $R^{(d)}$ (i.e. ample line bundle $\mathscr{L}$ by some power $\mathscr{L}^{\otimes d}$), since R can happen not to be a *pre-unary* graded C-algebra.

We'll distinguish these cases by saying that X is *pre-unary* or *unary projective* generalized scheme over $S = \operatorname{Spec} C$, if $X = \operatorname{Proj} R$ for some pre-unary (resp. unary) graded C-algebra R. For example, $\mathbb{P}^N_S$ is unary projective over S, and any "closed" subscheme of $\mathbb{P}^N_S$ is pre-unary projective over S.

6.6.41. (Further properties.) Some more classical properties of projective schemes and morphisms can be transferred to our case. However, we don't seem to have a reasonable notion of properness, and cohomological properties can be partially transferred only with the aid of homotopic algebra of Chapters **8** and **9**, so there are remain not so many classical properties not discussed yet.

7 Arakelov geometry

Now we want to discuss possible applications of generalized schemes and rings to Arakelov geometry, thus fulfilling some of the promises given in Chapters **1** and **2**. Our first aim is to construct $\widehat{\operatorname{Spec}\mathbb{Z}}$, the "compactification" of $\operatorname{Spec}\mathbb{Z}$. After that we are going discuss models $\mathscr{X}/\widehat{\operatorname{Spec}\mathbb{Z}}$ of algebraic varieties $X/\mathbb{Q}$.

We also present some simple applications, such as the classical formula for the height of a rational point P on a projective variety $X/\mathbb{Q}$ as the arithmetic degree $\deg\sigma_P^*\mathscr{O}_{\mathscr{X}}(1)$ of the pullback of the Serre line bundle $\mathscr{O}_{\mathscr{X}}(1)$ on a projective model $\mathscr{X}/\widehat{\operatorname{Spec}\mathbb{Z}}$ of X with respect to the section $\sigma_P : \widehat{\operatorname{Spec}\mathbb{Z}} \to \mathscr{X}$ inducing given point P on the generic fiber.

7.1. (Construction of $\widehat{\operatorname{Spec}\mathbb{Z}}$.)

7.1.1. (Notation: generalized rings A_N and B_N.) In order to construct $\widehat{\operatorname{Spec}\mathbb{Z}}$ we'll need two generalized subrings $A_N \subset B_N \subset \mathbb{Q}$, defined for any integer $N > 1$. Namely, we put $B_N := \mathbb{Z}[1/N] = \mathbb{Z}[N^{-1}]$ (so B_N is actually a classical ring), and $A_N := B_N \cap \mathbb{Z}_{(\infty)}$. Therefore,

$$A_N(n) = \bigl\{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Q}^n \,:\, \lambda_i \in B_N, \sum_i |\lambda_i| \leq 1\bigr\}, \quad \text{i.e.} \tag{7.1.1.1}$$

$$A_N(n) = \Bigl\{\Bigl(\frac{u_1}{N^k}, \ldots, \frac{u_n}{N^k}\Bigr) \,:\, k \geq 0, u_i \in \mathbb{Z}, \sum_i |u_i| \leq N^k\Bigr\} \tag{7.1.1.2}$$

In particular, $|A_N| = A_N(1) = \{\lambda \in B_N \,:\, |\lambda| \leq 1\}$, e.g. $1/N \in |A_N|$.

Another useful notation: we write $M \mid N^\infty$ if M divides N^k for some $k \geq 0$, i.e. the prime decomposition of M consists only of prime divisors of N. Of course, we write $M \mid N$ when M is a divisor of N, and $M \nmid N$ otherwise.

7.1.2. (B_N is a localization of A_N.) Image of $f := 1/N \in |A_N|$ under canonical embedding $A_N \to B_N$ is obviously invertible in $|B_N|$, hence we get a generalized ring homomorphism $A_N[f^{-1}] = A_N[(1/N)^{-1}] \to B_N$.

Claim. *Map $A_N[(1/N)^{-1}] \to B_N$ is an isomorphism, i.e. $A_N[(1/N)^{-1}] = B_N$ inside $\mathbb{Q}$.*

Proof. Since $A_N \to B_N$ is a monomorphism, i.e. all $A_N(n) \to B_N(n)$ are injective, the same is true for induced map of localizations $A_N[f^{-1}] \to B_N[f^{-1}] = B_N$ (this is a general property of localizations). Since the composite map $A_N \to A_N[f^{-1}] \to B_N$ is also a monomorphism, the same holds for the first arrow $A_N \to A_N[f^{-1}]$ as well, so we can identify all generalized rings involved with their images in $\mathbb{Q}$: $A_N \subset A_N[f^{-1}] \subset B_N \subset \mathbb{Q}$. We have to show that $A_N[f^{-1}] = B_N$, i.e. that any $\lambda = (\lambda_1, \ldots, \lambda_n) \in B_N(n) = (B_N)^n$

belongs to $A_N[f^{-1}]$. This is clear: choose large integer $k \geq 0$, such that $\sum_i |\lambda_i| \leq N^k$ (we use $N > 1$ here). Then $\mu := f^k\lambda = N^{-k}\lambda$ belongs to $A_N(n)$, hence $\lambda = \mu/f^k$ belongs to $A_N[f^{-1}](n)$, q.e.d. (cf. **6.1.23** for a similar proof).

Remark. Notice that the same reasoning shows that $A_N[(1/d)^{-1}] = B_N$ for any divisor $d > 1$ of N.

7.1.3. (Construction of $\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$.) Since $B_N = A_N[(1/N)^{-1}] = \mathbb{Z}[N^{-1}]$, we see that both $\operatorname{Spec}\mathbb{Z}$ and $\operatorname{Spec} A_N$ have principal open subsets isomorphic to $\operatorname{Spec} B_N$, regardless of the localization theory $\mathcal{T}^?$ chosen to construct spectra involved. Therefore, we can construct a generalized scheme $\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$ by gluing together $\operatorname{Spec}\mathbb{Z}$ and $\operatorname{Spec} A_N$ along their open subschemes isomorphic to $\operatorname{Spec} B_N$. Obviously $\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$ is quasicompact and quasiseparated. We are going to describe its points, at least for the minimal localization theory $\mathcal{T}^u$ (i.e. if $\operatorname{Spec} = \operatorname{Spec}^u = \operatorname{Spec}^p$ are the prime spectra).

7.1.4. (Prime ideals of A_N.) Let us describe all prime ideals of A_N. First of all, we have ideal $\mathfrak{p}_\infty$, preimage of $\mathfrak{m}_{(\infty)} \subset \mathbb{Z}_{(\infty)}$ under the canonical embedding $A_N \to \mathbb{Z}_{(\infty)}$. Clearly, $\mathfrak{p}_\infty = \{\lambda \in |B_N| : |\lambda| < 1\}$. Its complement $|A_N| - \mathfrak{p}_\infty$ consists of $\{\pm 1\}$, the invertible elements of $|A_N|$, hence A_N is local with maximal ideal $\mathfrak{p}_\infty$. Next, for any prime $p \nmid N$ we get a prime ideal $\mathfrak{p}_p \subset |A_N|$, the preimage of $pB_N \subset B_N$ under canonical map $A_N \to B_N$. Finally, we have the zero ideal $0 = \{0\} \subset |A_N|$.

We claim that this list consists of all prime ideals of A_N, i.e. $\operatorname{Spec}^p A_N = \{0, \mathfrak{p}_p, \ldots, \mathfrak{p}_\infty\}_{p\nmid N}$, where p runs over all primes $p \nmid N$. Indeed, since $B_N = A_N[(1/N)^{-1}]$, the set $D(1/N)$ of prime ideals of A_N not containing $1/N$ is in one-to-one correspondence to $\operatorname{Spec} B_N = \{0, (p), \ldots\}$, for the same values of p. Therefore, all we have to check is that $\mathfrak{p}_\infty$ is the only prime ideal of A_N, which contains $1/N$.

So let $\mathfrak{p} \subset |A_N|$ be any prime ideal, such that $1/N \in \mathfrak{p}$. Clearly, $\mathfrak{p} \subset \mathfrak{p}_\infty$. Conversely, suppose that $\lambda \in \mathfrak{p}_\infty$, i.e. $\lambda \in B_N$ and $|\lambda| < 1$. Then $|\lambda|^k < 1/N$ for a sufficiently large integer $k > 0$. Now put $\mu := N\lambda^k$. Then $\mu \in B_N$ and $|\mu| \leq 1$, hence $\mu \in |A_N|$, hence $\lambda^k = \mu \cdot (1/N) \in |A_N| \cdot \mathfrak{p} = \mathfrak{p}$, $\mathfrak{p}$ being an ideal in $|A_N|$. Since $\mathfrak{p}$ is prime, $\lambda^k \in \mathfrak{p}$ implies $\lambda \in \mathfrak{p}$, so we have proved opposite inclusion $\mathfrak{p}_\infty \subset \mathfrak{p}$, q.e.d.

7.1.5. (Prime spectrum of A_N.) We have just shown that $\operatorname{Spec}^u A_N = \operatorname{Spec}^p A_N = \{\xi, \infty, p, \ldots\}$, where p runs over the set of primes $p \nmid N$. Of course, in this notation ξ corresponds to the zero ideal 0, p corresponds to $\mathfrak{p}_p$, and ∞ to $\mathfrak{p}_\infty$. The topology on this set is described as follows. Point ξ is its generic point, and point ∞ is its only closed point, A_N being local. The complement $\operatorname{Spec}^p A_N - \{\infty\}$ of this closed point is homeomorphic to $\operatorname{Spec} B_N = \operatorname{Spec}\mathbb{Z}[1/N]$, hence each point p is closed in this complement, hence $\overline{\{p\}} = \{p, \infty\}$ in $\operatorname{Spec}^p A_N$. In other words, a non-empty subset $U \subset \operatorname{Spec}^p A_N$ is open iff its complement is either empty or consists of ∞ and of finitely many points p. In this respect $\operatorname{Spec}^p A_N$ looks like the spectrum of a (classical) two-dimensional regular local ring.

7.1.6. (Topology of $\widehat{\operatorname{Spec}^u\mathbb{Z}}^{(N)}$.) Once we have a complete description of topological spaces $\operatorname{Spec}^u\mathbb{Z} = \{\xi, p, \ldots\}_{p\in\mathbb{P}}$, $\operatorname{Spec}^u A_N = \{\xi, p, \ldots, \infty\}_{p\nmid N}$ and $\operatorname{Spec}^u B_N = \{\xi, p, \ldots\}_{p\nmid N}$, we can describe $\widehat{\operatorname{Spec}^u\mathbb{Z}}^{(N)}$ as well. We see that $\widehat{\operatorname{Spec}^u\mathbb{Z}}^{(N)} = \{\xi, p, \ldots, \infty\}_{p\in\mathbb{P}} = \operatorname{Spec}\mathbb{Z} \cup \{\infty\}$, i.e. $\widehat{\operatorname{Spec}^u\mathbb{Z}}^{(N)}$ has an additional point ∞, corresponding to the archimedian valuation of $\mathbb{Q}$. The topology of $\widehat{\operatorname{Spec}^u\mathbb{Z}}^{(N)}$ can be described as follows. Point ξ is generic, so it is contained in any non-empty open subset of $\widehat{\operatorname{Spec}^u\mathbb{Z}}^{(N)}$. Points ∞ and p for $p \mid N$ are closed, and the remaining prime points $p \nmid N$ are not closed since $\overline{\{p\}} = \{p, \infty\}$. A non-empty subset $U \subset \widehat{\operatorname{Spec}^u\mathbb{Z}}^{(N)}$ is open iff it contains ξ, its complement is finite, and if it either doesn't contain ∞ or contains all $p \nmid N$.

7.1.7. (Local rings of $S_N := \widehat{\operatorname{Spec}^u\mathbb{Z}}^{(N)}$.) Let's compute the stalks of $\mathscr{O} = \mathscr{O}_{S_N}$, where $S_N := \widehat{\operatorname{Spec}^u\mathbb{Z}}^{(N)}$. Since points ξ and p lie in open subscheme $\operatorname{Spec}\mathbb{Z} \subset S_N$, we have $\mathscr{O}_{S_N,\xi} = \mathbb{Q}$ and $\mathscr{O}_{S_N,p} = \mathbb{Z}_{(p)}$. On the other hand, $\operatorname{Spec} A_N$ is an open neighborhood of ∞, hence $\mathscr{O}_{S_N,\infty} = A_{N,\mathfrak{p}_\infty} = A_N$, because $\mathfrak{p}_\infty$ is the maximal ideal of local ring A_N. In particular, $\mathscr{O}_{S_N,\infty} \neq \mathbb{Z}_{(\infty)}$, contrary to what one might have expected.

7.1.8. (Morphisms $f = f_N^{NM} : \widehat{\operatorname{Spec}\mathbb{Z}}^{(NM)} \to \widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$.) Let N, $M > 1$ be two integers. Denote by U_1 and U_2 the two open subschemes of $\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$ isomorphic to $\operatorname{Spec}\mathbb{Z}$ and $\operatorname{Spec} A_N$, respectively. Similarly, denote by U_1' and U_2' the open subschemes of $\widehat{\operatorname{Spec}\mathbb{Z}}^{(NM)}$ isomorphic to $\operatorname{Spec}\mathbb{Z}$ and $\operatorname{Spec} A_{NM}$, respectively. By construction $\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)} = U_1 \cup U_2$ and $\widehat{\operatorname{Spec}\mathbb{Z}}^{(NM)} = U_1' \cup U_2'$.

Next, consider principal open subscheme $W = \operatorname{Spec} B_N = \operatorname{Spec}\mathbb{Z}[N^{-1}]$ of $U_1' = \operatorname{Spec}\mathbb{Z}$, and put $V_1 := U_1'$, $V_2 := U_2' \cup W$. We are going to construct generalized scheme morphisms $f_i : V_i \to U_i$, $i := 1, 2$, such that $f_1|_{V_1\cap V_2} = f_2|_{V_1\cap V_2}$, and $f_i^{-1}(U_1 \cap U_2) = V_1 \cap V_2$. Since $\widehat{\operatorname{Spec}\mathbb{Z}}^{(NM)} = V_1 \cup V_2$, these two f_i will define together a generalized scheme morphism $f : \widehat{\operatorname{Spec}\mathbb{Z}}^{(NM)} \to \widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$, such that $f^{-1}(U_i) = V_i$, $f_{U_i} = f_i : V_i \to U_i$, $i = 1, 2$.

(a) Since both $U_1 = \operatorname{Spec}\mathbb{Z}$ and $V_1 = U_1' = \operatorname{Spec}\mathbb{Z}$, we can take $f_1 := \mathrm{id}_{\operatorname{Spec}\mathbb{Z}}$.

(b) Since $U_2 = \operatorname{Spec} A_N$, generalized scheme morphisms $f_2 : V_2 \to U_2 =$

$\operatorname{Spec} A_N$ are in one-to-one correspondence with generalized ring homomorphisms $\varphi : A_N \to \Gamma(V_2, \mathscr{O})$. Since $V_2 = U_2' \cup W$, and $U_2' \cap W = U_2' \cap U_1' \cap W = D(NM) \cap D(N) = D(NM)$, we see that $\Gamma(V_2, \mathscr{O})$ equals the fibered product of $\Gamma(U_2', \mathscr{O}) = A_{NM}$ and $\Gamma(W, \mathscr{O}) = B_N$ over $\Gamma(D(NM), \mathscr{O}_{\operatorname{Spec}\mathbb{Z}}) = B_{NM}$, hence $\Gamma(V_2, \mathscr{O}) = A_{NM} \times_{B_{NM}} B_N = A_{NM} \times_{\mathbb{Q}} B_N = A_{NM} \cap B_N = \mathbb{Z}_{(\infty)} \cap B_{NM} \cap B_N = \mathbb{Z}_{(\infty)} \cap B_N = A_N$, so we can put $\varphi := \mathrm{id}_{A_N} : A_N \xrightarrow{\sim} \Gamma(V_2, \mathscr{O})$.

(c) Notice that $V_1 \cap V_2 = U_1' \cap (U_2' \cup W) = (U_1' \cap U_2') \cup (U_1' \cap W) = D_{U_1'}(NM) \cup W = W$, and $f_1^{-1}(U_1 \cap U_2) = f_1^{-1}(D_{U_1}(N)) = D_{U_1'}(N) = W = V_1 \cap V_2$. Furthermore, the map $(f_1)_{U_1 \cap U_2} : W \to U_1 \cap U_2$ obtained from f_1 by base change $U_1 \cap U_2 \to U_1$ is obviously the identity map of $W = D_{U_1'}(N) = \operatorname{Spec} B_N = U_1 \cap U_2$.

(d) Now compute $f_2^{-1}(U_1 \cap U_2) = f_2^{-1}(D_{U_2}(1/N)) = (V_2)_{1/N} = D_{U_2'}(1/N) \cup D_W(1/N) = \operatorname{Spec} A_{NM}[(1/N)^{-1}] \cup W = \operatorname{Spec} B_{NM} \cup \operatorname{Spec} B_N = \operatorname{Spec} B_N = W = V_1 \cap V_2$. Here we have used **7.1.2**, which claims that $A_{NM}[(1/N)^{-1}] = B_{NM}$.

(e) It remains to check that the map $(f_2)_{U_1 \cap U_2} : W \to U_1 \cap U_2$ equals $(f_1)_{U_1 \cap U_2} = \mathrm{id}_W$. Notice for this that according to the construction of f_2 given in (b), the restriction $f_2|_W : W = \operatorname{Spec} B_N \to U_2 = \operatorname{Spec} A_N$ is induced by canonical embedding $A_N \to B_N = A_N[(1/N)^{-1}]$, hence it is (up to a canonical isomorphism) the open immersion of $W = D_{\operatorname{Spec}\mathbb{Z}}(N)$ into $U_2 = \operatorname{Spec} A_N$. This implies $(f_2)_{U_1 \cap U_2} = \mathrm{id}_W$, thus concluding the proof of existence of $f : \widehat{\operatorname{Spec}\mathbb{Z}}^{(NM)} \to \widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$ with required properties.

(f) Notice that f is quasicompact and quasiseparated, just because both f_1 and f_2 are.

The reasoning used above to construct f is valid regardless of the localization theory $\mathcal{T}^?$ chosen.

7.1.9. (Map f is not an isomorphism.) Clearly, if $M \mid N^\infty$, numbers NM and N have same prime divisors, hence $A_{NM} = A_N$, $B_{NM} = B_N$, $\widehat{\operatorname{Spec}\mathbb{Z}}^{(NM)} = \widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$, and $f = f_M^{NM} : \widehat{\operatorname{Spec}\mathbb{Z}}^{(NM)} \to \widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$ is the identity map.

We claim that *if $M \nmid N^\infty$, then $f = f_N^{NM}$ is not an isomorphism.* If we use unary localization theory $\mathcal{T}^u$, we can just observe that f is a continuous bijective map of underlying topological spaces, but not a homeomorphism since any prime $p \mid M$, $p \nmid N$ defines a point, which is closed in $\widehat{\operatorname{Spec}^u\mathbb{Z}}^{(NM)}$, but not in $\widehat{\operatorname{Spec}^u\mathbb{Z}}^{(N)}$. However, f cannot be an isomorphism even if we choose another localization theory to construct our generalized schemes, for example because $\operatorname{Pic}(\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)})$ is a free abelian group of rank equal to the number of distinct prime divisors of N, regardless of the choice of $\mathcal{T}^?$ (cf. **7.1.35**), and NM has more prime divisors than N.

7.1.10. (Map f is not affine.) Furthermore, *if $M \nmid N^\infty$, then f is not affine.*

Indeed, suppose that $f = f_N^{NM}$ is affine. Then $f_2 = f_{U_2} : V_2 \to U_2 = \operatorname{Spec} A_N$ is affine, hence V_2 must be also affine. But induced map $A_N \to \Gamma(V_2, \mathscr{O}) = A_N$ is the identity map by **7.1.8**,(b), hence f_2 is an isomorphism. Now $f_1 = f_{U_1} : V_1 \to U_1$ is always an isomorphism, so we conclude that f is an isomorphism, which is absurd by **7.1.9**.

7.1.11. (Projective system $\widehat{\operatorname{Spec}\mathbb{Z}}^{(\cdot)}$.) The collection of all $\{\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}\}_{N>1}$, together with transition maps f_M^N, defined by **7.1.8** whenever M divides N (when $N = M$, we put $f_N^N := \mathrm{id}$), clearly constitutes a filtering projective system of generalized schemes over the set of integers $N > 1$ ordered by divisibility relation.

7.1.12. (Definition of $\widehat{\operatorname{Spec}\mathbb{Z}}$.) Now we are tempted to define the "true" compactification of $\operatorname{Spec}\mathbb{Z}$ by putting $\widehat{\operatorname{Spec}\mathbb{Z}} := \varprojlim_{N>1} \widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$. However, we need to discuss the meaning of such a projective limit before doing this. We suggest two possible approaches:

7.1.13. ($\widehat{\operatorname{Spec}\mathbb{Z}}$ as pro-generalized scheme.) We can consider $\widehat{\operatorname{Spec}\mathbb{Z}} :=$ "$\varprojlim$"$_{N>1} \widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$ as a pro-generalized scheme, i.e. a pro-object in the category of generalized schemes (cf. SGA 4 I for a discussion of pro-objects). This means that we put formally $\operatorname{Hom}(T, \widehat{\operatorname{Spec}\mathbb{Z}}) := \varprojlim_{N>1} \operatorname{Hom}(T, S_N)$ and $\operatorname{Hom}(\widehat{\operatorname{Spec}\mathbb{Z}}, T) := \varinjlim_{N>1} \operatorname{Hom}(S_N, T)$, where $S_N := \widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$, and T is any generalized scheme.

Furthermore, we can extend this to any stack $\mathcal{C}$ defined over the category of generalized schemes by putting $\mathcal{C}(\widehat{\operatorname{Spec}\mathbb{Z}}) := \varinjlim\mathrm{Lim}_{N>1} \mathcal{C}(S_N)$. Informally speaking, an object of $\mathcal{C}(\widehat{\operatorname{Spec}\mathbb{Z}})$ is just a couple (N, X), where $N > 1$ and $X \in \operatorname{Ob}\mathcal{C}(S_N)$, and such a couple (N, X) is identified with all its pullbacks $(NM, (f_N^{NM})^* X)$. In other words, we expect all objects of $\mathcal{C}(\widehat{\operatorname{Spec}\mathbb{Z}})$ to come from a finite stage of the projective limit.

However, this seems to be a reasonable approach only for stacks $\mathcal{C}$, which already have similar property with respect to filtered projective limits of affine generalized schemes, i.e. such that $\mathcal{C}(\operatorname{Spec} \varinjlim A_\alpha) \cong \underrightarrow{\mathrm{Lim}}\, \mathcal{C}(\operatorname{Spec} A_\alpha)$. For example, $\mathcal{C} = \mathbf{QCOH}$ doesn't have this property, but stacks of finitely presented objects (e.g. finitely presented sheaves of modules, finitely presented schemes, or finitely presented sheaves over finitely presented schemes) usually have this property, essentially by the classical reasoning of EGA IV 8. Therefore, we can say that a finitely presented $\mathscr{O}_{\widehat{\operatorname{Spec}\mathbb{Z}}}$-module $\mathscr{F}$ is just a couple $(N, \mathscr{F}_N)$, consisting of an integer $N > 1$ and a finitely presented $\mathscr{O}_{S_N}$-module $\mathscr{F}_N$. In such a situation we'll write $\mathscr{F} = f_N^*(\mathscr{F}_N)$, where $f_N : \widehat{\operatorname{Spec}\mathbb{Z}} \to S_N$ is the canonical projection (in the category of pro-generalized schemes). Similarly, a finitely presented scheme $\mathscr{X}$ over $\widehat{\operatorname{Spec}\mathbb{Z}}$, and a finitely presented $\mathscr{O}_{\mathscr{X}}$-module $\mathscr{F}$ can be described as "formal pullbacks" of corresponding objects X_N, $\mathscr{F}_N$, defined over some finite stage $S_N = \widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$ of the projective limit.

7.1.14. ($\widehat{\operatorname{Spec}\mathbb{Z}}$ as generalized ringed space.) Another appoach is to compute $\widehat{\operatorname{Spec}\mathbb{Z}} := \varprojlim_{N>1} S_N$, where $S_N = \widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$, in the category of generalized ringed spaces. This means that we first construct topological space $\widehat{\operatorname{Spec}\mathbb{Z}}$ as the projective limit of corresponding topological spaces S_N (one can also work on the level of corresponding sites or topoi, cf. SGA 4, but topological spaces are sufficient for our purpose), and then define the structural sheaf by $\mathscr{O}_{\widehat{\operatorname{Spec}\mathbb{Z}}} := \varinjlim_{N>1} f_N^{-1}\mathscr{O}_{S_N}$, where $f_N : \widehat{\operatorname{Spec}\mathbb{Z}} \to S_N$ are the natural projection maps, and f_N^{-1} denotes the "set-theoretic" pullback of sheaves of generalized rings.

This approach has its advantages. We obtain a reasonable topological space, and even a generalized ringed space, so we can study for example line bundles, vector bundles or finitely presented sheaves of modules over $\widehat{\operatorname{Spec}\mathbb{Z}}$. However, this generalized ringed space $\widehat{\operatorname{Spec}\mathbb{Z}}$ is not a generalized scheme, so in order to study schemes over $\widehat{\operatorname{Spec}\mathbb{Z}}$ in this approach we would need to develop the notion of a "relative generalized scheme". This is possible, but complicates everything considerably, so we prefer not to do this, and adopt the pro-generalized scheme approach instead, whenever we need to study schemes over $\widehat{\operatorname{Spec}\mathbb{Z}}$.

7.1.15. (Structure of generalized ringed space $\widehat{\operatorname{Spec}^u\mathbb{Z}}$.) Let us discuss the structure of $\widehat{\operatorname{Spec}^u\mathbb{Z}}$, considered as a generalized ringed space. First of all, each $S_N = \widehat{\operatorname{Spec}^u\mathbb{Z}}^{(N)} = \operatorname{Spec}\mathbb{Z} \cup \{\infty\}$ as a set, and all $f_N^{NM} : S_{NM} \to S_N$ induce identity maps on underlying sets, hence the underlying set of $\widehat{\operatorname{Spec}^u\mathbb{Z}}$ is also equal to $\operatorname{Spec}\mathbb{Z} \cup \{\infty\}$. Furthermore, $\widehat{\operatorname{Spec}^u\mathbb{Z}}$ contains $\operatorname{Spec}\mathbb{Z}$ as an open (generalized ringed) subspace, simply because this is true for each $S_N = \widehat{\operatorname{Spec}^u\mathbb{Z}}^{(N)}$. This determines all neighborhoods in $\widehat{\operatorname{Spec}^u\mathbb{Z}}$ of points $\neq \infty$; as to the neighborhoods of ∞, by definition of projective limit of topological spaces, a set $\infty \in U \subset \widehat{\operatorname{Spec}^u\mathbb{Z}}$ is a neighborhood of ∞ iff it is one inside some $\widehat{\operatorname{Spec}^u\mathbb{Z}}^{(N)}$. Applying our previous results of **7.1.6**, we see that *ξ is the generic point of $\widehat{\operatorname{Spec}^u\mathbb{Z}}$, and all other points of $\widehat{\operatorname{Spec}^u\mathbb{Z}}$ are closed. Non-empty open subsets $U \subset \widehat{\operatorname{Spec}^u\mathbb{Z}}$ are exactly the subsets containing ξ and having a finite complement.* In this respect $\widehat{\operatorname{Spec}^u\mathbb{Z}}$ is much more similar to an algebraic curve, where all points except the generic one are closed, than

$\widehat{\operatorname{Spec}^u\mathbb{Z}}^{(N)}$, where all primes $p \nmid N$ are "entangled" with ∞. Furthermore, the stalk of $\mathscr{O}_{\widehat{\operatorname{Spec}^u\mathbb{Z}}}$ at ∞ equals $\varinjlim_{N>1} \mathscr{O}_{\widehat{\operatorname{Spec}^u\mathbb{Z}}^{(N)},\infty} = \varinjlim_{N>1} A_N = \mathbb{Z}_{(\infty)}$, and the complement of ∞ is isomorphic to $\operatorname{Spec}\mathbb{Z}$ as a generalized ringed space.

In this way $\widehat{\operatorname{Spec}^u\mathbb{Z}}$ looks very much like one's idea of the "correct" compactification of $\operatorname{Spec}\mathbb{Z}$, apart from (not) being a generalized scheme.

7.1.16. (Finitely presented sheaves of $\mathscr{O}_{\widehat{\operatorname{Spec}^u\mathbb{Z}}}$-modules.) Let $\mathscr{F}$ be a finitely presented sheaf over $\widehat{\operatorname{Spec}^u\mathbb{Z}}$. Since the complement of ∞ is isomorphic to $\operatorname{Spec}\mathbb{Z}$, and $\mathscr{F}|_{\operatorname{Spec}\mathbb{Z}} = \tilde{M}$ for some finitely generated $\mathbb{Z}$-module M, only the structure of $\mathscr{F}$ near ∞ is interesting. By definition of finite presentation we can find an open neighborhood U of ∞ and a right exact diagram

$$\mathscr{O}_U(m) \overset{u,v}{\rightrightarrows} \mathscr{O}_U(n) \xrightarrow{p} \mathscr{F}|_U \tag{7.1.16.1}$$

Here $m, n \geq 0$, and $u, v \in \operatorname{Hom}_{\mathscr{O}_U}(\mathscr{O}_U(m), \mathscr{O}_U(n)) = \mathscr{O}_U(n)^m$ are two $m \times n$-matrices over $\mathscr{O}_U$. By definition of projective limit of topological spaces we may assume, making U smaller if necessary, that $U = (f_N)^{-1}(U_N)$ for some open neighborhood U_N of ∞ in $S_N := \widehat{\operatorname{Spec}^u\mathbb{Z}}^{(N)}$. Next, $u, v \in \mathscr{O}_U(n)^m = \varinjlim_M f_{MN}^{-1}\mathscr{O}_{(f_N^{MN})^{-1}(U_N)}(n)^m$, so replacing N by a suitable multiple NM, we may assume that u and v come from some u_N, $v_N \in \mathscr{O}_{U_N}(n)^m$. Consider finitely presented $\mathscr{O}_{U_N}$-module $\mathscr{F}'_N := \operatorname{Coker}(u_N, v_N : \mathscr{O}_{U_N}(m) \rightrightarrows \mathscr{O}_{U_N}(n))$. By construction $f_N^*\mathscr{F}'_N \cong \mathscr{F}|_U$, and $\mathscr{F}'_N$ agrees with $\mathscr{F}|_{\operatorname{Spec}\mathbb{Z}}$ on $U_N \cap \operatorname{Spec}\mathbb{Z}$ since the restriction of f_N to $U \cap \operatorname{Spec}\mathbb{Z}$ is an isomorphism from $U \cap \operatorname{Spec}\mathbb{Z}$ to $U_N \cap \operatorname{Spec}\mathbb{Z}$. Therefore, $\mathscr{F}'_N$ and $\mathscr{F}|_{\operatorname{Spec}\mathbb{Z}}$ patch together into a finitely presented $\mathscr{O}_{S_N}$-module $\mathscr{F}_N$, such that $\mathscr{F} = f_N^*(\mathscr{F}_N)$.

We have just shown that *any finitely presented sheaf* $\mathscr{F}$ *over* $\widehat{\operatorname{Spec}\mathbb{Z}}$ *comes from a finitely presented sheaf* $\mathscr{F}_N$ *defined over a finite stage* S_N *of the projective limit* $\widehat{\operatorname{Spec}\mathbb{Z}} = \varprojlim_{N>1} S_N$. Therefore, the category of finitely presented sheaves over $\widehat{\operatorname{Spec}^u\mathbb{Z}}$ coincides with that constructed in **7.1.13** via the pro-object approach.

7.1.17. (Equivalence of the two approaches to $\widehat{\operatorname{Spec}\mathbb{Z}}$.) We see that the pro-generalized scheme approach of **7.1.13** and the generalized ringed space approach of **7.1.14** yield equivalent results when we study *finitely presented* sheaves of modules over $\widehat{\operatorname{Spec}\mathbb{Z}}$. This seems to be equally true for all "categories of finitely presented objects" (sheaves of modules, relative generalized schemes,…), so we can safely adopt any of these approaches (or use both at the same time) whenever we consider only finitely presented objects.

7.1.18. (Quasicoherent sheaves over $\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$.) Consider $\mathbf{QCOH}(S_N)$, the category of quasicoherent $\mathscr{O}_{S_N}$-modules, where $S_N := \widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$ for some localization theory $\mathcal{T}^?$. Since $S_N = U_1 \cup U_2 = \operatorname{Spec}\mathbb{Z} \cup \operatorname{Spec} A_N$ is an affine open cover of S_N with $U_1 \cap U_2 = \operatorname{Spec} B_N$, we see that a quasicoherent $\mathscr{O}_{S_N}$-module $\mathscr{F}$ is completely determined by $\mathbb{Z}$-module $M_\mathbb{Z} := \Gamma(U_1, \mathscr{F})$, A_N-module $M_N := \Gamma(U_2, \mathscr{F})$, and B_N-module isomorphism $\theta_M : M_\mathbb{Z} \otimes_\mathbb{Z} B_N \xrightarrow{\sim} M_N \otimes_{A_N} B_N$, obtained by restricting to $U_1 \cap U_2$ isomorphisms $\widetilde{M_\mathbb{Z}} \xrightarrow{\sim} \mathscr{F}|_{U_1}$ and $\mathscr{F}|_{U_2} \xrightarrow{\sim} \widetilde{M_N}$, composing them, and taking global sections.

Furthermore, we obtain in this way an *equivalence* of $\mathbf{QCOH}(S_N)$ with the category $\bar{\mathcal{C}}_N$ of triples $(M_\mathbb{Z}, M_N, \theta_M)$, consisting of a $\mathbb{Z}$-module M, an A_N-module M_N, and a B_N-module isomorphism $\theta_N : (M_\mathbb{Z})_{(B_N)} = M_\mathbb{Z}[N^{-1}] \xrightarrow{\sim} (M_N)_{(B_N)} = M_N[(1/N)^{-1}]$. In particular, $\mathbf{QCOH}(\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)})$ *doesn't depend on the choice of localization theory* $\mathcal{T}^?$.

Pullback of quasicoherent sheaves with respect to $f_N^{NN'} : S_{NN'} \to S_N$ induces a functor $\bar{\mathcal{C}}_N \to \bar{\mathcal{C}}_{NN'}$, easily seen to transform $(M_\mathbb{Z}, M_N, \theta)$ into $(M_\mathbb{Z}, (M_N)_{(A_{NN'})}, \theta_{(B_{NN'})})$.

7.1.19. (Finitely presented sheaves over $\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$.) The essential image of the full subcategory of $\mathbf{QCOH}(S_N)$ consisting of finitely presented $\mathscr{O}_{S_N}$-modules, where again $S_N = \widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$, under the equivalence of categories $\mathbf{QCOH}(S_N) \to \bar{\mathcal{C}}_N$, is obviously equal to the full subcategory $\mathcal{C}_N$ of $\bar{\mathcal{C}}_N$, consisting of triples $(M_\mathbb{Z}, M_N, \theta_N)$ as above, with $M_\mathbb{Z}$ a finitely presented $\mathbb{Z}$-module, and M_N a finitely presented A_N-module. In particular, this category doesn't depend on the choice of $\mathcal{T}^?$, hence the same is true for the category of finitely presented sheaves of modules over $\widehat{\operatorname{Spec}\mathbb{Z}} = \varprojlim_{N>1} S_N$ (for any understanding of this projective limit), this category being equivalent to $\varinjlim_{N>1} \mathcal{C}_N$.

7.1.20. ("Point at ∞" of $\widehat{\operatorname{Spec}\mathbb{Z}}$.) Notice that for each $N > 1$ we have a canonical embedding $A_N \to \mathbb{Z}_{(\infty)}$, which induces a morphism $\operatorname{Spec}\mathbb{Z}_{(\infty)} \to \operatorname{Spec} A_N = U_2 \subset \widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$. Let us denote by η_N the natural morphism $\operatorname{Spec}\mathbb{Z}_{(\infty)} \to S_N := \widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$. It is easy to see that $\eta_N = f_N^{NM} \circ \eta_{NM}$ for any $M \geq 1$, i.e. $\hat{\eta} = (\eta_N)_{N>1}$ is a morphism from $\operatorname{Spec}\mathbb{Z}_{(\infty)}$ into the projective system $(S_N)_{N>1}$. Therefore, it defines a morphism $\hat{\eta} : \operatorname{Spec}\mathbb{Z}_{(\infty)} \to \widehat{\operatorname{Spec}\mathbb{Z}}$, both in the category of pro-generalized schemes, and in the category of generalized ringed spaces. If we adopt the generalized ringed space approach, and use $\mathcal{T}^? = \mathcal{T}^u$, then $\hat{\eta} : \operatorname{Spec}\mathbb{Z}_{(\infty)} \to \widehat{\operatorname{Spec}^u\mathbb{Z}}$ is just the map with image $\{\xi, \infty\}$, inducing identity on the local ring $\mathscr{O}_{\widehat{\operatorname{Spec}^u\mathbb{Z}},\infty} = \mathbb{Z}_{(\infty)} = \mathscr{O}_{\operatorname{Spec}\mathbb{Z}_{(\infty)},\infty}$.

7.1.21. (Stalk at ∞ of a finitely presented $\mathscr{O}_{\widehat{\operatorname{Spec}\mathbb{Z}}}$-module.) Let $\mathscr{F}$ be a finitely presented $\mathscr{O}_{\widehat{\operatorname{Spec}\mathbb{Z}}}$-module. Then $\hat{\eta}^*(\mathscr{F})$ is a finitely presented quasicoherent $\mathscr{O}_{\operatorname{Spec}\mathbb{Z}_{(\infty)}}$-module, hence $\mathscr{F}_\infty := \Gamma(\operatorname{Spec}\mathbb{Z}_{(\infty)}, \hat{\eta}^*(\mathscr{F}))$ is a finitely presented $\mathbb{Z}_{(\infty)}$-module. We will say that $\mathscr{F}_\infty$ *is the stalk of* $\mathscr{F}$ *at infinity*, regardless of the choice of $\mathcal{T}^?$ and of the understanding of $\widehat{\operatorname{Spec}\mathbb{Z}}$.

In the pro-generalized scheme approach, $\hat{\eta}^*(\mathscr{F})$ actually means $\eta_N^*(\mathscr{F}_N)$, if $\mathscr{F}$ is given by couple $(N, \mathscr{F}_N)$. On the other hand, in the generalized ringed space approach $\mathscr{F}_\infty$ is indeed the stalk of $\mathscr{F}$ at ∞, provided we use $\mathcal{T}^? = \mathcal{T}^u$.

7.1.22. (Category of finitely presented $\mathscr{O}_{\widehat{\operatorname{Spec}\mathbb{Z}}}$-modules.) Let $\mathscr{F}$ be a finitely presented $\mathscr{O}_{\widehat{\operatorname{Spec}\mathbb{Z}}}$-module. It defines a finitely generated $\mathbb{Z}$-module $M_\mathbb{Z} := \Gamma(\operatorname{Spec}\mathbb{Z}, \mathscr{F})$, a finitely presented $\mathbb{Z}_{(\infty)}$-module $M_\infty := \mathscr{F}_\infty = \hat{\eta}^*\mathscr{F}$, together with canonical isomorphism of $\mathbb{Q}$-vector spaces $\theta_M : M_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Q} \xrightarrow{\sim} M_\infty \otimes_{\mathbb{Z}_{(\infty)}} \mathbb{Q}$, constructed for example by computing the pullback of $\mathscr{F}$ with respect to the "generic point" $\hat{\xi} : \operatorname{Spec}\mathbb{Q} \to \widehat{\operatorname{Spec}\mathbb{Z}}$ in two different ways, using factorizations $\hat{\xi} : \operatorname{Spec}\mathbb{Q} \to \operatorname{Spec}\mathbb{Z} \to \widehat{\operatorname{Spec}\mathbb{Z}}$ and $\hat{\xi} : \operatorname{Spec}\mathbb{Q} \to \operatorname{Spec}\mathbb{Z}_{(\infty)} \xrightarrow{\hat{\eta}} \widehat{\operatorname{Spec}\mathbb{Z}}$. Of course, $\mathbb{Q}$-vector space $\mathscr{F}_\xi := \hat{\xi}^*\mathscr{F} \cong \mathscr{F}_\infty \otimes_{\mathbb{Z}_{(\infty)}} \mathbb{Q} \cong \Gamma(\operatorname{Spec}\mathbb{Z}, \mathscr{F}) \otimes_\mathbb{Z} \mathbb{Q}$ is called the *generic fiber of* $\mathscr{F}$.

In this way we obtain a functor from the category of finitely presented $\mathscr{O}_{\widehat{\operatorname{Spec}\mathbb{Z}}}$-modules into the category $\mathcal{C}$ of triples $(M_\mathbb{Z}, M_\infty, \theta)$, consisting of a finitely generated $\mathbb{Z}$-module $M_\mathbb{Z}$, finitely presented $\mathbb{Z}_{(\infty)}$-module M_∞, and a $\mathbb{Q}$-vector space isomorphism $\theta : M_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Q} \xrightarrow{\sim} M_\infty \otimes_{\mathbb{Z}_{(\infty)}} \mathbb{Q}$. It is easy to check that *this functor is in fact an equivalence of categories.* Let us show for example the essential surjectivity of this functor. Since the category of finitely presented $\mathscr{O}_{\widehat{\operatorname{Spec}\mathbb{Z}}}$-modules is equivalent to $\varinjlim_{N>1} \mathcal{C}_N$ (cf. **7.1.19**, **7.1.13** and **7.1.16**), it suffices to show that any triple $M := (M_\mathbb{Z}, M_\infty, \theta)$ as above comes from a triple $(M_\mathbb{Z}, M_N, \theta_N) \in \operatorname{Ob}\mathcal{C}_N$ for some $N > 1$ (i.e. $M_\infty = M_N \otimes_{A_N} \mathbb{Z}_{(\infty)}$, and $\theta = \theta_N \otimes 1_\mathbb{Q}$). Indeed, M_∞ is a finitely presented module over filtering inductive limit $\mathbb{Z}_{(\infty)} = \varinjlim_{N>1} A_N$, hence one can find an $N > 1$ and a finitely presented A_N-module M_N, such that $M_\infty = M_N \otimes_{A_N} \mathbb{Z}_{(\infty)}$ (the reasoning here is essentially the classical one of EGA IV 8, cf. also **7.1.16**). Next, θ is an isomorphism between $M_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Q} =: M_\mathbb{Q} = (M_\mathbb{Z} \otimes_\mathbb{Z} B_N) \otimes_{B_N} \mathbb{Q}$ and $M_\infty \otimes_{\mathbb{Z}_{(\infty)}} \mathbb{Q} = (M_N \otimes_{A_N} B_N) \otimes_{B_N} \mathbb{Q}$. Since $\mathbb{Q} = \varinjlim_{N' \geq 1} B_{NN'}$, this isomorphism between finitely presented $\mathbb{Q}$-modules comes from an isomorphism $\theta_{NN'} : (M_\mathbb{Z} \otimes_\mathbb{Z} B_N) \otimes_{B_N} B_{NN'} \xrightarrow{\sim} (M_N \otimes_{A_N} B_N) \otimes_{B_N} B_{NN'}$ for some $N' \geq 1$. Then $(M_\mathbb{Z}, M_N \otimes_{A_N} A_{NN'}, \theta_{NN'}) \in \operatorname{Ob}\mathcal{C}_{NN'}$ is a triple from $\operatorname{Ob}\mathcal{C}_{NN'}$ inducing original triple $M \in \operatorname{Ob}\mathcal{C}$ after base change, q.e.d.

7.1.23. (Finitely presented schemes over $\widehat{\operatorname{Spec}\mathbb{Z}}$.) This argument can be extended to other categories of (relative) finitely presented objects. For example, the category of (relative) finitely presented schemes $\bar{\mathscr{X}}/\widehat{\operatorname{Spec}\mathbb{Z}}$ is equivalent to the category of triples $(\mathscr{X}, \mathscr{X}_\infty, \theta)$, where $\mathscr{X}$ is a finitely presented scheme over $\operatorname{Spec}\mathbb{Z}$, $\mathscr{X}_\infty$ is a finitely presented scheme over $\operatorname{Spec}\mathbb{Z}_{(\infty)}$, and $\theta : \mathscr{X}_{(\mathbb{Q})} \xrightarrow{\sim} \mathscr{X}_{\infty,(\mathbb{Q})}$ is an isomorphism of $\mathbb{Q}$-schemes. Indeed, any such triple comes from a triple $(\mathscr{X}, \mathscr{X}_N, \theta_N)$, defined over a finite stage S_N of the projective limit $\widehat{\operatorname{Spec}\mathbb{Z}} = \varprojlim S_N$, and then $\mathscr{X} \to \operatorname{Spec}\mathbb{Z} = U_1 \subset S_N$ and $\mathscr{X}_N \to \operatorname{Spec} A_N = U_2 \subset S_N$ can be patched together into a finitely presented generalized scheme $\bar{\mathscr{X}}_N \to \widehat{\operatorname{Spec}\mathbb{Z}}^{(N)} = S_N$.

7.1.24. (Implication for finitely presented models.) An immediate consequence is that *constructing a finitely presented $\widehat{\operatorname{Spec}\mathbb{Z}}$-model $\bar{\mathscr{X}}$ of a finitely presented $\mathbb{Q}$-scheme X is the same thing as constructing a finitely presented $\mathbb{Z}$-model $\mathscr{X}$ and a finitely presented $\mathbb{Z}_{(\infty)}$-model $\mathscr{X}_\infty$ of given X.* Since the first half (existence and different properties of models over $\operatorname{Spec}\mathbb{Z}$) is quite well known, we'll concentrate our efforts on the second half (models over $\mathbb{Z}_{(\infty)}$). This agrees with our considerations of **1.4**, arising from classical Arakelov geometry, provided we want to study only *finitely presented* $\mathbb{Z}_{(\infty)}$-models.

7.1.25. ($\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$ is finitely presented.) Before going on, let us show that all $\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$ are finitely presented (absolutely, i.e. over $\mathbb{F}_\varnothing$, or equivalently, over $\mathbb{F}_1 = \mathbb{F}_\varnothing[0^{[0]}]$). This will enable us to speak about finitely presented schemes $\mathscr{X}_N$ over $\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$ without mentioning explicitly whether $\mathscr{X}_N$ is finitely presented absolutely, i.e. over $\mathbb{F}_\varnothing$, or relatively, i.e. over $\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$.

Since $S_N := \widehat{\operatorname{Spec}\mathbb{Z}}^{(N)} = U_1 \cup U_2 = \operatorname{Spec}\mathbb{Z} \cup \operatorname{Spec} A_N$, with $U_1 \cap U_2 = \operatorname{Spec} B_N$, we see that S_N is quasicompact and quasiseparated. Next, $\mathbb{Z} = \mathbb{F}_{\pm1}[+^{[2]} \mid \boldsymbol{e} + 0 = \boldsymbol{e} = 0 + \boldsymbol{e},\ \boldsymbol{e} + (-\boldsymbol{e}) = 0]$ is finitely presented over $\mathbb{F}_{\pm1} = \mathbb{F}_1[-^{[1]} \mid -^2 = \boldsymbol{e}]$, hence also over $\mathbb{F}_1$, so our statement is reduced to the following:

Theorem 7.1.26 *Generalized ring $A_N = \mathbb{Z}_{(\infty)} \cap \mathbb{Z}[1/N]$ is finitely presented over $\mathbb{F}_{\pm1}$, hence also over $\mathbb{F}_1$ and $\mathbb{F}_\varnothing$. More precisely, it is generated by the set of "averaging operations" $s_p \in A_N(p)$ given by*

$$s_p(\{1\}, \{2\}, \ldots, \{p\}) = \frac{1}{p}\{1\} + \frac{1}{p}\{2\} + \cdots + \frac{1}{p}\{p\} \qquad (7.1.26.1)$$

where p runs over the finite set of all prime divisors of $N > 1$, modulo finite list of relations (7.1.26.4)–(7.1.26.6) *(for each of these operations s_p) listed below.*

Proof. (a) Let us denote by $s_n \in \mathbb{Z}_{(\infty)}(n)$ the *n-th averaging operation:*

$$s_n := \frac{1}{n}\{1\} + \frac{1}{n}\{2\} + \cdots + \frac{1}{n}\{n\} \in \mathbb{Z}_{(\infty)}(n) \subset \mathbb{Z}_\infty(n) \subset \mathbb{R}^n \qquad (7.1.26.2)$$

We have already seen in **5.7.3** that these operations $\{s_n\}_{n>1}$ generate $\mathbb{Z}_{(\infty)}$, since any operation $\lambda = (m_1/n)\{1\} + \cdots + (m_k/n)\{1\} \in \mathbb{Z}_{(\infty)}(k)$, where n, $m_i \in \mathbb{Z}$, $\sum_i |m_i| \leq n > 0$, can be re-written as s_n, applied to the list containing $\pm\{i\}$ exactly $|m_i|$ times, and with the remaining $n - \sum |m_i|$ arguments set to zero. Furthermore, if $\lambda \in A_N(k)$, then we can take $n = N^t$ for some integer $t > 0$, hence A_N is generated by $\{s_{N^t}\}_{t>0}$. Next, the operation s_{nm} can be expressed in terms of s_n and s_m:

$$s_{nm} = s_n(s_m(\{1\}, \ldots, \{m\}), s_m(\{m+1\}, \ldots, \{2m\}), \ldots) \qquad (7.1.26.3)$$

Therefore, A_N is generated by one operation s_N, or by finite set of operations $\{s_p\}_{p|N}$.

Notice that each s_n satisfies following *idempotency, symmetry* and *cancellation* relations:

$$s_n(\{1\}, \{1\}, \ldots, \{1\}) = \{1\} \qquad (7.1.26.4)$$

$$s_n(\{1\}, \{2\}, \ldots, \{n\}) = s_n(\{\sigma(1)\}, \ldots, \{\sigma(n)\}), \quad \forall \sigma \in \mathfrak{S}_n \qquad (7.1.26.5)$$

$$s_n(\{1\}, \ldots, \{n-1\}, -\{n-1\}) = s_n(\{1\}, \ldots, \{n-2\}, 0, 0) \qquad (7.1.26.6)$$

Furthermore, these relations for s_{nm} follow from similar relations for s_n and s_m, once we take into account (7.1.26.3) and the implied commutativity relation between s_n and s_m. This means that it suffices to impose the above relations only for $\{s_p\}_{p|N}$.

(b) Let us denote by A'_N the (commutative) finitely presented $\mathbb{F}_{\pm 1}$-algebra generated by operations $S_p \in A'_N(p)$ for $p \mid N$, subject to the above relations. It will be convenient to define $S_n \in A'_N(n)$ for all $n \mid N^\infty$ by induction in n, using formula (7.1.26.3) for this, with $S_1 := \boldsymbol{e}$ as the base of induction. One checks that the RHS of (7.1.26.3) doesn't change when we interchange m and n, using commutativity between S_m and S_n together with (7.1.26.5), so this inductive procedure does define operations $\{S_n\}_{n|N^\infty}$, satisfying (7.1.26.3) and (7.1.26.4)–(7.1.26.6).

Denote by $\varphi : A'_N \to A_N$ the generalized $\mathbb{F}_{\pm 1}$-algebra homomorphism transforming S_p into s_p for all primes $p \mid N$, hence also S_n into s_n for any $n \mid N^\infty$. It is a strict epimorphism ("surjective map") since the $\{s_p\}_{p|N}$ generate A_N over $\mathbb{F}_{\pm 1}$. Now we are going to check that φ is a monomorphism ("injective"), since this would imply that $A_N \cong A'_N$ is finitely presented with the lists of generators and relations given above.

Now we need the following lemma:

Lemma 7.1.27 *Any element $\lambda \in A'_N(k)$ can be written as $S_n(u_1, \ldots, u_n)$, where each u_i belongs to $\mathbb{F}_{\pm 1}(k) \subset A'_N(k)$, i.e. equals 0 or $\pm\{x_i\}$ with $1 \leq x_i \leq k$.*

Proof. Denote by $A''_N(k) \subset A'_N(k)$ the set of all k-ary operations of A'_N, which can be represented in the above form. We have to check that $\|A''_N\| = \bigsqcup_k A''_N(k)$ contains all basis elements $\{x\} \in A'_N(k)$, $1 \leq x \leq k$, an obvious statement since $\{x\} = S_1(\{x\})$, and that $\|A''_N\|$ is stable under all maps $A'_N(\varphi)$, $\varphi : \mathbf{k} \to \mathbf{k}'$, an obvious condition as well, and finally, that $\|A''_N\|$ is stable under the action of any of generators $0^{[0]}$, $-^{[1]}$ and $S_p^{[p]}$, $p \mid N$, of A'_N.

- As to the zero $0^{[0]}$, it obviously lies in $A''_N(0)$, as well as in all $A''_N(k)$.
- Stability under $-$ is also immediate once we use the (implied) commutativity relation between S_n and $-$:

$$- S_n(u_1, \ldots, u_n) = S_n(-u_1, \ldots, -u_n), \tag{7.1.27.1}$$

  combined with equalities $-(-u) = u$ and $-0 = 0$.
- Stability under S_p is a bit more tricky. Let $z_1, \ldots, z_p \in A''_N(k)$ be the list of arguments to S_p. We have to show that $z := S_p(z_1, \ldots, z_p)$ lies in $A''_N(k)$ as well. By definition of set $A''_N(k)$ each z_i can be written as S_{n_i} applied to some arguments $u_{ij} \in \mathbb{F}_{\pm 1}(k)$, for some $n_i \mid N^\infty$. Notice that any n_i can be replaced by any its multiple mn_i with $m \mid N^\infty$: all we need is to write the idempotency relations (7.1.26.4): $u_{ij} = S_m(u_{ij}, \ldots, u_{ij})$, substitute these expressions into $z_i = S_{n_i}(u_{i1}, \ldots, u_{i,n_i})$, and apply (7.1.26.3). Therefore, we may replace n_i by their product and assume all n_i to be equal to some $n \mid N^\infty$. But then $z = S_p(z_1, \ldots, z_p) = S_p(S_n(u_{11}, \ldots, u_{1n}), \ldots, S_n(u_{p1}, \ldots, u_{pn}))$ can be rewritten as S_{pn} applied to the list of all u_{ij}, hence z belongs to $A''_N(k)$, q.e.d.

7.1.28. *(End of proof of* **7.1.26***.)* Let z, $z' \in A'_N(k)$ be two elements with $\varphi(z) = \varphi(z')$ in $A_N(k)$. According to the above lemma, we can write $z = S_n(u_1, \ldots, u_n)$ and $z = S_m(u'_1, \ldots, u'_m)$ for some m, $n \mid N^\infty$, u_i, $u'_j \in \mathbb{F}_{\pm 1}(k)$. Furthermore, we can replace m and n by their product and assume $m = n$, by the same argument as in the proof of lemma. Next, using (7.1.26.6) and (7.1.26.5), we can assume that $\{x\}$ and $-\{x\}$ do not occur in the list u_1, u_2, $\ldots$, u_n simultaneously, simply because if $u_i = -u_j$, we can replace both u_i and u_j by zero. Now we've obtained a *reduced* representation $z = S_n(u_1, \ldots, u_n)$. Notice that $\varphi(z) = (m_1/n)\{1\} + \cdots + (m_k/n)\{k\}$, where $|m_x|$ is the multiplicity of $\pm\{x\}$ in the list $u_1, \ldots, u_n$, and the sign is chosen depending on whether $\{x\}$ or $-\{x\}$ is present in this list. Since $\varphi(z) = \varphi(z')$, we see that $m_x = m'_x$ for all $1 \leq x \leq k$, hence the lists of arguments to S_n in our expressions for z and z' coincide (up to a permutation), hence $z = z'$ by (7.1.26.5). This proves the injectivity of $\varphi : A'_N \to A_N$, hence also **7.1.26**.

Corollary 7.1.29 *Morphisms* $f_N^{NM} : \widehat{\operatorname{Spec}\mathbb{Z}}^{(NM)} \to \widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$ *are finitely presented.*

Proof. This follows from the classical statement: "$g \circ f$ finitely presented, g of finite type $\Rightarrow$ f finitely presented", valid for generalized schemes as well.

Remark 7.1.30 (a) Notice that the definition of $A_N = \mathbb{Z}_{(\infty)} \cap \mathbb{Z}[1/N]$, used before only for $N > 1$, can be extended to $N = 1$, yielding $A_1 = \mathbb{Z}_{(\infty)} \cap \mathbb{Z} = \mathbb{F}_{\pm 1}$. Furthermore, **7.1.26** holds for $N = 1$ as well: $A_1 = \mathbb{F}_{\pm 1}$ is a finitely presented $\mathbb{F}_{\pm 1}$-algebra, with an empty list of generators and relations.

(b) Theorem **7.1.26** holds for $N = \infty$, i.e. for $\mathbb{Z}_{(\infty)} = \varinjlim_{N>1} A_N$ as well. Indeed, this filtered inductive limit representation immediately implies that $\mathbb{Z}_{(\infty)}$ is generated over $\mathbb{F}_{\pm 1}$ by all averaging operations $\{s_n\}_{n>1}$, or just by all $\{s_p\}_{p \in \mathbb{P}}$, subject to relations (7.1.26.4)–(7.1.26.6).

(c) Essentially the same reasoning as in the proof of **7.1.26** shows that A_N is generated by *one* averaging operation s_N, subject to relations (7.1.26.4)–(7.1.26.6).

(d) Another interesting observation is that we've got a p-ary generator s_p of A_p over $\mathbb{F}_{\pm 1}$. While A_2 and A_3 can be shown not to be generated by operations of lower arities, A_5 can be shown to be generated by quaternary operation $(2/5)\{1\} + (1/5)\{2\} + (1/5)\{3\} + (1/5)\{4\}$, so the beautiful system of generators given in **7.1.26** is not necessarily the one which minimizes the arity of operations involved.

(e) Notice that $\mathbb{Z}_{(\infty)}$ is already generated over $\mathbb{F}_{\pm 1}$ by its *binary* operations $\lambda\{1\} + (1-\lambda)\{1\}$, $\lambda \in [0,1] \cap \mathbb{Q}$, similarly to $\mathbb{Z}_\infty$, so the phenomena mentioned above in (d) are special for $N < \infty$.

7.1.31. (Special case: $N = 2$.) In particular, we obtain a presentation of $A_2 = \mathbb{Z}_{(\infty)} \cap \mathbb{Z}[1/2]$. It is generated by binary averaging operation s_2, which will be also denoted by $*$, when written in infix form. Thus $[*] = s_2 = (1/2)\{1\} + (1/2)\{2\}$; informally, $x * y = (x+y)/2$. We obtain $A_2 = \mathbb{F}_{\pm 1}[*^{[2]} \mid x * x = x,\ x * y = y * x,\ x * (-x) = 0]$. In this respect A_2 is somewhat similar to $\mathbb{Z} = \mathbb{F}_{\pm 1}[+^{[2]} \mid x + 0 = x = 0 + x,\ x + (-x) = 0]$: both are generated over $\mathbb{F}_{\pm 1}$ by one binary operation.

7.1.32. (Generalized rings and representations of braid groups.) Let A be a (commutative) generalized ring, $[*] \in A(2)$ be a binary operation, such that $\boldsymbol{e} * \boldsymbol{e} = \boldsymbol{e}$. For example, we can take $A = A_2$, or $A = \mathbb{Q}[q]$, $[*] = \{1\} * \{2\} := q\{1\} + (1-q)\{2\}$, or the universal case $A = \mathbb{F}_\varnothing[*^{[2]} \,|\, \boldsymbol{e} * \boldsymbol{e} = \boldsymbol{e}]$. Fix any integer $n \geq 2$, and denote by $t_k : A(n) \to A(n)$, where $1 \leq k < n$, the endomorphism of free A-module $A(n)$ defined by

$$t_k(\{i\}) = \begin{cases} \{i\}, & \text{if } i \neq k, k+1 \\ \{k+1\}, & \text{if } i = k \\ \{k\} * \{ k+1\}, & \text{if } i = k+1 \end{cases} \tag{7.1.32.1}$$

One checks directly that these matrices $t_k \in \mathrm{End}_A(A(n)) \cong M(n \times n; A) = A(n)^n$ satisfy the braid relation:

$$t_k \circ t_{k+1} \circ t_k = t_{k+1} \circ t_k \circ t_{k+1}, \quad \text{for } 1 \leq k \leq n-2. \tag{7.1.32.2}$$

Since t_k and t_{k+1} act non-identically only on basis elements $\{k\}$, $\{k+1\}$ and $\{k+2\}$, it suffices to check this relation for $k = 1$, $n = 3$. We get $(t_1 \circ t_2 \circ t_1)(x, y, z) = (t_1 \circ t_2)(y, x*y, z) = t_1(y, z, (x*y)*z) = (z, y*z, (x*y)*z)$ and $(t_2 \circ t_1 \circ t_2)(x, y, z) = (t_2 \circ t_1)(x, z, y * z) = t_2(z, x * z, y * z) = (x, y * z, (x * z) * (y * z))$, so we are reduced to showing $(x * y) * z = (x * z) * (y * z)$, which is immediate from relation $z = z * z$ and the commutativity relation $(x * y) * (z * w) = (x * z) * (y * w)$.

In this way we (almost) obtain a representation of the *braid group* $B_n = \langle t_1, \ldots, t_{n-1} \,|\, t_k t_{k+1} t_k = t_{k+1} t_k t_{k+1} \rangle$ by $n \times n$-matrices over generalized ring A. The only problem is that our matrices t_i may be not invertible; this can be circumvented by considering the matrix localization $A[Z^{-1}]$ of A with respect to matrix $Z : (x, y) \mapsto (y, x * y)$, cf. **6.1.28**. One easily checks that $A_2[Z^{-1}] = B_2 = \mathbb{Z}[1/2]$; as to the universal case $A = \mathbb{F}_\varnothing[*^{[2]} \,|\, \boldsymbol{e} * \boldsymbol{e} = \boldsymbol{e}]$, I don't know much about $A[Z^{-1}]$.

Among other things, this example illustrates the complexity of non-unary algebras: even $A = \mathbb{F}_\varnothing[*^{[2]} \,|\, \boldsymbol{e} * \boldsymbol{e} = \boldsymbol{e}]$ (or rather $A[Z^{-1}]$) is complicated enough to contain representations of the braid groups. It is interesting whether one can generalize this construction to $p > 2$, using the special properties of operation s_p instead of s_2.

Proposition 7.1.33 $\mathrm{Pic}(\mathrm{Spec}\, A_N) = 0$, *i.e. any line bundle over* $\mathrm{Spec}\, A_N$ *is trivial. More precisely, if P is a finitely generated projective A_N-module, such that* $\dim_{\mathbb{Q}} P_{(\mathbb{Q})} = 1$, *then* $P \cong |A_N|$.

Proof. (a) Notice that if $\mathscr{L}$ is a line bundle over $\mathrm{Spec}\, A_N$, then $\mathscr{L} = \tilde{P}$ for some finitely generated projective A_N-module P by **6.5.28**, and the generic fiber $\mathscr{L}_\xi \cong P_{(\mathbb{Q})}$ is obviously one-dimensional, so the last statement of the proposition does imply the other ones. So let P be as above. Choose a surjection $\pi : A_N(n) \twoheadrightarrow P$, and a section $j : P \to A_N(n)$ of π, existing because of projectivity of P. Let $p := j \circ \pi : A_N(n) \to A_N(n)$ be the corresponding projector. Denote by P', $\pi' : B_N(n) \twoheadrightarrow P'$, $j' : P' \to B_N(n)$ and $p' : B_N(n) \to B_N(n)$ the module and homomorphisms, obtained by scalar extension to B_N. Now consider the following diagram:

$$\begin{tikzcd} A_N(n) \arrow[r, "\xi_{A_N(n)}"] \arrow[d, bend right, "\pi"'] & B_N(n) \arrow[d, bend right, "\pi'"'] \\ P \arrow[r, "\xi_P"] \arrow[u, bend right, "j"'] & P' \arrow[u, bend right, "j'"'] \end{tikzcd} \tag{7.1.33.1}$$

Here $\xi_M : M \to M_{(B_N)}$ denotes the natural embedding of an A_N-module M into its scalar extension to B_N.

Notice that $\xi_{A_N(n)}$ is just the map $\varphi_n : A_N(n) \to B_N(n)$ coming from the natural embedding $\varphi : A_N \to B_N$, hence it is injective, hence the same is true for its retract ξ_P.

(b) Let us identify $A_N(n)$ with its image under $\xi_{A_N(n)}$ in $B_N(n)$, which in turn can be identified with a subset of $\mathbb{Q}^n$. Identify P' with $j'(P') = p'(B_N(n))$, and P with $j(P) = p(A_N(n)) \subset A_N(n) \subset B_N(n)$. Then all modules involved are identified with subsets of $B_N(n) \subset \mathbb{Q}^n$, and we have $P = P' \cap A_N(n) = P' \cap \mathbb{Z}_{(\infty)}(n)$.

Denote by $\|\cdot\|$ the L_1-norm on $\mathbb{Q}^n$, given by formula $\|(\lambda_1, \ldots, \lambda_n)\| = \sum_i |\lambda_i|$. Then $A_N(n) = \{\lambda \in B_N^n : \|\lambda\| \leq 1\}$ and $P = \{\lambda \in P' : \|\lambda\| \leq 1\}$. Let $u_i := p(\{i\})$, $1 \leq i \leq n$, be the finite set of generators of A_N-module P. Put $C := \max_i \|u_i\|$. Clearly $0 < C \leq 1$, since $P \neq 0$ and $P \subset A_N(n)$. Since $p(\lambda) = \sum_i \lambda_i u_i$ for any $\lambda = (\lambda_1, \ldots, \lambda_n) \in A_N(n)$, we get $\|p(\lambda)\| \leq C \cdot \sum_i |\lambda_i| = C \cdot \|\lambda\| \leq C$ for any $\lambda \in A_N(n)$, i.e. $C = \max_{x \in P} \|x\|$ since $P = p(A_N(n))$. We may assume that $C = \|u_1\|$; otherwise we would just renumber the basis elements of $A_N(n)$ and the generators u_i of P.

(c) We claim that $C = 1$. Indeed, suppose that $C < 1$. Then one can find a fraction v/N^k, such that $1 < v/N^k < C^{-1}$ (we use $N > 1$ here). Then $\mu := v/N^k \in |B_N|$, hence $\mu u_1 \in P'$, P' being a B_N-module, and by construction $\|\mu u_1\| \leq 1$, i.e. $\mu u_1 \in A_N(n)$, hence $\mu u_1 \in P' \cap A_N(n) = P$ with $\|\mu u_1\| > \|u_1\| = C$, which is absurd.

(d) Now we claim that u_1 freely generates A_N-module P, i.e. that any $\lambda \in P$ can be uniquely written as $\lambda = c \cdot u_1$ with $c \in |A_N|$. Uniqueness is clear, since we can always extend scalars to $\mathbb{Q}$, and there everything follows from $u_1 \neq 0$. This argument also shows existence of such $c \in \mathbb{Q}$, since $P_{(\mathbb{Q})} \cong \mathbb{Q}$ is a line in $A(n)_{(\mathbb{Q})} = \mathbb{Q}^n$. We have to show that this c actually lies in $|A_N|$. But this is clear: since $\|u_1\| = C = 1$ and $\lambda = c \cdot u_1$, we have $c = \pm\|\lambda\| = \pm\sum_i \lambda_i \in B_N$, because $\lambda \in P \subset B_N(n)$, i.e. all components $\lambda_i \in B_N$, and $|c| = \|\lambda\| \leq 1$, just because $\lambda \in P \subset A_N(n)$. We have just shown that $c \in |B_N| \cap |\mathbb{Z}_{(\infty)}| = |A_N|$, q.e.d.

Corollary 7.1.34 $\mathrm{Pic}(\mathrm{Spec}\,\mathbb{Z}_{(\infty)}) = 0$. *More precisely, any finitely generated projective $\mathbb{Z}_{(\infty)}$-module P, such that $\dim_{\mathbb{Q}} P_{(\mathbb{Q})} = 1$, is free of rank one.*

Proof. Since any finitely generated projective module is finitely presented (being a direct factor $P = \mathrm{Coker}(p, \mathrm{id})$, $p = p^2 \in M(n, \mathbb{Z}_{(\infty)})$, of some $\mathbb{Z}_{(\infty)}(n)$), the usual inductive limit argument shows that P comes from some finitely generated projective A_N-module P_N, for a suitable $N > 1$. Then $P_{N,(\mathbb{Q})} \cong P_{(\mathbb{Q})} \cong \mathbb{Q}$, hence P_N is free of rank one by **7.1.33**, hence this must be also true for $P = P_{N,(\mathbb{Z}_{(\infty)})}$.

7.1.35. (Line bundles over $\widehat{\mathrm{Spec}\,\mathbb{Z}}^{(N)}$.) Let $\mathscr{L}$ be a line bundle over $S_N := \widehat{\mathrm{Spec}\,\mathbb{Z}}^{(N)} = U_1 \cup U_2 = \mathrm{Spec}\,\mathbb{Z} \cup \mathrm{Spec}\,A_N$. Since all line bundles over $\mathrm{Spec}\,\mathbb{Z}$ and $\mathrm{Spec}\,A_N$ are trivial (cf. **7.1.33**), we can choose trivializations $\varphi_1 : \mathscr{O}_{\mathrm{Spec}\,\mathbb{Z}} \xrightarrow{\sim} \mathscr{L}|_{U_1}$ and $\varphi_2 : \mathscr{O}_{\mathrm{Spec}\,A_N} \xrightarrow{\sim} \mathscr{L}|_{U_2}$. Comparing these two trivializations over $U_1 \cap U_2 = \mathrm{Spec}\,B_N$, we obtain an invertible element $\lambda \in B_N^\times$, such that $\varphi_2|_{U_1\cap U_2} = \lambda \cdot \varphi_1|_{U_1\cap U_2}$. This element λ determines $\mathscr{L}$ up to isomorphism, and we can always construct a line bundle $\mathscr{L} = \mathscr{O}(\log\lambda)$ starting from any element $\lambda \in B_N^\times$, simply by taking trivial line bundles over U_1 and U_2, and gluing them over $U_1 \cap U_2$ with the aid of λ.

In this way we get a *surjection* of abelian groups $B_N^\times \twoheadrightarrow \mathrm{Pic}(S_N)$. It is not injective only because the trivializations φ_1 and φ_2 are not canonical: they are defined up to multiplication by some invertible elements $s_1 \in \Gamma(U_1, \mathscr{O}_{U_1}^\times) = \mathbb{Z}^\times = \{\pm 1\}$ and $s_2 \in \Gamma(U_2, \mathscr{O}_{U_2}^\times) = |A_N|^\times = \{\pm 1\}$, i.e. up to sign, hence λ is also defined up to sign. Identifying $B_N^\times/\{\pm 1\}$ with the group $B_{N,+}^\times$ of positive invertible elements in $B_N = \mathbb{Z}[N^{-1}]$, we obtain the following statement:

Proposition. *The Picard group* $\mathrm{Pic}(\widehat{\mathrm{Spec}\,\mathbb{Z}}^{(N)})$ *is canonically isomorphic to* $\log B_{N,+}^\times$*, the abelian group of positive invertible elements of* $B_N = \mathbb{Z}[N^{-1}]$*, written in additive form.*

If $p_1, \ldots, p_r$ is the set of all distinct prime divisors of N, then $B_{N,+}^\times$ consists of rational numbers $p_1^{k_1} \cdots p_r^{k_r}$, where k_i are arbitrary integers, hence $\log B_{N,+}^\times$ is a free abelian group with basis $\{\log p_i\}$, and $\mathrm{Pic}(\widehat{\mathrm{Spec}\,\mathbb{Z}}^{(N)})$ is also isomorphic to $\mathbb{Z}^r$, with basis $\{\mathscr{O}(\log p_i)\}_{1\leq i\leq r}$. Notice that these line bundles

$\mathscr{O}(\log p_i)$ are in one-to-one correspondence with $p \mid N$, the only primes "not entangled" with ∞ in $\widehat{\operatorname{Spec}^u \mathbb{Z}}^{(N)}$ (cf. **7.1.6**).

Among other things, this result completes the proof of **7.1.9** for arbitrary localization theory $\mathcal{T}^?$.

7.1.36. (Line bundles over $\widehat{\operatorname{Spec}\mathbb{Z}}$.) Since $\widehat{\operatorname{Spec}\mathbb{Z}} = \varprojlim_{N>1} S_N$, where $S_N = \widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$, we see that the category of line bundles over $\widehat{\operatorname{Spec}\mathbb{Z}}$ is $\underrightarrow{\mathrm{Lim}}$ of categories of line bundles over S_N, hence $\operatorname{Pic}(\widehat{\operatorname{Spec}\mathbb{Z}}) = \varinjlim_{N>1} \operatorname{Pic}(S_N) = \varinjlim_{N>1} \log B_{N,+}^{\times} = \log \mathbb{Q}_+^{\times}$, i.e. any line bundle over $\widehat{\operatorname{Spec}\mathbb{Z}}$ is isomorphic to exactly one line bundle of form $\mathscr{O}(\log \lambda)$, with λ a positive rational number. In other words, $\operatorname{Pic}(\widehat{\operatorname{Spec}\mathbb{Z}})$ is a free abelian group generated by all $\mathscr{O}(\log p)$, $p \in \mathbb{P}$.

7.1.37. (Degree of a line bundle over $\widehat{\operatorname{Spec}\mathbb{Z}}$.) Let $\mathscr{L}$ be a line bundle over $\widehat{\operatorname{Spec}\mathbb{Z}}$. According to **7.1.36**, it is isomorphic to $\mathscr{O}(\log \lambda)$ for exactly one $\lambda \in \mathbb{Q}_+^{\times}$. We will say that $\log \lambda \in \log \mathbb{Q}_+^{\times}$ is the *(arithmetic) degree of* $\mathscr{L}$, and write $\deg \mathscr{L} = \log \lambda$. Thus $\deg : \operatorname{Pic}(\widehat{\operatorname{Spec}\mathbb{Z}}) \to \log \mathbb{Q}_+^{\times}$ is the isomorphism between $\operatorname{Pic}(\widehat{\operatorname{Spec}\mathbb{Z}})$ and $\mathbb{Q}_+^{\times}$ constructed in **7.1.36**.

Now we want to obtain an explicit description of $\deg \mathscr{L}$.

Recall that, according to **7.1.22**, a line bundle $\mathscr{L}$ over $\widehat{\operatorname{Spec}\mathbb{Z}}$ is essentially the same thing as a triple $L = (L_{\mathbb{Z}}, L_\infty, \theta_L)$, where $L_{\mathbb{Z}}$ is a free $\mathbb{Z}$-module of rank one, L_∞ is a free $\mathbb{Z}_{(\infty)}$-module of rank one (we apply here **7.1.34**, $\tilde{L}_\infty = \hat{\eta}^* \mathscr{L}$ being a line bundle over $\operatorname{Spec}\mathbb{Z}_{(\infty)}$), and $\theta_L : L_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} L_\infty \otimes_{\mathbb{Z}_{(\infty)}} \mathbb{Q}$ is the isomorphism of one-dimensional $\mathbb{Q}$-vector spaces, arising from canonical isomorphisms of both sides to the generic fiber $\mathscr{L}_\xi = \hat{\xi}^* \mathscr{L}$.

Now choose a free $\mathbb{Z}$-generator e_1 of $L_{\mathbb{Z}}$ and a free $\mathbb{Z}_{(\infty)}$-generator e_2 of L_∞. Both are defined uniquely up to sign, since $\mathbb{Z}^\times = \{\pm 1\} = \mathbb{Z}_{(\infty)}^\times$. Then both $\theta_L(e_1 \otimes 1)$ and $e_2 \otimes 1$ are generators of one-dimensional $\mathbb{Q}$-vector space $L_\infty \otimes_{\mathbb{Z}_{(\infty)}} \mathbb{Q} \cong \mathscr{L}_\xi$, hence $e_2 \otimes 1 = \lambda \cdot \theta_L(e_1 \otimes 1)$ for some $\lambda \in \mathbb{Q}^\times$, defined uniquely up to sign. We can make λ unique by requiring $\lambda > 0$. Then $\log \lambda \in \log \mathbb{Q}_+^\times$ is exactly the degree of $\mathscr{L}$ (cf. **7.1.35** and **7.1.22**).

7.1.38. ($\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$ is similar to $(\mathbb{P}^1)^r$.) Let $N > 1$, $S_N := \widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$ be as before. Denote by $p_1, \ldots, p_r$ all distinct prime divisors of N. We have just seen that $\operatorname{Pic}(S_N) \cong \mathbb{Z}^r$, similarly to the Picard group $\operatorname{Pic}((\mathbb{P}^1)^r)$ of the product of r projective lines. In this respect S_N is quite similar to $P^r := (\mathbb{P}^1)^r$. Notice, however, the following:

(a) The automorphism group $\operatorname{Aut}(P^r)$ acts transitively on the canonical basis elements of $\operatorname{Pic}(P^r) = \mathbb{Z}^r$, i.e. all these line bundles $\mathscr{L}_i :=$

$\mathscr{O}(0,\ldots,1,\ldots,0)$ over P^r have the same properties, while $\mathrm{Aut}(S_N)=1$ doesn't act transitively on the generators $\mathscr{L}'_i := \mathscr{O}(\log p_i)$ of $\mathrm{Pic}(S_N)$.

(b) This analogy between Picard groups shouldn't extend to higher Chow groups: we expect that $c_1 : \mathrm{Pic}(S_N) \to CH^1(S_N)$ is an isomorphism for S_N, similarly to c_1 over P^r, but $CH^i(S_N) = 0$ for $i > 1$, in contrast with $CH^i(P^r) \cong \bigwedge^i \mathrm{Pic}(P^r)$.

7.1.39. (Global sections of $\mathscr{O}_{S_N}$.) Let us compute the global sections of the structural sheaf of $S_N = \widehat{\mathrm{Spec}\,\mathbb{Z}}^{(N)}$. Since $S_N = U_1 \cup U_2$, with $\Gamma(U_1,\mathscr{O}) = \mathbb{Z}$, $\Gamma(U_2,\mathscr{O}) = A_N$ and $\Gamma(U_1\cap U_2,\mathscr{O}) = B_N$, the sheaf condition for $\mathscr{O}$ yields $\Gamma(S_N,\mathscr{O}) = \mathbb{Z}\times_{B_N} A_N = \mathbb{Z}\cap A_N = \mathbb{F}_{\pm1}$, where the intersection is computed inside $B_N \subset \mathbb{Q}$. In order to check $\mathbb{Z}\cap A_N = \mathbb{F}_{\pm1}$ we write $\mathbb{Z}\cap A_N = \mathbb{Z}\cap B_N\cap\mathbb{Z}_{(\infty)} = \mathbb{Z}\cap\mathbb{Z}_{(\infty)}$, and $(\mathbb{Z}\cap\mathbb{Z}_{(\infty)})(n) = \mathbb{Z}(n)\cap\mathbb{Z}_{(\infty)}(n) = \{\lambda\in\mathbb{Z}^n : \|\lambda\|\leq 1\} = \mathbb{F}_{\pm1}(n)$, where $\|\cdot\|$ is the L_1-norm on $\mathbb{Q}^n$ as before. Actually this formula $\Gamma(\widehat{\mathrm{Spec}\,\mathbb{Z}},\mathscr{O}) = \mathbb{Z}\cap\mathbb{Z}_\infty = \mathbb{F}_{\pm1}$ was exactly our original motivation for defining $\mathbb{F}_{\pm1}$ the way we've defined it.

An immediate implication is that we can study all S_N and their projective limit $\widehat{\mathrm{Spec}\,\mathbb{Z}}$ as (pro-)generalized schemes over $S_1 := \mathrm{Spec}\,\mathbb{F}_{\pm1}$, or over $\mathrm{Spec}\,\mathbb{F}_1$ or $\mathrm{Spec}\,\mathbb{F}_\varnothing$, but these cases almost exhaust all reasonable possibilities.

7.1.40. (Global sections of $\mathscr{O}(\log\lambda)$.) Let $N>1$, S_N be as above, $\lambda\in B^\times_{N,+}$. Let us compute the set of global sections $\Gamma(S_N,\mathscr{O}(\log\lambda))$. Using trivializations φ_1 and φ_2 as in **7.1.35**, differing by multiplication by λ over $U_1\cap U_2$, together with the sheaf condition for $\mathscr{O}(\log\lambda)$ with respect to the cover $S_N = U_1\cap U_2$, we see that $\Gamma(S_N,\mathscr{O}(\log\lambda)) \cong \{(x,y)\in\mathbb{Z}\times|A_N| : x = \lambda y\} \cong \{x\in\mathbb{Z} : |\lambda^{-1}x|\leq 1\} = \mathbb{Z}\cap\lambda|A_N| = \mathbb{Z}\cap[-\lambda,\lambda]$.

For example, if $M\geq 1$ is an integer, such that $M \mid N^\infty$, then $\mathscr{O}(\log M)$ admits $2M+1$ global sections, corresponding to integers from $-M$ to M. The global sections corresponding to ±1 are invertible over $\mathrm{Spec}\,\mathbb{Z}$, while those corresponding to $\pm M$ are invertible over $\mathrm{Spec}\,A_N$, or more precisely, outside the primes $p\mid M$.

Notice that $\Gamma(S_N,\mathscr{O}(\log\lambda))$ doesn't change when we replace N by any its multiple N', hence $\Gamma(\widehat{\mathrm{Spec}\,\mathbb{Z}},\mathscr{O}(\log\lambda)) = \mathrm{Hom}(\mathscr{O}(0),\mathscr{O}(\log\lambda))$ equals $\varinjlim_{N'\geq1}\mathrm{Hom}_{\mathscr{O}_{S_{NN'}}}(\mathscr{O}(0),\mathscr{O}(\log\lambda)) = \varinjlim_{N'\geq1}\Gamma(S_{NN'},\mathscr{O}(\log\lambda)) \cong \mathbb{Z}\cap[-\lambda,\lambda]$ as well.

7.1.41. (Global sections of twisted vector bundles.) For any $\mathscr{O}_{S_N}$-module $\mathscr{F}$ and any $\lambda\in B^\times_{N,+}$, we denote by $\mathscr{F}(\log\lambda)$ the corresponding *Serre twist* $\mathscr{F}\otimes_{\mathscr{O}}\mathscr{O}(\log\lambda)$. If $\mathscr{F}$ was a vector bundle, the same is true for $\mathscr{F}(\log\lambda)$; in particular, we can start from the trivial vector bundle $\mathscr{E}_0 := \mathscr{O}^{(r)}_{S_N} = L_{\mathscr{O}}(r)$ of rank $r\geq 0$, and consider $\mathscr{E} := \mathscr{E}_0(\log\lambda) = L_{\mathscr{O}}(r)\otimes_{\mathscr{O}}\mathscr{O}(\log\lambda)$.

Let us compute the global sections of such twisted trivial vector bundles. The same reasoning as in **7.1.40** yields $\Gamma(\mathscr{O}_{S_N}, L_{\mathscr{O}}(r) \otimes_{\mathscr{O}} \mathscr{O}(\log \lambda)) = \mathbb{Z}^r \cap \lambda A_N(r) = \mathbb{Z}^n \cap \lambda \cdot \mathbb{Z}_{(\infty)}(r) = \{\mu = (\mu_1, \ldots, \mu_r) \in \mathbb{Z}^n : \|\mu\| \le \lambda\}$. In other words, global sections of such vector bundles correspond to integral points $(\mu_1, \ldots, \mu_r) \in \mathbb{Z}^n$, lying inside octahedron $\sum_i |\mu_i| \le \lambda$.

7.1.42. (Ample line bundles over $\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$.) Recall that the ample line bundles over $P^r = (\mathbb{P}^1)^r$ are $\mathscr{O}(k_1, \ldots, k_r)$ with all $k_i > 0$, so by analogy between $\operatorname{Pic}(S_N)$ and $\operatorname{Pic}(P^r)$ we might expect that the ample line bundles over $S_N = \widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$ are $\mathscr{O}(\log M)$ with $M = p_1^{k_1} \cdots p_r^{k_r}$, where all $k_i > 0$, and p_i are the distinct prime divisors of N.

Since the above condition on M is equivalent to $M \mid N^\infty$ and $N \mid M^\infty$, and for such M and N we have canonical isomorphisms $S_N \cong S_{NM} \cong S_M$, we are reduced to checking whether $\mathscr{O}(\log N)$ is ample over S_N.

7.1.43. (Graded ring $\Gamma_*(\mathscr{O}_{S_N})$ with respect to $\mathscr{L} := \mathscr{O}_{S_N}(\log N)$.) According to our general definitions **6.6.39**, we are to construct the graded generalized ring $R := \Gamma_*(\mathscr{O}_{S_N})$, where $\Gamma_d(\mathscr{O}_{S_N})(n) = \Gamma(S_N, L_{\mathscr{O}}(n) \otimes_{\mathscr{O}} \mathscr{O}(d \log N))$, and check whether the natural map $S_N \to \operatorname{Proj} R$ is an isomorphism.

We know already that $R_d(n) = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \mid \sum_i |\lambda_i| \le N^d\}$. It will be convenient to write elements of $R_d(n)$ in form $T^d(\lambda_1, \ldots, \lambda_n)$, or even $\lambda_1 T^d\{1\} + \cdots + \lambda_n T^d\{n\}$, thus embedding R into graded $\mathbb{Z}$-algebra $\mathbb{Z}[T]$; this embedding is actually compatible with the unit and multiplication maps $\mu_{n,e}^{(k,d)}$, so it completely determines the structure of generalized graded ring on R.

Sometimes it is convenient to think about formal variable T as if it were equal to $1/N$.

7.1.44. (Two special elements of degree one.) Notice that $|R| = R(1) = \bigsqcup_{d \ge 0} R_d(1)$ consists of expressions uT^d, with $d \ge 0$, $u \in \mathbb{Z}$, $|u| \le N^d$. In particular, we have two special elements of degree one: $f_1 := T$ and $f_2 := NT$.

Denote by $\mathfrak{a} \subset |R|_+$ the graded ideal generated by f_1 and f_2. It is clearly equal to the image of graded R-homomorphism $R(2)[-1] \to |R|^+$, which maps $\{1\} \mapsto f_1$, $\{2\} \mapsto f_2$. Since $R_{d-1}(2) = \{uT^{d-1}\{1\} + vT^{d-1}\{2\} : |u| + |v| \le N^{d-1}\}$, we see that $\mathfrak{a}_d \subset |R|_d$, where $d \ge 1$, consists of elements wT^d, which can be written in form $w = u + Nv$ for some u, $v \in \mathbb{Z}$ with $|u| + |v| \le N^{d-1}$. On the other hand, $|R|_{+,d} = |R|_d$ consists of all wT^d with $|w| \le N^d$, so we cannot expect $\mathfrak{a}_d = |R|_{+,d}$ for $N > 2$: indeed, $w = N^d - 1$ cannot be written as $w = u + Nv$ with integer u, v, such that $|u| + |v| \le N^{d-1}$.

However, we claim that $\mathfrak{r}(\mathfrak{a}) = |R|_+$. Indeed, let $f := wT^d$, $d > 0$, $|w| \le N^d$ be any element of $|R|_+$. If $|w| = N^d$, we can take $u = 0$, $v = w/N = \pm N^{d-1}$ and write $f = wT^d = vT^{d-1} \cdot NT = vT^{d-1} \cdot f_2 \in \mathfrak{a}_d$, and we are done.

If $|w| < N^d$, we can find an integer $k \geq 1$, such that $|N^{-d}w|^k \leq 1 - 1/N$ and $dk \geq 3$. Then $f^k = w^k T^{dk}$ with $|w^k| \leq (1 - 1/N)N^{dk} \leq N^{dk} - N^2$. Now write $w^k = vN + u$ with $u, v \in \mathbb{Z}$, $|u| < N$, $uw^k \geq 0$. Then $|v| \leq |w^k|/N \leq N^{dk-1} - N$, hence $|u| + |v| \leq N^{dk-1}$, so $f^k = w^k T^{dk}$ lies in $\mathfrak{a}$, and we are done.

7.1.45. (Localizations of R with respect to f_1, f_2 and $f_1 f_2$.) Now we want to compute generalized rings $R_{(f_1)}$, $R_{(f_2)}$ and $R_{(f_1 f_2)}$.

(a) $R_{(f_1)}(n) = R_{(T)}(n)$ consists of fractions $(u_1 T^d\{1\} + \cdots + u_n T^d\{n\})/T^d$ with $u_i \in \mathbb{Z}$, $\sum_i |u_i| \leq N^d$, for all $d \geq 0$. Therefore, $R_{(f_1)}(n)$ consists of all expressions $u_1\{1\} + \cdots + u_n\{n\}$ with $u_i \in \mathbb{Z}$, i.e. $R_{(f_1)}(n) = \mathbb{Z}^n = \mathbb{Z}(n)$ and $R_{(f_1)} = \mathbb{Z}$.

(b) $R_{(f_2)}(n) = R_{(NT)}(n)$ consists of $(u_1 T^d\{1\} + \cdots + u_n T^d\{n\})/(N^d T^d)$ with $u_i \in \mathbb{Z}$, $\sum |u_i| \leq N^d$, i.e. of expressions $v_1\{1\} + \cdots + v_n\{n\}$ with $v_i \in \mathbb{Z}[N^{-1}] = B_N$, $\sum_i |v_i| \leq 1$, hence $R_{(f_2)} = B_N \cap \mathbb{Z}_{(\infty)} = A_N$.

(c) Finally, $R_{(f_1 f_2)} = R_{(NT^2)}$ is isomorphic to $B_N = \mathbb{Z}[N^{-1}]$, either by a similar direct computation, or because of formula $R_{(f_1 f_2)} \cong R_{(f_1)}[(f_2/f_1)^{-1}] = \mathbb{Z}[N^{-1}]$ shown in **6.6.16**.

7.1.46. (Projectivity of $\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)} = \operatorname{Proj} R$.) We see that $\operatorname{Proj} R = D_+(f_1) \cup D_+(f_2)$ by **6.6.20** and **7.1.44**. Furthermore, $D_+(f_1) = \operatorname{Spec} R_{(f_1)} = \operatorname{Spec}\mathbb{Z}$, $D_+(f_2) = \operatorname{Spec} R_{(f_2)} = \operatorname{Spec} A_N$, and $D_+(f_1) \cap D_+(f_2) = D_+(f_1 f_2) = \operatorname{Spec} R_{(f_1 f_2)} = \operatorname{Spec} B_N$, i.e. $\operatorname{Proj} R$ can be constructed by gluing together $\operatorname{Spec}\mathbb{Z}$ and $\operatorname{Spec} A_N$ along their open subsets isomorphic to $\operatorname{Spec} B_N$.

In other words, we have just shown that $\operatorname{Proj} R$ is isomorphic to $\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$. This proves ampleness of $\mathscr{O}(\log N)$ over $S_N := \widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$, as well as projectivity of S_N (absolute or over $\mathbb{F}_{\pm 1}$).

Notice, however, that $S_N := \widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$ is a *non-unary* projective generalized scheme, cf. **6.6.40**. In particular, even if we know that $\Gamma(S_N, \mathscr{O}(\log N))$ is a free $\mathbb{F}_{\pm 1}$-module of rank N, we still shouldn't expect S_N to be isomorphic to a "closed" subscheme of $\mathbb{P}^{N-1}_{\mathbb{F}_{\pm 1}}$.

7.1.47. (Square of $\widehat{\operatorname{Spec}\mathbb{Z}}$.) We have seen in **5.1.22** that $\mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}$, when the tensor product (i.e. coproduct of generalized commutative algebras) is taken over $\mathbb{F}_\varnothing$, $\mathbb{F}_1$ or $\mathbb{F}_{\pm 1}$, hence $\operatorname{Spec}\mathbb{Z} \times_{\operatorname{Spec}\mathbb{F}_1} \operatorname{Spec}\mathbb{Z} = \operatorname{Spec}\mathbb{Z}$, i.e. $\operatorname{Spec}\mathbb{Z} \to \operatorname{Spec}\mathbb{F}_1$ is a monomorphism of generalized schemes.

Now consider $S_N := \widehat{\operatorname{Spec}\mathbb{Z}}^{(N)} = U_1 \cup U_2$, where $U_1 = \operatorname{Spec}\mathbb{Z}$ and $U_2 = \operatorname{Spec} A_N$ as before. We claim that $S_N \times_{\operatorname{Spec}\mathbb{F}_{\pm 1}} S_N = S_N$, *i.e.* $S_N \to \operatorname{Spec}\mathbb{F}_{\pm 1}$ *is a monomorphism of generalized schemes.* Since $\widehat{\operatorname{Spec}\mathbb{Z}} = \varprojlim_{N>1} S_N$, we obtain $\widehat{\operatorname{Spec}\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{F}_{\pm 1}} \widehat{\operatorname{Spec}\mathbb{Z}} = \widehat{\operatorname{Spec}\mathbb{Z}}$ (e.g. in the category of pro-generalized schemes) as well.

So let us show that $\Delta := \Delta_{S_N} : S_N \to S_N \times S_N$ (all products here are to be computed over $\operatorname{Spec}\mathbb{F}_{\pm1}$) is an isomorphism. Since $\Delta^{-1}(U_i \times U_j) = U_i \cap U_j$, and open subschemes $U_i \times U_j$, $1 \leq i, j \leq 2$, cover $S_N \times S_N$, we are reduced to showing that canonical morphisms $U_i \cap U_j \to U_i \times U_j$ are in fact isomorphisms. If $i = j = 1$, this is equivalent to $\mathbb{Z} \otimes_{\mathbb{F}_{\pm1}} \mathbb{Z} = \mathbb{Z}$, valid by **5.1.22**. If $i = j = 2$, we have to show $A_N \otimes_{\mathbb{F}_{\pm1}} A_N \cong A_N$. Since A_N is generated over $\mathbb{F}_{\pm1}$ by one averaging operation s_N (cf. **7.1.30**,(c); this is the only part of the proof valid over $\mathbb{F}_{\pm1}$, but not over $\mathbb{F}_1$ or $\mathbb{F}_\varnothing$), all we have to check is that whenever we have *two* commuting operations s_N, s'_N, both satisfying (7.1.26.4)–(7.1.26.6), then necessarily $s'_N = s_N$. This can be shown exactly in the same way as in **5.7.3**, applying commutativity relation between s_N and s'_N to a Latin square filled by basis elements $\{1\}, \ldots, \{N\}$.

Now only case $i \neq j$ remains. Since $U_1 \cap U_2 = \operatorname{Spec} B_N$, we are to check that $A_N \otimes \mathbb{Z} = B_N$. Put $\Sigma := A_N \otimes \mathbb{Z}$, and denote by f the unary operation of A_N and of Σ, given by $f := s_N(\boldsymbol{e}, 0, \ldots, 0)$, i.e. $f = 1/N$. One easily checks that $f \cdot N = \boldsymbol{e}$ in $|\Sigma|$, by applying commutativity relation for s_N and the N-tuple addition operation $\{1\} + \cdots + \{N\}$ to $N \times N$-matrix (x_{ij}) with $x_{ii} = \boldsymbol{e}$, $x_{ij} = 0$ for $i \neq j$. This means that $f = 1/N$ is invertible in Σ, hence $\Sigma = \Sigma[(1/N)^{-1}] = (A_N \otimes \mathbb{Z})[(1/N)^{-1}] = A_N[(1/N)^{-1}] \otimes \mathbb{Z} = B_N \otimes \mathbb{Z} = B_N \otimes_{\mathbb{Z}} \mathbb{Z} \otimes \mathbb{Z} = B_N \otimes_{\mathbb{Z}} \mathbb{Z} = B_N$. We've used here $A_N[(1/N)^{-1}] = B_N$, valid by **7.1.2**, as well as $\mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}$, shown in **5.1.22**.

This reasoning is a modification of that of **5.7.5**, used there to prove $\mathbb{Z}_{(\infty)} \otimes \mathbb{Z} = \mathbb{Q}$ and $\mathbb{Z}_\infty \otimes \mathbb{Z} = \mathbb{R}$.

7.1.48. (Projectivity of f_N^{NM}. $\widehat{\operatorname{Spec}\mathbb{Z}}$ as an infinite resolution of singularities.) Since $f_N^{MN} : S_{NM} \to S_N$ is a morphism between two projective generalized schemes, it is very reasonable to expect f_N^{NM} to be a projective morphism. Let's accept this without proof for the time being. Then we get the following picture. Each S_N is something like a curve, admitting $\operatorname{Spec}\mathbb{Z}$ as an open subset, and all projective maps f_N^{NM} are isomorphisms over this open subset, i.e. they change something only over ∞, so they can be thought of as a sort of blow-up (or a series of blow-ups) over ∞. Since the complement of $\operatorname{Spec}\mathbb{Z}$ after this "blow-up" still consists of one point (at least for $\mathcal{T}^u$), this looks very much like blowing up a complicated cusp-like singularity over a curve. This "cusp" appears to have infinite order, since it cannot be resolved in any finite number of steps; we have an infinite sequence of "blow-ups", and the "smooth model" $\widehat{\operatorname{Spec}\mathbb{Z}}$ appears only as a pro-object.

I would like to mention here that A. Smirnov, when we discussed possible meanings of $\widehat{\operatorname{Spec}\mathbb{Z}}$ several months ago, and this work was only in its initial stage, mentioned to me that he expects something like an infinite resolution of singularities over a cusp-like point ∞ to be necessary to construct the "correct" (or "smooth") compactification $\widehat{\operatorname{Spec}\mathbb{Z}}$. At that time I was somewhat skeptical about this. However, this completely intuitive perception of $\widehat{\operatorname{Spec}\mathbb{Z}}$ turned out to agree almost completely with the rigorous theory constructed here! This remains a mystery for me.

7.2. (Models over $\widehat{\operatorname{Spec}\mathbb{Z}}$, $\mathbb{Z}_\infty$ and $\mathbb{Z}_{(\infty)}$.) Now we are going to discuss models of algebraic varieties $X/\mathbb{Q}$, showing in particular that any such X admits at least one finitely presented model $\mathscr{X}$ over $\widehat{\operatorname{Spec}\mathbb{Z}}$. We also discuss models over $\mathbb{Z}_{(\infty)}$, $\mathbb{Z}_{(\infty)}$ and $\bar{\mathbb{Z}}_\infty$.

Then we are going to discuss the relation to Banach and even Kähler metrics, e.g. by showing that the Fubiny–Study metric on $\mathbb{P}^N(\mathbb{C})$ corresponds to a certain model $\bar{\mathbb{P}}^N/\bar{\mathbb{Z}}_\infty$ of $\mathbb{P}^N$. Unfortunately, models corresponding to "nice" metrics usually turn out to have "bad" algebraic properties (e.g. the model $\bar{\mathbb{P}}^N/\bar{\mathbb{Z}}_\infty$ just discussed is not finitely presented), and conversely. This probably explains the complexity of (classical) Arakelov geometry, which might be compared to the study of not necessarily finitely presented models $\mathscr{X}/\mathbb{Z}_p$ of an algebraic variety $X/\mathbb{Q}$. This also explains why we can construct our variant of Arakelov geometry in a simpler fashion, just by considering finitely presented models whenever possible.

7.2.1. (Reduction from $\widehat{\operatorname{Spec}\mathbb{Z}}$ to $\operatorname{Spec}\mathbb{Z}_{(\infty)}$.) Recall that constructing a finitely presented model $\bar{\mathscr{X}}/\widehat{\operatorname{Spec}\mathbb{Z}}$ of an algebraic variety $X/\mathbb{Q}$ is essentially the same thing as constructing finitely presented models $\mathscr{X}/\mathbb{Z}$ and $\mathscr{X}_\infty/\mathbb{Z}_{(\infty)}$ of this X (cf. **7.1.24**), so we need to concentrate our efforts only on finitely presented models over $\mathbb{Z}_{(\infty)}$, assuming the first part (models over $\mathbb{Z}$) to be well-known.

7.2.2. (Notations: $V \subset K$.) Let us fix some field K and an "archimedian valuation ring" $V \subset K$ in the sense of **5.7.13**. Usually we consider either $V = \mathbb{Z}_{(\infty)}$, $K = \mathbb{Q}$, or $V = \mathbb{Z}_\infty$, $K = \mathbb{R}$, or $V = \bar{\mathbb{Z}}_\infty$, $K = \mathbb{C}$. Denote by $|\cdot|$ the archimedian valuation of K corresponding to V, and let $\tilde{\mathbb{Q}} \subset K$ be the closure of $\mathbb{Q} \subset K$. Obviously, $\tilde{\mathbb{Q}}$ is canonically isomorphic to a subfield of $\mathbb{R}$. We'll consider only couples (V, K), which satisfy the following extra condition: $|\lambda|$ *belongs to* $\tilde{\mathbb{Q}} \subset \mathbb{R}$ *for any* $\lambda \in K$. This enables us to consider $|\lambda|$ as an element of K.

Notice that for any $f \in V$ with $0 < |f| < 1$ we have $V[f^{-1}] = K$, hence $\operatorname{Spec} K$ is isomorphic to a principal open subset of $\operatorname{Spec} V$. The image of the only point of $\operatorname{Spec} K$ in $\operatorname{Spec} V$ is the *generic* or *open point* ξ of $\operatorname{Spec} V$. Any $\mathscr{X}/V$ contains its generic fiber $\mathscr{X}_\xi = \mathscr{X}_{(K)} = \mathscr{X} \otimes_V K$ as an open generalized subscheme.

Now we are going to discuss models $\mathscr{X}/V$ of a finitely presented scheme X/K, i.e. an algebraic variety over K.

7.2.3. (Norm on $K[T_1, \ldots, T_k]$.) We denote by $\|\cdot\|$ the (coefficientwise) L_1-norm on the polynomial algebra $A = K[T_1, \ldots, T_k]$, defined as follows:

$$\|P\| = \sum_{\alpha \in \mathbb{N}_0^k} |c_\alpha|, \quad \text{if } P = \sum_{\alpha \in \mathbb{N}_0^k} c_\alpha T^\alpha \tag{7.2.3.1}$$

Here we use the multi-index notation: $T^\alpha = T_1^{\alpha_1} \cdots T_k^{\alpha_k}$. Furthermore, for any element $\lambda = \sum_{i=1}^n \lambda_i \{i\} \in A(n) = A^n$ we put $\|\lambda\| := \sum_i \|\lambda_i\|$. Then generalized polynomial ring $B := V[T_1, \ldots, T_k] \subset K[T_1, \ldots, T_k]$ can be described as follows: $B(n) = \{\lambda \in A(n) : \|\lambda\| \leq 1\}$.

Notice that our assumptions on (V, K) imply $\|\lambda\| \in K$ for any $\lambda \in A(n)$, hence we can write any such $\lambda \neq 0$ as $c \cdot \mu$, where $c := \|\lambda\| \in K^\times$, and $\mu := c^{-1}\lambda \in A(n)$ is of norm one, and in particular belongs to $B(n)$.

7.2.4. (Finitely presented models of affine varieties.) Let X be an affine variety over K, i.e. a finitely presented affine scheme over $\operatorname{Spec} K$. Then $X = \operatorname{Spec} A$, where $A = K[T_1, \ldots, T_k]/(f_1, \ldots, f_m)$, for some finite number of polynomials $f_j \in A_0 := K[T_1, \ldots, T_k]$, which can be assumed to be $\neq 0$. Furthermore, we are free to multiply f_j by non-zero scalars from K, so we can make $\|f_j\| \leq 1$, or even $\|f_j\| = 1$, i.e. $f_j \in |B_0|$, where $B_0 := V[T_1, \ldots, T_k] \subset A_0$ is the (generalized unary) polynomial algebra over V.

Put $B := B_0/\langle f_1 = 0, \ldots, f_m = 0\rangle = V[T_1^{[1]}, \ldots, T_k^{[1]} \,|\, f_1 = 0, \ldots, f_m = 0]$. Then $\mathscr{X} := \operatorname{Spec} B$ is an affine finitely presented generalized scheme over V, such that $\mathscr{X}_{(K)} = \operatorname{Spec}(B \otimes_V K) = \operatorname{Spec} A = X$, i.e. *any affine variety X/K admits at least one finitely presented affine model $\mathscr{X}/V$.*

7.2.5. (Finitely presented models of projective varieties.) Similarly, let $X = \operatorname{Proj} A$ be a projective variety over K. Then $A = K[T_0, \ldots, T_k]/(f_1, \ldots, f_m)$ for some homogeneous polynomials $f_j \neq 0$. Multiplying f_j by suitable constants from $K^\times$, we can assume $\|f_j\| = 1$, i.e. f_j lie in the generalized graded polynomial algebra $R := V[T_0, \ldots, T_k]$, with all free generators T_i unary of degree one. Put $B := R/\langle f_1 = 0, \ldots, f_m = 0\rangle$. This is a finitely presented unary graded algebra over V, such that $|B|_+$ is generated by finitely many elements of degree one. We conclude that $\mathscr{X} := \operatorname{Proj} B$ is a finitely presented projective model of X over V. In particular, *any projective variety X/K admits at least one projective finitely presented model $\mathscr{X}/V$,* and *any projective variety $X/\mathbb{Q}$ admits a finitely presented model $\bar{\mathscr{X}}$ over $\widehat{\operatorname{Spec}\mathbb{Z}}$.*

7.2.6. (Presence of V-torsion in finitely presented models.) Notice that the finitely presented models $\mathscr{X}$ constructed above usually have some V-torsion, i.e. natural homomorphisms $\mathscr{O}_\mathscr{X} \to \mathscr{O}_\mathscr{X} \otimes_V K$ are not necessarily monomorphisms of sheaves of generalized rings. In fact, almost all finitely presented schemes over $\operatorname{Spec}\mathbb{Z}_{(\infty)}$, with the notable exception of $\mathbb{A}^n_{\mathbb{Z}_{(\infty)}}$ and $\mathbb{P}^n_{\mathbb{Z}_{(\infty)}}$, do have some "torsion over ∞", hence the same is true for finitely presented models $\bar{\mathscr{X}}/\widehat{\operatorname{Spec}\mathbb{Z}}$: we can eliminate torsion over finite primes p if we want, but in general not over ∞.

One might think about this "torsion over ∞" in models $\mathscr{X}$ as a sort of "built-in analytic torsion". This explains why we expect the Grothendieck–Riemann–Roch formula to hold for such (finitely presented) models without any complicated correction terms: all analytic torsion is already accounted for by this torsion over ∞, built into the structure of the model chosen.

7.2.7. (Existence of torsion-free models.) Suppose that our variety X/K can be embedded as a closed subvariety into some variety P/K, which admits a torsion-free model $\bar{P}/V$. This condition is always fulfilled for any projective variety X, since we can take $P := \mathbb{P}^N_K$, $\bar{P} := \mathbb{P}^N_V$ for a suitable $N \geq 0$.

Then we can construct a torsion-free model $\bar{X}/V$ of X/K, simply by taking the "scheme-theoretic closure" of $X \subset P = \bar{P}_\xi \subset \bar{P}$ inside $\bar{P}$. More formally, let $i : X \to P$ be the closed immersion of X into P, $j : P \to \bar{P}$ be the open immersion of the generic fiber of $\bar{P}$ into $\bar{P}$. Consider following homomorphism of sheaves of generalized $\mathscr{O}_{\bar{P}}$-algebras: $\varphi : \mathscr{O}_{\bar{P}} \to j_*j^*\mathscr{O}_{\bar{P}} = j_*\mathscr{O}_P \twoheadrightarrow j_*i_*\mathscr{O}_X$. The first arrow here is the adjointness morphism for j_* and j^*, and the second one is obtained by applying j_* to $\mathscr{O}_P \twoheadrightarrow i_*\mathscr{O}_X$. Since j is quasicompact and quasiseparated (being affine), all $\mathscr{O}_{\bar{P}}$-algebras involved are quasicoherent, hence the same is true for $\mathscr{O}_{\bar{X}} := \varphi(\mathscr{O}_{\bar{P}}) \subset j_*i_*\mathscr{O}_X$. Now let $\bar{X}$ be the "closed" subscheme of $\bar{P}$ defined by this quasicoherent strict quotient $\mathscr{O}_{\bar{P}} \twoheadrightarrow \mathscr{O}_{\bar{X}}$. It is immediate that this $\bar{X}$ is indeed a torsion-free model of X.

Notice, however, that in general $\bar{X}$ will be of finite type over V, but not finitely presented. Therefore, such torsion-free models over $\mathbb{Z}_{(\infty)}$ of varieties $X/\mathbb{Q}$ cannot be used (together with some flat model $\mathscr{X}/\mathbb{Z}$) to construct models $\bar{\mathscr{X}}$ of X over $\widehat{\operatorname{Spec}\mathbb{Z}}$. All we can do is to write $\bar{X}$ as a projective limit of finitely presented models $\bar{X}_\alpha$ of X over V, use these $\bar{X}_\alpha$ to obtain a projective system of finitely presented models $\bar{\mathscr{X}}_\alpha/\widehat{\operatorname{Spec}\mathbb{Z}}$ of X, and put $\bar{\mathscr{X}} :=$ "$\varprojlim$" $\bar{\mathscr{X}}_\alpha$. In this way we obtain a torsion-free ("flat") finitely presented model $\bar{\mathscr{X}}/\widehat{\operatorname{Spec}\mathbb{Z}}$ of X in the category of pro-generalized schemes. In the fancy language of **7.1.48** one might say that in order to eliminate "embedded analytic torsion" of a finitely presented model over $\widehat{\operatorname{Spec}\mathbb{Z}}$ we have to perform an infinite resolution of singularities.

7.2.8. (Extending rational points to sections.) One expects that any rational point $P \in X(\mathbb{Q})$ of a *projective* variety $X/\mathbb{Q}$ extends to a unique section $\sigma_P : \widehat{\operatorname{Spec}\mathbb{Z}} \to \bar{\mathscr{X}}$, for any projective model $\bar{\mathscr{X}}/\widehat{\operatorname{Spec}\mathbb{Z}}$ of X. According to **7.1.23**, finding such a σ_P is equivalent to finding a section σ'_P of a model $\mathscr{X}/\mathbb{Z}$, and a section σ''_P of model $\mathscr{X}_\infty/\mathbb{Z}_{(\infty)}$, both inducing P on the generic fiber. Existence and uniqueness of σ'_P is very well known, so we are reduced to the case of $\mathbb{Z}_{(\infty)}$-models.

In order to lift K-rational points of $\mathscr{X}/V$ to V-rational points one usually has to apply the valuative criterion of properness in the easy direction. We would like, therefore, to discuss whether valuative criteria still work (at least in the easy direction) over archimedian valuation rings $V \subset K$ of **7.2.2**.

Definition 7.2.9 *We say that a generalized scheme morphism $f : X \to S$ satisfies the* valuative criterion of separability *(resp.* properness*)* (with respect to (V, K))*, if, given any $u \in X(K)$ and $v \in S(V)$ fitting into commutative diagram*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "u"] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} V \arrow[r, "v"] \arrow[ur, dashed, "w"] & S \end{tikzcd} \tag{7.2.9.1}$$

one can find at most one (resp. exactly one) $w \in X(V)$ completing the above diagram.

It is clear that the class of morphisms satisfying the valuative criterion of separability (resp. properness) with respect to (V, K) is stable under composition and base change.

Notice that, given a diagram as above, we can always replace X by $X_{(\operatorname{Spec} V)} = X \times_S \operatorname{Spec} V$, f by $f_{(\operatorname{Spec} V)}$, and reduce the above "lifting problem" to the case $S = \operatorname{Spec} V$, $v = \mathrm{id}_S$.

Lemma 7.2.10 *If $f : X \to S$ is "separated" in the sense of* **6.5.23***, i.e. if $\Delta_{X/S}$ is a "closed" immersion (cf.* loc. cit.*), then f satisfies the valuative criterion of separability with respect to (V, K).*

Proof. First of all, we can replace $f : X \to S$ by its base change with respect to $v : \operatorname{Spec} V \to S$ from (7.2.9.1), and assume $S = \operatorname{Spec} V$, $v = \mathrm{id}_S$. Let w, $w' : S \to X$ be two sections of $f : X \to S$, coinciding on $\operatorname{Spec} K \subset S$, i.e. having the same generic fibers $w_{(K)} = w'_{(K)}$. Let $T \subset S$ be the pullback of $\Delta_{X/S} : X \to X \times_S X$ with respect to $(w, w') : S \to X \times_S X$. On one hand, $T \subset S$ is a "closed" subscheme of S, since $f : X \to S$ has been supposed to be "separated". On the other hand, $T = \operatorname{Ker}(w, w' : S \rightrightarrows X)$, hence T is the largest subobject of S, such that $w|_T = w'|_T$, and the construction of T commutes with base change. Now $w_{(K)} = w'_{(K)}$ implies $T_{(K)} = \operatorname{Ker}(w_{(K)}, w'_{(K)}) = S_{(K)}$. Applying lemma **7.2.11** below, we see that $S = T$, hence $w = w'$, q.e.d.

Lemma 7.2.11 *If T is a "closed" subscheme of $S = \operatorname{Spec} V$, such that $T_{(K)} = S_{(K)}$, then $T = S$. Furthermore, this statement holds for any monomorphism of generalized rings $V \to K$, not only for embeddings of an archimedian valuation ring V into its fraction field K.*

Proof. By definition a "closed" subscheme $T \subset S = \operatorname{Spec} V$ equals $\operatorname{Spec} W$ for some strict quotient W of V. On the other hand, condition $T_{(K)} = S_{(K)}$ means that $\operatorname{Spec} K \to \operatorname{Spec} V$ factorizes through $T = \operatorname{Spec} W$, i.e. that monomorphism $V \hookrightarrow K$ factorizes through strict epimorphism $\pi : V \twoheadrightarrow W$. This means that π is both a strict epimorphism and a monomorphism, hence an isomorphism, i.e. $W = V$ and $T = S$.

Lemma 7.2.12 *Any "closed" immersion $f : X \to Y$ satisfies the valuative criterion of properness with respect to (V, K).*

Proof. Applying base change with respect to $\operatorname{Spec} V \xrightarrow{v} Y$ as before, we are reduced to case $Y = S = \operatorname{Spec} V$, $X = T \subset S$ a "closed" subscheme of S. We have to check that any point $u \in T_S(K)$ lifts to exactly one point $w \in T_S(V)$. If $T_S(K) = \varnothing$, then $T_S(V) = \varnothing$ and the statement is trivial. If $T_S(K) \neq \varnothing$, then $T_S(K) \subset S_S(K)$ is a one-element set; this means that $\operatorname{Spec} K \to S$ factorizes through T, i.e. $T_{(K)} = S_{(K)}$. By **7.2.11** we obtain $T = S$, hence $T_S(V) = S_S(V) = \{\mathrm{id}_S\}$, and we are done.

Theorem 7.2.13 *Let $\mathscr{F}$ be a quasicoherent $\mathscr{O}_S$-module of finite type over some generalized scheme S, and X be a "closed" subscheme of $\mathbb{P}_S(\mathscr{F})$. Then $X \to S$ satisfies the valuative criterion of properness with respect to (V, K).*

Proof. (a) By base change we can again assume $S = \operatorname{Spec} V$, $v = \mathrm{id}_S$. Then $\mathscr{F} = \tilde{M}$ for a finitely generated V-module M, so we can find a strict epimorphism $\pi : V(n+1) \twoheadrightarrow M$, which induces a strict epimorphism of unary graded V-algebras $S_V(V(n+1)) \twoheadrightarrow S_V(M)$, hence a "closed" immersion $\mathbb{P}_S(\mathscr{F}) \to \mathbb{P}_S(V(n+1)) = \mathbb{P}^n_S$. Now $X \to \mathbb{P}^n_S$ is a "closed" immersion, hence it satisfies the valuative criterion of properness for (V, K) by **7.2.12**, so we are reduced to checking that $P := \mathbb{P}^n_S \to S$ satisfies the valuative criterion of properness with respect to (V, K), and we can still assume $v = \mathrm{id}_S$.

In other words, we are to check that the canonical map $\mathbb{P}^n(V) \to \mathbb{P}^n(K)$ is bijective, for any $n \geq 0$.

(b) Notice that $\operatorname{Pic}(V) = 0$, for example because any non-trivial finitely generated V-submodule $P \subset K$ is free of rank one: indeed, if $u_1, \ldots, u_n \in P$ are the V-generators of P, we just choose u_k with maximal norm $|u_k|$, and notice that P is already generated by u_k, since $u_i/u_k \in |V|$ for any other i.

(c) Applying **6.6.38**, we see that elements of $\mathbb{P}^n(V)$ are in one-to-one correspondence with strict quotients L of $V(n+1)$, isomorphic to $|V|$, and similarly for $\mathbb{P}^n(K)$. So let σ be an element of $\mathbb{P}^n(V)$, represented by a strict quotient $p : V(n+1) \twoheadrightarrow P$, where $P \cong |V|$. Let $p_K := p_{(K)} : K(n+1) = K^{n+1} \twoheadrightarrow P_K := P_{(K)} \cong K$ be its scalar extension to K; it defines the image of σ in $\mathbb{P}^n(K)$. Now consider the following commutative diagram:

$$\begin{tikzcd} V(n+1) \arrow[r, "\xi_{V(n+1)}"] \arrow[d, "p"] & K^{n+1} \arrow[d, "p_K"] \\ P \arrow[r, "\xi_P"] & P_K \end{tikzcd} \tag{7.2.13.1}$$

Here $\xi_M : M \to M_{(K)}$ denotes the natural map from a V-module M into its scalar extension to K. Notice that the horizontal arrows are injective, since $P \cong |V|$ and $V \to K$ is a monomorphism of generalized rings, and the vertical arrows are strict epimorphisms, i.e. surjective.

Therefore, the strict quotient $p : V(n+1) \twoheadrightarrow P$ is completely determined by p_K, since P can be identified with the image of $p_K \circ \xi_{V(n+1)}$. Furthermore, if we start from any point of $\mathbb{P}^n(K)$, i.e. from a strict epimorphism $p_K : K^{n+1} \twoheadrightarrow K$, we can put $P := \operatorname{Im} p_K \circ \xi_{V(n+1)} = p_K(V(n+1))$. This is a non-trivial finitely generated V-submodule of K, hence it is free of rank one by (b), so we get a strict quotient $V(n+1) \twoheadrightarrow P \cong |V|$, easily seen to coincide with original p_K after base change to K. In other words, we have just shown that any point of $\mathbb{P}^n(K)$ uniquely lifts to a point of $\mathbb{P}^n(V)$, q.e.d.

Corollary 7.2.14 *(a) Let $\mathscr{X}_\infty/V$ be a pre-unary projective model of a projective variety X/K (this means exactly that $\mathscr{X}_\infty$ can be embedded as a "closed" subscheme in $\mathbb{P}^n_V$, cf.* **6.6.40**)*. Then any rational point $P \in X(K)$ extends to a unique section $\sigma_P : \operatorname{Spec} V \to \mathscr{X}_\infty$.*

(b) Let $\bar{\mathscr{X}}/\widehat{\operatorname{Spec}\mathbb{Z}}$ be a finitely presented pre-unary projective model of a projective variety $X/\mathbb{Q}$. Then any point $P \in X(\mathbb{Q})$ extends to a unique section $\sigma_P : \widehat{\operatorname{Spec}\mathbb{Z}} \to \bar{\mathscr{X}}$.

Proof. (a) is an immediate consequence of **7.2.13**, and (b) follows from (a), together with classical valuative criterion of properness, which enables us to extend rational points to sections of proper models over $\operatorname{Spec}\mathbb{Z}$, and **7.2.8**, q.e.d.

Notice that this corollary is applicable in particular to the finitely presented projective models $\bar{\mathscr{X}}/\widehat{\operatorname{Spec}\mathbb{Z}}$ of projective varieties $X/\mathbb{Q}$ constructed in **7.2.5**.

7.3. ($\mathbb{Z}_\infty$-models and metrics.) We want to sketch briefly the relationship between models over $\mathbb{Z}_\infty$ or $\widehat{\operatorname{Spec}\mathbb{Z}}$ of an algebraic variety X and a vector bundle $\mathscr{E}$ on X, on one side, and metrics on complex points $X(\mathbb{C})$, $\mathscr{E}(\mathbb{C})$ of this algebraic variety and vector bundle, on the other side. This will relate our theory to "classical" Arakelov geometry, based on consideration of metrics.

7.3.1. (Case of $\mathbb{Z}_\infty$-lattices.) Suppose that E is a $\mathbb{Z}_\infty$-lattice, i.e. a $\mathbb{Z}_\infty$-module, such that $E \to E_{(\mathbb{R})}$ is injective and has compact image in finite-dimensional real vector space $E_{(\mathbb{R})}$. Then the same reasoning as in part (c) of the proof of **7.2.13** shows that $\mathbb{P}_{\mathbb{Z}_\infty}(E) \to \operatorname{Spec}\mathbb{Z}_\infty$ satisfies the valuative criterion of properness with respect to $(\mathbb{Z}_\infty, \mathbb{R})$. Indeed, the only difference is that, in the notation of (7.2.13.1), where we use E and $E_{(\mathbb{R})}$ instead of $V(n+1)$ and K^{n+1}, now $P := p_K(E) = p_{\mathbb{R}}(E) \subset \mathbb{R}$ is not a finitely generated non-trivial $\mathbb{Z}_\infty$-submodule of $\mathbb{R}$; however, it is non-trivial and compact, E being a $\mathbb{Z}_\infty$-lattice, and any non-trivial compact $\mathbb{Z}_\infty$-submodule of $\mathbb{R}$ is freely generated by any its element with maximal absolute value.

Reasoning further as in the proof of **7.2.13**, we see that, given any "closed" subscheme $\mathscr{X}$ of a projective bundle $\mathbb{P}_{\mathbb{Z}_\infty}(E)$ defined by a $\mathbb{Z}_\infty$-lattice E, we obtain a bijection between sections $\sigma : \operatorname{Spec}\mathbb{Z}_\infty \to \mathscr{X}$, and real points of projective variety $X := \mathscr{X}_{(\mathbb{R})} \subset \mathbb{P}_{\mathbb{R}}^{\dim E_{(\mathbb{R})}-1}$.

7.3.2. (Case of $\bar{\mathbb{Z}}_\infty$-lattices.) Similarly, any non-trivial compact $\bar{\mathbb{Z}}_\infty$-submodule of $\mathbb{C}$ is freely generated by any its element with maximal absolute value, so we can extend the above results to "closed" subschemes $\mathscr{X}$ of $\mathbb{P}_{\bar{\mathbb{Z}}_\infty}(E)$ for any $\bar{\mathbb{Z}}_\infty$-lattice E, i.e. $\mathscr{X}(\bar{\mathbb{Z}}_\infty) \to \mathscr{X}(\mathbb{C})$ is a bijection for any such $\mathscr{X}$.

Now consider projective complex algebraic variety $X := \mathscr{X}_{(\mathbb{C})}$. If $\mathscr{E}$ is a vector bundle over X, which can be extended to a vector bundle (or maybe just a finitely presented sheaf) $\bar{\mathscr{E}}$ on the whole of $\mathscr{X} \supset X$ (i.e. $\bar{\mathscr{E}}$ is a "model" of $\mathscr{E}$), then we can start from any point $P \in X(\mathbb{C})$, extend it to a section $\sigma_P : \operatorname{Spec}\bar{\mathbb{Z}}_\infty \to X$, and compute $\sigma_P^*\mathscr{E}$. This will be a finitely presented $\mathscr{O}_{\operatorname{Spec}\bar{\mathbb{Z}}_\infty}$-module, with the generic fiber canonically isomorphic to $\mathbb{C}$-vector space $\mathscr{E}(P)$. In other words, we've got a $\bar{\mathbb{Z}}_\infty$-structure on each fiber $\mathscr{E}(P)$, i.e. essentially a *complex Banach norm* on $\mathscr{E}(P)$ (cf. Chapter **2**). This explains the relationship between $\bar{\mathbb{Z}}_\infty$-models in our sense, and metrics on vector bundles, which are used instead of $\bar{\mathbb{Z}}_\infty$-structures in the classical approach to Arakelov geometry.

7.3.3. (Fubini–Study metric.) Let $Q(z_0, \ldots, z_n) := \sum_i |z_i|^2$ be the standard hermitian form on $\mathbb{C}^{n+1}$, and consider corresponding $\bar{\mathbb{Z}}_\infty$-lattice $E := \{z \in \mathbb{C}^{n+1} : Q(z) \leq 1\}$. Put $\tilde{\mathbb{P}}^n := \mathbb{P}_{\bar{\mathbb{Z}}_\infty}(E)$. Then $\tilde{\mathbb{P}}^n$ is a $\bar{\mathbb{Z}}_\infty$-model of complex projective space $\mathbb{P}^n_{\mathbb{C}}$, and according to **7.3.2**, any point $x \in \mathbb{P}^n(\mathbb{C})$ extends to a unique section $\sigma_x \in \Gamma(\tilde{\mathbb{P}}^n / \operatorname{Spec} \bar{\mathbb{Z}}_\infty)$.

Suppose that we are able to construct a non-trivial (co)metric on $\mathbb{P}^n(\mathbb{C})$, by pulling back with respect to σ_x some natural model of $\Omega^1_{\mathbb{P}^n/\mathbb{Z}}$, similarly to what we did in **2.12.2** for $\mathbb{A}^1_{\mathbb{Z}_\infty} = \operatorname{Spec} \mathbb{Z}_\infty[T]$ (cf. also **2.13.14**). Then the only possibility for this metric is to be the Fubiny–Study metric, up to multiplication by some constant. Indeed, this metric must be equivariant under the action of $\operatorname{Aut}(\tilde{\mathbb{P}}^n/\bar{\mathbb{Z}}_\infty) \supset \operatorname{Aut}_{\bar{\mathbb{Z}}_\infty}(E) = U(n+1)$, and the only metric (up to a constant) on $\mathbb{P}^n(\mathbb{C})$ equivariant under the (transitive) action of unitary group $U(n+1)$ is the Fubini–Study metric.

This argument extends to "closed" subvarieties $\mathscr{X} \subset \tilde{\mathbb{P}}^n$: if we manage to construct a metric on $\mathscr{X}(\mathbb{C})$, it must coincide (at least on the smooth locus of $X := \mathscr{X}_{(\mathbb{C})}$) with the metric on X, induced from the Fubini–Study metric on $\mathbb{P}^n_{\mathbb{C}} \supset X$. (It makes sense to construct models of such $X \subset \mathbb{P}^n_{\mathbb{C}}$ by taking the "scheme-theoretic closure" of X in $\tilde{\mathbb{P}}^n$, cf. **7.2.7**.)

After discussing this relationship between our "algebraic" approach to Arakelov geometry, based on generalized schemes and models over $\widehat{\operatorname{Spec} \mathbb{Z}}$, and the classical approach, based on introducing suitable metrics on the complex points of all varieties and vector bundles involved, we would like to discuss a purely arithmetic application.

7.4. (Heights of rational points.)

7.4.1. (Heights of rational points in projective space.) Let $x \in \mathbb{P}^n(\mathbb{Q})$ be a rational point of the n-dimensional projective space $\mathbb{P}^n$. Denote by $(x_0 : x_1 : \ldots : x_n)$ the homogeneous coordinates of x. These x_i are rational numbers, not all equal to zero, so we can define the *height* $H(x) = H_{\mathbb{P}^n}(x)$ of x by

$$H(x) := \prod_{v \in \mathbb{P} \cup \{\infty\}} \max\bigl(|x_0|_v, |x_1|_v, \ldots, |x_n|_v\bigr) \tag{7.4.1.1}$$

Here v runs over the set of all valuations of $\mathbb{Q}$, with the p-adic valuation $|\cdot|_p$ normalized by the usual requirement $|p|_p = p^{-1}$. Notice that this infinite product makes sense, because $|x_i|_v = 1$ for almost all (i.e. all but finitely many) valuations v. Furthermore, this product doesn't depend on the choice of homogeneous coordinates of x: if we multiply $(x_0 : x_1 : \ldots : x_n)$ by any $\lambda \in \mathbb{Q}^\times$, the product (7.4.1.1) is multiplied by $\prod_v |\lambda|_v$, equal to one by the product formula (1.1.6.1).

One can describe $H(x)$ in another way. Namely, dividing the x_i by their g.c.d., we can assume that the homogeneous coordinates x_i of point x are coprime integers. Then all terms in (7.4.1.1) for $v \neq \infty$ vanish, and we obtain

$$H(x) = \max\bigl(|x_0|, \ldots, |x_n|\bigr), \quad \text{if } (x_i) \text{ are coprime integers.} \tag{7.4.1.2}$$

In particular, $H(x)$ is a positive integer for any $x \in \mathbb{P}^n(\mathbb{Q})$.

7.4.2. (Heights of rational points on a projective variety.) Now suppose we have a projective variety $X/\mathbb{Q}$, embedded as a closed subvariety into $\mathbb{P}^n_{\mathbb{Q}}$. Then we put $H_X(x) := H_{\mathbb{P}^n}(x)$ for any $x \in X(\mathbb{Q}) \subset \mathbb{P}^n(\mathbb{Q})$.

Notice that this height $H_X(x)$ actually depends not only on $X/\mathbb{Q}$ and $x \in X(\mathbb{Q})$, but also on the choice of closed embedding $X \to \mathbb{P}^n$, or equivalently, on the choice of an ample line bundle $\mathscr{L}$ on X and a finite family of global sections of $\mathscr{L}$. However, it is a classical result that two such choices corresponding to the same ample line bundle give rise to logarithmic height functions $\log H_X$, $\log H'_X : X(\mathbb{Q}) \to \mathbb{R}_{\geq 0}$ that differ by a bounded amount.

7.4.3. (Logarithmic heights.) One usually defines *logarithmic height* functions $h_X : X(\mathbb{Q}) \to \mathbb{R}_{\geq 0}$ simply by putting $h_X := \log H_X$.

Theorem 7.4.4 *Let $\mathscr{X}$ be a finitely presented "closed" subscheme of $\mathscr{P} := \mathbb{P}^n_{\widehat{\operatorname{Spec}\mathbb{Z}}}$. Denote by $X \subset \mathbb{P}^n_{\mathbb{Q}}$ the generic fiber of $\mathscr{X}$, a closed subvariety of projective space $\mathbb{P}^n_{\mathbb{Q}}$. Let $j : \mathscr{X} \to \mathscr{P}$ be the "closed" immersion, and denote the ample (Serre) line bundle $j^*\mathscr{O}_{\mathscr{P}}(1)$ by $\mathscr{O}_{\mathscr{X}}(1)$. Let $p : \mathscr{X} \to \widehat{\operatorname{Spec}\mathbb{Z}}$ be the structural morphism. Then:*

(i) Any rational point $x \in X(\mathbb{Q}) \subset \mathbb{P}^n(\mathbb{Q})$ extends in a unique way to a section σ_x of $\mathscr{X}$ over $\widehat{\operatorname{Spec}\mathbb{Z}}$.

(ii) Degree of line bundle $\mathscr{L} := \sigma_x^*\mathscr{O}_{\mathscr{X}}(1)$ over $\widehat{\operatorname{Spec}\mathbb{Z}}$ equals $h_X(x) = \log H_X(x) = \log H_{\mathbb{P}^n}(x)$, the logarithmic height of x:

$$h_X(x) = \deg \sigma_x^*\mathscr{O}_{\mathscr{X}}(1) \tag{7.4.4.1}$$

Proof. (i) has been already shown in **7.2.14**, so we have to prove (ii). Notice that $j\sigma_x$ is the section of $\mathscr{P}/\widehat{\operatorname{Spec}\mathbb{Z}}$ extending $x \in X(\mathbb{Q}) \subset \mathbb{P}^n(\mathbb{Q})$, and $\sigma_x^*\mathscr{O}_{\mathscr{X}}(1) = \sigma_x^*j^*\mathscr{O}_{\mathscr{P}}(1) = (j\sigma_x)^*\mathscr{O}_{\mathscr{P}}(1)$. Therefore, we may assume $\mathscr{X} = \mathscr{P} = \mathbb{P}^n_{\widehat{\operatorname{Spec}\mathbb{Z}}}$ and $X = \mathbb{P}^n_{\mathbb{Q}}$.

(a) Recall that sections of $\mathscr{P} := \mathbb{P}^n_{\widehat{\operatorname{Spec}\mathbb{Z}}}$ over $\widehat{\operatorname{Spec}\mathbb{Z}}$ are in one-to-one correspondence with strict quotients $\mathscr{L}$ of trivial vector bundle $\mathscr{E} := \mathscr{O}^{(n+1)}_{\widehat{\operatorname{Spec}\mathbb{Z}}}$, which are line bundles. Furthermore, this correspondence transforms a section $\sigma : \widehat{\operatorname{Spec}\mathbb{Z}} \to \mathscr{P}$ into $\sigma^*(\pi)$, where $\pi : p^*\mathscr{E} \twoheadrightarrow \mathscr{O}_{\mathscr{P}}(1)$ is the canonical surjection over $\mathscr{P}$ (cf. **6.6.37**). In particular, section σ_x corresponds to strict epimorphism $\varphi := \sigma_x^*(\pi) : \mathscr{E} = \sigma_x^*p^*\mathscr{E} \twoheadrightarrow \sigma_x^*\mathscr{O}_{\mathscr{P}}(1) = \mathscr{L}$, i.e. this strict quotient of $\mathscr{E}$ is actually the line bundle $\mathscr{L}$ from the statement of the theorem.

(b) Let us study this strict epimorphism $\varphi : \mathscr{E} = \mathscr{O}^{(n+1)}_{\widehat{\operatorname{Spec}\mathbb{Z}}} \twoheadrightarrow \mathscr{L}$ in more detail. Put $L_{\mathbb{Z}} := \Gamma(\operatorname{Spec}\mathbb{Z}, \mathscr{L})$, $L_\infty := \mathscr{L}_\infty$, $L_{\mathbb{Q}} := \mathscr{L}_\xi$, and define $E_{\mathbb{Z}}$, …,

$\varphi_{\mathbb{Q}} : E_{\mathbb{Q}} \to L_{\mathbb{Q}}$ similarly. We obtain the following commutative diagram:

$$\begin{tikzcd} E_{\mathbb{Z}} \arrow[r, "u_E"] \arrow[d, "\varphi_{\mathbb{Z}}"] & E_{\mathbb{Q}} \arrow[d, "\varphi_{\mathbb{Q}}"] & E_\infty \arrow[l, "v_E"'] \arrow[d, "\varphi_\infty"] \\ L_{\mathbb{Z}} \arrow[r, "u_L"] & L_{\mathbb{Q}} & L_\infty \arrow[l, "v_L"'] \end{tikzcd} \tag{7.4.4.2}$$

All horizontal arrows are natural embeddings of $\mathbb{Z}$- or $\mathbb{Z}_{(\infty)}$-modules into their scalar extensions to $\mathbb{Q}$. Since all modules involved are free, and both $\mathbb{Z} \to \mathbb{Q}$ and $\mathbb{Z}_{(\infty)} \to \mathbb{Q}$ are monomorphisms of generalized rings, all horizontal arrows are injective. On the other hand, the vertical arrows are strict epimorphisms, i.e. surjective maps.

(c) Notice that $E_{\mathbb{Q}} = \mathbb{Q}^{n+1}$, and $L_{\mathbb{Q}} \cong \mathbb{Q}$. Fix any such isomorphism $L_{\mathbb{Q}} \cong \mathbb{Q}$; then the images $x_i := \varphi_L(e_i)$ of the standard basis elements $e_i \in \mathbb{Q}^{n+1}$, $0 \leq i \leq n$, are exactly the homogeneous coordinates $(x_0 : \ldots : x_n)$ of our original point $x \in \mathbb{P}^n(\mathbb{Q})$, just by definition of homogeneous coordinates (cf. **6.6.38**). We can always fix isomorphism $L_{\mathbb{Q}} \cong \mathbb{Q}$ so as to make all x_i coprime integers.

(d) Now identify $E_{\mathbb{Z}}$ and E_∞ with their images $\mathbb{Z}^n$ and $\mathbb{Z}_{(\infty)}(n)$ in $E_{\mathbb{Q}} = \mathbb{Q}^n$, and similarly $L_{\mathbb{Z}}$ and L_∞ with their images in $L_{\mathbb{Q}}$. Furthermore, identify $L_{\mathbb{Q}} = \mathbb{Q}$ as in (c), so as to make the homogeneous coordinates x_i coprime integers. Then $\varphi_{\mathbb{Q}} : \mathbb{Q}^n \to \mathbb{Q}$ is the map $e_i \mapsto x_i$, $L_{\mathbb{Z}} \subset \mathbb{Q}$ is identified with $\varphi_{\mathbb{Q}}(\mathbb{Z}^n)$, and $L_\infty \subset \mathbb{Q}$ is identified with $\varphi_{\mathbb{Q}}(\mathbb{Z}_{(\infty)}(n))$.

(e) Notice that $L_{\mathbb{Z}} \subset \mathbb{Q}$ is the $\mathbb{Z}$-submodule of $\mathbb{Q}$, generated by coordinates x_i. Since these coordinates are coprime integers, we get $L_{\mathbb{Z}} = \mathbb{Z}$. Similarly, $L_\infty \subset \mathbb{Q}$ is the $\mathbb{Z}_{(\infty)}$-submodule of $\mathbb{Q}$, generated by $x_0, \ldots, x_n$. Recall that any non-trivial finitely generated $\mathbb{Z}_{(\infty)}$-submodule of $\mathbb{Q}$ is freely generated by its generator of largest absolute value (cf. part (c) of the proof of **7.2.13**), hence $L_\infty = \lambda \cdot |\mathbb{Z}_{(\infty)}|$ for $\lambda := \max(|x_0|, \ldots, |x_n|)$.

(f) According to the description of $\deg \mathscr{L}$ given in **7.1.37**, we have to choose a free generator f_1 of $L_{\mathbb{Z}}$, a free generator f_2 of L_∞, compute f_2/f_1 in the generic fiber $L_{\mathbb{Q}}$, and then $\deg \mathscr{L} = \log(f_2/f_1)$. In our case $f_1 = 1$, $f_2 = \lambda$, hence $\deg \mathscr{L} = \log \lambda$. On the other hand, $\lambda = H_{\mathbb{P}^n}(x)$ by (7.4.1.2), hence $\deg \sigma_x^* \mathscr{O}_{\mathscr{X}}(1) = \deg \mathscr{L} = \log \lambda = \log H_X(x) = h_X(x)$, q.e.d.

7.4.5. (Comparison to classical Arakelov geometry.) Classical Arakelov geometry has a similar formula relating height $\hat{h}(x)$ of a point $x \in \mathbb{P}^n(\mathbb{Q})$ with arithmetic degree of the pullback of corresponding ample (Serre) line bundle $\mathscr{O}(1)$ on $\mathbb{P}^n(\mathbb{Q})$, equipped with its natural ("Fubini–Study") metric. However, the logarithmic height $\hat{h}(x)$ thus obtained is defined by formula

$$\hat{h}(x) = \log\bigl(|x_0|^2 + \cdots + |x_n|^2\bigr)^{1/2}, \quad \text{if } (x_i) \text{ are coprime integers.} \tag{7.4.5.1}$$

On the other hand, we obtain in **7.4.4** the "classical" height $h(x)$ of (7.4.1.2), used everywhere else in arithmetic geometry. Of course, this distinction is not very significant, since $\hat{h}(x)$ and $h(x)$ differ by a bounded amount, but it is quite amusing nonetheless.

8 Homological and homotopical algebra

The aim of this chapter is to develop a variant of homological algebra for the categories of modules over generalized rings. This achievement will be extended in the next chapter to the categories of (sheaves of) modules over a generalized scheme or even a generalized ringed topos. Such homological (or rather homotopical) algebra will be intensively used in the last chapter of this work to construct a reasonable arithmetic intersection theory in the framework of generalized schemes, suitable e.g. for a discussion of arithmetic Riemann–Roch theorem.

8.0. (Tale of the Great Plan.) Let us present a brief sketch of the layout of the remainder of this chapter. We also discuss some of ideas that will be used in the following two chapters. Thus one might consider this sketch as a common introduction to the three "homotopical" chapters of this work.

8.0.1. (Homotopical algebra and model categories.) It is quite clear that classical homological algebra cannot work properly in a non-additive setup, and that one should use homotopical algebra instead, based on simplicial objects, model and homotopic categories, and weak equivalences, rather than on chain complexes, derived categories and quasi-isomorphisms. Therefore, we recall the basic definitions and constructions of homotopical algebra before using them. Our basic reference here is the foundational work of Quillen ([Quillen]) that seems to be quite well suited for our purpose. For example, the results of *loc.cit.* enable us to construct a simplicial closed model category structure on $s(\Sigma\text{-}Mod)$ almost immediately. However, we sometimes insist on using a more modern terminology (e.g. we say "acyclic fibrations" instead of Quillen's "trivial fibrations").

8.0.2. (Comparison with the classical case: Dold–Kan and Eilenberg–Zilber theorems.) Of course, we would like to compare our constructions and definitions with the classical ones when the generalized rings under consideration turn out to be additive. The key tool here is the *Dold–Kan correspondence* between the category $s\mathcal{A}$ of simplicial objects over an abelian category $\mathcal{A}$ and the category $\mathrm{Ch}(\mathcal{A}) = \mathrm{Ch}_{\geq 0}(\mathcal{A})$ of non-negative chain complexes over $\mathcal{A}$. More precisely, these two categories turn out to be equivalent, and chain homotopies in $\mathrm{Ch}(\mathcal{A})$ correspond exactly to the simplicial homotopies in $s\mathcal{A}$.

We'd like to mention here the way the equivalence $K : \mathrm{Ch}(\mathcal{A}) \to s\mathcal{A}$ acts on objects:

$$(KC.)_n = \bigoplus_{\eta:[n]\twoheadrightarrow[p]} C_p \tag{8.0.2.1}$$

Here $[n] = \{0, 1, \ldots, n\}$, and η runs over all monotone surjective maps from

$[n]$ onto $[p]$. The quasi-inverse to K is the *normalization functor* $N : s\mathcal{A} \to \mathrm{Ch}(\mathcal{A})$; it admits an explicit description as well.

Another important tool here, especially useful when dealing with bisimplicial objects and chain bicomplexes over an abelian category, is the *Eilenberg–Zilber theorem* that asserts the following. Given a bisimplicial object $K_{..}$ over an abelian category $\mathcal{A}$, we can construct a chain bicomplex $N_I N_{II} K_{..}$ and consider its totalization $\mathrm{Tot}(N_I N_{II} K_{..})$; of course, this totalization corresponds to a certain simplicial object $\hat{K}_.$. Now the Eilenberg–Zilber theorem asserts that $\hat{K}_.$ is actually homotopy equivalent to the diagonal simplicial object $\tilde{K}_.$ of $K_{..}$, defined simply by $\tilde{K}_n := K_{nn}$. Therefore, the diagonal simplicial object of a bisimplicial object is something like the totalization of a bicomplex, and we exploit this observation in the non-additive case as well.

8.0.3. (Model category structure on $s(\Sigma\text{-}Mod)$.) For any algebraic monad Σ (i.e. a non-commutative generalized ring) we can obtain a simplicial closed model category structure on $s(\Sigma\text{-}Mod)$, by a direct application of a theorem of [Quillen]. We check that when Σ is additive (i.e. $\Sigma\text{-}Mod$ is abelian), $\Sigma = \mathbb{F}_\varnothing$ or $\mathbb{F}_1$, we recover the well-known model category structures on $s\mathcal{A}$ ($\mathcal{A}$ an abelian category), simplicial sets, and simplicial sets with a marked point. In particular, for an additive Σ the weak equivalences $f : X_. \to Y_.$ correspond via the Dold–Kan correspondence to the quasi-isomorphisms of non-negative chain complexes. Therefore, the corresponding homotopic category $\mathrm{Ho}\, s(\Sigma\text{-}Mod)$, obtained by localizing $s(\Sigma\text{-}Mod)$ with respect to all weak equivalences, is a natural replacement for the subcategory $\mathcal{D}^{\leq 0}(\Sigma\text{-}Mod)$ of the derived category $\mathcal{D}^-(\Sigma\text{-}Mod)$; if necessary, we can recover an analogue of $\mathcal{D}^-(\Sigma\text{-}Mod)$ itself by formally inverting the translation functor in $\mathcal{D}^{\leq 0}$. We put $\mathcal{D}^{\leq 0}(\Sigma) := \mathrm{Ho}\, s(\Sigma\text{-}Mod)$.

Moreover, for any homomorphism of algebraic monads $\varphi : \Sigma \to \Sigma'$ we show that the arising adjoint pair of functors φ^*, φ_* between $s(\Sigma\text{-}Mod)$ and $s(\Sigma'\text{-}Mod)$ constitute a Quillen functor. Therefore, we obtain the corresponding derived functors $\mathbb{L}\varphi^* : \mathcal{D}^{\leq 0}(\Sigma) \to \mathcal{D}^{\leq 0}(\Sigma')$ and $\mathbb{R}\varphi_*$ in the opposite direction, adjoint to each other. When both Σ and Σ' are additive, these derived functors reduce via the Dold–Kan correspondence to their classical counterparts.

8.0.4. (Derived tensor products, symmetric and exterior powers.) Of course, we can derive the tensor product $\otimes_\Sigma$, thus obtaining a derived tensor product $\underline{\underline{\otimes}} : \mathcal{D}^{\leq 0}(\Sigma) \times \mathcal{D}^{\leq 0}(\Sigma) \to \mathcal{D}^{\leq 0}(\Sigma)$. By Eilenberg–Zilber this derived tensor product turns out to coincide with the usual one for an additive Σ. It has almost all properties one expects from it, e.g. commutativity, associativity, compatibility with $\mathbb{L}\varphi^*$, with the notable exception of the projection formula for a non-unary φ.

Moreover, when Σ is alternating, we can try derive the exterior powers, thus obtaining some functors $\mathbb{L}\bigwedge^r$, essentially in the same way Dold and Puppe constructed their homology of non-additive functors. However, it turns out to be technically more convenient to derive the *symmetric powers* instead, similarly to what Dold and Puppe actually did (notice, however, that they had a simpler situation since they had to derive *non-additive functors* between *additive categories*). Among other things, this makes sense for any generalized rings, not just for alternating $\mathbb{F}_{\pm 1}$-algebras.

We show that these derived symmetric powers $\mathbb{L}S^r$ satisfy at least some of their classical properties. These properties suffice to introduce a pre-λ-ring structure on K_0 of perfect complexes of modules later in Chapter **10**.

8.0.5. (Global situation.) Of course, we'd like to generalize the above constructions to the case of sheaves (quasicoherent or not) of $\mathscr{O}_X$-modules on a generalized scheme X, or even on a generalized ringed topos $(X, \mathscr{O}_X)$. The obvious problem is that we don't have any reasonable model category structure on $s(\mathscr{O}_X\text{-}Mod)$. This is due to the fact we don't have enough projectives in these categories. However, we can still define the set of weak equivalences in such categories: for example, if $(X, \mathscr{O}_X)$ has enough points (e.g. is a topological space), then $f : K. \to L.$ is a weak equivalence iff $f_p = p^*f$ is a weak equivalence in $s(\mathscr{O}_{X,p}\text{-}Mod)$ for any point p of X. For simplicial quasicoherent sheaves on a generalized scheme X it suffices to require $\Gamma(U, f)$ to be a weak equivalence of simplicial $\Gamma(U, \mathscr{O}_X)$-modules for all U from some affine cover of X. In the general case we have to consider the pro-object of all hypercoverings of X instead; this presents some technical but not conceptual complications.

In any case, once we have a class of weak equivalences, we can define $\mathcal{D}^{\leq 0}(X, \mathscr{O}_X)$ to be the localization of $s(\mathscr{O}_X\text{-}Mod)$ with respect to this class of morphisms; the only complication is to prove that we still get an $\mathcal{U}$-category, i.e. that all sets of morphisms in $\mathcal{D}^{\leq 0}$ remain small. This might be achieved by considering only hypercoverings composed from objects of a small generating family for topos X.

After that we can construct a derived functor $\mathbb{L}f^* : \mathcal{D}^{\leq 0}(Y, \mathscr{O}_Y) \to \mathcal{D}^{\leq 0}(X, \mathscr{O}_X)$ for any morphism of generalized ringed topoi $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$, as well as the derived tensor product on these categories. For these constructions we need to consider the categories of presheaves as well: if $X = \tilde{\mathcal{S}}$ for some small site $\mathcal{S}$, then the category of presheaves of $\mathscr{O}_X$-modules on $\mathcal{S}$ does have a closed model category structure. Notice that we cannot obtain any $\mathbb{R}f_*$ in this setup, so the projection formula cannot even be written down.

8.0.6. (Perfect complexes.) Consider the case of a classical ring Σ first, so that we have the Dold–Kan correspondence. One checks directly that a chain complex $C.$ consists of finitely generated projective Σ-modules iff the same holds for the corresponding simplicial object $(KC.)$., and that $C.$ has bounded homology iff the natural map $\mathrm{sk}_N KC. \to KC.$ is a weak equivalence for $N \gg 0$. Therefore, both these conditions can be stated on the level of simplicial objects, and we can define the category of perfect complexes of Σ-modules (or rather perfect simplicial Σ-modules) for any algebraic monad Σ. We can extend these definitions to the case of a generalized scheme X as well. The only potential problem here is whether we want to consider the locally projective or just the locally free $\mathscr{O}_X$-modules, since in general we don't expect these two classes to coincide.

8.0.7. (K-theory.) After that we are ready to construct $K_0(\mathrm{Perf}\, X)$. Of course, this will be the abelian group generated by weak equivalence classes of perfect simplicial $\mathscr{O}_X$-modules modulo some relations. Clearly, we must require $[A. \oplus B.] = [A.] + [B.]$. A less trivial relation is this: if $A.$ and $B.$ are two perfect complexes, such that A_n is isomorphic to B_n for each $n \geq 0$ (no compatibility with morphisms $A_n \to A_m$ required!), then $[A.] = [B.]$. These relations turn out to be sufficient in the additive case, so we use them in the general case as well.

The derived tensor product defines a product on $K^\bullet(X) := K_0(\mathrm{Perf}\, X)$, and $K^\bullet(X)$ depends contravariantly on X. After that we use derived symmetric powers $\mathbb{L}S^r$ to define a pre-λ-ring structure on $K^\bullet(X)$. We would like to check that this pre-λ-ring is actually a λ-ring. However, we didn't manage to prove all necessary relations for our derived symmetric powers so far, so we replace $K^\bullet(X)$ with its largest λ-ring quotient $K^\bullet(X)_\lambda$ instead.

This means that we can define on $K^\bullet(X)_\lambda$ the γ-operations and the γ-filtration in the way known since Grothendieck, and define the Chow ring of X by $\mathrm{Ch}^\cdot(X) := \mathrm{gr}_\gamma K^\bullet(X)_{\lambda,\mathbb{Q}}$. Grothendieck's argument gives us immediately a theory of Chern classes with values in $\mathrm{Ch}^\cdot(X)$ enjoying all the usual formal properties. An alternative approach, due to Soulé, is to define $\mathrm{Ch}^i(X)$ as the weight i part of $K^\bullet(X)_{\lambda,\mathbb{Q}}$ with respect to Adams operations. We show that these two approaches yield the same result whenever the γ-filtration is finite and separated.

8.0.8. (Geometric properties: projective bundles, regular embeddings and Gysin maps.) Of course, we'd like to prove the usual geometric properties of $\mathrm{Ch}^\cdot(X)$ and our Chern classes, e.g. the projective bundle theorem, existence of Gysin maps and so on. Such considerations are important for proving any sort of Riemann–Roch theorem; however, we postpone these "geometric" considerations until a subsequent work.

8.0.9. (Covariant data and cosimplicial objects.) Up to this point we've been constructing only contravariant data, similar to the cohomology ring, e.g. the perfect complexes, Chern classes, Chow rings and so on. However, we'd like to have something covariant as well, similar to the homology groups, e.g. the complexes with bounded coherent cohomology, $K.(X) := K_0(\operatorname{Coh} X)$ etc. It is natural to require these covariant objects to have an action of corresponding contravariant objects, similar to the $\cap$-product action of $H^*(X)$ on $H_*(X)$ or to the action of $K^{\cdot}(X)$ on $K.(X)$, given by the derived tensor product in the classical case. Among other things, this would enable us to write down a projection formula.

We see that we need something like cochain complexes. In the additive case the dual of Dold–Kan correspondence assures us that the category of non-negative cochain complexes over an abelian category $\mathcal{A}$ is equivalent to the category $c\mathcal{A}$ of *cosimplicial* objects over $\mathcal{A}$. So we are tempted to use $c(\Sigma\text{-}Mod)$ and $c(\mathscr{O}_X\text{-}Mod)$ in the general case as well. Indeed, the action of $s(\Sigma\text{-}Mod)$ on $c(\Sigma\text{-}Mod)$ would be given by the derived inner Hom $\mathbb{R}\,\mathbf{Hom}_\Sigma$, and similarly on any generalized ringed topos $(X, \mathscr{O}_X)$, and we can even hope to obtain a projection formula just from the adjointness of $\otimes$ and $\mathbf{Hom}$, and f^* and f_*, *provided we are able to define a model category structure on $c\mathcal{A}$ and construct the derived direct image $\mathbb{R}f_*$.*

Here is the problem. We cannot construct a reasonable model category structure on $c(\Sigma\text{-}Mod)$ or $c(\mathscr{O}_X\text{-}Mod)$ since these categories don't have enough injectives, and cannot derive f_* for the same reason. However, something can still be done. First of all, we can try to derive $\mathbf{Hom}$: $s(\Sigma\text{-}Mod)^{op}\times c(\Sigma\text{-}Mod) \to c(\Sigma\text{-}Mod)$ using only the model category structure on the first argument. Next, in the additive case we can compute $\mathbb{R}\Gamma$ and $\mathbb{R}f_*$ by means of Verdier theorem (cf. [SGA4], V) that asserts that all cohomology groups can be computed by something like Čech cohomology provided we consider all hypercoverings (which are simplicial objects themselves), not just the coverings, so we can mimic this construction in the non-additive case as well.

8.0.10. (Homotopy groups.) Since any $A.$ in $s(\Sigma\text{-}Mod)$ can be considered as a simplicial set, we can compute its homotopy invariants $\pi_n(A.) = \pi_n(\mathbb{R}\Gamma_\Sigma A.)$, $n \geq 0$, at least if we are given a base point in A_0, e.g. if Σ has a zero constant. Of course, $\pi_0(A)$ is just a (pointed) set, $\pi_1(A)$ is a group, and $\pi_n(A)$ is an abelian group for $n \geq 2$. Moreover, we get a Σ-structure on each of these sets, compatible with the group structure for $n \geq 1$. Recall that the category of abelian groups is the category of modules over $\mathbb{Z}$, and the category of groups is defined by some non-commutative algebraic monad $\mathbb{G}$ as well. All this means that $\pi_0(A)$ belongs to $\Sigma\text{-}Mod$, $\pi_1(A)$ to $(\Sigma\otimes\mathbb{G})\text{-}Mod$, and $\pi_n(A)$ to $(\Sigma\otimes\mathbb{Z})\text{-}Mod$. For example, if Σ is additive, then $\Sigma\otimes\mathbb{G} \cong \Sigma\otimes\mathbb{Z} \cong \Sigma$, and the Σ-modules $\pi_n(A)$ are canonically isomorphic to the homology groups $H_n(NA)$ of corresponding chain complex NA. Another example: if $\Sigma = \mathbb{Z}_\infty$, then $\pi_0(A)$ is just a $\mathbb{Z}_\infty$-module, $\pi_1(A)$ is a $\mathbb{Z}_\infty \otimes \mathbb{G}$-module, and all higher $\pi_n(A)$, $n \geq 2$, are $\mathbb{R}$-vector spaces.

Notice that the model category structure on $s(\Sigma\text{-}Mod)$ is defined in such a way that $f : A \to B$ is a weak equivalence iff all $\pi_n(f)$ are isomorphisms (more precisely, $\pi_n(f, x) : \pi_n(A, x) \to \pi_n(B, f(x))$ for all $n \geq 0$ and all choices of base point $x \in A_0$). However, two simplicial Σ-modules with isomorphic homotopy groups need not be weakly isomorphic or isomorphic in the homotopic category; such examples are very well known even in the additive case, e.g. $\Sigma = \mathbb{Z}$. In our case the homotopy groups carry even less information compared to the corresponding object of the homotopic category. For example, the "cones" of $\mathbb{Z}_\infty \xrightarrow{1/2} \mathbb{Z}_\infty$ and $\mathbb{Z}_\infty \xrightarrow{1/3} \mathbb{Z}_\infty$ in $\mathcal{D}^{\leq 0}(\mathbb{Z}_\infty)$ turn out to have the same homotopy groups ($\pi_n = 0$ for $n > 0$, $\pi_0 = \mathbb{Z}_\infty/\mathfrak{m}_\infty$), but they don't seem to be isomorphic in $\mathcal{D}^{\leq 0}(\mathbb{Z}_\infty)$…

All this means that we shouldn't expect to extract anything like the Euler characteristic of a simplicial Σ-module just from its homotopy groups.

8.0.11. (Euler characteristics.) Of course, we'd like to be able to define and compute Euler characteristics of the (co)simplicial objects we construct, since this is the natural way to obtain arithmetic intersection numbers in our setup. Let's consider the case, say, of a perfect simplicial object $A.$ over a generalized ring Σ. Let ℓ be an additive function from the set of isomorphism classes of finitely generated projective modules over Σ into, say, a $\mathbb{Q}$-vector space V, additivity meaning here just $\ell(P \oplus Q) = \ell(P) + \ell(Q)$.

If Σ is additive, we can consider the corresponding normalized chain complex $C. := NA.$, and then $A. = KC.$ can be recovered according to formula (8.0.2.1); by additivity we get

$$\ell(A_n) = \sum_{p=0}^{n} \binom{n}{p} \cdot \ell(C_p) \tag{8.0.11.1}$$

We can transform this identity formally to obtain

$$\ell(C_n) = \sum_{p=0}^{n} (-1)^{n-p} \binom{n}{p} \cdot \ell(A_p) \tag{8.0.11.2}$$

Now nothing prevents us from defining $\ell(C_n)$ by these formulas in the non-additive case and putting $\chi(A) := \sum_{n\geq 0}(-1)^n \ell(C_n)$, provided we are able to prove that $\ell(C_n) = 0$ for $n \gg 0$. Of course, in order to obtain a reasonable Euler characteristic $\chi : K^\bullet(\Sigma) \to V$ we'll have to check that $\chi(A) = \chi(A')$ for any weak equivalence $A \to A'$ as well.

We can consider the Euler–Poincaré series of A and C, defined by the usual formula $P_t(A) := \sum_{n\geq 0} \ell(A_n)t^n$, and similarly for C. We get $P_t(A) = P_{t/(1-t)}(C)$ and $P_u(C) = P_{u/(u+1)}(A)$, hence $\chi(A) = P_{-1}(C) = P_\infty(A)$. This seems absurd, but in our situation we expect $P_u(C)$ to be a polynomial, hence $P_t(A)$ to be a rational function, so we can indeed evaluate it at infinity. We see that some sort of regularization process is already being used here.

Another approach consists in considering the Hilbert polynomial $H_A(t)$ of A, defined by the requirement $\ell(A_n) = H_A(n)$ for all integer $n \geq 0$. In our case we have $H_A(x) = \sum_{p=0}^{N} \ell(C_p)\binom{x}{p}$, so this is indeed a polynomial. On the other hand, $\binom{-1}{p} = (-1)^p$ for any $p \geq 0$, so we get $\chi_A = H_A(-1)$, i.e. $\chi(A)$ is something like "$\ell(A_{-1})$", and the Euler characteristic computation turns out to be a certain extrapolation problem.

The Hilbert polynomial point of view is also extremely useful. Suppose for example that we have a polynomial map $Q_r : V \to V$, such that $\ell(\wedge^r P) = Q_r(\ell(P))$ for any projective P of finite type. Then we get $H_{\lambda^r A}(n) = Q_r(H_A(n))$ for all $n \geq 0$, hence $H_{\lambda^r A}(t) = Q_r(H_A(t))$ and $\chi(\lambda^r A) = Q_r(\chi(A))$.

8.0.12. (Intersection numbers and Euler characteristics.) We'd like to be able to compute some intersection numbers, say, for any $a \in \mathrm{Ch}^d(X)$, $d = \dim X$. In order to do this we need a dimension theory on X, so as to be able to define d, as well as something like the "fundamental class of variety X", $[X] \in H_{2d}(X)$; then the number we want to compute would be given by the $\cap$-product.

In our situation a can be represented by a perfect simplicial object $A_.$. And the most natural way to fix a dimension theory, at least for classical noetherian schemes, is to fix a dualizing (cochain) complex $I^\cdot$, i.e. some cosimplicial object. Notice that in the classical case $I^\cdot$ provides both a dimension theory, since for any $x \in X$ we can define $d(x)$ to be the only index i, for which $\mathrm{Ext}^i(\kappa(x), I^\cdot) \neq 0$, and something like a representative of the fundamental class, since $Z^\cdot := \mathbb{R}\,\mathrm{Hom}(A_., I^\cdot)$ is a complex (or a cosimplicial set), trivial in degrees $\neq d = \dim X$, and having $\ell(H^d(Z)) = \pm\chi(Z^\cdot)$ equal to the intersection number we are looking for.

It seems quite reasonable to mimic this construction in the non-additive case as well: fix some "dualizing cosimplicial object" $I^\cdot$ in $c(\mathscr{O}_X\text{-}Mod)$, and compute the cosimplicial set $Z^\cdot := \mathbb{R}\mathrm{Hom}(A_., I^\cdot) = \mathbb{R}\Gamma\,\mathbb{R}\mathbf{Hom}(A_., I^\cdot)$ for any $A_.$ from a suitable step of the γ-filtration of $K^\bullet(X)$. Then, if we are lucky, $\pm\chi(Z^\cdot)$ will give us the intersection number we are looking for. Of course, we have to choose $I^\cdot$ in a reasonable way; if we think about X as a smooth variety, then $\mathscr{O}_X$ concentrated in degree 0, i.e. constant cosimplicial object with value $\mathscr{O}_X$, would do.

We refer to the above process as *the arithmetic integration*. Maybe we should even denote the resulting intersection number by $\int_{I.} A.$.

8.0.13. (Euler characteristics of cosimplicial sets.) First of all, we have to introduce a simplicial closed model category structure on the category *cSets* of cosimplicial sets. We might do this with the aid of the dual of Quillen theorem already used to obtain a model category structure on $s(\Sigma\text{-}Mod)$, since *Sets* has sufficiently many injectives (something we don't have in an arbitrary Σ-*Mod*), and any set can be embedded into a group (e.g. the corresponding free group). After that, given a cosimplicial set $Z.$, we can try to find a finite fibrant-cofibrant replacement $A.$ of $Z.$, using the model category structure on *cSets*. Since we work in the dual situation, in order to compute Euler characteristics we'll need a map ℓ from the category of finite sets into a vector space V, such that $\ell(X \times Y) = \ell(X) + \ell(Y)$. The natural choice would be $V := \mathbb{R}$, $\ell(X) := \log|X|$ (another option: $V = \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}^{\times}_{>0}$). Then we can compute the Euler characteristic $\chi(Z.) = \chi(A.)$ by the procedure sketched above; it'll be a logarithm of a positive rational number. This definition makes sense at least for algebraic varieties X over finite fields $\mathbb{F}_q$: the arithmetic intersection numbers we obtain turn out to be equal to corresponding geometric intersection numbers multiplied by $\log q$.

8.0.14. (Mystery of good intersection numbers.) It seems a bit strange that we obtain only logarithms of rational numbers in this way. However, this is not so strange once we recall that we cannot choose metrics on our arithmetic varieties and vector bundles arbitrarily. In fact, we have already seen in Chapter **2** that arithmetic varieties of finite presentation over $\widehat{\operatorname{Spec}\mathbb{Z}}$ tend to have quite exotic singular (co)metrics, and, while more classical metrics, like the restriction of the Fubiny–Study metric on a closed subvariety of a projective space, can be described in our approach, they are not of finite presentation over $\widehat{\operatorname{Spec}\mathbb{Z}}$, so we'll either need some regularization process while computing our Euler characteristics, or we'll have to represent the given metric as a limit of "finitely presented metrics" and compute the limit of intersection numbers. In any case we cannot expect the resulting number to be the logarithm of a rational number. This is something like inscribing convex polyhedra with rational vertices into the unit sphere and taking the limit of their volumes: their volumes will be rational, while the limit will be not.

8.1. (Model categories.) We would like to recall here the basic definitions and constructions of homotopic algebra that we are going to use later. Our basic reference here is [Quillen]; however, our terminology is sometimes a bit different from that of Quillen. For example, all model categories we'll consider will be closed, so the axioms we give below for a model category (cf. e.g. [Hovey]) are actually equivalent to Quillen's axioms for a closed model category.

Definition 8.1.1 *A* (closed) model category $\mathcal{C}$ *is a category* $\mathcal{C}$*, together with three distinguinshed classes of morphisms, called* **fibrations**, **cofibrations** *and* **weak equivalences**, *subject to conditions (CM1)–(CM5) listed below.*

8.1.2. (Acyclic (co)fibrations, retracts and lifting properties.) Before listing the axioms of a model category, let's introduce some terminology. First of all, we say that a morphism $f : X \to Y$ in a category $\mathcal{C}$ with three distinguished classes of morphisms as above is an *acyclic fibration* (resp. *acyclic cofibration*) if it is both a fibration (resp. cofibration) and a weak equivalence.

Secondly, given two morphisms $i : A \to B$ and $f : X \to Y$ in an arbitrary category $\mathcal{C}$, we say that i has the *left lifting property (LLP) with respect to* f, or that f has the *right lifting property (RLP) with respect to* i, if for any two morphisms $u : A \to X$ and $v : B \to Y$, such that $v \circ i = f \circ u$, one can find a morphism $h : B \to X$, such that $h \circ i = u$ and $f \circ h = v$:

$$\begin{tikzcd} A \arrow[r, dashed, "u"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r, dashed, "v"] \arrow[ur, dotted, "\exists h"] & Y \end{tikzcd} \tag{8.1.2.1}$$

Finally, we say that a morphism $g : Z \to T$ is a *retract* of another morphism $f : X \to Y$ iff there exist morphisms $(i, j) : g \to f$ and $(p, q) : f \to g$, such that $(p, q) \circ (i, j) = \mathrm{id}$, i.e. if there are morphisms $i : Z \to X$, $j : T \to Y$, $p : X \to Z$, $q : Y \to T$, such that $f \circ i = j \circ g$, $g \circ p = q \circ f$, $p \circ i = \mathrm{id}_Z$ and $q \circ j = \mathrm{id}_T$:

$$\begin{tikzcd} Z \arrow[r, "i"'] \arrow[rr, bend left, "\mathrm{id}_Z"] \arrow[d, "g"] & X \arrow[r, "p"'] \arrow[d, "f"] & Z \arrow[d, "g"] \\ T \arrow[r, "j"] \arrow[rr, bend right, "\mathrm{id}_T"'] & Y \arrow[r, "q"] & T \end{tikzcd} \tag{8.1.2.2}$$

These notions have some obvious properties, e.g. if i' is a retract of i, and i has the LLP with respect to some f, then i' has the same LLP as well. Another simple property: the class of morphisms having the RLP with respect to some fixed morphism i is stable under base change and composition.

8.1.3. (Axioms of a closed model category.)

(CM1) The category $\mathcal{C}$ is closed under inductive and projective limits. (Finite inductive and projective limits actually suffice for most applications.)

(CM2) Each of the three distinguished classes of morphisms in $\mathcal{C}$ is closed under retracts.

(CM3) ("2 out of 3") Given two morphisms $f : X \to Y$ and $g : Y \to Z$, such that any two of f, g and $g \circ f$ are weak equivalences, then so is the third.

(CM4) (Lifting axiom) Any fibration has the RLP with respect to all acyclic cofibrations, and any cofibration has the LLP with respect to all acyclic fibrations, i.e. we can always complete a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd} \tag{8.1.3.1}$$

provided i is a cofibration, f is a fibration, and either i or f is a weak equivalence.

(CM5) (Factorization) Any morphism $f : X \to Y$ can be factored both into an acyclic cofibration followed by a fibration, or into a cofibration followed by an acyclic fibration: $f = p \circ j = q \circ i$, with i, j cofibrations, p, q fibrations, and j, q weak equivalences.

8.1.4. (Immediate consequences.) One observation is that the axioms are self-dual, if we interchange fibrations and cofibrations, so the opposite $\mathcal{C}^0$ of a model category $\mathcal{C}$ has a natural model category structure itself.

Another observation is that each of the lifting properties listed in (CM4) determines completely one distinguished class of morphisms in terms of another. For example, the acyclic cofibrations are *exactly* those morphisms of $\mathcal{C}$ that have the LLP with respect to all fibrations, and so on. This means that *any two of the three distinguished classes of morphisms in a model category completely determine the third*, i.e. the model category data is redundant. For example, once the fibrations and weak equivalences are known, the cofibrations are recovered as the morphisms with the LLP with respect to all acyclic fibrations. Similarly, weak equivalences can be described as those morphisms that can be decomposed into an acyclic cofibration followed by an acyclic cofibration, hence they are determined once one knows all fibrations and cofibrations.

Notice that any class of morphisms characterized by the RLP with respect to some other class of morphisms is automatically closed under retracts, base change and composition; this applies in particular to fibrations and acyclic fibrations. Similarly, cofibrations and acyclic cofibrations are stable under retracts, composition and pushouts.

8.1.5. (Fibrant and cofibrant objects.) An object X of a model category $\mathcal{C}$ is called *fibrant* if the morphism $X \to e_{\mathcal{C}}$ from X into the final object is fibrant; dually, X is called *cofibrant* if $\varnothing_{\mathcal{C}} \to X$ is cofibrant. Given any object X, a *cofibrant replacement* for X is a weak equivalence $Z \to X$ with Z cofibrant; dually, a *fibrant replacement* for X is a weak equivalence $X \to Z$ with a fibrant Z. Fibrant and cofibrant replacements always exist because of (CM5) applied to $X \to e_{\mathcal{C}}$ and $\varnothing_{\mathcal{C}} \to X$, and we see that a cofibrant replacement $Z \to X$ for a fibrant X can be always chosen in such a way that Z is both fibrant and cofibrant, hence for an arbitrary X we can find weak equivalences $X \to Y \leftarrow Z$ with a fibrant-cofibrant Z.

8.1.6. (Example.) One of the most important examples for us is that of the category $\mathrm{Ch}(\mathcal{A})$ of *non-negative chain complexes* $K. = (\cdots \xrightarrow{\partial_1} K_1 \xrightarrow{\partial_0} K_0)$ over an abelian category $\mathcal{A}$ with sufficiently many projective objects, e.g. $\mathcal{A} = R\text{-}\mathit{Mod}$, R a classical ring. In this case the weak equivalences are just the quasi-isomorphisms $f : K. \to L.$, i.e. chain maps, such that $H_n(f)$ is an isomorphism for all $n \geq 0$. Moreover, f is a fibration iff all f_n, $n > 0$, are epimorphic, and f is a cofibration iff all $f_n : K_n \to L_n$ are monomorphisms with $\operatorname{Coker} f_n$ a projective object, i.e. f_n is actually the embedding $K_n \to L_n \cong K_n \oplus P_n$ with a projective P_n. In this case all objects of $\mathrm{Ch}(\mathcal{A})$ turn out to be fibrant, the cofibrant objects are exactly the chain complexes with projective components, and a cofibrant replacement $P. \to K.$ of a complex $K.$ is nothing else than a projective resolution of $K.$. This illustrates the importance of cofibrant replacements for the computation of left derived functors.

We'll construct later a counterpart of this model category over an arbitrary generalized ring Σ, so as to recover $\mathrm{Ch}(\Sigma\text{-}\mathit{Mod})$ (together with its model structure just described) when Σ is a classical ring.

Definition 8.1.7 *(Homotopy category of a model category.) Given any model category $\mathcal{C}$, its* model category $\operatorname{Ho}\mathcal{C}$ *is the localization of $\mathcal{C}$ with respect to the class of weak equivalences.*

In other words, we have a *localization functor* $\gamma : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$, such that $\gamma(f)$ is an isomorphism whenever f is a weak equivalence, and this functor is universal among all functors $\mathcal{C} \to \mathcal{D}$ with this property. It is well-known that the localization of a category with respect to any set of morphisms always exists; however, in general one might need to enlarge the universe $\mathcal{U}$ since the set of morphisms between two objects of the localization is described as equivalence classes of paths in a certain graph, so this set needn't be $\mathcal{U}$-small. This is not the case with the model categories: if $\mathcal{C}$ is $\mathcal{U}$-small, the same is true for $\mathrm{Ho}\,\mathcal{C}$. This follows immediately from the "homotopic" description of $\mathrm{Ho}\,\mathcal{C}$ that we are going to recall later.

For example, $\mathrm{Ho}\,\mathrm{Ch}(\mathcal{A})$ for an abelian category $\mathcal{A}$ with sufficiently many projectives is equivalent to the full subcategory $\mathcal{D}^{\le 0}(\mathcal{A})$ of the derived category $\mathcal{D}^{-}(\mathcal{A})$.

Actually, the weak equivalences in a model category are crucial for defining the corresponding homotopy category and derived functors, while fibrations and cofibrations should be thought of as some means for auxiliary constructions in $\mathcal{C}$ and explicit computations of derived functors.

8.1.8. Notation. $\mathrm{Hom}_{\mathrm{Ho}\,\mathcal{C}}(X,Y)$ is sometimes denoted by $[X,Y]$; if X and Y are objects of $\mathcal{C}$, then $[X,Y]$ means $\mathrm{Hom}_{\mathrm{Ho}\,\mathcal{C}}(\gamma X,\gamma Y)$.

8.1.9. Since our model categories are always closed, *a morphism $f : X \to Y$ in $\mathcal{C}$ is a weak equivalence if and only if $\gamma(f)$ is an isomorphism in* $\mathrm{Ho}\,\mathcal{C}$ (cf. [Quillen, 1.5], prop. 1).

8.1.10. (Cylinder and path objects.) A *cylinder object* $A\times I$ for an object A of a model category $\mathcal{C}$ is a diagram $A \sqcup A \xrightarrow{\langle\partial_0,\partial_1\rangle} A\times I \xrightarrow{\sigma} A$ with $\sigma\circ\langle\partial_0,\partial_1\rangle = \nabla_A = \langle \mathrm{id}_A, \mathrm{id}_A\rangle$, such that $\langle\partial_0,\partial_1\rangle$ is a cofibration and σ is a weak equivalence. Dually, a *path object* B^I for an object B is a diagram $B \xrightarrow{s} B^I \xrightarrow{(d_0,d_1)} B\times B$ with $(d_0,d_1)\circ s = \Delta_B = (\mathrm{id}_B,\mathrm{id}_B)$, where s is a weak equivalence and (d_0,d_1) is a fibration.

Notice that there can be non-isomorphic cylinder objects for the same object A; in particular, $A\times I$ does *not* depend functorially on A, and $A\times I$ is not the product of A and another object I; it is just a notation. Same applies to path objects.

8.1.11. (Left and right homotopies.) Given two maps $f,g : A \rightrightarrows B$, we say that f *is left-homotopic to* g and write $f \overset{\ell}{\sim} g$ if there is a morphism $h : \tilde{A} \to B$ from a cylinder object $\tilde{A}$, $A\sqcup A \xrightarrow{\langle\partial_0,\partial_1\rangle} \tilde{A} \xrightarrow{\sigma} A$, such that $h\circ\langle\partial_0,\partial_1\rangle = \langle f,g\rangle : A\sqcup A \to B$, i.e. $f = h\circ\partial_0$ and $g = h\circ\partial_1$. Dually, we say that f *is right-homotopic to* g and write $f \overset{r}{\sim} g$ if there is a morphism $h : A \to \tilde{B}$ into a path object $\tilde{B}$ for B, such that $d_0\circ h = f$ and $d_1\circ h = g$. In both cases we say that h is the corresponding (left or right) homotopy between f and g.

For any A and B we denote by $\pi^\ell(A,B)$ (resp. by $\pi^r(A,B)$) the quotient of $\mathrm{Hom}(A,B)$ with respect to the equivalence relation generated by $\overset{\ell}{\sim}$ (resp. $\overset{r}{\sim}$).

Let us list some basic properties of homotopies (cf. [Quillen, 1.1]). If A is cofibrant, then $\overset{\ell}{\sim}$ is an equivalence relation on $\mathrm{Hom}_{\mathcal{C}}(A,B)$, and $f \overset{\ell}{\sim} g$

implies $f \stackrel{r}{\sim} g$. Moreover, in this case the composition induces a well-defined map $\pi^r(B,C) \times \pi^r(A,B) \to \pi^r(A,C)$. If A is cofibrant and B is fibrant, $\stackrel{r}{\sim}$ and $\stackrel{\ell}{\sim}$ coincide on $\mathrm{Hom}(A,B)$; the resulting equivalence relation can be denoted simply by $\sim$, and $\pi(A,B) := \mathrm{Hom}(A,B)/\sim$. In this case given any $f \sim g$ and any cylinder object $A \times I$ for A, we can find a left homotopy $h : A \times I \to B$ between f and g, and similarly for path objects B^I and right homotopies $h : A \to B^I$.

Another simple but important observation is that any of $f \stackrel{\ell}{\sim} g$ or $f \stackrel{r}{\sim} g$ implies $\gamma(f) = \gamma(g)$ in $\mathrm{Ho}\,\mathcal{C}$. A slightly more complicated statement is that for cofibrant A and fibrant B the converse is true: $f \sim g$ iff $\gamma(f) = \gamma(g)$.

8.1.12. (Homotopy category and homotopies.) Given a model category $\mathcal{C}$, let us denote by $\mathcal{C}_c$, $\mathcal{C}_f$ and $\mathcal{C}_{cf}$ its full subcategories consisting of all cofibrant, fibrant and fibrant-cofibrant objects, respectively, and by $\mathrm{Ho}\,\mathcal{C}_c$, $\mathrm{Ho}\,\mathcal{C}_f$ and $\mathrm{Ho}\,\mathcal{C}_{cf}$ the localizations of these categories with respect to those morphisms that are weak equivalences in $\mathcal{C}$.

Denote by $\pi\mathcal{C}_c$ the category with the same objects as $\mathcal{C}_c$, but with morphisms given by $\mathrm{Hom}_{\pi\mathcal{C}_c}(A,B) = \pi^r(A,B)$, and define similarly $\pi\mathcal{C}_f$ and $\pi\mathcal{C}_{cf}$ (in the latter case $\mathrm{Hom}_{\pi\mathcal{C}_{cf}} = \pi(A,B)$). Since $f \stackrel{\ell}{\sim} g$ and $f \stackrel{r}{\sim} g$ both imply $\gamma(f) = \gamma(g)$, we get canonical functors $\bar{\gamma}_c : \pi\mathcal{C}_c \to \mathrm{Ho}\,\mathcal{C}_c$, $\bar{\gamma}_f : \pi\mathcal{C}_f \to \mathrm{Ho}\,\mathcal{C}_f$, and $\bar{\gamma} : \pi\mathcal{C}_{cf} \to \mathrm{Ho}\,\mathcal{C}$.

Theorem. ([Quillen, 1.1], th. 1) *Functors* $\mathrm{Ho}\,\mathcal{C}_c \to \mathrm{Ho}\,\mathcal{C}$, $\mathrm{Ho}\,\mathcal{C}_f \to \mathrm{Ho}\,\mathcal{C}$ *and* $\bar{\gamma} : \pi\mathcal{C}_{cf} \to \mathrm{Ho}\,\mathcal{C}$ *are equivalences of categories.*

In other words, any object of $\mathcal{C}$ becomes isomorphic to a fibrant-cofibrant object in the homotopy category (this is clear since any object can be connected by a chain of weak equivalences to the fibrant replacement of its cofibrant replacement), any morphism $\bar{f} \in [A,B]$ is representable by a morphism $f : A \to B$ if A and B are fibrant-cofibrant, and in this case $\bar{f} = \bar{g}$ iff $f \sim g$, i.e. $[A,B] \cong \pi(A,B)$. It is actually sufficient to require here A to be cofibrant and B to be fibrant; cf. [Quillen, 1.1], cor. 1 of th. 1.

This theorem shows in particular that $\mathrm{Ho}\,\mathcal{C}$ is a $\mathcal{U}$-category whenever $\mathcal{C}$ is one.

8.1.13. (Example.) In the situation $\mathcal{C} = \mathrm{Ch}(\mathcal{A})$, $\mathcal{A}$ an abelian category with sufficiently many projectives, the above theorem actually tells us that $\mathcal{D}^{\leq 0}(\mathcal{A})$ is equivalent to the category of non-negative chain complexes consisting of projective objects with morphisms considered up to (chain) homotopy.

Definition 8.1.14 *Given a functor* $F : \mathcal{C} \to \mathcal{D}$ *between two model categories, its* left derived *functor* $\mathbb{L}F : \mathrm{Ho}\,\mathcal{C} \to \mathrm{Ho}\,\mathcal{D}$ *is the functor closest from the left to making the obvious diagram commutative. More formally, we must have a natural transformation* $\varepsilon : \mathbb{L}F \circ \gamma_{\mathcal{C}} \to \gamma_{\mathcal{D}} \circ F$, *such that for any other* *functor* $G : \mathrm{Ho}\,\mathcal{C} \to \mathrm{Ho}\,\mathcal{D}$ *and natural transformation* $\zeta : G \circ \gamma_{\mathcal{C}} \to \gamma_{\mathcal{D}} \circ F$ *there is a unique natural transformation* $\theta : G \to \mathbb{L}F$, *for which* $\zeta = \varepsilon \circ (\theta \star \gamma_{\mathcal{C}})$. *The* right derived functor $\mathbb{R}F : \mathrm{Ho}\,\mathcal{C} \to \mathrm{Ho}\,\mathcal{D}$, $\eta : \gamma_{\mathcal{D}} \circ F \to \mathbb{R}F \circ \gamma_{\mathcal{C}}$ *is defined similarly.*

Of course, the derived functors are defined uniquely by their universal properties, but the question of their existence is more subtle. Notice that we need only weak equivalences to define the derived functors.

Definition 8.1.15 *(Quillen functors.) Given two adjoint functors* $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ *between two model categories, we say that* (F, G) *is a* Quillen pair, *or that* F *is a* Quillen functor *if* F *preserves weak equivalences between cofibrant objects of* $\mathcal{C}$, *and* G *preserves weak equivalences between fibrant objects of* $\mathcal{D}$.

The importance of Quillen functors is illustrated by the following theorem.

Theorem 8.1.16 (cf. [Quillen, 1.4], prop. 2) *Suppose that* $F : \mathcal{C} \to \mathcal{D}$ *and* $G : \mathcal{C} \to \mathcal{D}$ *constitute a Quillen pair. Then both derived functors* $\mathbb{L}F : \mathrm{Ho}\,\mathcal{C} \to \mathrm{Ho}\,\mathcal{D}$ *and* $\mathbb{R}G : \mathrm{Ho}\,\mathcal{D} \to \mathrm{Ho}\,\mathcal{C}$ *exist and are adjoint to each other. Moreover,* $\mathbb{L}F(\gamma_{\mathcal{C}}X)$ *is canonically isomorphic to* $\gamma_{\mathcal{D}}F(Z)$, *where* X *is any object of* $\mathcal{C}$ *and* $Z \to X$ *is any cofibrant replacement of* X. *A similar description in terms of fibrant replacements can be given for* $\mathbb{R}G$ *as well.*

For example, if $\rho : R \to S$ is any ring homomorphism, we have the base change and scalar restriction functors $\rho^* : \mathrm{Ch}(R\text{-}Mod) \to \mathrm{Ch}(S\text{-}Mod)$ and $\rho_* : \mathrm{Ch}(S\text{-}Mod) \to \mathrm{Ch}(R\text{-}Mod)$. They constitute a Quillen pair, so the derived functors $\mathbb{L}\rho^*$ and $\mathbb{R}\rho_*$ exist and are adjoint to each other. We see that $\mathbb{R}\rho_*$ can be computed componentwise since all objects of $\mathrm{Ch}(S\text{-}Mod)$ are already fibrant, and to compute $\mathbb{L}\rho^*(X_.)$ we need to apply ρ^* componentwise to a cofibrant replacement, i.e. a projective resolution $P_. \to X_.$. Therefore, these functors coincide with those known from classical homological algebra. Of course, we want to generalize these results later to the case of an arbitrary algebraic monad homomorphism $\rho : \Sigma \to \Xi$.

Let us mention a criterion for existence of derived functors that doesn't require existence of adjoints:

Proposition 8.1.17 ([Quillen, 1.4], cor. of prop. 1) a) *Let* $F : \mathcal{C} \to \mathcal{C}'$ *be a functor between model categories that transforms weak equivalences between cofibrant objects of* $\mathcal{C}$ *into weak equivalences in* $\mathcal{C}'$. *Then* $\mathbb{L}F : \mathrm{Ho}\,\mathcal{C} \to \mathrm{Ho}\,\mathcal{C}'$ *exists, and for any cofibrant* P *the morphism* $\varepsilon_P : \mathbb{L}F(\gamma_{\mathcal{C}}P) \to \gamma_{\mathcal{C}'}F(P)$ *is an isomorphism, hence* $\mathbb{L}F(\gamma_{\mathcal{C}}X)$ *can be computed as* $\gamma_{\mathcal{C}'}F(P)$ *for any cofibrant replacement* $P \to X$.

b) If $F' : \mathcal{C}' \to \mathcal{C}''$ is another functor preserving weak equivalences between cofibrant objects, and if F transforms cofibrant objects of $\mathcal{C}$ into cofibrant objects of $\mathcal{C}'$, then derived functors $\mathbb{L}(F' \circ F)$, $\mathbb{L}F'$ and $\mathbb{L}F$ exist, and the functorial morphism $\mathbb{L}F' \circ \mathbb{L}F \to \mathbb{L}(F' \circ F)$ arising from the universal property of $\mathbb{L}(F' \circ F)$ is an isomorphism.

Proof. The first part is proved in [Quillen, 1.4], cor. of prop. 1, while the second one follows immediately from the description of derived functors given above.

8.1.18. (Suspension and loop functors; cf. [Quillen, 1.2]) Given any two morphisms $f, g : A \rightrightarrows B$ with a cofibrant A and a fibrant B, we can define a notion of a left homotopy H between two left homotopies $h : A \times I \to B$ and $h' : A \times I' \to B$ from f to g. This defines a homotopy relation between homotopies from f to g, that turns out to be an equivalence relation, thus defining a set $\pi_1^\ell(A, B; f, g)$ of homotopy classes of homotopies $h : f \sim g$. We have a dual construction for right homotopies, but $\pi_1^r(A, B; f, g)$ turns out to be canonically isomorphic to $\pi_1^\ell(A, B; f, g)$, so we can write simply $\pi_1(A, B; f, g)$. Furthermore, we have natural composition maps $\pi_1(A, B; f_2, f_3) \times \pi_1(A, B; f_1, f_2) \to \pi_1(A, B; f_1, f_3)$, coming from composition of homotopies; in particular, any $\pi_1(A, B; f, f)$ is a group, and the collection of all $\pi_1(A, B; f, g)$ defines a groupoid.

Now suppose that $\mathcal{C}$ is a pointed category, i.e. it has a zero object $0 = 0_{\mathcal{C}}$. Then for any A and B we have a zero map $0 = 0_{AB} : A \to 0 \to B$, and we put $\pi_1(A, B) := \pi_1(A, B; 0, 0)$; this is a group for any cofibrant A and fibrant B. Next, we can extend $\pi_1(A, B)$ to a functor $A, B \mapsto [A, B]_1$, $(\operatorname{Ho}\mathcal{C})^0 \times \operatorname{Ho}\mathcal{C} \to$ *Grps*, by requiring $[A, B]_1 = \pi_1(A, B)$ whenever A is cofibrant and B is fibrant.

Theorem. ([Quillen, 1.2], th. 2) There are two functors $\operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{C}$, called the *suspension functor* Σ and the *loop functor* Ω, such that

$$[\Sigma A, B] \cong [A, B]_1 \cong [A, \Omega B] \tag{8.1.18.1}$$

Recall that the suspension ΣA of a cofibrant A can be computed as follows: choose any cylinder object $A \times I$ for A and take the cofiber (i.e. the pushout with respect to the map $A \sqcup A \to 0$) of the map $\langle \partial_0, \partial_1 \rangle : A \sqcup A \to A \times I$. Similarly, ΩB is computed for a fibrant B as the fiber of $(d_0, d_1) : B^I \to B \times B$.

Recall that $\Sigma^n X$ is a cogroup object and $\Omega^n X$ is a group object in $\operatorname{Ho}\mathcal{C}$ for $n \geq 1$, which is commutative for $n \geq 2$. In particular, $\pi_n(X) := [\Sigma^n(*), X] \cong [*, \Omega^n X]$ is a group for $n \geq 1$, commutative for $n \geq 2$, for any choice of $* \in \operatorname{Ob}\operatorname{Ho}\mathcal{C}$.

In the case $\mathcal{C} = \mathrm{Ch}(\mathcal{A})$ the suspension functor Σ is just the degree translation functor: $(\Sigma K.)_n = K_{n-1}$, $(\Sigma K.)_0 = 0$, i.e. $\Sigma K. = K.[1]$, while its right adjoint Ω is given by the truncation of the opposite translation functor: $\Omega K. = \sigma_{\leq 0}(K.[-1])$, i.e. $(\Omega K.)_n = K_{n+1}$ for $n > 0$, $(\Omega K.)_0 = \mathrm{Ker}(\partial_0 : K_1 \to K_0)$.

8.1.19. (Cofibrantly generated model categories.) Suppose that I is a set of cofibrations and J is a set of acyclic cofibrations of a model category $\mathcal{C}$, such that the fibrations are exactly the morphisms with the RLP with respect to all morphisms from J, and the acyclic fibrations are exactly the morphisms with the RLP with respect to I. Then the model category structure of $\mathcal{C}$ is completely determined by these two sets I and J, and we say that they *generate* the model structure of $\mathcal{C}$.

We say that an object X of $\mathcal{C}$ is *(sequentially) small* if $\mathrm{Hom}(X, -)$ commutes with sequential inductive limits, i.e.

$$\mathrm{Hom}_{\mathcal{C}}(X, \varinjlim_{n\geq 0} Y_n) \cong \varinjlim_{n\geq 0} \mathrm{Hom}_{\mathcal{C}}(X, Y_n) \quad .$$

In particular, any finitely presented object X (characterized by the property of $\mathrm{Hom}(X, -)$ to commute with filtered inductive limits) is small.

Finally, we say that a model category $\mathcal{C}$ is *cofibrantly generated* if it admits generating sets I and J consisting of morphisms with small sources. The importance of this notion is due to the fact that in a cofibrantly generated model category the factorizations of (CM5) can be chosen to be functorial in f.

Consider for example the case $\mathcal{C} = \mathrm{Ch}(R\text{-}Mod)$, with R a classical ring. Denote by $D(n, R)$ the complex $K.$ with $K_n = K_{n+1} = R$, $\partial_n = \mathrm{id}_R$, and all other $K_i = 0$, and by $S(n, R)$ the complex $R[n]$ consisting of R placed in (chain) degree n. Then $I = \{S(n, R) \to D(n, R) \,|\, n \geq 0\}$ and $J = \{0 \to D(n, R) \,|\, n \geq 0\}$ generate the model structure of $\mathcal{C}$, hence $\mathrm{Ch}(R\text{-}Mod)$ is cofibrantly generated.

8.2. (Simplicial and cosimplicial objects.) Since the simplicial and cosimplical objects and categories will play a crucial role in the remaining part of this work, we'd like to list their basic properties and fix some notation. Our main reference here is [GZ].

Definition 8.2.1 *The* (non-empty finite) ordinal number category $\mathbf{\Delta}$ *is by definition the full subcategory of the category of ordered sets, consisting of all standard finite non-empty ordered sets* $[n] = \{0, 1, \ldots, n\}$, $n \geq 0$.

In other words, the objects $[n]$ of $\mathbf{\Delta}$ are in one-to-one correspondence with non-negative integers, and the morphisms $\varphi : [n] \to [m]$ are simply the order-preserving maps: $\varphi(x) \le \varphi(y)$ whenever $0 \le x \le y \le n$.

We consider two subcategories $\mathbf{\Delta}_+$ and $\mathbf{\Delta}_-$ of $\mathbf{\Delta}$ as well. They have the same objects as $\mathbf{\Delta}$, but $\mathrm{Hom}_{\mathbf{\Delta}_+}([n],[m])$ consists of all injective order-preserving maps $\varphi : [n] \to [m]$, while $\mathrm{Hom}_{\mathbf{\Delta}_-}([n],[m])$ consists of all surjective order-preserving maps.

8.2.2. (Face and degeneracy maps.) For any integers $0 \le i \le n$ we define two morphisms in $\mathbf{\Delta}$:

- The *face map* $\partial_n^i : [n-1] \to [n]$, the increasing injection that doesn't take value $i \in [n]$, $n > 0$;
- The *degeneracy map* $\sigma_n^i : [n+1] \to [n]$, the non-decreasing surjection that takes twice the value $i \in [n]$.

When no confusion can arise, we write simply ∂^i and σ^i.

The compositions of these maps are subject to following relations:

$$\partial_{n+1}^j \partial_n^i = \partial_{n+1}^i \partial_n^{j-1}, \quad 0 \le i < j \le n+1 \tag{8.2.2.1}$$

$$\sigma_n^j \sigma_{n+1}^i = \sigma_n^i \sigma_{n+1}^{j+1}, \quad 0 \le i \le j \le n \tag{8.2.2.2}$$

$$\sigma_{n-1}^j \partial_n^i = \begin{cases} \partial_{n-1}^i \sigma_{n-2}^{j-1}, & i < j \\ \mathrm{id}_{[n-1]}, & j \le i \le j+1 \\ \partial_{n-1}^{i-1} \sigma_{n-2}^j, & i > j+1 \end{cases} \tag{8.2.2.3}$$

It is well-known that any morphism $\varphi : [n] \to [m]$ can be uniquely decomposed into a non-decreasing surjection $p : [n] \twoheadrightarrow [s]$ followed by an increasing injection $i : [s] \hookrightarrow [m]$. More precisely, any morphism $\varphi : [n] \to [m]$ can be uniquely written in form

$$\varphi = \partial_m^{i_{m-s}} \partial_{m-1}^{i_{m-s-1}} \cdots \partial_{s+1}^{i_1} \sigma_s^{j_{n-s}} \sigma_{s+1}^{j_{n-s-1}} \cdots \sigma_{n-1}^{j_1} \quad ,$$
$$\text{where } m \ge i_{m-s} > \cdots > i_1 \ge 0, \quad 0 \le j_{n-s} < \cdots < j_1 < n \tag{8.2.2.4}$$

An immediate consequence of this lemma is that $\mathbf{\Delta}$ can be identified with the category generated by the face and degeneracy maps subject to conditions listed above (cf. [GZ, 2.2.2]). Notice that the category generated by the face maps ∂_n^i subject to relations (8.2.2.1) coincides with $\mathbf{\Delta}_+$, and the category generated by the degeneracy maps σ_n^i subject to relations (8.2.2.2) coincides with $\mathbf{\Delta}_-$.

Definition 8.2.3 *Let $\mathcal{C}$ be an arbitrary category. A* simplicial object *of $\mathcal{C}$ is a functor $X : \mathbf{\Delta}^0 \to \mathcal{C}$. The category of all simplicial objects of $\mathcal{C}$ will* *be denoted by* $s\mathcal{C} := Funct(\mathbf{\Delta}^0, \mathcal{C})$. *Dually,* a cosimplicial object of $\mathcal{C}$ *is a functor* $Y : \mathbf{\Delta} \to \mathcal{C}$, *and the category of all such cosimplicial objects will be denoted by* $c\mathcal{C} := Funct(\mathbf{\Delta}, \mathcal{C})$.

Usually we write X_n instead of $X([n])$, and for any $0 \leq i \leq n$ put $d_i^{n,X} := X(\partial_n^i) : X_n \to X_{n-1}$ (if $n > 0$) and $s_i^{n,X} := X(\sigma_n^i) : X_n \to X_{n+1}$. When no confusion can arise, we write simply d_i and s_i. In this way a simplicial object X of $\mathcal{C}$ is simply a collection of objects X_n, $n \geq 0$, together with some morphisms $d_i^{n,X} : X_n \to X_{n-1}$, called *face operators*, and $s_i^{n,X} : X_n \to X_{n+1}$, called *degeneracy operators*, subject to the dual of relations of **8.2.2**:

$$d_i^n d_j^{n+1} = d_{j-1}^n d_i^{n+1}, \quad 0 \leq i < j \leq n+1 \tag{8.2.3.1}$$

$$s_i^{n+1} s_j^n = s_{j+1}^{n+1} s_i^n, \quad 0 \leq i \leq j \leq n \tag{8.2.3.2}$$

$$d_i^n s_j^{n-1} = \begin{cases} s_{j-1}^{n-2} d_i^{n-1}, & i < j \\ \mathrm{id}_{X_{n-1}}, & j \leq i \leq j+1 \\ s_j^{n-2} d_{i-1}^{n-1}, & i > j+1 \end{cases} \tag{8.2.3.3}$$

Similarly, we write Y^n instead of $Y([n])$ and put $\partial_{n,Y}^i := Y(\partial_n^i) : Y_{n-1} \to Y_n$, $\sigma_{n,Y}^i := Y(\sigma_n^i) : Y_{n+1} \to Y_n$; we sometimes write simply ∂^i and σ^i. In this way a cosimplicial object Y is a collection of objects Y^n, $n \geq 0$, together with face and degeneracy operators $\partial_{n,Y}^i$, $\sigma_{n,Y}^i$, subject to relations **8.2.2**.

Clearly, $c(\mathcal{C}^0) = (s\mathcal{C})^0$ and $c\mathcal{C} = (s(\mathcal{C}^0))^0$, so statements about simplicial and cosimplicial objects can be obtained from each other by dualizing.

8.2.4. (Skeleta and coskeleta.) Consider the full subcategory $\mathbf{\Delta}_n \subset \mathbf{\Delta}$ consisting of objects $[p]$ with $p \leq n$. Functors $\mathbf{\Delta}_n^0 \to \mathcal{C}$ and $\mathbf{\Delta}_n \to \mathcal{C}$ are called *truncated (co)simplicial objects of* $\mathcal{C}$. Clearly, any (co)simplicial object can be truncated; if $I_n : \mathbf{\Delta}_n \to \mathbf{\Delta}$ is the embedding functor, then the truncation functor is equal to $I_n^* : X \mapsto X \circ I_n$. Therefore, we can study its left and right adjoints, i.e. left and right Kan extensions $I_{n,!}$ and $I_{n,*}$ of I_n. Of course, in general they exist only if arbitrary inductive, resp. projective limits exist in $\mathcal{C}$. In our case it suffices to require existence of finite inductive, resp. projective limits, since $\mathbf{\Delta}_n/[m]$ and $[m]\backslash\mathbf{\Delta}_n$ are finite categories:

$$(I_{n,!}X)(m) = \varinjlim_{([m]\backslash\mathbf{\Delta}_n)^0} X \tag{8.2.4.1}$$

Since I_n is fully faithful, the same is true for its Kan extensions, hence $X \to I_n^* I_{n,!} X$ and $I_n^* I_{n,*} X \to X$ are isomorphisms for any truncated (co)simplicial object X. We define the *n-th skeleton* (resp. *coskeleton*) of a (co)simplicial object X by $\mathrm{sk}_n X := I_{n,!} I_n^* X$ (resp. $\mathrm{cosk}_n X := I_{n,*} I_n^* X$). We have canonical morphisms $\mathrm{sk}_n X \to X$ and $X \to \mathrm{cosk}_n X$, that are isomorphisms in degrees $\leq n$; in fact, $\mathrm{sk}_n X \to X$ has a universal property among all morphisms $Y \to X$ that are isomorphisms in degrees $\leq n$, and similarly for $\mathrm{cosk}_n X$.

8.2.5. (Simplicial sets.) A very important example is given by the *category of simplicial sets* $sSets = Funct(\mathbf{\Delta}^0, Sets)$. Given a simplicial set X, we say that elements $x \in X_n$ are the *n-simplices of X*; n is called the *degree* or *dimension* of x. If x doesn't come from any X_m with $m < n$, i.e. if x doesn't lie in the union of the images of the degeneracy operators $s_i^{n,X} : X_{n-1} \to X_n$, we say that x is a *non-degenerate n-simplex of X.* Finally, we say that X is a *finite simplicial set* if the set of its non-degenerate simplices (in all degrees) is finite. It is easy to check that finite simplicial sets are exactly the finitely presented objects of $sSets$. Notice that the n-th skeleton $\mathrm{sk}_n X$ of a simplicial set X is its smallest subobject containing all simplices of X of dimension $\leq n$. This explains why we say that X *is of dimension* $\leq n$ whenever $\mathrm{sk}_n X$ coincides with X; this is equivalent to saying that all simplices of X in dimensions $> n$ are degenerate.

A simplicial set X is finite iff it has bounded dimension and all X_n are finite. An important property is that *the product of two finite simplicial sets is finite.*

8.2.6. (Examples of simplicial sets.) Let us list some important examples of (finite) simplicial sets. First of all, for any $n \geq 0$ we have the corresponding representable functor $\Delta(n) := \mathrm{Hom}_{sSets}(-, [n])$, called the *standard n-simplex.* By Yoneda $\mathrm{Hom}_{sSets}(\Delta(n), X) \cong X_n$. In this way we obtain a fully faithful functor $\Delta : \mathbf{\Delta} \to sSets$. It is easy to see that finite simplicial sets can be described as finite inductive limits of standard simplices.

Another important example is the *boundary* of the standard n-simplex: $\dot{\Delta}(n) := \mathrm{sk}_{n-1} \Delta(n) \subset \Delta(n)$. It is the largest simplicial subset of $\Delta(n)$ that doesn't contain $\mathrm{id}_{[n]} \in \Delta(n)_n$. Other description:

$$\dot{\Delta}(n)_m = \{\text{non-decreasing maps } \varphi : [m] \to [n] \text{ with } \varphi([m]) \neq [n]\} \quad (8.2.6.1)$$

In particular, $\dot{\Delta}(0) = \varnothing$.

We can define the *horns* $\Lambda_k(n) \subset \dot{\Delta}(n) \subset \Delta(n)$, $0 \leq k \leq n$, by removing one of the faces of $\dot{\Delta}(n)$. More precisely, $\Lambda_k(n)_m$ consists of all order-preserving maps $\varphi : [m] \to [n]$, such that the complement of their image $[n] - \varphi([m])$ is distinct from both $\varnothing$ and $\{k\}$.

8.2.7. (Geometric realization and singular simplices.) For any set S we denote by $\Delta^{(S)}$ the convex hull of the standard base of $\mathbb{R}^{(S)}$, considered as a topological space. This is a covariant functor from *Sets* into *Top*. Restricting it to $\mathbf{\Delta} \subset Sets$ we obtain a covariant functor $\Delta^{(-)} : \mathbf{\Delta} \to Top$, i.e. a cosimplicial topological space. Now we define the *singular functor* $\mathrm{Sing} : Top \to sSets$

by putting $(\operatorname{Sing} X)([n]) := \operatorname{Hom}_{Top}(\Delta^{([n])}, X)$; clearly, $(\operatorname{Sing} X)_n$ is just the set of singular n-simplices of X. We define the *geometric realization functor* $|-| : sSets \to Top$ as the left adjoint to Sing; its uniqueness is evident, and its existence follows from the fact that $|\Delta(n)|$ must be equal to $\Delta^{([n])}$, that any simplicial set can be written down as an inductive limit of standard simplices, and that $|-|$ must commute with arbitrary inductive limits. This yields a formula for $|X|$:

$$|X| = \varinjlim_{\mathbf{\Delta}/X} \Delta^{(-)} \tag{8.2.7.1}$$

The geometric realization functor takes values in the full subcategory of *Top* consisting of Kelley spaces (i.e. compactly generated Hausdorff spaces); if we consider it as a functor from *sSets* into this subcategory, it preserves finite projective and arbitrary inductive limits (cf. [GZ, 3.3]).

8.2.8. (Closed model category structure on *Top*.) There is a closed model category structure on the category of topological spaces, characterized by its classes of fibrations and weak equivalences: the fibrations are the Serre fibrations and weak equivalences are maps $f : X \to Y$ that preserve all homotopy groups, i.e. $\pi_n(f) : \pi_n(X, x) \to \pi_n(Y, f(x))$ has to be an isomorphism for all $n \geq 0$ and all base points $x \in X$ (cf. [Quillen, 2.3]).

8.2.9. (Closed model category structure on *sSets*.) The category of simplicial sets has a model category structure as well, that can be described as follows. The fibrations in *sSets* are the Kan fibrations, cofibrations are just the injective maps, and $f : X \to Y$ is a weak equivalence iff its geometric realization $|f|$ is one. In particular, all simplicial sets are cofibrant, but usually not fibrant; recall that $\operatorname{Ch} \mathcal{A}$ had the opposite property.

Moreover, adjoint functors $|-|$ and Sing constitute a Quillen pair, and their derived functors $\mathbb{L}|-|$ and $\mathbb{R}\operatorname{Sing}$ are adjoint equivalences of categories; thus Ho *sSets* and Ho *Top* are naturally equivalent (cf. [Quillen, 2.3]).

8.2.10. (Cofibrant generators for *sSets*.) The model structure of *sSets* happens to be cofibrantly generated. More precisely, $I := \{\dot{\Delta}(n) \to \Delta(n) \,|\, n \geq 0\}$ is a generating set of cofibrations with small sources, while $J := \{\Lambda_k(n) \to \Delta(n) \,|\, 0 \leq k \leq n > 0\}$ is a generating set of acyclic cofibrations.

Let's apply this to describe the acyclic fibrations $p : E \to B$, i.e. the maps with the RLP with respect to all inclusions $\dot{\Delta}(n) \to \Delta(n)$. We know that $\operatorname{Hom}(\Delta(n), X) \cong X_n$, and using adjointness of sk_{n-1} and $\operatorname{cosk}_{n-1}$ we get $\operatorname{Hom}(\dot{\Delta}(n), X) = \operatorname{Hom}(\operatorname{sk}_{n-1} \Delta(n), X) \cong \operatorname{Hom}(\Delta(n), \operatorname{cosk}_{n-1} X) \cong (\operatorname{cosk}_{n-1} X)_n$. We see that $p : E \to B$ is an acyclic fibration iff $E_n \to B_n \times_{(\operatorname{cosk}_{n-1} B)_n} (\operatorname{cosk}_{n-1} E)_n$ is surjective for all $n > 0$, and $p_0 : E_0 \to B_0$ is surjective as well. In particular, X is fibrant iff $X_0 \neq \varnothing$ and $X_n \to (\operatorname{cosk}_{n-1} X)_n$ is surjective for all $n > 0$.

8.2.11. (Simplicial groups.) The category *sGrps* of simplical groups has a model category structure as well. Now we'd like to mention that $f : G \to H$ is a fibration in *sGrps* iff it is a fibration in *sSets*, and that all simplicial groups are fibrant, both in *sGrps* and in *sSets*.

8.3. (Simplicial categories.) There are several ways of defining a simplicial structure on a category $\mathcal{C}$, all of them more or less equivalent when arbitrary inductive and projective limits exist in $\mathcal{C}$. We'll use a definition based on the notion of an external $\otimes$-action.

8.3.1. (ACU $\otimes$-structure on *sSets*.) First of all, notice that the direct product defines an ACU $\otimes$-structure on *Sets*, hence also on *sSets*. We have $(K \otimes L)_n = (K \times L)_n = K_n \times L_n$ for this $\otimes$-structure.

Definition 8.3.2 *A* simplicial structure *on a category $\mathcal{C}$ is a right $\otimes$-action $\otimes : \mathcal{C} \times sSets \to \mathcal{C}$, $(X, K) \mapsto X \otimes K$ (cf.* **3.1.1***) of sSets with the ACU $\otimes$-structure just considered on $\mathcal{C}$, required to commute with arbitrary inductive limits in K. A* simplicial category *is just a category with a simplicial structure. A* simplicial functor *$F : \mathcal{C} \to \mathcal{D}$ between two simplicial categories is simply a* lax *external $\otimes$-functor in the sense of* **3.1.3**. *Recall that this involves giving a collection of morphisms $F(X) \otimes K \to F(X \otimes K)$, subject to certain relations.*

Notice that the $\otimes$-structure on *sSets* is commutative, so the distinction between left and right external $\otimes$-actions of *sSets* is just a matter of convention.

8.3.3. (A variant.) Since any simplicial set is an inductive (even filtered inductive) limit of finite simplicial sets, and the category of finite simplicial sets is closed under products, we might have defined a simplicial structure as an external $\otimes$-action of the ACU $\otimes$-category of finite simplicial sets, required to commute with finite inductive limits. This definition is completely equivalent to the one given above when arbitrary inductive limits exist in $\mathcal{C}$.

Notice that any simplicial set is an inductive limit of standard simplices $\Delta(n)$, so it might suffice to define all $X \otimes \Delta(n)$. However, the associativity conditions $(X \otimes \Delta(n)) \otimes \Delta(m) \cong X \otimes (\Delta(n) \times \Delta(m))$ cannot be naturally expressed in terms of such tensor products since $\Delta(n) \times \Delta(m)$ is not a standard simplex itself.

8.3.4. (Simplicial sets of homomorphisms.) Let us fix two objects X and Y of a simplicial category $\mathcal{C}$. The functor $\mathrm{Hom}_{\mathcal{C}}(X \otimes -, Y)$, $(sSets)^0 \to Sets$, transforms arbitrary inductive limits into corresponding projective limits; therefore, it is representable by an object $\mathbf{Hom}(X, Y) = \mathbf{Hom}_{\mathcal{C}}(X, Y)$:

$$\mathrm{Hom}_{sSets}(K, \mathbf{Hom}_{\mathcal{C}}(X, Y)) \cong \mathrm{Hom}_{\mathcal{C}}(X \otimes K, Y) \tag{8.3.4.1}$$

In this way we get a functor $\mathbf{Hom}_{\mathcal{C}} : \mathcal{C}^0 \times \mathcal{C} \to sSets$. These simplicial sets $\mathbf{Hom}(X, Y)$ will be called the *simplicial sets of homomorphisms*. Clearly, $\mathbf{Hom}_{\mathcal{C}}(X, Y)_n = \mathrm{Hom}_{\mathcal{C}}(X \otimes \Delta(n), Y)$; in particular, the unit axiom for an external action implies that $\mathrm{Hom}_{\mathcal{C}}(X, Y) \cong \mathbf{Hom}_{\mathcal{C}}(X, Y)_0$ functorially in X and Y. A standard argument yields the existence of canonical *composition maps*, satisfying natural associativity and unit conditions:

$$\circ : \mathbf{Hom}_{\mathcal{C}}(Y, Z) \times \mathbf{Hom}_{\mathcal{C}}(X, Y) \to \mathbf{Hom}_{\mathcal{C}}(X, Z) \quad . \tag{8.3.4.2}$$

Quillen has defined in [Quillen] a simplicial category as a category $\mathcal{C}$ together with a functor $\mathbf{Hom}_{\mathcal{C}}$, a collection of composition maps $\circ$ and functorial isomorphisms $\mathbf{Hom}_{\mathcal{C}}(X, Y)_0 \cong \mathrm{Hom}_{\mathcal{C}}(X, Y)$, subject to some relations (essentially the associativity and unit relations just mentioned). Of course, $\mathbf{Hom}_{\mathcal{C}}$ determines the external action $\otimes$ uniquely up to a unique isomorphism, so Quillen's approach is essentially equivalent to one adopted here.

Yet another approach consists in describing the simplicial structure in terms of the functor $M : \mathcal{C}^0 \times (sSets)^0 \times \mathcal{C} \to Sets$, $M(X, K; Y) := \mathrm{Hom}_{\mathcal{C}}(X \otimes K, Y)$...

8.3.5. (Exponential objects.) Given an object $Y \in \mathrm{Ob}\,\mathcal{C}$ and a simplicial set K, we define the corresponding *exponential object* Y^K as the object representing the functor $\mathrm{Hom}_{\mathcal{C}}(- \otimes K, Y)$, whenever it exists. Therefore,

$$\mathrm{Hom}_{\mathcal{C}}(X \otimes K, Y) \cong \mathrm{Hom}_{sSets}(K, \mathbf{Hom}_{\mathcal{C}}(X, Y)) \cong \mathrm{Hom}_{\mathcal{C}}(X, Y^K) \tag{8.3.5.1}$$

Clearly, $(Y^K)^L \cong Y^{K \times L}$, and $Y^{\Delta(0)} \cong Y$; of course, the functor $(Y, K) \mapsto Y^K$ determines completely the simplicial structure (actually this functor is an external $\otimes$-action of $(sSets)^0$ on $s\mathcal{C}$), so we might have written the axioms in terms of this functor as well.

8.3.6. (Simplicial structure on categories of simplicial objects.) Given any object X of a category $\mathcal{C}$ and any set E, we denote by $X \times E$ the direct sum (i.e. coproduct) of E copies of X, i.e. $\mathrm{Hom}_{\mathcal{C}}(X \times E, Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, Y)^E$.

Consider now the category $s\mathcal{C} = Funct(\mathbf{\Delta}^0, \mathcal{C})$ of simplicial objects over a category $\mathcal{C}$, supposed to have arbitrary inductive and projective limits. Then $s\mathcal{C}$ has a natural simplicial structure, given by

$$(X \otimes K)_n := X_n \times K_n, \quad \text{for any } X \in \mathrm{Ob}\, s\mathcal{C},\ K \in \mathrm{Ob}\, sSets. \tag{8.3.6.1}$$

Moreover, one can prove (cf. [Quillen, 2.1]) that in this case all exponential objects Y^K are representable. Notice that the simplicial extension $sF : s\mathcal{C} \to s\mathcal{D}$ of an arbitrary functor $F : \mathcal{C} \to \mathcal{D}$ turns out to be automatically simplicial with respect to this simplicial structure by means of canonical morphisms $F(X_n) \times K_n \to F(X_n \times K_n)$.

If $\mathcal{C}$ doesn't have any inductive or projective limits, this simplicial structure can be still described in terms of functors $\mathbf{Hom}_{\mathcal{C}}$ or M as explained before.

8.3.7. (Simplicial structure on categories of cosimplicial objects.) Notice that the notion of a simplicial structure on a category $\mathcal{C}$ is essentially self-dual, if we don't forget to interchange the arguments of $\mathbf{Hom}$, and $X \otimes K$ with X^K, i.e. the opposite of a simplicial category has a natural simplicial structure itself.

Let $\mathcal{C}$ be an arbitrary category with arbitrary inductive and projective limits. Since $c\mathcal{C} = \mathit{Funct}(\mathbf{\Delta}, \mathcal{C}) = (s(\mathcal{C}^0))^0$, we must have a simplicial structure on $c\mathcal{C}$ as well. After dualizing and interchanging $X \otimes K$ and X^K, we see that this simplicial structure is easily characterized in terms of X^K. Namely, for any cosimplicial object X and any simplicial set K we have

$$(X^K)^n = \mathrm{Hom}(K_n, X^n) = (X^n)^{K_n} \tag{8.3.7.1}$$

After this $\mathbf{Hom}_{c\mathcal{C}}(X, Y)$ and $X \otimes K$ are completely determined by (8.3.5.1).

8.3.8. It seems a bit strange that the categories of simplicial and cosimplicial sets act on each other in an asymmetric way: *sSets* acts both on itself and on *cSets*, while *cSets* doesn't act on anything. This can be explained in the following fancy way. If we think of simplicial sets as chain complexes of $\mathbb{F}_\varnothing$-modules, and of cosimplicial sets as cochain complexes, then the action of *sSets* on itself is essentially the same thing as the tensor product over $\mathbb{F}_\varnothing$ of chain complexes, while the action of *sSets* on *cSets* is something like $\mathrm{Hom}_{\mathbb{F}_\varnothing}$ from a chain complex into a cochain complex, yielding a cochain complex.

8.4. (Simplicial model categories.)

Definition 8.4.1 *We say that a category $\mathcal{C}$ is a simplicial (closed) model category if it has both a simplicial and a closed model category structure, these two compatible in the following sense (cf. [Quillen, 2.2]):*

(SM7) Whenever $i : A \to B$ is a cofibration and $p : X \to Y$ is a fibration in $\mathcal{C}$, the following map of simplicial sets is a fibration in *sSets*:

$$\mathbf{Hom}(B, X) \xrightarrow{(i^*, p_*)} \mathbf{Hom}(i, p) := \mathbf{Hom}(A, X) \times_{\mathbf{Hom}(A,Y)} \mathbf{Hom}(B, X) \tag{8.4.1.1}$$

We require this fibration to be acyclic when either i or p is a weak equivalence.

This axiom is easily shown to be equivalent to any of the following:

(SM7a) If $p : X \to Y$ is a fibration in $\mathcal{C}$ and $s : K \to L$ is a cofibration in *sSets*, then

$$X^L \to X^K \times_{Y^K} Y^L \tag{8.4.1.2}$$

is a fibration in $\mathcal{C}$, acyclic if either p or s is a weak equivalence.

(SM7b) If $i : A \to B$ is a cofibration in $\mathcal{C}$ and $s : K \to L$ is a cofibration in *sSets*, then

$$i \,\Box\, s : A \otimes L \sqcup_{A\otimes K} B \otimes K \to B \otimes L \tag{8.4.1.3}$$

is a cofibration in $\mathcal{C}$, acyclic if either i or s is a weak equivalence.

It is actually sufficient to require (SM7a) or (SM7b) when $s : K \to L$ runs through a system of generators for (acyclic or all) cofibrations in *sSets*.

8.4.2. (Example.) The category of simplicial sets *sSets* happens to be a simplical model category.

8.4.3. (Simplicial homotopies.) Given two objects X and Y of an arbitrary simplicial category $\mathcal{C}$, we say that $h \in \mathbf{Hom}_{\mathcal{C}}(X,Y)_1$ is a *simplicial homotopy* between two morphisms $f, g \in \mathrm{Hom}_{\mathcal{C}}(X,Y) = \mathbf{Hom}_{\mathcal{C}}(X,Y)_0$ if $d_0(h) = f$ and $d_1(h) = g$. In this case we say that f *and* g *are strictly simplicially homotopic* and write $f \overset{ss}{\sim} g$. The equivalence relation on $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ generated by $\overset{ss}{\sim}$ will be denoted by $\overset{s}{\sim}$; if $f \overset{s}{\sim} g$, we say that f *and* g *are simplicially homotopic.* Notice that giving an element $h \in \mathbf{Hom}(X,Y)_1$ is equivalent to giving a morphism $h' : X \otimes \Delta(1) \to Y$ or $h'' : X \to Y^{\Delta(1)}$.

The quotient $\mathrm{Hom}(X,Y)/\overset{s}{\sim}$ will be denoted by $\pi_0 \mathrm{Hom}(X,Y)$, or simply by $\pi_0(X,Y)$. For any full subcategory $\mathcal{C}' \subset \mathcal{C}$ we denote by $\pi_0\mathcal{C}'$ the category with the same objects, but with morphisms given by $\pi_0(X,Y)$.

More generally, for any simplicial set X we denote by $\pi_0 X$ the cokernel of the two face operators $d_0, d_1 : X_1 \rightrightarrows X_0$. We say that $\pi_0 X$ is *the set of components* of X.

8.4.4. (Homotopies in simplicial model categories.) Recall the following proposition (cf. [Quillen, 2.2], prop. 5):

Proposition. *If $f, g : X \rightrightarrows Y$ are two morphisms in a simplicial model category $\mathcal{C}$, then $f \overset{s}{\sim} g$ implies both $f \overset{\ell}{\sim} g$ and $f \overset{r}{\sim} g$. If X is cofibrant and Y is fibrant, then the homotopy relations $\overset{r}{\sim}$, $\overset{\ell}{\sim}$, $\overset{s}{\sim}$ and $\overset{ss}{\sim}$ on* $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ *coincide. In particular,* $\mathrm{Ho}\,\mathcal{C} \cong \pi\mathcal{C}_{cf} \cong \pi_0\mathcal{C}_{cf}$, *i.e. the homotopy category of $\mathcal{C}$ can be computed with the aid of simplicial homotopies.*

The proof involves a very simple but useful observation. Namely, the face and degeneracy maps induce two acyclic cofibrations $\Delta(d_0), \Delta(d_1) : \Delta(0) \rightrightarrows \Delta(1)$ and a weak equivalence $\Delta(s_0) : \Delta(1) \to \Delta(0)$. Using (SM7) in its equivalent forms we see that induced diagrams $X \oplus X \to X \otimes \Delta(1) \to X$ and $Y \to Y^{\Delta(1)} \to Y \times Y$ give us a canonical way to construct a cofibrant cylinder for any cofibrant X, resp. a fibrant path object for any fibrant Y. If we put $I := \Delta(1)$ ("the simplicial segment"), then this cylinder and path object will be indeed denoted by $X \times I = X \otimes I$, resp. Y^I; this explains our previous notation for them.

8.4.5. (Simplicial model category structure on $s\mathcal{C}$.) Let $\mathcal{C}$ be a category closed under arbitrary inductive and projective limits. We would like to discuss some conditions under which it is possible to endow the simplicial category $s\mathcal{C}$ with a compatible model category structure in a natural way (cf. [Quillen, 2.4]).

Recall that $f : X \to Y$ is said to be a *strict epimorphism* if it is the cokernel of its kernel pair $X \times_Y X \rightrightarrows X$. An object $P \in \operatorname{Ob}\mathcal{C}$ is *projective* if $\operatorname{Hom}_{\mathcal{C}}(P, -)$ transforms strict epimorphisms into surjections, and *finitely presented* if $\operatorname{Hom}_{\mathcal{C}}(P, -)$ preserves filtered inductive limits. We say that $\mathcal{C}$ *has sufficiently many projectives* if for any object X there is a strict epimorphism $P \to X$ with a projective P. A set $\mathcal{G} \subset \operatorname{Ob}\mathcal{C}$ is a set of *generators* of $\mathcal{C}$ if for any object X of $\mathcal{C}$ one can find a strict epimorphism from a direct sum of objects of $\mathcal{G}$ into X.

Now we are able to state the following important theorem:

Theorem. ([Quillen, 2.4], th. 4) *Let $\mathcal{C}$ be a category closed under arbitrary projective and inductive limits, $s\mathcal{C}$ be the corresponding category of simplicial objects. Define a morphism f in $s\mathcal{C}$ to be a fibration (resp. weak equivalence) iff* $\operatorname{Hom}(P, f)$ *is a fibration (resp. weak equivalence) in sSets for each projective object P of $\mathcal{C}$, and a cofibration if it has the LLP with respect to all acyclic fibrations. Then $s\mathcal{C}$ becomes a simplicial model category provided one of the following conditions holds:*

a) Every object of $s\mathcal{C}$ is fibrant.

b) Any object of $\mathcal{C}$ is a strict quotient of a cogroup object.

c) $\mathcal{C}$ has a small set of finitely presented projective generators.

Moreover, these conditions are not independent: b) implies a).

Notice that if $\mathcal{G}$ is a set of projective generators of $\mathcal{C}$, then any projective object is a direct factor of a direct sum of objects from $\mathcal{G}$, hence f is a fibration (resp. weak equivalence) in $s\mathcal{C}$ iff $\operatorname{Hom}(P, f)$ is a fibration (resp. weak equivalence) for all $P \in \mathcal{G}$. (We use here that any product of weak equivalences in *sSets* is again a weak equivalence!)

Another interesting observation is that in case c) the model structure of $s\mathcal{C}$ turns out to be cofibrantly generated. More precisely, if $\mathcal{G} \subset \mathrm{Ob}\,\mathcal{C}$ is a set of finitely presented projective generators as in c), and I and J are the cofibrant generators for *sSets*, then $\mathcal{G} \otimes I = \{P \otimes f \mid P \in \mathcal{G},\ f \in I\}$ and $\mathcal{G} \otimes J$ are cofibrant generators for $s\mathcal{C}$.

8.4.6. (First applications.) The above theorem is of course applicable to $\mathcal{C} = \mathit{Sets}$, $\mathbf{1}$ being a finitely presented projective generator of *Sets*. The model structure we get coincides with the one we have already, since $\mathrm{Hom}(\mathbf{1}, f) = f$.

Another application: Let R be a (classical) associative ring, $\mathcal{C} := R\text{-}\mathit{Mod}$. Then R is a finitely presented projective generator of $\mathcal{C}$, so the condition c) of the theorem is fulfilled. Actually the codiagonal $\nabla_C : C \to C \oplus C$ defines a cogroup structure on any object of R-*Mod*, so conditions b) and a) are fulfilled as well.

8.4.7. (Application: category of cosimplicial sets.) A less trivial example is given by the category Sets^0. In this case condition b), hence also a), is fulfilled since the strict monomorphisms of *Sets* are just the injective maps, and any set can be embedded into a group, e.g. the corresponding free group. Therefore, $s(\mathit{Sets}^0) = (c\mathit{Sets})^0$ has a simplicial model category structure. By dualizing we obtain such a structure on the category *cSets* of cosimplicial sets. Since $\mathbf{2} = \{1, 2\}$ is an injective cogenerator of *Sets*, we see that *a map $f : X \to Y$ of cosimplicial sets is a cofibration (resp. a weak equivalence) iff the map of simplicial sets $f^* : \mathbf{2}^Y \to \mathbf{2}^X$ is a fibration (resp. a weak equivalence)*, and that *all cosimplicial sets are cofibrant.*

8.4.8. (Category of simplicial Σ-modules.) Now let Σ be an algebraic monad, $\mathcal{C} := \Sigma\text{-}\mathit{Mod}$. Clearly, $|\Sigma| = \Sigma(1)$ is a finitely presented projective generator for $\mathcal{C}$, so Theorem **8.4.5**,c) is applicable, hence *the category $s(\Sigma\text{-}\mathit{Mod})$ of simplicial Σ-modules admits a natural simplicial model category structure.* Moreover, *this model structure is cofibrantly generated by $L_\Sigma(I)$ and $L_\Sigma(J)$, where I and J are any cofibrant generators for sSets.* Since $\mathrm{Hom}_\Sigma(|\Sigma|, -)$ is nothing else than the forgetful functor $\Gamma_\Sigma : \Sigma\text{-}\mathit{Mod} \to \mathit{Sets}$, we see that *a morphism f of simplicial Σ-modules is a fibration (resp. weak equivalence) iff this is true for $\Gamma_\Sigma(f)$, i.e. iff f is a fibration (resp. weak equivalence) when considered as a map of simplicial sets.*

When Σ is a classical ring, we recover the simplical model category structure discussed in **8.4.6**. When $\Sigma = \mathbb{F}_\varnothing$ we get the usual simplicial model category structure on $s(\mathbb{F}_\varnothing\text{-}\mathit{Mod}) = s\mathit{Sets}$, and when $\Sigma = \mathbb{F}_1$ we get the simplicial model category of pointed simplicial sets. Moreover, if we put $\Sigma = \mathbb{G}$, we recover the usual model category structure on the category of simplicial groups.

8.4.9. (Lawvere theorem and Morita equivalence.)
Quillen refers in [Quillen, 2.4], rem. 1, to a theorem of Lawvere that asserts that a category closed under inductive limits and possessing a finitely presented projective generator is essentially a category of universal algebras, and conversely. Let us restate Lawvere theorem in our terms:

Theorem 8.4.10 (Lawvere, cf. [Lawvere]) *Let $(\mathcal{C}, P)$ be a couple consisting of a category $\mathcal{C}$ closed under arbitrary inductive limits and a finitely presented projective generator P of $\mathcal{C}$. Then the functor $\Gamma := \mathrm{Hom}_{\mathcal{C}}(P, -) : \mathcal{C} \to Sets$ admits a left adjoint L, thus defining an* algebraic *monad $\Sigma := \Gamma L$ and a comparison functor $I : \mathcal{C} \to \Sigma\text{-}Mod$, transforming P into $|\Sigma|$. This comparison functor is fully faithful; if all equivalence relations in $\mathcal{C}$ are effective, i.e. if $R \cong X \times_{X/R} X$ for any equivalence relation $R \rightrightarrows X$ in $\mathcal{C}$, then I is an equivalence. Conversely, all these conditions hold for $(\Sigma\text{-}Mod, |\Sigma|)$, for an arbitrary algebraic monad Σ.*

In other words, Lawvere theorem essentially gives us an algebraic description of categories with one finitely presented projective generator. Quillen uses this theorem as a source of situations where theorem **8.4.5** is applicable; we see that these situations correspond *exactly* to algebraic monads!

As to the proof of this theorem, it goes essentially in the same way we have shown in **2.14.14** that $\mathbb{Z}_\infty$-*Fl.Mod* is equivalent to a full subcategory of $\mathbb{Z}_\infty$-*Mod*; actually we might have proved Lawvere theorem first and apply it afterwards to deduce **2.14.14**.

Observe that if $(\mathcal{C}, P)$ is a couple satisfying the conditions of Lawvere theorem, $(\mathcal{C}, P^{(n)})$, $P^{(n)} = P^{\oplus n}$, is another. Therefore, we can start with an arbitrary algebraic monad Σ and put $(\mathcal{C}, P) := (\Sigma\text{-}Mod, \Sigma(n))$; Lawvere theorem yields another algebraic monad $M_n\Sigma$, given by $(M_n\Sigma)(m) = \mathrm{Hom}_\Sigma(\Sigma(n), \Sigma(nm)) \cong \Sigma(nm)^n$, and asserts that $M_n\Sigma$-*Mod* is equivalent to Σ-*Mod*. We've got a generalized version of Morita equivalence here!

8.4.11. (Free simplicial objects and morphisms.) We can translate further Quillen results of [Quillen, 2.4] into the language of algebraic monads as well.

Namely, we say that a morphism $i : X \to Z$ in $s(\Sigma\text{-}Mod)$ is *free* if the degeneracy diagram of Z, i.e. the restriction of Z to $\mathbf{\Delta}_- \subset \mathbf{\Delta}$, is isomorphic to the direct sum of the degeneracy diagram of X and some free degeneracy diagram $L_\Sigma(C)$, $C : \mathbf{\Delta}_-^0 \to Sets$, and if the morphism of degeneracy diagrams $i|_{\mathbf{\Delta}_-} : X|_{\mathbf{\Delta}_-} \to Z|_{\mathbf{\Delta}_-}$ can be identified with the natural embedding $X_{\mathbf{\Delta}_-} \to X_{\mathbf{\Delta}_-} \oplus L_\Sigma(C)$.

Any free morphism $i : X \to Z$ is a cofibration in $s(\Sigma\text{-}Mod)$; moreover, any morphism $f : X \to Y$ of this category can be factored into a free morphism followed by an acyclic fibration, and any cofibration is a retract of a free morphism. In particular, any cofibrant simplicial Σ-module is a rectract of a "free" simplicial Σ-module F, if we agree to say that F is free wheneven its degeneracy diagram is isomorphic to some $L_\Sigma(C)$, $C : \mathbf{\Delta}^0_- \to \mathit{Sets}$.

8.5. (Chain complexes and simplicial objects over abelian categories.) Let $\mathcal{A}$ be an abelian category, $\mathrm{Ch}(\mathcal{A})$ be the category of non-negative chain complexes over $\mathcal{A}$, and $s\mathcal{A} = \mathit{Funct}(\mathbf{\Delta}^0, \mathcal{A})$ be the category of simplicial objects of $\mathcal{A}$. The main result here is the *Dold–Kan correspondence* that actually establishes an equivalence between these two categories (cf. [Dold]).

8.5.1. (Unnormalized chain complex defined by a simplicial object.) The easiest way to obtain a chain complex $CX.$ from a simplicial object $X.$ is the following. Put $(CX)_n := X_n$ and define $\partial_n^{CX} : X_n \to X_{n-1}$ by

$$\partial_n^{CX} := \sum_{i=0}^{n} (-1)^i d_i^{n,X} \quad . \tag{8.5.1.1}$$

Relation $\partial_n^{CX} \partial_{n+1}^{CX} = 0$ follows immediately from (8.2.3.1), and in this way we obtain the *unnormalized chain complex* of X.

The whole construction is very classical. For example, $H_n(CL_{\mathbb{Z}}X)$ computes the simplicial homology of a simplicial set X, considered here as a CW-complex, and $H_n(CL_{\mathbb{Z}} \operatorname{Sing} X)$ computes the singular homology $H_n(X, \mathbb{Z})$ of a topological space X.

8.5.2. (Normalized chain complex of a simplicial object.) There are several other ways of obtaining a chain complex out of a simplicial object X. Consider the *degenerate subcomplex* $DX \subset CX$, generated in each dimension by the images of degeneracy maps:

$$(DX)_n := \sum_{i=0}^{n-1} s_i^{n-1,X}(X_{n-1}), \quad \text{where } n > 0. \tag{8.5.2.1}$$

We put $(DX)_0 = 0$. One checks directly from (8.2.3.3) that DX is indeed a subcomplex of CX. Moreover, it has an increasing filtration by subcomplexes $F_pDX \subset DX$, defined by taking the sum of images of degeneracy maps s_i with $i \leq p$. One can check that all $F_pDX/F_{p-1}DX$ are acyclic, hence DX is acyclic, hence $H_n(CX) \cong H_n(CX/DX)$ for all $n \geq 0$.

We put $N'X := CX/DX$; this is one of the equivalent descriptions of the *normalized chain complex* of X. Another one is given by the formula

$$(NX)_n := \bigcap_{i=1}^{n} \operatorname{Ker} d_i^{n,X} \subset X_n \tag{8.5.2.2}$$

One can check that NX is indeed a subcomplex of CX, and that DX is the complement of NX in CX, i.e. $X_n = (DX)_n \oplus (NX)_n$ for all $n \geq 0$. Therefore, $NX \cong N'X$, and it is this complex that is usually called the normalized chain complex of X. Notice that ∂_n^{NX} actually coincides with the restriction of $d_0^{X,n} : X_n \to X_{n-1}$ to $(NX)_n \subset X_n$, so we have a simpler formula in this case, and of course $H_n(NX) \xrightarrow{\sim} H_n(CX)$ for all $n \geq 0$.

8.5.3. (Simplicial object defined by a non-negative chain complex.) We have just constructed two functors $C, N : s\mathcal{A} \to \mathrm{Ch}(\mathcal{A})$. Now we'd like to define a functor $K : \mathrm{Ch}(\mathcal{A}) \to s\mathcal{A}$ in the opposite direction. Let's start with a non-negative chain complex $A.$, with differentials $\partial_n^A : A_n \to A_{n-1}$, $n \geq 1$. First of all, we construct a semisimplicial object $\tilde{A} : \mathbf{\Delta}_+^0 \to \mathcal{A}$ by putting

$$\tilde{A}_n := A_n, \quad d_k^{n,A} := \begin{cases} \partial_n^A, & \text{if } k = 0 \\ 0, & \text{if } k > 0 \end{cases} \tag{8.5.3.1}$$

Since $\mathbf{\Delta}_+$ is generated by the face maps ∂_n^k, this completely determines $\tilde{A}$, provided relations (8.2.3.1) hold. This is indeed the case since the product of two face operators $d_k^{n,A} d_l^{n+1,A}$ is zero for all choices of k and l.

Now we define KA to be $J_!\tilde{A}$, where $J_!$ is the left Kan extension of the embedding $J : \mathbf{\Delta}_+ \to \mathbf{\Delta}$. This implies that $(KA)_n = (J_!\tilde{A})([n]) = \varinjlim_{([n]\backslash\mathbf{\Delta}_+)^0} \tilde{A}$. Using the fact that any morphism $\varphi : [n] \to [m]$ in $\mathbf{\Delta}$ uniquely decomposes into a surjection $\eta : [n] \twoheadrightarrow [p]$ followed by an injection $i : [p] \to [m]$, i.e. a morphism of $\mathbf{\Delta}_+$, we see that the category $[n]\backslash\mathbf{\Delta}_+$ is a disjoint union of its subcategories, parametrized by the corresponding value of η. Each of these subcategories has an initial object, namely, η itself, and we arrive to the following classical formula:

$$(KA)_n = \bigoplus_{\eta : [n] \twoheadrightarrow [p]} A_p \tag{8.5.3.2}$$

The left Kan extension description given above defines the face and degeneracy operators on KA as well. More precisely, given a morphism $\varphi : [m] \to [n]$ in $\mathbf{\Delta}$ and a component $A_{p,\eta} = A_p$ of $(KA)_n$ indexed by some $\eta : [n] \twoheadrightarrow [p]$, the restriction of $(KA)(\varphi) : (KA)_n \to (KA)_m$ maps $A_{p,\eta}$ into $A_{q,\zeta}$, where $[m] \overset{\zeta}{\twoheadrightarrow} [q] \overset{\psi}{\hookrightarrow} [p]$ is the surjective-injective decomposition of $\eta \circ \varphi$. The map $A_{p,\eta} \to A_{q,\zeta}$ itself is taken to be equal to $\tilde{A}(\psi)$, i.e. it is the identity if $p = q$, it is ∂_p^A if $q = p - 1$ and $\psi = \partial_p^0$, and it is zero otherwise.

8.5.4. (Lower degrees of KA.) So if we start from $A = (\cdots \to A_2 \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} A_0)$, we get

$$\cdots A_0 \oplus A_1 \oplus A_1 \oplus A_2 \overset{d_i^2}{\Rrightarrow} A_0 \oplus A_1 \overset{d_i^1}{\rightrightarrows} A_0 \tag{8.5.4.1}$$

Here $s_0^0 = \binom{1}{0} : A_0 \to A_0 \oplus A_1$, $d_0^1 = (1, \partial_1^A)$, and $d_1^1 = (1, 0) : A_0 \oplus A_1 \to A_0$.

In general A_p occurs in $(KA)_n$ exactly $\binom{n}{p}$ times.

Theorem 8.5.5 (Dold–Kan, cf. [Dold]) *For any abelian category $\mathcal{A}$ the functors $N : s\mathcal{A} \rightleftarrows \mathrm{Ch}_{\geq 0}(\mathcal{A}) : K$ are adjoint equivalences of categories. In particular, $NKA. \cong A.$ for any $A. \in \mathrm{Ch}\,\mathcal{A}$. Moreover, two chain maps $f, g : A \rightrightarrows B$ are chain homotopic iff $K(f), K(g) : KA \rightrightarrows KB$ are simplicially homotopic.*

8.5.6. An immediate consequence is that $A \mapsto DKA$ is a universal construction that transforms any chain complex A over any abelian category $\mathcal{A}$ into an acyclic complex. One can show by applying this construction to an appropriate "universal" complex that this necessarily implies that DKA is (chain) homotopic to zero, and, moreover, a chain homotopy $H : 1_{DKA} \to 0$ can be defined by means of a universal formula. Therefore, DX is (chain) homotopic to zero, and CX is homotopic to NX for any simplicial object X of $\mathcal{A}$.

8.5.7. (Dual Dold–Kan correspondence.) Of course, we have a dual result: the category of cosimplicial objects $c\mathcal{A}$ is equivalent to the category of non-negative cochain complexes over $\mathcal{A}$. The dual of N' seems to be more convenient than that of N: we get $(N'X^{\cdot})^n = \bigcap_{k=0}^{n-1} \mathrm{Ker}\,\sigma_{n-1,X}^k \subset X^n$, with $d_{N'X}^n$ given by the restriction of $\sum_{k=0}^{n+1}(-1)^k \partial_{n+1,X}^k$.

8.5.8. (Model category structure on $\mathrm{Ch}\,\mathcal{A}$.) Now suppose that $\mathcal{A}$ is an abelian category satisfying the conditions of Quillen theorem **8.4.5**, e.g. $\mathcal{A} = R\text{-}\mathit{Mod}$ for a classical ring R. Then we have a natural simplicial model category structure on $s\mathcal{A}$, and we can transfer it to $\mathrm{Ch}\,\mathcal{A}$ by means of the Dold–Kan correspondence.

It turns out that a chain map $f : K \to L$ in $\mathrm{Ch}\,\mathcal{A}$ is a weak equivalence iff all $H_n(f)$, $n \geq 0$, are isomorphisms, a fibration iff all f_n, $n > 0$, are epimorphisms, and a cofibration iff all f_n, $n \geq 0$, are monomorphisms with a projective cokernel. This actually defines a model category structure on $\mathrm{Ch}\,\mathcal{A}$ whenever $\mathcal{A}$ is an abelian category with sufficiently many projective objects.

Moreover, we have a similar model category structure on the category $\mathrm{Ch}_{\gg -\infty}(\mathcal{A})$ of chain complexes bounded from below. In this case the weak equivalences are again the quasi-isomorphisms, the fibrations are the (componentwise) epimorphic maps, and the cofibrations are again the componentwise monic maps with projective cokernels.

8.5.9. (Homology as homotopy groups.) Given any simplicial group G, one define its normalization NG by $(NG)_n := \bigcap_{i>0} \mathrm{Ker}\, d_i^n$, and define maps $d = d_n : (NG)_n \to (NG)_{n-1}$ by taking the restriction of d_0^n to $(NG)_n \subset G_n$.

We obtain a complex of (non-abelian) groups NG, so we can compute the *(Moore) homotopy groups* of G by

$$\pi_n(G) := \mathrm{Ker}\big(d : (NG)_n \to (NG)_{n-1}\big) \big/ \mathrm{Im}\big(d : (NG)_{n+1} \to (NG)_n\big) \tag{8.5.9.1}$$

One can show that these groups are indeed canonically isomorphic to the homotopy groups $\pi_n(G, e) = \pi_n(|G|, e)$ of the underlying simplicial set of G. Notice that in this case all $\pi_n(G)$ are groups, abelian if $n > 0$.

Now suppose A is a simplicial abelian group, or more generally, a simplicial R-module, with R a classical ring. We obtain immediately $\pi_n(A, 0) \cong \pi_n(A) \cong H_n(NA) \cong H_n(CA)$, i.e. *the Dold–Kan correspondence induces a canonical isomorphism between homology groups of a chain complex and homotopy groups of the corresponding simplicial R-module.*

8.5.10. (Bisimplicial objects and bicomplexes.) Now let's consider the categories of non-negative chain bicomplexes over an abelian category $\mathcal{A}$, and the category $ss\mathcal{A} = \mathit{Funct}(\mathbf{\Delta}^0 \times \mathbf{\Delta}^0, \mathcal{A})$ of bisimplicial objects of $\mathcal{A}$. Such bisimplicial objects $X_{..}$ can be described as collections $\{X_{pq}\}_{p,q \geq 0}$ of objects of $\mathcal{A}$, together with "vertical" and "horizontal" face and degeneracy operators $d_i^{I,pq} : X_{pq} \to X_{p-1,q}$, $d_i^{II,pq} : X_{pq} \to X_{p,q-1}$ and so on, vertical and horizontal operators being required to commute between themselves.

Clearly, the Dold–Kan correspondence extends to this situation (e.g. we can treat a bicomplex as a complex over the category of complexes, and similarly for the simplicial objects). In this case we have two commuting normalization functors N_I and N_{II}, and the "true" normalization functor is their composite: $N = N_I N_{II} = N_{II} N_I$.

8.5.11. (Diagonal simplicial objects and totalizations of bicomplexes.) Let's start with a bisimplicial object $A = A_{..}$ over $\mathcal{A}$. We can compute the corresponding chain bicomplex CA or NA, and take its totalization $\mathrm{Tot}(CA)$ or $\mathrm{Tot}(NA)$; these two chain complexes are homotopy equivalent. On the other hand, we can construct the *diagonal simplicial object* $\mathrm{diag}\, A$ of A by composing $A : \mathbf{\Delta}^0 \times \mathbf{\Delta}^0 \to \mathcal{A}$ with the diagonal functor $\mathbf{\Delta} \to \mathbf{\Delta} \times \mathbf{\Delta}$. Clearly, $(\mathrm{diag}\, A)_n = A_{nn}$, and $d_i^{\mathrm{diag}\, A} = d_i^I d_i^{II}$. After that we can consider the corresponding unnormalized or normalized chain complex $C(\mathrm{diag}\, A)$ or $N(\mathrm{diag}\, A)$.

Theorem 8.5.12 (Eilenberg–Zilber; cf. [DoldPuppe, 2]) *Let A be a bisimplicial object over an abelian category $\mathcal{A}$. Then chain complexes $C(\mathrm{diag}\, A)$ and $\mathrm{Tot}\, CA$ are canonically quasi-isomorphic and even homotopy equivalent; in particular, their homology is canonically isomorphic: $\pi_n \,\mathrm{diag}\, A \cong H_n(C(\mathrm{diag}\, A)) \cong H_n(\mathrm{Tot}\, CA)$ for all $n \geq 0$.*

Morally, this means that computing the diagonal of a bisimplicial object corresponds to computing the totalization of a bicomplex.

8.5.13. (Alexander–Whitney and shuffle maps.) It's useful to have explicit constructions for the quasi-isomorphisms (even homotopy equivalences, because of the universality of all constructions) $f : C(\operatorname{diag} A) \to \operatorname{Tot}(CA)$ and $\nabla : \operatorname{Tot}(CA) \to C(\operatorname{diag} A)$ implied in the above theorem.

The map $f : C(\operatorname{diag} A) \to \operatorname{Tot}(CA)$, $f_n : A_{nn} \to \bigoplus_{p+q=n} A_{pq}$ is called the *Alexander–Whitney map* (cf. [Weibel, 8.5.4] or [DoldPuppe, 2]). Its component $f_{pq} : A_{nn} \to A_{pq}$ equals $d^I_{p+1} \cdots d^I_n d^{II}_0 \cdots d^{II}_0$. The map $\nabla : \operatorname{Tot}(CA) \to C(\operatorname{diag} A)$ is called the *shuffle map*. Its components $\nabla_{pq} : A_{pq} \to A_{nn}$, $n = p + q$, are given by the sum

$$\nabla_{pq} = \sum_{\mu} \operatorname{sgn}(\mu) s^I_{\mu(n-1)} \cdots s^I_{\mu(p)} s^{II}_{\mu(p-1)} \cdots s^{II}_{\mu(0)} \tag{8.5.13.1}$$

over all (p, q)-shuffles μ, i.e. all permutations μ of the set $\{0, 1, \ldots, n-1\}$, such that $\mu(0) < \mu(1) < \cdots < \mu(p-1)$ and $\mu(p) < \mu(p+1) < \cdots < \mu(n-1)$.

It is possible to check that f and ∇ are indeed chain maps, inducing isomorphisms between homologies of corresponding complexes, inverse to each other. Once this is shown, the universality of all constructions involved implies that f and ∇ are homotopy equivalences between $C(\operatorname{diag} A)$ and $\operatorname{Tot}(CA)$, homotopy inverse to each other, so one might try to construct explicit homotopies $f\nabla \sim 1$, $\nabla f \sim 1$ instead, and deduce the Eilenberg–Zilber theorem from this (cf. [DoldPuppe, 2]).

8.5.14. (Application: derived tensor product.) Suppose we have two nonnegative chain complexes X and Y over abelian category $\mathcal{A} = R\text{-}Mod$, R a commutative ring, and we want to compute the derived tensor product $X \underline{\underline{\otimes}} Y$. Classically we have to take projective resolutions, i.e. cofibrant replacements $P \to X$ and $Q \to Y$, compute the bicomplex $P \otimes Q$ given by $(P \otimes Q)_{ij} := P_i \otimes Q_j$, and put $X \underline{\underline{\otimes}} Y := \operatorname{Tot}(P \otimes Q)$.

However, Eilenberg–Zilber theorem implies that we can compute derived tensor products in another way. Namely, let's start from two simplicial R-modules $\tilde{X} = KX$ and $\tilde{Y} = KY$, choose some cofibrant replacements $\tilde{P} \to \tilde{X}$ and $\tilde{Q} \to \tilde{Y}$. Then $C\tilde{P}$ is chain homotopic to $N\tilde{P}$, and similarly for $C\tilde{Q}$, hence $\operatorname{Tot}(C\tilde{P} \otimes C\tilde{Q})$ computes $X \underline{\underline{\otimes}} Y$ as well. Applying Eilenberg–Zilber we see that CT with $T := \operatorname{diag}(\tilde{P} \otimes \tilde{Q})$ computes this derived tensor product as well, i.e. we just have to apply $\otimes_R$ componentwise: $T_n := \tilde{P}_n \otimes_R \tilde{Q}_n$. This gives us a very convenient way of deriving bifunctors, additive or not, just by applying them componentwise to appropriate fibrant or cofibrant replacements (cf. [DoldPuppe]).

8.6. (Simplicial Σ-modules.) Let Σ be any algebraic monad, e.g. a generalized ring. Recall that the category $s(\Sigma\text{-}Mod)$ of simplicial Σ-modules has a cofibrantly generated simplicial (closed) model category structure, characterized as follows: a morphism $f : X \to Y$ in $s(\Sigma\text{-}Mod)$ is a fibration (resp. a weak equivalence) iff $\Gamma_\Sigma(f)$ is a fibration (resp. a weak equivalence) of underlying simplicial sets, and f is a cofibration iff it has the LLP with respect to all acyclic fibrations (cf. **8.4.8**). When Σ is a classical ring, $s(\Sigma\text{-}Mod)$ is equivalent to the category of non-negative chain complexes of Σ-modules (cf. **8.5.5**).

The corresponding homotopy category $\operatorname{Ho} s(\Sigma\text{-}Mod)$ will be also denoted by $\mathcal{D}^{\leq 0}(\Sigma\text{-}Mod)$ or $\mathcal{D}^{\leq 0}(\Sigma)$; when Σ is a classical ring, this category is indeed equivalent to the subcategory of $\mathcal{D}(\Sigma\text{-}Mod)$ consisting of (cochain) complexes concentrated in non-positive degrees.

8.6.1. (Algebraic simplicial monads.) Notice that the simplicial extension $s\Sigma : sSets \to sSets$ of any monad Σ on *Sets* is a monad again, so we can consider the category $(s\Sigma)\text{-}Mod = (sSets)^{s\Sigma}$. However, since $((s\Sigma)(X))_n = \Sigma(X_n)$, a $s\Sigma$-structure on a simplicial set X is exactly a collection of Σ-structures on each X_n, compatible with the face and degeneracy operators, i.e. $(s\Sigma)\text{-}Mod \cong s(\Sigma\text{-}Mod)$, so we can write simply $s\Sigma\text{-}Mod$.

One can define *algebraic simplicial monads* Ξ on *sSets* by requiring Ξ to commute with filtered inductive limits of simplicial sets, and by imposing the following "continuity condition": the n-th truncation of $\Xi(X)$ should depend only on the n-th truncation of X, i.e. $\operatorname{sk}_n \Xi(\operatorname{sk}_n X) \to \operatorname{sk}_n \Xi(X)$ has to be an isomorphism for all simplicial sets X and all $n \geq 0$.

For example, $s\Sigma$ is an algebraic simplicial monad, for any algebraic monad Σ on *Sets*. More generally, any simplicial algebraic monad defines an algebraic simplicial monad, but not the other way around.

We might extend to the case of algebraic simplicial monads Ξ the theory developed in previous parts of this work for algebraic monads, e.g. define "simplicial sets of morphisms" $\mathbf{Hom}_\Xi(X, Y)$, tensor products (for a commutative Ξ), as well as construct a natural simplicial model category structure on $(sSets)^\Xi$ and prove some variants of most results that follow. We won't pursue this approach further, because it is too general for our purpose; however, some trace of it can be found in our notations, e.g. $\operatorname{Hom}_{s\Sigma}$ for the simplicial set of homomorphisms of simplicial Σ-modules.

Notice that this approach might be useful for arithmetic geometry, even if it is beyond the scope of present work. For example, considering the cone of $\mathbb{Z}_\infty \xrightarrow{f} \mathbb{Z}_\infty$ as a new algebraic simplicial monad is a natural way to construct a quotient $\mathbb{Z}_\infty/(f)$ different from $\mathbb{Z}_\infty/\mathfrak{m}_\infty$. Therefore, if we want to consider divisors like $-\log|f| \cdot [\infty]$ as effective divisors given by closed subschemes of

$\operatorname{Spec}\mathbb{Z}_\infty$, we'll need such a way of constructing "fine" quotients.

Now let's return to our much more specific situation.

Proposition 8.6.2 *Let $\rho : \Sigma \to \Xi$ be an algebraic monad homomorphism. Consider the pair of adjoint functors $\rho^* = s\rho^* : s(\Sigma\text{-}Mod) \to s(\Xi\text{-}Mod)$ and $\rho_* : s(\Xi\text{-}Mod) \to s(\Sigma\text{-}Mod)$. We claim that this is a pair of Quillen functors (cf.* **8.1.15***), hence their derived functors $\mathbb{L}\rho^* : \mathcal{D}^{\leq 0}(\Sigma) \to \mathcal{D}^{\leq 0}(\Xi)$ and $\mathbb{R}\rho_* : \mathcal{D}^{\leq 0}(\Xi) \to \mathcal{D}^{\leq 0}(\Sigma)$ exist and are adjoint to each other; they can be computed by applying ρ^*, resp. ρ_* to arbitrary cofibrant, resp. fibrant replacements (cf.* **8.1.16***). Actually ρ_* preserves all weak equivalences, hence $\mathbb{R}\rho_* = \rho_*$.*

Proof. First of all, the definition of model category structure on $s(\Sigma\text{-}Mod)$ together with equality $\Gamma_\Xi = \Gamma_\Sigma \circ \rho_*$ imply that ρ_* preserves all fibrations and weak equivalences, hence also acyclic fibrations. Using adjointness of ρ^* and ρ_* we deduce that ρ^* preserves arbitrary cofibrations and acyclic cofibrations, since $\rho^*(f)$ has the required LLP for any cofibration, resp. acyclic cofibration $f : X \to Y$ in $s(\Sigma\text{-}Mod)$. Now it remains to check that ρ^* preserves weak equivalences between cofibrant objects. We know this already for acyclic cofibrations, so we can finish the proof by applying the following lemma:

Lemma 8.6.3 *a) Let $F : \mathcal{C} \to \mathcal{D}$ be a functor from a model category $\mathcal{C}$ that transforms acyclic cofibrations between cofibrant objects into isomorphisms. Then the same is true for all weak equivalences between cofibrant objects of $\mathcal{C}$.*

b) Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between two model categories that transforms acyclic cofibrations between cofibrant objects of $\mathcal{C}$ into weak equivalences in $\mathcal{D}$. Then F preserves all weak equivalences between cofibrant objects of $\mathcal{C}$, hence F has a left derived functor $\mathbb{L}F$ by **8.1.17**.

Proof. b) follows immediately from a) applied to $\gamma_D \circ F : \mathcal{C} \to \operatorname{Ho}\mathcal{D}$ and from the fact that φ is a weak equivalence in $\mathcal{D}$ iff $\gamma_D(\varphi)$ is an isomorphism, all our model categories being closed, so let's prove a).

So let $p : Q \to P$ be a weak equivalence between two cofibrant objects. By (CM5) and (CM3) we can decompose p into an acyclic cofibration $i : Q \to P'$ followed by an acyclic fibration $\pi : P' \to P$. Clearly, P' is itself cofibrant, and $F(i)$ is an isomorphism by assumption, so we are reduced to proving that $F(\pi)$ is an isomorphism for any acyclic fibration $\pi : P' \to P$ between cofibrant objects.

Since P is cofibrant and π is an acyclic fibration, any morphism $P \to P$ can be lifted to a morphism $P \to P'$; applying this to id_P we obtain a section $\sigma : P' \to P$. Clearly, σ is a weak equivalence since both π and $\pi \circ \sigma = \mathrm{id}_P$

are, so it can be decomposed into an acyclic cofibration $\tau : P \to P''$ followed by an acyclic fibration $\pi' : P'' \to P'$, and we have $\pi \circ \pi' \circ \tau = \mathrm{id}_P$. Since P is cofibrant and $\tau : P \to P''$ is an acyclic cofibration, P'' is also cofibrant and $F(\tau)$ is an isomorphism by assumption, hence $F(\pi) \circ F(\pi') = F(\tau)^{-1}$ is an isomorphism as well.

Now $\pi' : P'' \to P'$ being an acyclic fibration between cofibrant objects, we can repeat the same construction again, obtaining a cofibrant object P''', an acyclic fibration $\pi'' : P''' \to P''$ and an acyclic cofibration $\tau' : P' \to P'''$, such that $\pi' \circ \pi'' \circ \tau' = \mathrm{id}_{P'}$, and $F(\tau')$ is an isomorphism, hence $F(\pi') \circ F(\pi'')$ as well. We see that both $F(\pi) \circ F(\pi')$ and $F(\pi') \circ F(\pi'')$ are isomorphisms, hence $F(\pi)$ is an isomorphism by Lemma **8.6.4** below, q.e.d.

Lemma 8.6.4 *If* $\cdot \xrightarrow{u} \cdot \xrightarrow{v} \cdot \xrightarrow{w} \cdot$ *are three composable morphisms of an arbitrary category, such that* vu *and* wv *are isomorphisms, then* u, v *and* w *are isomorphisms themselves.*

Proof. Indeed, v has both a left inverse $(wv)^{-1}w$ and a right inverse $u(vu)^{-1}$, hence it is invertible with $v^{-1} = (wv)^{-1}w = u(vu)^{-1}$, hence $u = v^{-1}(vu)$ and $w = (wv)v^{-1}$ are invertible as well.

8.6.5. (Functoriality of derived base change.) Now suppose that $\sigma : \Xi \to \Lambda$ is another algebraic monad homomorphism. Since ρ^* transforms cofibrant objects into cofibrant objects, we see that $\mathbb{L}(\sigma \circ \rho)^* = \mathbb{L}(\sigma^* \circ \rho^*) \cong \mathbb{L}\sigma^* \circ \mathbb{L}\rho^*$ (cf. **8.1.17**,b), and similarly $\mathbb{R}(\sigma \circ \rho)_* = \mathbb{R}(\rho_* \circ \sigma_*) \cong \mathbb{R}\rho_* \circ \mathbb{R}\sigma_*$.

8.6.6. (Application: abelianization and cohomology.) Let $\mathcal{C}$ be a model category, $\mathcal{C}_{ab}$ be the category of abelian group objects in $\mathcal{C}$, and suppose that the inclusion functor $i : \mathcal{C}_{ab} \to \mathcal{C}$ has a left adjoint ab, and that $\mathcal{C}_{ab}$ has a nice model category structure, satifying the conditions of [Quillen, 2.5]. Recall that these conditions are fulfilled if $\mathcal{C}_{ab}$ is equivalent to $s\mathcal{A}$ for an abelian category $\mathcal{A}$, and if the derived functors $\mathbb{L}ab$ and $\mathbb{R}i$ exist. Then Quillen defines the *generalized cohomology* $H^q(X, A)$ of an object X of $\operatorname{Ho}\mathcal{C}$ with coefficients in an object A of $\operatorname{Ho}\mathcal{C}_{ab}$ by

$$H^q(X, A) := [\mathbb{L}ab(X), \Omega^{q+N}\Sigma^N A], \quad \text{for any } N \gg 0.$$

Now if $\mathcal{C} = s(\Sigma\text{-}\mathit{Mod})$, then $\mathcal{C}_{ab} = s(\Sigma \otimes \mathbb{Z})\text{-}\mathit{Mod}$, and the existence of ab, i and their derived follows from our general results applied to algebraic monad homomorphism $\Sigma \to \Sigma \otimes \mathbb{Z}$.

8.6.7. (Base change and suspension.) Recall that for any pointed model category we have a suspension functor $\Sigma : \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{C}$; the suspension of a cofibrant X can be computed as the cofiber of $X \sqcup X \to X \times I$, where $X \times I$ is any cylinder object for X. If $\mathcal{C}$ is simplicial, $X \otimes \Delta(1)$ is a cylinder object for an acyclic X, so we get a functorial way of computing ΣX by means of the following cocartesian square:

$$\begin{tikzcd} X \sqcup X \arrow[r] \arrow[d] & X \otimes \Delta(1) \arrow[d, dashed] \\ 0 \arrow[r, dashed] & \Sigma X \end{tikzcd} \tag{8.6.7.1}$$

Let's apply this to $s(\Sigma\text{-}Mod)$, for any algebraic monad Σ with zero. Then $(X \otimes \Delta(1))_n$ is the sum of $n+2$ copies of X_n, parametrized by monotone maps $\eta : [n] \to [1]$, and the embeddings $X_n \rightrightarrows (X \otimes \Delta(1))_n$ are just the embeddings into the components corresponding to constant η's. We deduce that

$$(\Sigma X)_n \cong \bigoplus_{\eta:[n] \twoheadrightarrow [1]} X_n \quad . \tag{8.6.7.2}$$

Now if $\rho : \Sigma \to \Xi$ is a homomorphism of algebraic monads with zero, the base change functor ρ^* preserves direct sums and cofibrant objects; hence $\Sigma\rho^* X \cong \rho^*\Sigma X$ for any cofibrant X in $s(\Sigma\text{-}Mod)$, and $\Sigma \circ \mathbb{L}\rho^* \cong \mathbb{L}\rho^* \circ \Sigma$ on the level of derived functors (we might write Σ for the functor on $s(\Sigma\text{-}Mod)$ defined by (8.6.7.1), and $\mathbb{L}\Sigma$ for the suspension functor on the homotopic category). By adjointness we get $\mathbb{R}\rho_* \circ \Omega \cong \Omega \circ \mathbb{R}\rho_*$. This is actually a consequence of a general fact: Theorem 3 of [Quillen, 1.4] implies that $\mathbb{L}\rho^*$ preserves suspensions and cofibration sequences, while $\mathbb{R}\rho_*$ preserves loop objects and fibration sequences.

8.6.8. (Mapping cones and cylinders.) Let $f : X \to Y$ be a morphism in any simplicial model category $\mathcal{C}$. We define the *(mapping) cylinder* $Cyl(f)$ of f by the following cocartesian square:

$$\begin{tikzcd} X \otimes \{0\} \arrow[r, "f"] \arrow[d] & Y \otimes \{0\} \arrow[d, dashed] \\ X \otimes \Delta(1) \arrow[r, dashed] & Cyl(f) \end{tikzcd} \tag{8.6.8.1}$$

Here $\{e\}$, $e = 0, 1$, denotes the corresponding vertex of $\Delta(1)$, i.e. the subobject defined by $\Delta(\sigma_1^e) : \Delta(0) \to \Delta(1)$. Clearly, in $s(\Sigma\text{-}Mod)$ we get $Cyl(f)_n \cong Y_n \oplus X_n^{\oplus(n+1)}$; we basically construct $(X \otimes \Delta(1))_n$ and replace the first copy of X_n with Y_n.

When $\mathcal{C}$ is pointed, i.e. it has a zero object, we define the *(mapping) cone* $C(f)$ of f as the cofiber of $X \cong X \otimes \{1\} \to X \otimes \Delta(1) \to Cyl(f)$. For $s(\Sigma\text{-}Mod)$ we get $C(f)_n \cong Y_n \oplus X_n^{\oplus(n)}$. One can check that these definitions of cone and cylinder correspond to those known from homological algebra via the Dold–Kan correspondence when Σ is additive.

The above construction is especially useful when X and f are cofibrant; we can extend it for any morphism $f : X \to Y$ in $\operatorname{Ho}\mathcal{C}$ by replacing X and Y by appropriate cofibrant representatives. Then $Y \to Cyl(f)$ is a weak equivalence for any f, and $X \to Y \to C(f)$ is a cofibration sequence in $\operatorname{Ho}\mathcal{C}$ for any morphism f, so for any $Z \in \operatorname{Ob}\operatorname{Ho}\mathcal{C}$ we get a long exact sequence of homotopy:

$$\cdots \to [\Sigma^2 X, Z] \to [\Sigma C(f), Z] \to [\Sigma Y, Z] \to \\ \to [\Sigma X, Z] \to [C(f), Z] \to [Y, Z] \to [X, Z] \to 0 \quad (8.6.8.2)$$

Now an obvious but important statement is that $\mathbb{L}\rho^*$ *preserves mapping cones and cylinders.*

8.6.9. (Computation of π_0.) Recall that for any simplicial set A we can compute its "set of connected components" $\pi_0 A = \pi_0(|A|)$ as the cokernel of $A_1 \rightrightarrows A_0$ in the category of sets. We know that any weak equivalence induces an isomorphism of π_0's, so we obtain a functor $\pi_0 : \operatorname{Ho} sSets \to Sets$.

Now suppose that A is a simplicial Σ-module. Since weak equivalences of simplicial Σ-modules are defined in terms of underlying simplicial sets, we obtain a well-defined functor $\pi_0 : \operatorname{Ho} s(\Sigma\text{-}Mod) \to Sets$. We claim that the surjection $A_0 \twoheadrightarrow \pi_0 A = A_0/\overset{s}{\sim}$ is always compatible with the Σ-structure on A_0, so that $\pi_0 A$ inherits a canonical Σ-structure, and we get a functor $\pi_0 : \operatorname{Ho} s(\Sigma\text{-}Mod) \to \Sigma\text{-}Mod$. Indeed, for any operation $t \in \Sigma(n)$, and any elements $x_1 \overset{ss}{\sim} y_1$, $x_2 = y_2$, $\ldots$, $x_n = y_n$ of A_0 one can find by definition elements $h_i \in A_1$ with $d_0(h_i) = x_i$, $d_1(h_i) = y_i$ (when $x_i = y_i$ we take $h_i := s_0(x_i)$), put $h := t(h_1, \ldots, h_n)$, and obtain $d_0(h) = x := t(x_1, \ldots, x_n)$, $d_1(h) = y := t(y_1, \ldots, y_n)$ since the d_k are Σ-homomorphisms. We conclude that $x \overset{ss}{\sim} y$, so the equivalence relation $\overset{s}{\sim}$ generated by $\overset{ss}{\sim}$ is compatible with the Σ-structure as well.

On the other hand, any Σ-module A can be considered as a constant simplicial Σ-module that will be usually denoted by the same letter. This yields a functor $\Sigma\text{-}Mod \to s(\Sigma\text{-}Mod)$ in the opposite direction, clearly right adjoint to π_0, and $\pi_0 A \cong A$ for any constant simplicial set. We deduce that the functor $\Sigma\text{-}Mod \to s(\Sigma\text{-}Mod)$ is fully faithful. Notice that for any simplicial Σ-module we get a morphism $A \to \pi_0 A$.

One can check that the higher homotopic invariants $\pi_n A$, $n \geq 0$, also have a natural Σ-structure, commuting with their group structure, at least if Σ has a zero. One can use $\pi_n(A) \cong \pi_0(\Omega^n A)$ to show this.

8.6.10. Since $\pi_0 A = \operatorname{Coker}(A_1 \rightrightarrows A_0)$ both in $\Sigma\text{-}Mod$ and $Sets$, and ρ^* is right exact, we see that $\pi_0 \rho^* P \cong \rho^* \pi_0 P$ for any cofibrant simplicial Σ-

module P and any algebraic monad homomorphism $\rho : \Sigma \to \Xi$, hence $\pi_0 \circ \mathbb{L}\rho^* \cong \rho^* \circ \pi_0 : \mathcal{D}^{\leq 0}(\Sigma) \to \Xi\text{-}Mod$. In particular, for any Σ-module M we get $\pi_0 \mathbb{L}\rho^* M \cong \rho^* M$, i.e. we have a canonical map $\mathbb{L}\rho^* M \to \rho^* M$ inducing an isomorphism of π_0's. This map is a weak equivalence iff all $\pi_n(\mathbb{L}\rho^* M, x_0) = 0$, $n \geq 1$, for all choices of base point $x_0 \in \pi_0 \mathbb{L}\rho^* M = \rho^* \pi_0 M$; then we can write $\mathbb{L}\rho^* M \cong M$.

8.6.11. (Cohomological flatness.) We say that an algebraic monad homomorphism $\rho : \Sigma \to \Xi$ is *cohomologically flat* if ρ^* preserves weak equivalences, i.e. if $\mathbb{L}\rho^* = \rho^*$. This implies $\pi_n \mathbb{L}\rho^* M = 0$ for all $n \geq 1$ and all Σ-modules M. Notice that usual flatness (exactness of ρ^*) implies cohomological flatness. Indeed, since ρ^* preserves acyclic cofibrations, all we have to check is that it preserves acyclic fibrations. But $p : X \to Y$ is an acyclic fibration iff all maps $X_n \to (\operatorname{cosk}_{n-1} X)_n \times_{(\operatorname{cosk}_{n-1} Y)_n} Y_n$ are surjective, and $(\operatorname{cosk}_{n-1} X)_n \cong \operatorname{Hom}(\dot{\Delta}(n), X)$ can be expressed in terms of finite projective limits of components of X, $\dot{\Delta}(n)$ being a finite inductive limit of standard simplices. Therefore, the property of p to be an acyclic fibration is preserved by any exact functor.

8.6.12. (Stable homotopic category.) There are several ways of transforming $\operatorname{Ho} sSets = \mathcal{D}^{\leq 0}(\mathbb{F}_\varnothing)$ (more precisely, the corresponding pointed category $\mathcal{D}^{\leq 0}(\mathbb{F}_1)$) into a category on which Ω and Σ become autoequivalences, inverse to each other. The simplest way consists in considering the category of couples (X, n), $X \in \operatorname{Ho} sSets$, $n \in \mathbb{Z}$, with $\operatorname{Hom}((X,n),(Y,m)) := \varinjlim_{N \gg 0} [\Sigma^{n+N} X, \Sigma^{m+N} Y]$. If we apply this construction to $\mathcal{D}^{\leq 0}(R)$ for a classical ring R, we obtain $\mathcal{D}^-(R)$, so we use the same notation $\mathcal{D}^-(\Lambda)$ for the stable homotopy category of $s(\Lambda\text{-}Mod)$ for any algebraic monad Λ with zero.

Notice that $\mathcal{D}^-(\Lambda)$ will be a *triangulated* category for any Λ. The proof is exactly the same as for the stable homotopic category of simplicial sets: the abelian group structure on stable homomorphism sets comes from the abelian cogroup object structure on all $\Sigma^n X$, $n \geq 2$, and the distinguished triangles arise from cofibration sequences.

We get some *stable homotopy groups* $\pi_n(X) := [\Sigma^n \Delta(0), X]_{st}$; they constitute a homological functor on $\mathcal{D}^-(\Lambda)$ with values in $\mathbb{Z}\text{-}Mod$ (actually in $(\Lambda \otimes \mathbb{Z})\text{-}Mod$, if Λ is commutative). However, these stable homotopy groups are not so easy to compute, and they do not reflect all cohomological structure of $\mathcal{D}^-(\Lambda)$. For example, one can construct an object P in $\mathcal{D}^-(\mathbb{Z}_\infty)$ which has $\pi_n(P) = 0$ for all $n \in \mathbb{Z}$, but $P \neq 0$. (It is still unknown whether such examples exist over $\mathbb{F}_1$.)

8.7. (Derived tensor product.) The aim of this subsection is to construct the *derived tensor product* $\underline{\underline{\otimes}} = \mathbb{L}\otimes_\Lambda : \mathcal{D}^{\leq 0}(\Lambda) \times \mathcal{D}^{\leq 0}(\Lambda) \to \mathcal{D}^{\leq 0}(\Lambda)$ for any generalized ring, i.e. commutative algebraic monad Λ. Before doing this we have to make some general remarks about deriving bifunctors.

8.7.1. (Simplicial extension of a bifunctor.) Recall that any functor $F : \mathcal{C} \to \mathcal{D}$ has a natural simplicial extension $sF : s\mathcal{C} \to s\mathcal{D}$, defined by composing functors $\mathbf{\Delta}^0 \to \mathcal{C}$ with F. Now suppose that $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$, i.e. we have a bifunctor $F : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{D}$. Notice that $s(\mathcal{C}_1 \times \mathcal{C}_2) = \mathit{Funct}(\mathbf{\Delta}^0, \mathcal{C}_1 \times \mathcal{C}_2) \cong \mathit{Funct}(\mathbf{\Delta}^0, \mathcal{C}_1) \times \mathit{Funct}(\mathbf{\Delta}^0, \mathcal{C}_2) = s\mathcal{C}_1 \times s\mathcal{C}_2$, so the simplicial extension sF of F is indeed a functor $s\mathcal{C}_1 \times s\mathcal{C}_2 \to s\mathcal{D}$ as one would expect. Of course, we usually write just F instead of sF. For any two simplicial objects X and Y of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, we get $(sF(X,Y))_n = F(X_n, Y_n)$, with the face and degeneracy operators given by formulas like $d_i^{sF(X,Y)} = F(d_i^X, d_i^Y)$.

In particular, the tensor product $\otimes_\Lambda : \Lambda\text{-}\mathit{Mod} \times \Lambda\text{-}\mathit{Mod} \to \Lambda\text{-}\mathit{Mod}$ extends to a bifunctor $s\Lambda\text{-}\mathit{Mod} \times s\Lambda\text{-}\mathit{Mod} \to s\Lambda\text{-}\mathit{Mod}$.

8.7.2. (Product of model categories.) The next thing we need to derive a bifunctor $F : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{D}$, with $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{D}$ model categories, is a model category structure on the product $\mathcal{C}_1 \times \mathcal{C}_2$. The natural choice is to declare $f = (f_1, f_2) : (X_1, X_2) \to (Y_1, Y_2)$ a fibration (resp. cofibration, weak equivalence) in $\mathcal{C}_1 \times \mathcal{C}_2$ iff both its components $f_i : X_i \to Y_i$, $i = 1, 2$, are fibrations (resp. . .) in $\mathcal{C}_i$. One checks immediately that this is indeed a model category structure, and that $\mathrm{Ho}(\mathcal{C}_1 \times \mathcal{C}_2) \cong \mathrm{Ho}\,\mathcal{C}_1 \times \mathrm{Ho}\,\mathcal{C}_2$, so we can safely consider the left and right derived functors of F with respect to this model structure: they'll be some functors $\mathbb{L}F, \mathbb{R}F : \mathrm{Ho}\,\mathcal{C}_1 \times \mathrm{Ho}\,\mathcal{C}_2 \to \mathrm{Ho}\,\mathcal{D}$. Notice that if $\mathcal{C}_1$ and $\mathcal{C}_2$ are simplicial model categories, we can define a compatible simplicial structure on $\mathcal{C}_1 \times \mathcal{C}_2$ as well by setting $(X_1, X_2) \otimes K := (X_1 \otimes K, X_2 \otimes K)$ for any $X_i \in \mathrm{Ob}\,\mathcal{C}_i$ and $K \in \mathrm{Ob}\,\mathit{sSets}$.

In particular, the above considerations enable us to define the left derived tensor product $\underline{\underline{\otimes}} : \mathcal{D}^{\leq 0}(\Lambda) \times \mathcal{D}^{\leq 0}(\Lambda) \to \mathcal{D}^{\leq 0}(\Lambda)$. We just have to show its existence.

8.7.3. (Criterion for existence of left derived bifunctors.) Now we'd like to combine **8.1.17** and **8.6.3** with the above considerations. We obtain the following proposition:

Proposition. *Let $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{D}$ be model categories, $F : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{D}$ be a bifunctor. Then the following conditions are equivalent:*

a) F transforms weak equivalences between cofibrant objects of $\mathcal{C}_1 \times \mathcal{C}_2$ into weak equivalences in $\mathcal{D}$.

b) F transforms acyclic cofibrations between cofibrant objects of $\mathcal{C}_1 \times \mathcal{C}_2$ into weak equivalences in $\mathcal{D}$. In other words, if $f_i : X_i \to Y_i$ are acyclic *cofibrations between cofibrant objects of $\mathcal{C}_i$, then $F(f_1, f_2)$ is a weak equivalence.*

c) For any cofibrant object P of $\mathcal{C}_1$ the functor $F(P, -) : \mathcal{C}_2 \to \mathcal{D}$ transforms acyclic cofibrations between cofibrant objects into weak equivalences, and similarly for $F(-, Q) : \mathcal{C}_1 \to \mathcal{D}$, Q any cofibrant object of $\mathcal{C}_2$.

If the above conditions hold, the functor F admits a left derived $\mathbb{L}F : \operatorname{Ho}\mathcal{C}_1 \times \operatorname{Ho}\mathcal{C}_2 \to \operatorname{Ho}\mathcal{D}$, and $(\mathbb{L}F)(\gamma X, \gamma Y)$ can be computed for any $X \in \operatorname{Ob}\mathcal{C}_1$, $Y \in \operatorname{Ob}\mathcal{C}_2$ by taking $\gamma F(P, Q)$, where $P \to X$ and $Q \to Y$ are arbitrary cofibrant replacements.

Proof. Immediate from **8.1.17**, **8.6.3** and the description of model category structure on $\mathcal{C}_1 \times \mathcal{C}_2$ given in **8.7.2**.

Now we'd like to verify the above conditions for the tensor product. However, it pays out to study a more general situation.

Definition 8.7.4 *Let $\otimes$ be a tensor structure on a model category $\mathcal{C}$. We say that these two structures are* compatible *if the following version of (SM7b) holds:*

(TM) If $i : A \to B$ and $s : K \to L$ are cofibrations in $\mathcal{C}$, then

$$i \,\square\, s : A \otimes L \sqcup_{A \otimes K} B \otimes K \to B \otimes L \tag{8.7.4.1}$$

is a cofibration in $\mathcal{C}$ as well, acyclic if either i or s is acyclic.

In this case we say that $\mathcal{C}$ is a tensor model category.

Similarly, if $\oslash : \mathcal{D} \times \mathcal{C} \to \mathcal{D}$ is an external $\otimes$-action of a tensor model category $\mathcal{C}$ on another model category $\mathcal{D}$, we say that $\oslash$ is compatible with the model structure of $\mathcal{D}$ *if the following variant of (TM) holds:*

(TMe) If $i : A \to B$ is a cofibration in $\mathcal{D}$ and $s : K \to L$ is a cofibration in $\mathcal{C}$, then

$$A \oslash L \sqcup_{A \oslash K} B \oslash K \to B \oslash L \tag{8.7.4.2}$$

is a cofibration in $\mathcal{D}$ as well, acyclic if either i or s is acyclic.

8.7.5. (Examples.) Clearly, the $\otimes$-structure on *sSets* given by the direct product, as well as the $\otimes$-action of *sSets* on any simplicial model category $\mathcal{C}$ are compatible with the model structures involved, essentially by definition. Another example will be given by $\otimes_\Lambda$ on $s(\Lambda\text{-}Mod)$ for any generalized ring Λ after we check (TM) in that case.

8.7.6. (Deriving $\otimes$.) The importance of compatible tensor structures $\otimes$ on a model category $\mathcal{C}$ is due to the fact they can be derived, thus yielding a tensor structure on $\operatorname{Ho}\mathcal{C}$. More precisely:

Proposition. *Let $\otimes$ be a compatible tensor structure on a model category $\mathcal{C}$. Suppose that $\varnothing \otimes X \cong \varnothing \cong X \otimes \varnothing$ for any object X of $\mathcal{C}$, where $\varnothing$ denotes the initial object. Then $P \otimes Q$ is a cofibrant object for any two cofibrant objects P and Q of $\mathcal{C}$, and $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ satisfies the conditions of* **8.7.3**; *in particular, its left derived $\underline{\underline{\otimes}} = \mathbb{L}\otimes$ exists and it can be computed by taking the tensor product of cofibrant replacements of both arguments. It defines a tensor structure on $\operatorname{Ho}\mathcal{C}$, with the same constraints as the original tensor structure on $\mathcal{C}$.*

Similar statements can be made for a compatible $\otimes$-action $\oslash : \mathcal{D} \times \mathcal{C} \to \mathcal{D}$ that preserves initial objects in each argument: it can be derived, thus yielding an external $\otimes$-action of $\operatorname{Ho}\mathcal{C}$ on $\operatorname{Ho}\mathcal{D}$.

Proof. Applying (TM) to $\varnothing \to B$ and $\varnothing \to L$ we see immediately that $B \otimes L$ is cofibrant for any cofibrant B and L. Applying the same axiom to any acyclic cofibration $A \to B$ and any cofibration $\varnothing \to L$ we see that $A \otimes L \to B \otimes L$ is an acyclic cofibration as well. In particular, $-\otimes L$ transforms acyclic cofibrations between cofibrant objects into weak equivalences for any cofibrant L, and we have a similar result for $B \otimes -$, for any cofibrant B. So the condition **8.7.3**,c) holds and we can derive $\otimes$. The case of an external $\otimes$-action is treated similarly with (TM) replaced by (TMe).

8.7.7. (Compatibility and inner Homs.) Suppose we are given some $\otimes$-structure on a model category $\mathcal{C}$, such that for any A and X in $\mathcal{C}$ the functor $\operatorname{Hom}_{\mathcal{C}}(A \otimes -, X)$ is representable by some inner Hom object $\mathbf{Hom}_{\mathcal{C}}(A, X)$:

$$\operatorname{Hom}_{\mathcal{C}}(K, \mathbf{Hom}_{\mathcal{C}}(A, X)) \cong \operatorname{Hom}_{\mathcal{C}}(A \otimes K, X) \tag{8.7.7.1}$$

Then we can state the compatibility axiom (TM) in the following equivalent way:

(TMh) Whenever $i : A \to B$ is a cofibration and $p : X \to Y$ is a fibration in $\mathcal{C}$, the following morphism is a fibration in $\mathcal{C}$:

$$\mathbf{Hom}_{\mathcal{C}}(B, X) \xrightarrow{(i^*, p_*)} \mathbf{Hom}_{\mathcal{C}}(A, X) \times_{\mathbf{Hom}_{\mathcal{C}}(A,Y)} \mathbf{Hom}_{\mathcal{C}}(B, X) \tag{8.7.7.2}$$

We require this fibration to be acyclic when either i or p is a weak equivalence.

The proof of equivalence of (TM) and (TMh) proceeds in the same way as the classical proof of (SM7)$\Leftrightarrow$(SM7b), using just the various lifting properties and adjointness isomorphisms involved, so we won't provide any further details.

Similarly, if we are given an external action $\oslash : \mathcal{D} \times \mathcal{C} \to \mathcal{D}$, with both $\mathcal{C}$ and $\mathcal{D}$ model categories, and if the functor $\operatorname{Hom}_{\mathcal{D}}(A \otimes -, X)$ is representable by some $\mathbf{Hom}_{\mathcal{C}}(A, X)$, then the compatibility condition (TMe) has an equivalent form similar to (TMh) above, where $i : A \to B$ and $p : X \to Y$ are required now to be a cofibration and a fibration in $\mathcal{D}$.

8.7.8. (Deriving **Hom**.) Suppose the above conditions are fulfilled, so we have an inner Hom functor $\mathbf{Hom}_{\mathcal{C}} : \mathcal{D}^0 \times \mathcal{D} \to \mathcal{C}$. Let's assume that $\mathbf{Hom}_{\mathcal{C}}(\varnothing_{\mathcal{D}}, X) \cong e_{\mathcal{C}}$ and $\mathbf{Hom}_{\mathcal{C}}(X, e_{\mathcal{D}}) \cong e_{\mathcal{C}}$ for any object X of $\mathcal{D}$; actually the second relation holds automatically, while the first is equivalent to $\varnothing_{\mathcal{D}} \oslash K \cong \varnothing_{\mathcal{D}}$ for all K in $\mathcal{C}$. Then (TMh) implies the (variant of) condition **8.7.3**,c), sufficient for the existence of the right derived $\mathbb{R}\,\mathbf{Hom}_{\mathcal{C}} : \operatorname{Ho}\mathcal{D}^0 \times \operatorname{Ho}\mathcal{D} \to \operatorname{Ho}\mathcal{C}$, by the same reasoning as in **8.7.6**. In particular, $\mathbb{R}\,\mathbf{Hom}_{\mathcal{C}}(\gamma A, \gamma Y)$ can be computed as $\gamma\,\mathbf{Hom}_{\mathcal{C}}(B, X)$, where $A \to B$ is any cofibrant replacement, and $X \to Y$ is any fibrant replacement; in this case $\mathbf{Hom}_{\mathcal{C}}(B, X)$ will be automatically fibrant.

8.7.9. (Adjointness of $\underline{\underline{\oslash}} = \mathbb{L}\oslash$ and $\mathbb{R}\,\mathbf{Hom}$.) One can show that under the above conditions $\mathbb{R}\,\mathbf{Hom}$ is an inner Hom for $\underline{\underline{\oslash}} : \operatorname{Ho}\mathcal{D} \times \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{D}$, so we get

$$\operatorname{Hom}_{\operatorname{Ho}\mathcal{D}}(A \underline{\underline{\oslash}} K, X) \cong \operatorname{Hom}_{\operatorname{Ho}\mathcal{C}}(K, \mathbb{R}\,\mathbf{Hom}_{\mathcal{C}}(A, X)) \tag{8.7.9.1}$$

The proof goes as follows. We can assume A and K to be cofibrant and X to be fibrant. Then $A \underline{\underline{\oslash}} K = A \oslash K$ is again cofibrant and $\mathbb{R}\,\mathbf{Hom}_{\mathcal{C}}(A, X)$ is fibrant. Now let us fix a cofibrant A and consider the two adjoint functors $A \oslash - : \mathcal{C} \to \mathcal{D}$ and $\mathbf{Hom}_{\mathcal{C}}(A, -) : \mathcal{D} \to \mathcal{C}$. They are easily seen to be a Quillen pair, and for example $(\mathbb{L}(A \oslash -))(K)$ clearly coincides with $A \underline{\underline{\oslash}} K$, so the adjointness statement follows from Quillen's theorem **8.1.16**.

8.7.10. (Example.) In particular, for any simplicial model category $\mathcal{C}$ we can derive $\otimes : \mathcal{C} \times sSets \to \mathcal{C}$ and $\mathbf{Hom} : \mathcal{C}^0 \times \mathcal{C} \to sSets$.

8.7.11. (Inner Homs for $\otimes_\Lambda$.) Let Λ be a generalized ring, i.e. a commutative algebraic monad, and denote by $\otimes_\Lambda$ both the tensor product $\Lambda\text{-}Mod \times \Lambda\text{-}Mod \to \Lambda\text{-}Mod$ and its simplicial extension $s\otimes_\Lambda$. We'd like to show that *inner Homs* $\mathbf{Hom}_{s\Lambda}$ *exist for the simplicial extension of* $\otimes_\Lambda$, and that *the underlying simplicial set of* $\mathbf{Hom}_{s\Lambda}(A, X)$ *coincides with* $\mathbf{Hom}(A, X)$ *of* **8.3.4**.

Once these two statements are established, the rest is trivial: the compatibility condition (TMh) will follow from (SM7) and the fact that a morphism of simplicial Λ-modules is a fibration iff this is true for the underlying map of simplicial sets, and the additional conditions for $\otimes_\Lambda$ and $\mathbf{Hom}_\Lambda$ hold trivially, so we'll get the existence of $\underline{\underline{\otimes}}$ and $\mathbb{R}\,\mathbf{Hom}_\Lambda$, their adjointness and so on.

8.7.12. (Λ-structure on sets of homomorphisms.) First of all, for any two simplicial Λ-modules X and Y we get a Λ-structure on each set $\operatorname{Hom}_\Lambda(X_n, Y_n)$,

Λ being commutative (cf. **5.3.1**), thus obtaining an induced Λ-structure on $\mathrm{Hom}_{s\Lambda}(X,Y) \subset \prod_{n\geq 0} \mathrm{Hom}_\Lambda(X_n, Y_n)$. If Z is another simplicial Λ-module, the composition map $\mathrm{Hom}_{s\Lambda}(Y,Z) \times \mathrm{Hom}_{s\Lambda}(X,Y) \to \mathrm{Hom}_{s\Lambda}(X,Z)$ is clearly Λ-bilinear, this being true for individual composition maps $\mathrm{Hom}_\Lambda(Y_n, Z_n) \times \mathrm{Hom}_\Lambda(X_n, Y_n) \to \mathrm{Hom}_\Lambda(X_n, Z_n)$. In this sense $s\Lambda$-*Mod* becomes a "Λ-category".

8.7.13. (Λ-structure on simplicial sets of homomorphisms.) Let X and Y be two simplicial Λ-modules, and consider the following functor $\mathbf{\Delta}^0 \to \Lambda$-*Mod*:

$$[n] \mapsto \mathrm{Hom}_{s\Lambda}(X \otimes \Delta(n), Y) \quad . \tag{8.7.13.1}$$

This functor is a simplicial Λ-module that will be denoted by $\mathbf{Hom}_\Lambda(X,Y)$ or $\mathbf{Hom}_{s\Lambda}(X,Y)$. Notice that its underlying simplicial set coincides with the "simplicial set of homomorphisms" $\mathbf{Hom}(X,Y)$, characterized by the property $\mathrm{Hom}_{sSets}(K, \mathbf{Hom}(X,Y)) \cong \mathrm{Hom}_{s\Lambda}(X \otimes K, Y)$ (cf. **8.3.4**).

8.7.14. (Adjointness of $\mathbf{Hom}_{s\Lambda}$ and $s\otimes_\Lambda$.) For any two simplicial Λ-modules X and Y we have a canonical evaluation map of simplicial sets $\mathrm{ev}_{X,Y} : X \times \mathbf{Hom}_{s\Lambda}(X,Y) \to Y$, easily seen to be Λ-bilinear. Thus we get $\mathrm{ev}'_{X,Y} : X \otimes_\Lambda \mathbf{Hom}_{s\Lambda}(X,Y) \to Y$. Now we use $\mathrm{ev}'_{X,Y}$ to construct a canonical map for any simplicial Λ-module Z:

$$\mathrm{Hom}_{s\Lambda}(Z, \mathbf{Hom}_{s\Lambda}(X,Y)) \to \mathrm{Hom}_{s\Lambda}(X \otimes_\Lambda Z, Y) \tag{8.7.14.1}$$

We claim that the above arrow is a functorial isomorphism, hence that $\mathbf{Hom}_{s\Lambda}$ is indeed an inner Hom for $s\otimes_\Lambda$. First of all, this is true for $Z = L_\Lambda \Delta(n)$, since $\mathrm{Hom}_{s\Lambda}(L_\Lambda\Delta(n), X) \cong X_n$ and $X \otimes_\Lambda L_\Lambda K \cong X \otimes K$ for any simplicial set K, in particular for $K = \Delta(n)$. Next, the simplicial Λ-modules $L_\Lambda\Delta(n)$ constitute a set of generators for $s\Lambda$-*Mod*, so any Z can be written as an inductive limit of objects of this form; our claim follows now from the fact that $\otimes_\Lambda$ commutes with arbitrary inductive limits.

8.7.15. (Existence of $\underline{\underline{\otimes}} = \mathbb{L}\otimes_\Lambda$ and $\mathbb{R}\,\mathbf{Hom}_{s\Lambda}$.) Since the underlying simplicial set of $\mathbf{Hom}_{s\Lambda}(X,Y)$ coincides with the simplicial set of homomorphisms $\mathbf{Hom}(X,Y)$, and a morphism of simplicial Λ-modules is a fibration iff this is true for the underlying map of simplicial sets, we see that (SM7a) implies immediately (TMh), so $s\otimes_\Lambda$ *is a compatible ACU $\otimes$-structure on $s\Lambda$-Mod with inner Homs* $\mathbf{Hom}_{s\Lambda}$. All the additional assumptions like $\varnothing \otimes X \cong \varnothing \cong X \otimes \varnothing$ hold as well, so we can invoke our previous results and obtain the existence and adjointness of derived tensor products and inner Homs:

Proposition. *Let Λ be a generalized ring. Then the simplicial extension of the tensor product $\otimes_\Lambda$ to the category of simplicial Λ-modules is compatible* *with the model structure on this category, and* $\mathbf{Hom}_{s\Lambda}$ *is an inner Hom for this tensor structure. Both these functors can be derived. The derived tensor product* $\underline{\underline{\otimes}} = \mathbb{L}\otimes_\Lambda$ *induces an ACU* $\otimes$*-structure on the corresponding homotopy category* $\mathcal{D}^{\leq 0}(\Lambda)$*, and* $\mathbb{R}\,\mathbf{Hom}_\Lambda$ *is an inner Hom for the derived tensor product.*

Moreover, $\underline{\underline{\otimes}}$ *can be computed by applying* $\otimes_\Lambda$ *to the cofibrant replacements of its arguments, yielding a cofibrant object as well, and* $\mathbb{R}\,\mathbf{Hom}_\Lambda$ *can be computed by taking the cofibrant replacement of the first argument and fibrant replacement of the second argument and applying* $\mathbf{Hom}_{s\Lambda}$*.*

8.7.16. (Further properties of derived tensor products.) Now we can easily deduce some other natural properties of derived tensor products. For example, if Λ is a generalized ring with zero, $\underline{\underline{\otimes}}$ commutes with suspension in each variable: $\Sigma X \underline{\underline{\otimes}} Y \cong \Sigma(X \underline{\underline{\otimes}} Y) \cong X \underline{\underline{\otimes}} \Sigma Y$. If $\rho : \Lambda \to \Xi$ is a homomorphism of generalized rings, we have $\mathbb{L}\rho^*(X \underline{\underline{\otimes}}_\Lambda Y) \cong \mathbb{L}\rho^* X \underline{\underline{\otimes}}_\Xi \mathbb{L}\rho^* Y$. We get some formulas involving $\mathbb{R}\,\mathbf{Hom}$ as well: $\mathbb{R}\,\mathbf{Hom}_\Lambda(X, \mathbb{R}\rho_* Y) \cong \mathbb{R}\,\mathbf{Hom}_\Xi(\mathbb{L}\rho^* X, Y)$.

8.7.17. (Stable version.) Since both $\underline{\underline{\otimes}}$ and $\mathbb{L}\rho^*$ commute with suspension, we can construct their stable versions. Thus $\underline{\underline{\otimes}}$ defines an ACU $\otimes$-structure on $\mathcal{D}^-(\Lambda)$ for any generalized ring Λ with zero, and we get a $\otimes$-functor $\mathbb{L}\rho^* : \mathcal{D}^-(\Lambda) \to \mathcal{D}^-(\Xi)$ for any homomorphism $\rho : \Lambda \to \Xi$.

8.7.18. (Derived tensor powers.) Fix any integer $k \geq 0$ and consider the k-fold tensor power functor $T^k : X \mapsto X^{\otimes k}$ on Λ-*Mod* and $s\Lambda$-*Mod*. We see that this functor preserves cofibrant objects and weak equivalences between them, so it has a left derived $\mathbb{L}T^k$, that can be computed with the aid of cofibrant replacements. Clearly, $\mathbb{L}T^k$ coincides with the k-fold derived tensor product on $\mathcal{D}^{\leq 0}(\Lambda)$. However, T^k is not additive even in the classical case, so we've got here a natural example of a derived non-additive functor (cf. [DoldPuppe]). Notice that T^k is not "linear" in several other respects, for example $T^k \circ \Sigma \cong \Sigma^k \circ T^k$ and $T^k(X \otimes K) \cong T^k(X) \otimes (K^{\times n})$, i.e. T^k is "homogeneous of degree k".

8.7.19. (Classical case.) Notice that when the generalized ring Λ is additive, i.e. just a classical commutative ring, then the derived tensor product and derived pullbacks we constructed here coincide with their classical counterparts via Dold–Kan and Eilenberg–Zilber (cf. **8.5.14**).

9 Homotopic algebra over topoi

Now we'd like to extend our constructions to the case of a category of modules over a generalized ringed site or topos. The main idea here is to develop "local" variants of all homotopic notions discussed so far, and try to convert existing proofs in a more or less systematic fashion; this would enable us to re-use most statements and proofs without much work.

The dictionary to translate mathematical statements from the "global" or "point" context (based on constructions in *Sets*) to the "local" or "topos" context (based on similar constructions in an arbitrary topos $\mathcal{E}$) is given by the so-called *Kripke–Joyal semantics*, that essentially asserts that topoi are models of intuitionistic set theory.

We are going to present the main ideas of this semantics in an informal fashion after we give all necessary preliminary definitions; let us content ourselves for now with the following remark. While transferring from the "point" to the "topos" situation, we have to replace sets by sheaves of sets (i.e. objects of our topos $\mathcal{E}$), elements of sets by sections of sheaves, sets of maps by sheaves of maps (i.e. local Homs $\mathbf{Hom}_{\mathcal{E}}$), maps of sets by morphisms of sheaves, small categories by inner categories, and arbitrary categories by stacks and/or prestacks. Moreover, all constructions and situations we consider have to be compatible with arbitrary pullbacks $i_X^* : \mathcal{E} \to \mathcal{E}_{/X}$, for any object X of $\mathcal{E}$, and the properties of maps, sets etc. we consider have to be *local.* Finally, the last but not least important ingredient of Kripke–Joyal semantics is that we have to replace the classical logic in all proofs by the *intuitionistic logic.* This means that we cannot use the law of excluded middle ($A \vee \neg A$), the proofs based on *reductio ad absurdum* ($A \wedge \neg B \Rightarrow 0$ doesn't imply $A \Rightarrow B$, and $\neg\forall x.\neg A[x]$ doesn't imply $\exists x.A[x]$), as well as the axiom of choice. Our hope here is that most algebra is essentially intuitionistic (that's why the modules over a ringed topos have almost all properties of modules over a ring, for example), even if most calculus is not (as established by Brouwer and Weyl in their attempts to build all mathematics intuitionistically).

Since we always try to keep things as algebraic as possible, we might hope to be able to transfer the definitions and constructions of model categories to the topos case, thus obtaining the notion of a *model stack.*

9.1. (Generalities on stacks.) Now we are going to recall the basic definitions of fibered categories, cartesian functors, prestacks and stacks (cf. [SGA1, VI] and [Giraud]).

9.1.1. ($\mathcal{S}$-categories; cf. SGA 1.) When we are given a functor $p : \mathcal{C} \to \mathcal{S}$, we can say that $\mathcal{C}$ *is an $\mathcal{S}$-category,* or *a category over $\mathcal{S}$.* Then for any

$S \in \operatorname{Ob}\mathcal{S}$ we denote by $\mathcal{C}_S$ or $\mathcal{C}(S)$ the fiber of p over S, considered here as a point subcategory of $\mathcal{S}$. For any $\varphi : T \to S$, $X \in \operatorname{Ob}\mathcal{C}_T$, $Y \in \operatorname{Ob}\mathcal{C}_S$ we denote by $\operatorname{Hom}_\varphi(X, Y)$ the set of all *φ-morphisms* $f : X \to Y$, i.e. all morphisms $f : X \to Y$, such that $p(f) = \varphi$. If $\varphi = \mathrm{id}_S$, we write Hom_S instead of $\operatorname{Hom}_{\mathrm{id}_S}$; clearly, Hom_S is just the set of morphisms inside $\mathcal{C}_S$.

An *$\mathcal{S}$-functor* $F : \mathcal{C} \to \mathcal{C}'$ between two $\mathcal{S}$-categories is simply any functor, such that $p' \circ F = p$; an *$\mathcal{S}$-natural transformation* $\eta : F \to G$ between two $\mathcal{S}$-functors F, $G : \mathcal{C} \to \mathcal{C}'$ is a natural transformation $\eta : F \to G$, such that $p' \star \eta = \mathrm{Id}_p$. In this way we obtain a strictly associative 2-category $Cat/\mathcal{S}$, so we can define "$\mathcal{S}$-adjoint functors", "$\mathcal{S}$-equivalences" and so on with the usual properties (cf. [SGA1]).

Informally, one should think of an $\mathcal{S}$-category $\mathcal{C}$ as "collection of categories $\mathcal{C}_S$ parametrized by $\mathcal{S}$"; $\mathcal{S}$-functors $F : \mathcal{C} \to \mathcal{C}'$, $\mathcal{S}$-equivalences etc. should be thought of as "compatible" collections of fiberwise functors $F_S : \mathcal{C}_S \to \mathcal{C}'_S$ etc.

Given any functor $H : \mathcal{S}' \to \mathcal{S}$, we can consider the pullback $p' : \mathcal{C}' \to \mathcal{S}'$ of any $\mathcal{S}$-category $p : \mathcal{C} \to \mathcal{S}$; clearly, $\mathcal{C}'_S = \mathcal{C}_{H(S)}$.

9.1.2. (Fibered categories and cartesian functors; cf. SGA 1.) A φ-morphism $h : Y \to X$ in an $\mathcal{S}$-category $\mathcal{C}$, where $X \in \operatorname{Ob}\mathcal{C}_S$, $Y \in \operatorname{Ob}\mathcal{C}_T$, $\varphi : T \to S$, is said to be *cartesian* if it induces a bijection $h_* : \operatorname{Hom}_T(Z, Y) \to \operatorname{Hom}_f(Z, X)$, for all $Z \in \operatorname{Ob}\mathcal{C}_T$. We say that $\mathcal{C}$ *is a fibered category over* $\mathcal{S}$ if for any $\varphi : T \to S$ and any $X \in \operatorname{Ob}\mathcal{C}_S$ one can find a cartesian φ-morphism $Y \to X$ with target X, and if the composite of any two cartesian morphisms is again cartesian.

When $\mathcal{C}$ is fibered over $\mathcal{S}$, we choose for each $\varphi : T \to S$ and $X \in \operatorname{Ob}\mathcal{C}_S$ some cartesian φ-morphism $\varphi^* X \to X$, thus obtaining a *pullback functor* $\varphi^* : \mathcal{C}_S \to \mathcal{C}_T$, defined uniquely up to a unique isomorphism. We obtain natural isomorphisms $(\varphi\psi)^* \cong \psi^* \circ \varphi^* : \mathcal{C}_S \to \mathcal{C}_U$ for any $U \xrightarrow{\psi} T \xrightarrow{\varphi} S$, satisfying the pentagon axiom. In this way *a fibered category over $\mathcal{S}$ is essentially a contravariant 2-functor $\mathcal{S}^0 \to Cat$.* The difference is that we don't fix any particular choice of pullback functors φ^*.

An $\mathcal{S}$-functor $F : \mathcal{C} \to \mathcal{C}'$ between two fibered categories over $\mathcal{S}$ is called *cartesian* if it transforms cartesian morphisms of $\mathcal{C}$ into cartesian morphisms of $\mathcal{C}'$. This essentially means that we have a collection of functors $F_S : \mathcal{C}_S \to \mathcal{C}'_S$, commuting with all pullback functors φ^* in $\mathcal{C}$ and $\mathcal{C}'$ up to an isomorphism.

We denote by $Funct_{cart,\mathcal{S}}(\mathcal{C}, \mathcal{C}')$ or $Cart_{\mathcal{S}}(\mathcal{C}, \mathcal{C}')$ the category of all cartesian functors $\mathcal{C} \to \mathcal{C}'$. Since $\mathcal{S}$ is fibered over itself, we can consider "the category of cartesian sections" $\Gamma_{cart,\mathcal{S}}(\mathcal{C}) := Cart_{\mathcal{S}}(\mathcal{S}, \mathcal{C})$; we denote it by $\varprojlim_{\mathcal{S}} \mathcal{C}$ as well. Clearly, when $\mathcal{S}$ has a final object e, we have an equivalence

$\mathcal{C}_e \xrightarrow{\sim} \Gamma_{cart}(\mathcal{C}/\mathcal{S})$.

Notice that for any object F of $\mathcal{S}$, or, more generally, for any presheaf F over $\mathcal{S}$, the natural functor $\mathcal{S}_{/F} \to \mathcal{S}$ defines a fibered category over $\mathcal{S}$, so we can consider $Cart_{\mathcal{S}}(\mathcal{S}_{/F}, \mathcal{C})$; when F lies in $\mathcal{S}$, this category is equivalent to $\mathcal{C}_F$. If we replace each $\mathcal{C}_S$ with equivalent category $Cart_{\mathcal{S}}(\mathcal{S}_{/F}, \mathcal{S})$, we obtain an $\mathcal{S}$-equivalent $\mathcal{S}$-fibered category with a natural choice of pullback functors, such that $(\psi\varphi)^* = \varphi^*\psi^*$. Actually this construction, when considered for all presheaves $F \in \operatorname{Ob}\hat{\mathcal{S}}$, yields a canonical extension $\mathcal{C}^+/\hat{\mathcal{S}}$ of $\mathcal{C}/\mathcal{S}$. Notice that $S \mapsto \operatorname{Ob}\mathcal{C}_S^+$ and $S \mapsto \operatorname{Ar}\mathcal{C}_S^+$ are presheaves over $\mathcal{S}$, i.e. we obtain a presheaf of categories over $\mathcal{S}$, or an inner category in $\hat{\mathcal{S}}_{\mathcal{V}}$, for a suitable universe $\mathcal{V} \ni \mathcal{U}$.

Finally, if $H : \mathcal{S}' \to \mathcal{S}$ is any functor, then the pullback with respect to H transforms cartesian functors between $\mathcal{S}$-fibered categories into cartesian functors between $\mathcal{S}'$-fibered categories. For example, $\mathcal{C}^+ \times_{\hat{\mathcal{S}}} \mathcal{S}$ is $\mathcal{S}$-equivalent to $\mathcal{C}$. When $\mathcal{S}' = \mathcal{S}_{/F}$, we put $\mathcal{C}_{/F} := \mathcal{C} \times_{\mathcal{S}} \mathcal{S}_{/F}$.

Notation 9.1.3 Whenever a pullback functor $\varphi^* : \mathcal{C}_S \to \mathcal{C}_T$ in an $\mathcal{S}$-fibered category $\mathcal{C}$ has a left adjoint, it will be denoted by $\varphi_! : \mathcal{C}_T \to \mathcal{C}_S$. Similarly, the right adjoint to φ^* will be denoted by φ_*.

9.1.4. (Hom-presheaves.) Given a fibered $\mathcal{S}$-category $\mathcal{C}$ and two objects X, $Y \in \operatorname{Ob}\mathcal{C}_S$, $S \in \operatorname{Ob}\mathcal{C}$, we define the *local Hom* $\mathbf{Hom}_{\mathcal{C}_{/S}}(X, Y) : (\mathcal{S}_{/S})^0 \to$ *Sets* by $(T \xrightarrow{\varphi} S) \mapsto \operatorname{Hom}_{\mathcal{C}_T}(\varphi^*X, \varphi^*Y)$. Clearly, $\mathbf{Hom}_{\mathcal{C}_{/S}}(X, Y)$ is a presheaf over $\mathcal{S}_{/S}$, with global sections equal to $\operatorname{Hom}_{\mathcal{C}_S}(X, Y)$, and $\varphi^*\mathbf{Hom}_{\mathcal{C}_{/S}}(X, Y) \cong \mathbf{Hom}_{\mathcal{C}_{/T}}(\varphi^*X, \varphi^*Y)$, for any $\varphi : T \to S$ in $\mathcal{S}$.

9.1.5. (Prestacks.) We say that a fibered category $\mathcal{C}$ over a *site* $\mathcal{S}$ is a *prestack* if for any $S \in \operatorname{Ob}\mathcal{S}$, $X, Y \in \operatorname{Ob}\mathcal{C}_S$ the Hom-presheaf $\mathbf{Hom}_{\mathcal{C}_{/S}}(X, Y)$ is actually a sheaf over $\mathcal{S}_{/S}$. *Morphisms* of prestacks are just cartesian $\mathcal{S}$-functors.

Notice that the above definition applies when $\mathcal{S}$ is actually a topos $\widetilde{\mathcal{E},}$ considered as a site with respect to its canonical topology. In this case $\widetilde{\mathcal{S}_{/S}}$ is equivalent to $\mathcal{E}_{/S}$, so we get a local Hom-object $\mathbf{Hom}_{\mathcal{C}_{/S}}(X, Y) \in \operatorname{Ob}\mathcal{E}_{/S}$.

9.1.6. (Descent data.) Given any object $S \in \operatorname{Ob}\mathcal{S}$ and any sieve $R \subset S$, i.e. a subpresheaf of S in $\hat{\mathcal{S}}$, we obtain a natural functor for any fibered $\mathcal{S}$-category $\mathcal{C}$:

$$\mathcal{C}(S) = \mathcal{C}_S \cong Cart(\mathcal{S}_{/S}, \mathcal{C}) \to \mathcal{C}^+(R) = Cart(\mathcal{S}_{/R}, \mathcal{C}) \tag{9.1.6.1}$$

We say that $\mathcal{C}^+(R)$ is *the category of descent data with respect to* R, and denote it by $Desc(R; \mathcal{C})$. When R is generated by some cover $(S_\alpha \xrightarrow{f_\alpha} S)$, and double and triple fibered products of S_α over S do exist, we denote by $Desc((S_\alpha \to S); \mathcal{C})$ the category of descent data with respect to this cover;

by definition, an object of this category is a descent datum with respect to $(S_\alpha \to S)$ with values in $\mathcal{C}$, i.e. a collection of objects $X_\alpha \in \operatorname{Ob}\mathcal{C}(S_\alpha)$ and isomorphisms $\theta_{\alpha\beta} : \mathrm{pr}_1^* X_\alpha \xrightarrow{\sim} \mathrm{pr}_2^* X_\beta$ in $\mathcal{C}(S_\alpha \times_S S_\beta)$, subject to the usual cocycle relation in $\mathcal{C}(S_\alpha \times_S S_\beta \times_S S_\gamma)$. One checks immediately that the naturally arising functor $\mathit{Desc}(R;\mathcal{C}) \to \mathit{Desc}((S_\alpha \to S);\mathcal{C})$ is indeed an equivalence of categories. This justifies the above terminology.

Definition 9.1.7 (cf. [Giraud, II,1.2]) *A fibered category $\mathcal{C}$ over a site $\mathcal{S}$ is called a* stack *(resp.* prestack*) over $\mathcal{S}$ if for any object $S \in \operatorname{Ob}\mathcal{S}$ and any covering sieve R of S the functor $\mathcal{C}(S) \to \mathcal{C}^+(R)$ of* (9.1.6.1) *is an equivalence (resp. fully faithful), i.e. if all covers in $\mathcal{S}$ have the effective descent (resp. descent) property with respect to $\mathcal{C}$.*

One checks immediately that $\mathcal{C}(S) \to \mathcal{C}^+(R)$ is fully faithful iff all local Hom-presheaves $\mathbf{Hom}_{\mathcal{C}_{/S}}(X,Y)$ satisfy the sheaf condition for $R \subset S$; therefore, *a fibered category $\mathcal{C}/\mathcal{S}$ is a prestack iff all* $\mathbf{Hom}_{\mathcal{C}_{/S}}(X,Y)$ *are sheaves*, so **9.1.7** and **9.1.5** are compatible in this respect.

9.1.8. (Examples.) (a) For any site $\mathcal{S}$ we have a “stack of sets” (or sheaves) $\mathbf{SETS}_\mathcal{S}$, given by $\mathbf{SETS}_\mathcal{S}(X) := \widetilde{\mathcal{S}_{/X}}$, with the pullback functors given by restriction of sheaves. We might have denoted this stack by $\mathbf{SHEAVES}_\mathcal{S}$ as well, but we wanted to illustrate that this stack will indeed play the role of *Sets* in our further local considerations.

(b) For any category $\mathcal{E}$ with fibered products the category of arrows $\operatorname{Ar}\mathcal{E}$ together with the target functor $t : \operatorname{Ar}\mathcal{E} \to \mathcal{E}$ defines a fibered category structure, such that $(\operatorname{Ar}\mathcal{E})(X) = \mathcal{E}_{/X}$. When $\mathcal{E}$ is a topos, $\widetilde{\mathcal{E}_{/X}}$ is naturally equivalent to $\mathcal{E}_{/X}$, hence $\operatorname{Ar}\mathcal{E}$ is $\mathcal{E}$-equivalent to $\mathbf{SETS}_\mathcal{E}$; in particular, it is an $\mathcal{E}$-stack. Actually, we'll usually put $\mathbf{SETS}_\mathcal{E} := \operatorname{Ar}\mathcal{E}$ when working over a topos.

(c) When Σ is an algebraic monad over a topos $\mathcal{E}$, we can define the *stack of Σ-modules* $\Sigma\text{-}\mathbf{MOD} = \mathbf{MOD}_{\mathcal{E},\Sigma}$ by $\Sigma\text{-}\mathbf{MOD}(X) := \Sigma_{|X}\text{-}\mathit{Mod}$. Of course, we can do the same construction over a site $\mathcal{S}$, if we define $\Sigma\text{-}\mathbf{MOD}(X)$ to be the category of sheaves of $\Sigma_{|X}$-modules over $\mathcal{S}_{/X}$. When Σ is the constant algebraic monad $\mathbb{F}_\varnothing$, we recover $\mathbf{SETS}_\mathcal{E}$.

9.1.9. (Stacks over a site and over the corresponding topos.) If we have a stack $\mathcal{C}$ over a site $\mathcal{S}$, we can extend to a stack $\tilde{\mathcal{C}}$ over topos $\tilde{\mathcal{S}}$, simply by restricting $\mathcal{C}^+$ to $\tilde{\mathcal{S}}$: $\tilde{\mathcal{C}} := \mathcal{C}^+ \times_{\hat{\mathcal{S}}} \tilde{\mathcal{S}}$, i.e. $\tilde{\mathcal{C}}(F) = \mathcal{C}^+(F) = \mathit{Cart}(\mathcal{S}_{/F},\mathcal{C})$ for any sheaf F. Conversely, any stack $\mathcal{C}'$ over $\tilde{\mathcal{S}}$ defines a stack $\mathcal{C}' \times_{\tilde{\mathcal{S}}} \mathcal{S}$ over $\mathcal{S}$ by restriction. Now we claim that *these two functors establish a 2-equivalence between the 2-categories of stacks over $\mathcal{S}$ and over $\tilde{\mathcal{S}}$* (cf. [Giraud, II,3.3]). We'll use this observation to work with stacks over topoi whenever possible.

The main idea here is that any sheaf F can be covered in $\tilde{\mathcal{S}}$ by a small family of objects S_α, and each $S_\alpha \times_F S_\beta$ has a similar cover by some $S_{\alpha\beta}^{(\nu)}$ as well, hence for any stack $\mathcal{C}'/\tilde{\mathcal{S}}$ the category $\mathcal{C}'(F)$ can be described in terms of some sort of descent data, consisting of collections of objects $X_\alpha \in \mathcal{C}'(S_\alpha)$ and isomorphisms $\theta_{\alpha\beta}^{(\nu)}$ in $\mathcal{C}'(S_{\alpha\beta}^{(\nu)})$, hence $\mathcal{C}'$ is completely determined by its restriction to $\mathcal{S}$.

9.1.10. (Fibers over direct sums.) If $\mathcal{C}$ is a stack over a topos $\mathcal{E}$, then $\mathcal{C}(\bigsqcup_\alpha S_\alpha)$ is naturally equivalent to the product $\prod_\alpha \mathcal{C}(S_\alpha)$, just because this product is the category of descent data for the cover $(S_\alpha \to S)_\alpha$, $S := \bigsqcup S_\alpha$, direct sums in $\mathcal{E}$ being disjoint.

9.1.11. (Stacks over the point site and topos.) The above considerations apply in particular to the point site $\mathbf{1}$ (the final category) and the point topos *Sets*. We see that *stacks over Sets are essentially just categories.* More precisely, a stack $\mathcal{C}$ over *Sets* defines a category $\mathcal{C}(\mathbf{1})$, and conversely, any category $\mathcal{D}$ defines a stack $\mathcal{D}^+$ over *Sets* by $\mathcal{D}^+(I) := \mathcal{D}^I$ for any set I.

9.1.12. (Prestack associated to a fibered category.) Given a fibered category $\mathcal{C}$ over a site $\mathcal{S}$, the *stack* (resp. *prestack*) *associated to* $\mathcal{C}$ is defined by the following requirement: it is a stack (resp. prestack) $\bar{\mathcal{C}}/\mathcal{S}$ with a cartesian $\mathcal{S}$-functor $I : \mathcal{C} \to \bar{\mathcal{C}}$, such that for any stack (resp. prestack) $\mathcal{D}/\mathcal{S}$ the functor $I^* : \mathit{Cart}_{\mathcal{S}}(\bar{\mathcal{C}}, \mathcal{D}) \to \mathit{Cart}_{\mathcal{S}}(\mathcal{C}, \mathcal{D})$ is an equivalence.

The prestack $\bar{\mathcal{C}}$ associated to $\mathcal{C}$ can be constructed as follows. Consider the sheafifications $a\,\mathbf{Hom}_{\mathcal{C}_{/S}}(X,Y)$ of local Hom-presheaves, where $S \in \operatorname{Ob}\mathcal{S}$, $X, Y \in \mathcal{C}_S$. Since $a : \hat{\mathcal{S}} \to \tilde{\mathcal{S}}$ commutes with finite products, we get canonical maps $a\,\mathbf{Hom}_{\mathcal{C}_{/S}}(Y,Z) \times a\,\mathbf{Hom}_{\mathcal{C}_{/S}}(X,Y) \to a\,\mathbf{Hom}_{\mathcal{C}_{/S}}(X,Z)$ induced by composition. Now we put $\operatorname{Ob}\bar{\mathcal{C}}_S := \operatorname{Ob}\mathcal{C}_S$, $\operatorname{Hom}_{\bar{\mathcal{C}}_S}(X,Y) := \Gamma(S, a\,\mathbf{Hom}_{\mathcal{C}_{/S}}(X,Y))$, and define for any $\varphi : T \to S$ the pullback functors $\varphi^* : \bar{\mathcal{C}}_S \to \bar{\mathcal{C}}_T$ on morphisms $f \in \operatorname{Hom}_{\bar{\mathcal{C}}_S}(X,Y) = \mathscr{F}(S)$, $\mathscr{F} := a\,\mathbf{Hom}_{\mathcal{C}_{/S}}(X,Y)$, by means of $\mathscr{F}(\varphi) : \mathscr{F}(S) \to \mathscr{F}(T)$. One checks immediately that this defines indeed a fibered category $\bar{\mathcal{C}}/\mathcal{S}$, and by construction all $\mathbf{Hom}_{\bar{\mathcal{C}}_{/S}}(X,Y) = a\mathbf{Hom}_{\mathcal{C}_{/S}}(X,Y)$ are indeed sheaves, i.e. $\bar{\mathcal{C}}$ is a prestack. The universal property of the "identity" functor $I : \mathcal{C} \to \bar{\mathcal{C}}$ is also immediate, once we observe that any cartesian functor $F : \mathcal{C} \to \mathcal{D}$ induces maps of presheaves $F_{X,Y} : \mathbf{Hom}_{\mathcal{C}_{/S}}(X,Y) \to \mathbf{Hom}_{\mathcal{D}_{/S}}(F(X), F(Y))$.

9.1.13. (Stack associated to a fibered category.) Now we'd like to present a construction of the stack $\hat{\mathcal{C}}$ associated to a fibered category $\mathcal{C}$ over a site $\mathcal{S}$, different from that given in [Giraud]. First of all, we construct the associated prestack $\bar{\mathcal{C}}$ by sheafifying all Hom-presheaves as above, and observe that a stack associated to $\bar{\mathcal{C}}$ will be also associated to $\mathcal{C}$, i.e. we can assume $\mathcal{C}$ to be a prestack.

Next, we define $\hat{\mathcal{C}}(S)$ to be the category of descent data in $\mathcal{C}$ $(=\bar{\mathcal{C}})$ with respect to all covers of S. More formally, we consider the ordered set $J(S)$ of covering sieves of S as a category, and put

$$\hat{\mathcal{C}}(S) := \varinjlim_{R\in J(S)} \mathcal{C}^+(R) \quad . \tag{9.1.13.1}$$

Recall that for any fibered category $\mathcal{F}$ over another category $\mathcal{I}$ the pseudolimit $\varinjlim_{\mathcal{I}} \mathcal{F}$ can be constructed simply as the localization of $\mathcal{F}$ with respect to all cartesian morphisms; in particular, we get canonical functors $\mathcal{F}_i \to \varinjlim_{\mathcal{I}} \mathcal{F}$, $i \in \operatorname{Ob}\mathcal{I}$, commuting with the pullback functors up to some natural isomorphisms.

In our situation all pullback (=restriction) functors $\mathcal{C}^+(R) \to \mathcal{C}^+(R')$ will be fully faithful for any covering sieves $R' \subset R \subset S$, $\mathcal{C}$ being a prestack, hence the functors $\mathcal{C}^+(R) \to \hat{\mathcal{C}}(S)$ and $J_S : \mathcal{C}(S) \to \hat{\mathcal{C}}(S)$ will be fully faithful as well.

This construction defines a fibered category $\hat{\mathcal{C}}/S$ and a fully faithful cartesian functor $J : \mathcal{C} \to \hat{\mathcal{C}}$; we leave to the reader the verification of the universal property of J and of $\hat{\mathcal{C}}$ being a stack.

9.1.14. (Direct image of a stack.) Let $\mathcal{C}$ be a stack over a site $\mathcal{S}$, and $u : \mathcal{S} \to \mathcal{S}'$ be a morphism of sites, given by some pullback functor $u^* : \mathcal{S}' \to \mathcal{S}$ (if $\mathcal{S}$ and $\mathcal{S}'$ are closed under finite projective limits, this means simply that u^* is left exact and preserves covers). Then we can define a new fibered category $\mathcal{C}'$ over $\mathcal{S}'$, namely, the pullback $\mathcal{C}' := \mathcal{C} \times_{\mathcal{S}} \mathcal{S}'$ of $\mathcal{C}$ with respect to u^*. One checks immediately (at least when fibered products exist in both sites) that $\mathcal{C}'$ is a *stack* over $\mathcal{S}'$. It is usually called *the direct image of* $\mathcal{C}$ *with respect to* u and is denoted by $u_*^{st}\mathcal{C}$.

9.1.15. (Inverse image of a stack.) Given a morphism of sites $u : \mathcal{S} \to \mathcal{S}'$, and two stacks $\mathcal{C} \xrightarrow{p} \mathcal{S}$ and $\mathcal{C}' \xrightarrow{p'} \mathcal{S}'$, a *$u$-functor* $F : \mathcal{C}' \to \mathcal{C}$ is simply any functor, such that $p \circ F = u^* \circ p'$, and the induced functor $\bar{F} : \mathcal{C}' \to \mathcal{C} \times_{\mathcal{S}} \mathcal{S}' = u_*^{st}\mathcal{C}$ is *cartesian*. Informally, F is a collection of functors $F_{S'} : \mathcal{C}'(S') \to \mathcal{C}(u^*S')$, parametrized by S' in $\mathcal{S}'$.

If a u-functor $I : \mathcal{C}' \to u_{st}^*\mathcal{C}'$ has a universal property among all u-functors with source $\mathcal{C}'$, i.e. if for any stack $\mathcal{C}$ over $\mathcal{S}$ the functor $I^* : \mathit{Cart}(u_{st}^*\mathcal{C}', \mathcal{C}) \to \mathit{Funct}_u(\mathcal{C}', \mathcal{C})$ is an equivalence of categories, then we say that the stack $u_{st}^*\mathcal{C}'$ is the *inverse image* or *pullback* of $\mathcal{C}'$ with respect to u.

One can construct the inverse image $u_{st}^*\mathcal{C}'$ as follows. First of all, define a fibered category $u^\bullet\mathcal{C}'$ over $\mathcal{S}$ with the required universal property among all fibered categories $\mathcal{C}/\mathcal{S}$, by the "left Kan extension formula" $(u^\bullet\mathcal{C}')(T) :=$

$\varinjlim_{\mathcal{S}'_{/T}} \mathcal{C}'$, for all $T \in \operatorname{Ob}\mathcal{S}$. Then the stack associated to $u^{\bullet}\mathcal{C}'$ has the property required from $u^*_{st}\mathcal{C}'$.

Example. The pullback of *Sets*, considered here as a stack over the point topos/site, to an arbitrary topos $\mathcal{E}$, is the substack $\mathbf{LCSETS}_{\mathcal{E}} \subset \mathbf{SETS}_{\mathcal{E}}$ of locally constant objects of $\mathcal{E}$.

9.2. (Kripke–Joyal semantics.) Now we describe a version of Kripke–Joyal semantics, suitable for transferring intuitionistic statements about categories into statements about stacks over a site or topos.

We fix a site $\mathcal{S}$; in some cases we assume $\mathcal{S}$ to have finite projective limits. When $\mathcal{S}$ is a topos, we denote it by $\mathcal{E}$ as well.

9.2.1. (Contexts.) We define a *context* $\mathfrak{V}$ (over a "base object" S of $\mathcal{S}$) as a finite collection of *variables* (usually denoted by lowercase Latin letters), together with their *types* (usually denoted by uppercase letters) and *values* of appropriate type. Strictly speaking, the assignment of values is needed only for semantics ("evaluation"), but not for checking the syntax of an expression, so we should distinguish *signatures* $\mathfrak{V}_0$ (lists of variables together with their types, but without any values), and *contexts*, but usually we'll mix them for shortness.

Sometimes we have two contexts $\mathfrak{V}$ and $\mathfrak{W}$ over the same base object S, such that all variables of $\mathfrak{V}$ appear in $\mathfrak{W}$ with the same types and values; then we write $\mathfrak{W} \geq \mathfrak{V}$ or $\mathfrak{V} \leq \mathfrak{W}$ and say that $\mathfrak{W}$ *extends* $\mathfrak{V}$.

Notice that all variables, bound or free, must have some specified type; when x is of type X, we usually write $x : X$ or even $x \in X$.

Once we have a context $\mathfrak{V}$, we can construct different *terms* and *propositions*, involving the variables of $\mathfrak{V}$ and some basic logical operations, and evaluate them. Notice that all terms we construct have a well-defined type, and their values, when defined at all, are of corresponding type. As to the propositions, all we can say is whether they are (universally) valid in a context $\mathfrak{V}$ or not. When a proposition A is valid in context $\mathfrak{V}$, we write $\mathfrak{V} \models A$. The value of a term t in context $\mathfrak{V}$ will be denoted by $t(\mathfrak{V})$ or $t_{\mathfrak{V}}$; this applies to variables as well.

9.2.2. (Simple interpretation.) The main idea here is that types correspond to some sheaves over $\mathcal{S}_{/S}$, or to sets in classical case, while variables and terms correspond to sections of these sheaves, or to elements of sets in the classical case. As to the propositions, they correspond to *local* properties of these sheaves and sections.

9.2.3. (Pullback of contexts.) The most important property of contexts is that any context $\mathfrak{V}/S$ can be pulled back with respect to any morphism $\varphi : T \to S$ in $\mathcal{S}$, yielding a context $\varphi^*\mathfrak{V}/T$. The way this is done is usually immediate from the context. For example, if a type X corresponds to a sheaf $X_{\mathfrak{V}}$ over $\mathcal{S}_{/S}$, then the value $X_{\varphi^*\mathfrak{V}}$ is simply the restriction of $X_{\mathfrak{V}}$ to $\mathcal{S}_{/T}$, and if a variable $x : X$ has value $x_{\mathfrak{V}} \in \Gamma_S(X_{\mathfrak{V}}) = X_{\mathfrak{V}}(S)$, then the value $x_{\varphi^*\mathfrak{V}}$ is simply the restriction $\varphi^* x_{\mathfrak{V}} = (X_{\mathfrak{V}}(\varphi))(x_{\mathfrak{V}})$ of $x_{\mathfrak{V}}$ to T.

9.2.4. (Localness of all properties.) Any proposition A, depending on a context $\mathfrak{V}$ (of some fixed signature $\mathfrak{V}_0$, which lists all variables of A with appropriate type), defines a *local property* of the objects involved. This means the following:

- If A holds in $\mathfrak{V}$, it holds in all pullbacks $\varphi^*\mathfrak{V}$.
- If $\{S_\alpha \xrightarrow{\varphi_\alpha} S\}$ is a cover in $\mathcal{S}$, and $\mathfrak{V}$ is a context over S, such that A holds in all $\varphi_\alpha^*\mathfrak{V}$, then A holds in $\mathfrak{V}$ itself.

9.2.5. (Constants.) Apart from variables, we can also have some *constants*. These are essentially some variables with predefined values, always the same in all contexts we consider; these values come from "outside", and the constants are usually denoted by the same letter they've been denoted in the "outside" reasoning. For example, if we have somehow obtained a section $x \in X(S)$ of some sheaf X, we can use x as a constant of type X. An important thing about constants is that they are *never* considered free variables.

Notice that we have *constant types* as well, as illustrated by the constant type X given by sheaf X in the above example.

As to the *constant properties* $P(\mathfrak{W})$ of a context of signature $\mathfrak{W}_0$, the main requirement for them to be allowed to appear in our expressions is to be *local*. Then P can be used as a proposition of any signature $\geq \mathfrak{W}_0$.

9.2.6. (Types.) These are our constructions of *types*, in a context $\mathfrak{W}/S$. We list the "sheaf types" first, the values of which are sheaves over $\mathcal{S}_{/S}$:

- Any sheaf X over $\mathcal{S}$ or $\mathcal{S}_{/S}$ defines a *constant type*, denoted also by X.
- If X and Y are sheaf types, then Y^X or $X \to Y$ denotes the (sheaf) type $\mathbf{Hom}_{\mathcal{S}_{/S}}(X, Y)$.
- If A is a proposition depending on the variables of $\mathfrak{W}$ as well as on a variable $x : X$, then $\{x|A\}$ denotes the largest subsheaf $X' \subset X$, such that A holds for all pullbacks $\varphi^*\mathfrak{W}$, $\varphi : T \to S$, whenever we choose the value of x belonging to $X'(T) \subset X(T)$.
- If X and Y are sheaf types, then $X \times Y$ is another.
- One can extend this list by means of any sheaf construction compatible with pullbacks, using the same notation.

We have some additional constructions involving "large" types:

- $\mathrm{Ob}\,\mathcal{C}$ is a type for any stack $\mathcal{C}$ over $\mathcal{S}$. Values of terms of type $\mathrm{Ob}\,\mathcal{C}$ are objects of $\mathcal{C}(S)$.

- If x and y are terms of type $\mathrm{Ob}\,\mathcal{C}$, then $x \to y$ or $\mathbf{Hom}_{\mathcal{C}}(x,y)$ is the sheaf type given by $\mathbf{Hom}_{\mathcal{C}/S}(x_{\mathfrak{V}}, y_{\mathfrak{V}})$.

9.2.7. (Terms.) We write here $x : X$ to denote that x is a term (i.e. expression) of type X.

- Any variable $x : X$ is a valid term of type X.

- Any section $x \in X(S)$ defines a constant x of (sheaf) type X.

- Any object $x \in \mathrm{Ob}\,\mathcal{C}(S)$ defines a constant $x{:}\mathrm{Ob}\,\mathcal{C}$.

- Tuples: $(x_1, x_2, \ldots, x_n)$ is a term of type $X_1 \times X_2 \times \cdots \times X_n$, whenever $x_i{:}X_i$.

- Application of functions: $f\,x$ or $f(x)$ is a term of type Y for any $f : X \to Y$ and $x{:}X$.

- λ-abstraction: $\lambda x.t$ or $x \mapsto t$ is a term of type $X \to Y$ whenever t is a valid term of type Y in any context obtained from $\mathfrak{V}$ by pulling back and assigning to x any value of type X

- Hilbert's iota: $\iota_x A$ or $\iota x.A$ is a term of type X whenever A is a valid proposition in any context obtained from $\mathfrak{V}$ by pulling back and assigning to x any value of type X, provided $\mathfrak{V} \models \exists! x : X.A$.

- id_x is a term of type $x \to x$ for any $x : \mathrm{Ob}\,\mathcal{C}$.

- $g \circ f$ is a term of type $x \to z$ for any $f : x \to y$, $g : y \to z$, $x, y, z : \mathrm{Ob}\,\mathcal{C}$.

- We use the standard conventions for binary operations. For example, if $+ : X \times X \to X$, $x, y{:}X$, we write $x + y$ instead of $+(x, y)$.

9.2.8. (Propositions.) We explain here the *syntax* of propositions in a context $\mathfrak{V}$ (or a signature $\mathfrak{V}_0$); the semantics will be explained later.

- Conjuction $A\&B = A \wedge B$, disjunction $A \vee B$, implication $A \Rightarrow B$, equivalence $A \Leftrightarrow B$ and negation $\neg A$ are (valid) propositions whenever A and B are.

- Logical constants $\mathbf{1}$ (true) and $\mathbf{0}$ (false) are propositions.

- $x = y$ is a proposition whenever x and y are variables of the same sheaf type X.

- $(\forall x)A$, $(\forall x : X)A$, $\forall x.A$ and $\forall x : X.A$ are (equivalent) propositions for any proposition A, valid in the signature obtained from $\mathfrak{V}_0$ by adding a variable x of type X (if a variable named x is already present in $\mathfrak{V}_0$, it has to be removed first).

- $(\exists x)A$, $\exists x : X.A$ and so on are valid propositions for any A as above.

- Same applies to $(\exists! x : X)A$, $\exists! x.A$ etc.

9.2.9. (Semantics of propositions.) Now we describe when a proposition A holds in a context $\mathfrak{V}/S$ (notation: $\mathfrak{V} \models A$ or $A(\mathfrak{V})$), which assignes appropriate values to all variables from A.

Before listing the rules recall that all our propositions must be local, i.e.:

- If A holds in $\mathfrak{V}$, it holds in any pullback $\varphi^*\mathfrak{V}$.

- If $\{\varphi_\alpha : S_\alpha \to S\}$ is a cover in $\mathcal{S}$, and $\varphi_\alpha^*\mathfrak{V} \models A$ for all α, then $\mathfrak{V} \models A$.

Below we denote by $\mathfrak{V} \times_S T$ the pullback $\varphi^*\mathfrak{V}$ of $\mathfrak{V}$ with respect to a morphism $\varphi : T \to S$. If we have a cover $\{S_\alpha \to S\}$, we put $\mathfrak{V}_\alpha := \mathfrak{V} \times_S S_\alpha$. Now the rules:

- Conjunction: $A \& B = A \wedge B$ holds in $\mathfrak{V}$ iff both A and B hold in $\mathfrak{V}$.

- Disjunction: $A \vee B$ holds in $\mathfrak{V}/S$ iff there is a cover $\{S_\alpha \to S\}$, such that for each α at least one of A and B holds in $\mathfrak{V}_\alpha/S_\alpha$.

- Negation: $\neg A$ holds in $\mathfrak{V}/S$ iff A does not hold in any pullback $\varphi^*\mathfrak{V}$ of $\mathfrak{V}$.

- Implication: $A \Rightarrow B$ holds in $\mathfrak{V}/S$ iff in any pullback $\varphi^*\mathfrak{V}$, $\varphi^*\mathfrak{V} \models B$ whenever $\varphi^*\mathfrak{V} \models A$.

- Equivalence: $A \Leftrightarrow B$ holds in $\mathfrak{V}/S$ iff in any pullback of $\mathfrak{V}$ each of A and B holds whenever the other holds. In other words, $\mathfrak{V} \models A \Leftrightarrow B$ is equivalent to $\mathfrak{V} \models (A \Rightarrow B)\&(B \Rightarrow A)$.

- Constants: $\mathbf{1}$ is always true, and $\mathbf{0}$ is always false.

- $\mathfrak{V} \models x = y$, where x and y are terms of sheaf type X, iff $x_{\mathfrak{V}} = y_{\mathfrak{V}}$ in $X(S)$.

- Universality: $\mathfrak{V} \models (\forall x\!:\!X)A$ means that A holds in any context $\mathfrak{W}/T$, obtained by pulling $\mathfrak{V}$ back with respect to any morphism $\varphi : T \to S$, and assigning to a new variable $x\!:\!X$ an arbitrary value $x_{\mathfrak{W}} \in X(T)$.

- Existence: $\mathfrak{V} \models (\exists x\!:\!X)A$ means that there is a cover $\{S_\alpha \to S\}$ and some elements $x_\alpha \in X(S_\alpha)$, such that A holds in contexts $\mathfrak{W}_\alpha$, obtained by pulling $\mathfrak{V}$ back to S_α and assigning to $x\!:\!X$ the value x_α.

- Uniqueness: $(\exists! x : X)A[x]$ is equivalent to $(\exists x : X.A[x])\&(\forall x : X.\forall y : X.A[x]\&A[y] \Rightarrow x = y)$. The brackets [] are used here to point out some free variables entering in A, as well as to denote the result of substituting these variables.

- Closure: If A involves some free variables $x_1 : X_1, \ldots, x_n : X_n$, then $\mathfrak{V} \models A$ actually means $\mathfrak{V} \models \forall x_1\!:\!X_1 \ldots \forall x_n\!:\!X_n.A$.

- When no context is given, $\models A$ (i.e. "A holds") means that A holds in the empty context over any object S of $\mathcal{S}$ (or just over the final object).

9.2.10. (a) Notice that existence in Kripke–Joyal semantics always actually means *local existence*, and that universality means universality after arbitrary pullbacks.

(b) Example: if X and Y are sheaves, and $f : X \to Y$ is a morphism of sheaves, thus defining a constant f of type $X \to Y$, then the "surjectivity condition" $\forall y\!:\!Y.\exists x\!:\!X.f(x) = y$ means (in Kripke–Joyal semantics) that for any object S of $\mathcal{S}$ and any section $y \in Y(S)$ one can find a cover $\{S_\alpha \to S\}$ and sections $x_\alpha \in X(S_\alpha)$, such that $f_{S_\alpha}(x_\alpha) = y|_{S_\alpha}$, i.e. the surjectivity of f as a map of sheaves.

(c) Suppose that $\mathfrak{V} \models \exists! x : X.A[x]$, i.e. "such an $x \in X$ exists and is unique". First of all, existence $\exists x.A[x]$ means local existence, i.e. we have a cover $\{S_\alpha \to S\}$ and elements $x_\alpha \in X(S_\alpha)$, having the property expressed by A. Now the uniqueness implies that the restrictions of x_α and x_β to $S_\alpha \times_S S_\beta$ coincide, so we can glue them to section $x \in X(S)$ with property A, X being a sheaf and A being a local property. In other words, *we don't need to pass to a cover to find an object that exists and is unique.* This section $x \in X(S)$ is actually the value of term $\iota x\!:\!X.A[x]$ in context $\mathfrak{V}$.

(d) Notice that the proposition $A \vee \neg A$ doesn't usually hold, i.e. we have indeed to forget the law of excluded middle and its consequences. One can check that the logical laws still applicable in these situations are exactly those of intuitionistic logic. For example, we have the *modus ponens:* $(A \Rightarrow B)\&A \Rightarrow B$. We have also to forget about the axiom of choice and Hilbert's tau ($\tau_x A$ chooses any x, for which $A[x]$ holds, whenever this is possible, thus implying the axiom of choice). All we have is Hilbert's iota that singles out objects that exist and are unique.

(e) We abbreviate $\forall x_1 : X_1.\forall x_2 : X_2.\forall \ldots$ into $\forall x_1 : X_1, x_2 : X_2, \ldots$; and we write $x, y, z : X$ instead of $x : X, y : X, z : X$.

(f) Notice that our system is actually redundant. For example, $\lambda x : X.A$ can be replaced with $\iota f : X \to Y.\forall x : X.f(x) = A$. However, none of basic logical operations $\vee$, $\wedge$ and $\Rightarrow$ can be expressed in terms of the others (but $\neg A$ is actually equivalent to $A \Rightarrow \mathbf{0}$).

9.2.11. (Global and local notions.) When we need to distinguish the intuitionistic notions arising from Kripke–Joyal semantics from those used in classical sense, we call them *local*, e.g. local existence, local disjunction etc., as opposed to *global* notions: global existence, global disjuction etc.

Notice that $\mathfrak{V} \models A$ is a classical statement, obeying the rules of classical logic, while A itself is intuitionistic.

9.2.12. (Topos case.) Of course, Kripke–Joyal semantics can be applied directly to any topos $\mathcal{E}$, considered as a site with respect to its canonical topology. In this case $\widetilde{\mathcal{E}_{/S}} \cong \mathcal{E}_{/S}$, i.e. the "sheaf types" of $\mathcal{E}$ are actually objects of $\mathcal{E}$ or $\mathcal{E}_{/S}$. Some rules can be simplified by using the existence of (small) sums $\coprod_\alpha S_\alpha$, together with $X(\coprod_\alpha S_\alpha) \cong \prod_\alpha X(S_\alpha)$, for any sheaf X. For example, "local existence" $\exists x.A[x]$ in some context $\mathfrak{V}/S$ can be understood as follows: "there is an *epimorphism* $S' \twoheadrightarrow S$ and an element $x \in X(S')$ having the required property in the pullback $\mathfrak{V} \times_S S'$". Another example: $\mathfrak{V} \models A_1 \vee A_2$ iff there is an epimorphism $S_1 \sqcup S_2 \twoheadrightarrow S$, such that $\mathfrak{V} \times_S S_i \models A_i$, $i = 1, 2$.

9.2.13. (Localness of all propositions.) Notice that the localness of all properties expressed by propositions $A[x]$ of Kripke–Joyal semantics is actually a consequence of logical quantifier elimination and introduction rules. For example,

$$(\forall_R)\ \frac{B \Rightarrow A[a]}{B \Rightarrow \forall x : X.A[x]} \quad \text{if } a \text{ is free in } A \text{ and doesn't occur in } B \tag{9.2.13.1}$$

when applied to $B = A[x]$ and a new variable $z : \mathbf{1}$, yields $A[x] \Rightarrow \forall z.A[x]$, hence that $A[x]$ holds in all pullbacks of the original context whenever it holds in this context itself. The second localness condition follows similarly from $\exists z.A[x] \Rightarrow A[x]$, a consequence of rule

$$(\exists_L)\ \frac{A[a] \Rightarrow C}{(\exists x : X.A[x]) \Rightarrow C} \quad \text{if } a \text{ is free in } A \text{ and doesn't occur in } C \tag{9.2.13.2}$$

9.2.14. (Presheaves vs. sheaves.) Notice that almost all constructions of Kripke–Joyal semantics appear to be applicable to the case when the "sheaf types" are allowed to be arbitrary presheaves X. However, the localness of proposition $x = y$, where $x, y : X$, implies separability of presheaf X; and the axiom for the iota-symbol

$$(\iota)\ \frac{B \Rightarrow \exists! x : X . A[x]}{B \Rightarrow A[\iota x : X . A[x]]} \tag{9.2.14.1}$$

actually expresses the sheaf condition for X.

In other words, the sheaf condition is necessary to pick up elements of "sets" characterized by some property $A[x]$, once it is shown that such an element exists and is unique.

9.2.15. (Prestacks vs. stacks.) Similarly, one might think that we might use prestacks $\mathcal{C}$ instead of stacks in these considerations, since all we formally need is that all $\mathbf{Hom}_{\mathcal{C}/S}(x, y)$, $x, y \in \operatorname{Ob}\mathcal{C}_S$, be sheaves. However, the stack condition for $\mathcal{C}$ is actually equivalent to the ability to pick up objects of a "category" $\mathcal{C}$, characterized by some property $A[x]$ uniquely up to a *unique* isomorphism, once the (local) existence of such an x is shown.

Indeed, once we know that such an x exists (locally) in some context $\mathfrak{V}/S$, we can find a cover $\{S_\alpha \to S\}$ and objects $x_\alpha \in \mathcal{C}(S_\alpha)$ with property A. Using uniqueness, we obtain isomorphisms $\theta_{\alpha\beta}$ between pullbacks of x_α and x_β to $\mathcal{C}(S_\alpha \times_S S_\beta)$. Finally, pulling everything back to $S_\alpha \times_S S_\beta \times_S S_\gamma$, and using uniqueness of these isomorphisms, we obtain the cocycle relation for the θs, i.e. we've got a descent datum. Now if $\mathcal{C}$ is a stack, this descent datum is effective, and we get an object $x \in \mathcal{C}(S)$ with required property.

In this way the stack condition is required in our "intuitionistic category theory" to be able to pick up objects, defined uniquely up to a unique isomorphism. This is important since we don't have the axiom of choice or Hilbert's τ to do this in the usual way. For example, this ability is important to construct initial or final objects, adjoint functors, or to show that any equivalence (defined here as a fully faithful essentially surjective functor) admits a quasi-inverse (adjoint) equivalence.

9.2.16. (Transitivity of pullbacks and stacks.) The reader may have noticed a certain problem with our constructions. Namely, if a context $\mathfrak{V}$ involves some variable x of "large" or "stack" type $\operatorname{Ob}\mathcal{C}$, then the pullbacks are not transitive: $\psi^*\varphi^*\mathfrak{V} \neq (\varphi\psi)^*\mathfrak{V}$ since $(\varphi\psi)^*(x)$ is known just to be isomorphic to $\psi^*\varphi^*(x)$. Actually, a problem appears even if we construct one pullback $\varphi^*\mathfrak{V}$, since the pullback functors $\varphi^* : \mathcal{C}_S \to \mathcal{C}_T$, $\varphi : T \to S$, are defined only up to an isomorphism, and the stack structure of $\mathcal{C}$ doesn't provide a canonical choice of φ^*.

Of course, one might tackle with these problems by replacing $\mathcal{C}$ by the $\mathcal{S}$-equivalent stack $\mathcal{C}^+$, so as to have $(\varphi\psi)^* = \psi^*\varphi^*$. However, we don't like this solution for several reasons, one of them being that the embedding $\mathcal{C} \to \mathcal{C}^+$ is itself defined by a large-scale application of the axiom of choice.

Instead, we prefer to think of objects of $\mathcal{C}$ as "defined up to an isomorphism", using the fact that we cannot distinguish two isomorphic objects of a category by internal means: they must have exactly the same properties. We have already seen in **9.2.15** that this is a viable point of view. Therefore, all our considerations should not change if we replace some objects (i.e. values of some variables of type $\operatorname{Ob}\mathcal{C}$) by other objects isomorphic to them, for example if we choose $\varphi^*(x)$ in another way.

This is actually the reason why $x = y$ is *not* a proposition in our system when x and y are of type $\operatorname{Ob}\mathcal{C}$, while $x \simeq y$ ("x is isomorphic to y") is one: $x \simeq y := (\exists f : x \to y.\exists g : y \to x.(f \circ g = \mathrm{id}_y \,\&g \circ f = \mathrm{id}_x))$.

9.2.17. (Local properties of objects and morphisms of stacks.) Another consequence of this philosophy is that *local properties of objects of a stack should be stable under isomorphisms.* In other words, a subset $P \subset \operatorname{Ob}\mathcal{C}$ is a *local set* or *class of objects of* $\mathcal{C}$ if: (a) $\varphi^*(X) \in P$ whenever $X \in P$; (b) $X \in P$ whenever all $\varphi_\alpha^* X \in P$ for some cover $\{\varphi_\alpha : S_\alpha \to S\}$; and (c) $X \simeq Y$ in $\mathcal{C}_S$ and $X \in P$ implies $Y \in P$.

Indeed, without property (c) properties (a) and (b) do not make any sense, just because of different possible choices of $\varphi^* X$. On the other hand, if P satisfies these properties, we can easily transfer P to any stack $\mathcal{S}$-equivalent to $\mathcal{C}$.

Similar remarks apply to *local* sets of morphisms Q in fibers of $\mathcal{C}$: they have to be closed under isomorphisms as well, i.e. if f in $\operatorname{Ar}\mathcal{C}_S$ belongs to Q, and u, v are isomorphisms in $\mathcal{C}_S$, such that ufv is defined, then ufv has to belong to Q as well.

9.2.18. Of course, our description of Kripke–Joyal semantics is by no means rigorous and complete from a logician's point of view. For example, we didn't pay enough attention to distinguish between syntax and semantics, or between axioms, axiom schemata, and deduction rules, and we haven't in fact said anything about proofs at all (a suitable extension of Gentzen's LJ system combined with λ-calculus would do). Another obvious gap is that we haven't explained how one can incorporate into the system described above cartesian functors $F : \mathcal{C} \to \mathcal{D}$ (this is quite clear anyway: if $x : \operatorname{Ob}\mathcal{C}$, then $F(x) : \operatorname{Ob}\mathcal{D}$, and if $f : x \to y$, then $F(f) : F(x) \to F(y)$), and, more importantly, how one can construct new stacks from existing ones (e.g. $\mathcal{C} \times_{\mathcal{S}} \mathcal{D}$, corresponding to "product of intuitionistic categories", or $\mathit{Cart}_{\mathcal{S}}(\mathcal{C}, \mathcal{D})$, or $\mathcal{C}_{/x}$, $x \in \operatorname{Ob}\mathcal{C}_e \ldots$). We think these things are already clear enough, and that we'll be able to perform them by "external" means if necessary.

9.3. (Model stacks.) Now we are going to present the definition of a model stack over a site or topos, and the construction of its homotopic category. The main idea here is to deal with a stack as a "intuitionistic category", transfer Quillen's definitions and proofs to the intuitionistic case, and interpret the result in Kripke–Joyal semantics, thus regaining descriptions in terms of stacks over sites.

9.3.1. (Local classes of morphisms.) We fix a stack $\mathcal{C}$ over a site $\mathcal{S}$ for the most part of this subsection. Sometimes we'll assume for simplicity that $\mathcal{S}$ is closed under finite projective limits and its topology is subcanonical, even if this is inessential for most statements. Usually we'll have three *local* classes of morphisms in fibers of $\mathcal{C}$ (cf. **9.2.17**), called *fibrations*, *cofibrations* and *weak equivalences* (cf. **8.1.1**). Of course, a morphism $f : X \to Y$ in $\mathcal{C}_S$ is said to be an *acyclic fibration* (resp. *acyclic cofibration*) if it is both a fibration (resp. cofibration) and a weak equivalence, and we say that an object $X \in \operatorname{Ob}\mathcal{C}_S$ is *cofibrant* (resp. *fibrant*) if $\varnothing_{\mathcal{C}_S} \to X$ is a cofibration, resp. if $X \to e_{\mathcal{C}_S}$ is a fibration. Clearly, these are local classes of objects and morphisms in the sense of **9.2.17**.

9.3.2. (Retracts and local retracts.) Recall that a morphism $g : Z \to T$ is a retract (or a *global retract*) of another morphism $f : X \to Y$ in $\mathcal{C}_S$ iff there are morphisms $i : Z \to X$, $j : T \to Y$, $p : X \to Z$, $q : Y \to T$, such that $f \circ i = j \circ g$, $g \circ p = q \circ f$, $p \circ i = \mathrm{id}_Z$ and $q \circ j = \mathrm{id}_T$ (cf. (8.1.2.2)).

We can express this condition by formula $\exists i : Z \to X.\exists j : T \to Y.\exists p : X \to Z.\exists q : Y \to T.(f \circ i = j \circ g \& g \circ p = q \circ f \& p \circ i = \mathrm{id}_Z \& q \circ j = \mathrm{id}_T)$ of signature X, Y, Z, $T : \operatorname{Ob}\mathcal{C}$, $f : X \to Y$, $g : Z \to T$. Interpreting this formula in Kripke–Joyal semantics, we obtain the notion of a *local retract*: g is a local retract of f iff such morphisms i, j, p, q as above exist locally, i.e. iff $g|_{S_\alpha}$ is a retract of $f|_{S_\alpha}$ in $\mathcal{C}_{S_\alpha}$ for some cover $\{S_\alpha \to S\}$ in $\mathcal{S}$.

Usually we introduce the "local" or "intuitionistic" counterparts of usual (i.e. "global" or "classical") notions by a similar procedure, that needn't be explicit each time.

For example, if (P) is a local property of morphisms in fibers of $\mathcal{C}$, we can say that $f : X \to Y$ in $\mathcal{C}_S$ is a local retract of an (unspecified) morphism with property (P) iff there is some cover $\{S_\alpha \to S\}$, such that each $f|_{S_\alpha}$ is a retract in $\mathcal{C}_{S_\alpha}$ of a morphism $u_\alpha : Z_\alpha \to T_\alpha$ having property (P).

9.3.3. (Local lifting properties.) Similarly, given two morphisms $i : A \to B$ and $p : X \to Y$ in $\mathcal{C}_S$, we say that i *has the local left lifting property (local LLP or lLLP) with respect to* p, or that p *has the lRLP with respect to* i, if for any $T \to S$ in $\mathcal{S}$ and any morphisms $u : A|_T \to X|_T$ and $v : B|_T \to Y|_T$, such that $v \circ i|_T = p|_T \circ v$, there is a cover $\{T_\alpha \to T\}$ and morphisms $h_\alpha : B|_{T_\alpha} \to X|_{T_\alpha}$, such that $h_\alpha \circ i|_{T_\alpha} = u|_{T_\alpha}$ and $p|_{T_\alpha} \circ h_\alpha = v|_{T_\alpha}$. Again, this condition is obtained simply by interpreting the usual definition of lifting properties in Kripke–Joyal semantics. We can express it by means of the following diagram:

$$\begin{tikzcd} A \arrow[r, dashed, "\forall u"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, dashed, "\forall v"] \arrow[ur, dotted, "\exists h"] & Y \end{tikzcd} \tag{9.3.3.1}$$

This diagram is understood as follows. First, we are given or can construct the solid arrows (i and p in this case) in $\mathcal{C}_S$, without pulling anything back. Then we pull back with respect to any $T \to S$ and choose arbitrarily the dashed arrows u and v, so as to make the diagram commutative. After this the dotted arrow h exists only after some other pullbacks $\{T_\alpha \to T\}$; they must constitute a cover of T because of the $\exists$ sign.

Definition 9.3.4 *A stack $\mathcal{C}$ over a site $\mathcal{S}$ with three distinguished* local *classes of morphisms in fibers as in* **9.3.1** *is said to be a* **model stack** *if the conditions (MS1)–(MS5) hold:*

(MS1) All fiber categories $\mathcal{C}_S$ are closed under arbitrary (small) projective and inductive limits, and all pullback functors $\varphi^* : \mathcal{C}_S \to \mathcal{C}_T$ admit both left and right adjoints $\varphi_!, \varphi_* : \mathcal{C}_T \to \mathcal{C}_S$.

(MS2) Each distinguished class of morphisms is (local and) stable under global retracts (in each fiber $\mathcal{C}_S$).

(MS3) ("2-out-of-3") Given $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}_S$, such that two of f, g and gf are weak equivalence, then so is the third.

(MS4) (Lifting.) Any cofibration $i : A \to B$ in $\mathcal{C}_S$ has the local LLP with respect to all acyclic fibrations $p : X \to Y$, and any acyclic cofibration $i : A \to B$ has the local LLP with respect to all fibrations $p : X \to Y$. In other words, for any cofibration $i : A \to B$ and fibration $p : X \to Y$ in $\mathcal{C}_S$, one of them being acyclic, and any morphisms $u : A \to X$, $v : B \to Y$ in $\mathcal{C}_S$, such that $vi = pu$, one can *locally* find $h_\alpha : B|_{S_\alpha} \to X|_{S_\alpha}$, making (9.3.3.1) commutative.

(MS5) (Factorization.) Any morphism $f : X \to Y$ in $\mathcal{C}_S$ can be *globally* factorized into $X \xrightarrow{\alpha} Z \xrightarrow{\beta} Y$ and $X \xrightarrow{\gamma} W \xrightarrow{\delta} Y$ (in $\mathcal{C}_S$), where α and γ are cofibrations, β and δ are fibrations, and α and δ are weak equivalences.

9.3.5. Remarks. (a) One can show that (MS1) is actually equivalent to the more natural requirement of $\mathcal{C}$ to be closed under arbitrary *local* inductive and projective limits (taken over abritrary inner categories $\mathcal{I}$ in $\tilde{\mathcal{S}}$, i.e. essentially sheaves of small categories). However, the axiom (MS1) is technically simpler to verify, and it doesn't require any knowledge of local limits.

(b) Notice that (MS2) actually implies that all distinguished classes are closed under *local* retracts as well, and that for example even if $f : X \to Y$ is local retract of an unspecified fibration, then it is a fibration itself. Therefore, we might replace (MS2) by the more naturally looking condition of each distinguished class to be stable under local retracts.

(c) Similarly, the explicit description given in (MS4) seems too weak, because the statement "$i : A \to B$ has the local LLP with respect to all acyclic cofibrations" actually means that we are free to choose an acyclic cofibration $p : X \to Y$ after making an arbitrary pullback, and then u and v can be chosen after another arbitrary pullback. However, all these additional pullbacks are not necessary, once we know that all distinguished classes are local and in particular stable under pullbacks.

9.3.6. (d) The most interesting is our form of (MS5). Indeed, the natural local form would be a (strictly) weaker statement:

(MS5l) (Local factorization.) Any morphism $f : X \to Y$ in $\mathcal{C}_S$ can be *locally* factorized into $X|_{S_i} \xrightarrow{\alpha_i} Z_i \xrightarrow{\beta_i} Y|_{S_i}$ and $X|_{S_i} \xrightarrow{\gamma_i} W_i \xrightarrow{\delta_i} Y|_{S_i}$ (in $\mathcal{C}_{S_i}$ for some cover $\{S_i \to S\}$), where α_i and γ_i are cofibrations, β_i and δ_i are fibrations, and α_i and δ_i are weak equivalences.

The reason why we require the stronger axiom (MS5) is that in fact all model categories people really use (e.g. all cofibrantly generated model categories) admit functorial factorizations (e.g. constructed by means of Quillen's small object argument) in (M5), and actually some authors *require* the existence of such functorial factorizations in the definition of a model category (cf. e.g. [Hovey]). This means that all model stacks we consider actually satisfy a *stronger* version (MS5f) of (MS5), which in particular implies (MS5):

(MS5f) (Functorial factorization.) One can choose factorizations of (MS5) functorially in $f \in \operatorname{Ar} \mathcal{C}_S$, i.e. we actually have functors W, Z : $\operatorname{Ar} \mathcal{C}_S \to \mathcal{C}_S$, α, β, γ, $\delta : \operatorname{Ar} \mathcal{C}_S \to \operatorname{Ar} \mathcal{C}_S$, such that for any $f : X \to Y$ in $\operatorname{Ar} \mathcal{C}_S$ we get $X \xrightarrow{\alpha(f)} Z(f) \xrightarrow{\beta(f)} Y$, $X \xrightarrow{\gamma(f)} W(f) \xrightarrow{\delta(f)} Y$ with the properties listed in (MS5). Moreover, these functors are compatible with base change functors $\varphi^* : \mathcal{C}_S \to \mathcal{C}_T$, i.e. they extend to cartesian functors between appropriate stacks.

However, (MS5) seems to be sufficient for almost all our constructions, so we don't insist on requiring (MS5f). On the other hand, (MS5) allows us to construct (globally) fibrant and cofibrant replacements, something we wouldn't be able to do having only (MS5l). This somewhat simplifies the exposition and allows us to construct globally more than it would be possible by only local means.

9.3.7. (Basic properties of distinguished classes.) One checks, essentially in the classical way, that for example the acyclic cofibrations are *exactly* the morphisms in fibers of $\mathcal{C}$ that have the local LLP with respect to all fibrations (of course, if we fix some $i : A \to B$ in $\mathcal{C}(S)$, we are free to choose a fibration $p : X \to Y$ and morphisms $u : A \to X$ and $v : B \to Y$ after any pullback). Furthermore, it is easy to see that the class of morphisms P' in fibers of a stack $\mathcal{C}$, characterized by the local LLP with respect to some other *local* class of morphisms P, is itself local, and stable under composition, pushouts, *finite* direct sums and local retracts (but not sequential inductive limits!) in the fibers of $\mathcal{C}$, as well as under all pullback functors φ^*. This applies in particular to cofibrations and acyclic cofibrations in a model stack, and the classes of fibrations and acyclic fibrations have dual properties, e.g. stability under pullbacks, finite products, and φ^*.

Similarly, one sees immediately from (MS5) and (MS3) that the weak equivalences are exactly those morphisms that can be factorized into an acyclic cofibration followed by an acyclic fibration. Therefore, any two of distinguished local classes of a model stack $\mathcal{C}$ completely determine the remaining class.

9.3.8. (Absence of stability under sequential inductive limits.) Notice that the class of morphisms P' characterized by the local LLP with respect to some other local class P needn't be closed under sequential limits. In other words, if $A_0 \xrightarrow{i_0} A_1 \xrightarrow{i_1} A_2 \to \cdots$ is an inductive system in $\mathcal{C}(S)$, such that each $i_n : A_n \to A_{n+1}$ has the local LLP with respect to some $p : X \to Y$ in $\mathcal{C}(S)$ (and all its pullbacks), we cannot conclude that $i : A := A_0 \to B := \varinjlim_n A_n$ has the same property. Indeed, let us try to repeat the usual proof. Fix some $v : B \to Y$ and denote by $Z_n := \mathbf{Hom}_Y(A_n, X)$ the subsheaf of $\mathbf{Hom}(A_n, X)$ consisting of all local liftings of v, i.e. $Z_n = \{f : A_n \to X \,|\, p \circ f = v \circ j_n\}$ in Kripke–Joyal semantics, where $j_n : A_n \to B$ is the natural embedding. Put $Z := \mathbf{Hom}_Y(B, X) = \mathbf{Hom}_Y(\varinjlim_n A_n, X) = \varprojlim_n Z_n$. Then the local LLP of i_n means that each $i_n^* : Z_{n+1} \to Z_n$ is an epimorphism; and hypothetical local LLP of i would mean that $i^* : Z \to Z_0$ is an epimorphism as well.

In other words, we have a projective system $\cdots \to Z_2 \to Z_1 \to Z_0$ with epimorphic transition morphisms in a topos $\mathcal{E} = \tilde{\mathcal{S}}$, and we would like to conclude that $Z = \varprojlim_n Z_n \to Z_0$ is also epimorphic. However, the usual proof of this statement for $\mathcal{E} = \mathit{Sets}$ invokes the (dependent countable) axiom of choice, so it cannot be transferred to the topos case. In fact, one can construct examples where $Z \to Z_0$ is not epimorphic while all $Z_{n+1} \to Z_n$ are. Put $\mathcal{E} := \mathcal{B}_{\hat{\mathbb{Z}}}$; this is the topos of discrete $\hat{\mathbb{Z}}$-sets, i.e. sets X with an action of $\hat{\mathbb{Z}}$, such that the stabilizer of any point $x \in X$ is open in $\hat{\mathbb{Z}}$. Epimorphisms in this topos are just the surjective maps of discrete $\hat{\mathbb{Z}}$-sets, and projective limits can be computed by taking the subset of all points of the usual projective limit having an open stabilizer. Next, put $Z_n := \mathbb{Z}/p^n\mathbb{Z}$, and let the generator σ of $\hat{\mathbb{Z}}$ act on Z_n by adding one: $\sigma(x) = x + 1$. Clearly, these $\hat{\mathbb{Z}}$-sets Z_n together with canonical projections $Z_{n+1} \to Z_n$ define a projective system in $B_{\hat{\mathbb{Z}}}$ with epimorphic transition morphisms. However, the projective limit Z of this system in $\mathcal{B}_{\hat{\mathbb{Z}}}$ is an empty set, as well as the product $\prod_n Z_n \supset Z$, the stabilizer of any point of corresponding set-theoretical product being of infinite index in $\hat{\mathbb{Z}}$.

9.3.9. (Fiberwise dual of a model stack.) Given a fibered category $\mathcal{C} \to \mathcal{S}$, we define its *fiberwise dual* or *opposite* $\mathcal{C}^{fop} \to \mathcal{S}$ by replacing each fiber category $\mathcal{C}(S)$ by its opposite $\mathcal{C}(S)^0$ while preserving the pullback functors $\varphi^* : \mathcal{C}(S) \to \mathcal{C}(T)$. Of course, this definition doesn't actually depend on the choice of pullback functors φ^*. Now we see immediately that *the axioms of a model stack are self-dual*, i.e. $\mathcal{C}^{fop}$ is a model stack over $\mathcal{S}$ whenever $\mathcal{C}$ is one (of course, one has to interchange fibrations and cofibrations while dualizing). This remark enables us to deduce new statements about model stacks by duality.

9.3.10. (Flat stacks.) We say that a fibered category $\mathcal{C} \to \mathcal{S}$ satisfying (MS1) over a category $\mathcal{S}$ with fibered products is *flat* if the following condition is fulfilled:

(MS1+) For any cartesian square in $\mathcal{S}$

$$\begin{tikzcd} T' \arrow[r, "g"] \arrow[d, "q"] & T \arrow[d, "p"] \\ S' \arrow[r, "f"] & S \end{tikzcd} \tag{9.3.10.1}$$

the canonical morphism $q_!g^* \to f^*p_!$ is an isomorphism of functors $\mathcal{C}(T) \to \mathcal{C}(S')$, hence the same is true for its adjoint $p^*f_* \to g_*q^*$ as well.

Notice that we have actually *two* canonical morphisms $q_!g^* \to f^*p_!$. The first is deduced by adjointness from $g^* \to q^*f^*p_! \cong g^*p^*p_!$, itself obtained by applying $g^* \star -$ to the unit $\mathrm{Id}_{C(T)} \to p^*p_!$. The second one is deduced by adjointness from $f_!q_!g^* \cong p_!g_!g^* \to p_!$, obtained by applying $p_! \star -$ to the counit $g_!g^* \to \mathrm{Id}_{C(S')}$. In most cases it is clear that these two coincide; let us include for simplicity the equality of these two canonical morphisms $q_!g^* \to f^*p_!$ as an additional requirement in (MS1+).

9.3.11. (Equivalent formulations of flatness.) Let $\mathcal{C}$ be a stack over $\mathcal{S}$, and $T \xrightarrow{\varphi} S$ be a morphism in $\mathcal{S}$. We denote by $\mathcal{C}^{T/S}$ or $\mathcal{C}^T_{/S}$ (or simply $\mathcal{C}^T$

when $S = e_{\mathcal{S}}$) the direct image $\varphi_*^{st}\mathcal{C}_{/T}$ of stack $\mathcal{C}_{/T} := \mathcal{C} \times_{\mathcal{S}} \mathcal{S}_{/T}$ over $\mathcal{S}_{/T}$. Clearly, $\mathcal{C}^T_{/S}(S') = \mathcal{C}(T \times_S S')$ for any $S' \in \operatorname{Ob}\mathcal{S}_{/S}$, so the pullback functors $\varphi^*_{S'} : \mathcal{C}(S') \to \mathcal{C}(T \times_S S')$ for $\varphi_{S'} : T \times_S S' \to S'$ combine together to a *cartesian* functor $\boldsymbol{\varphi}^* : \mathcal{C}_{/S} \to \mathcal{C}^T_{/S}$ over $\mathcal{S}_{/S}$. Now the flatness condition (MS1+) can be interpreted as a requirement for all $\boldsymbol{\varphi}^*$ to admit *cartesian* $\mathcal{S}_{/S}$-adjoint functors $\boldsymbol{\varphi}_!$, $\boldsymbol{\varphi}_* : \mathcal{C}^T_{/S} \to \mathcal{C}_{/S}$. This adjointness can be interpreted in yet another way as a functorial $\mathcal{S}_{/S}$-isomorphism of local Hom-sheaves, i.e. an $\mathcal{S}_{/S}$-isomorphism of cartesian functors $(\mathcal{C}^T_{/S})^{fop} \times \mathcal{C}_{/S} \to \mathbf{SETS}_{\mathcal{S}_{/S}}$:

$$\begin{aligned}\mathbf{Hom}_{\mathcal{C}^T_{/S}|S'}(X, \boldsymbol{\varphi}^*Y) \cong \mathbf{Hom}_{\mathcal{C}_{/S}|S'}(\boldsymbol{\varphi}_!X, Y) \quad \text{for all } S' \in \operatorname{Ob}\mathcal{S}_{/S},\\ X \text{ in } \mathcal{C}^T_{/S}(S') = \mathcal{C}(T \times_S S'),\, Y \text{ in } \mathcal{C}_{/S}(S') = \mathcal{C}(S'). \quad (9.3.11.1)\end{aligned}$$

We obtain a similar “local adjointess” interpretation for $\boldsymbol{\varphi}^*$ and $\boldsymbol{\varphi}_*$ of course.

Notice that the Kripke–Joyal philosophy insists that we should always use Hom-sheaves, never Hom-sets; therefore, (9.3.11.1) is the only correct way to discuss adjoint functors to $\boldsymbol{\varphi}^*$ from this point of view. Therefore, it might be very natural to include (MS1+) into the list of axioms for model stacks and consider only flat model stacks.

From the intuitionistic point of view $\mathcal{C}_{/S}$ is an “intuitionistic category” (over base site $\mathcal{S}_{/S}$), $T \in \operatorname{Ob}\mathcal{C}_{/S}$ is an “intuitionistic set”, and $\mathcal{C}^T_{/S}$ is “the category of families of objects of $\mathcal{C}_{/S}$ indexed by T”, i.e. some sort of (local) product of categories. Furthermore, $\boldsymbol{\varphi}^* : \mathcal{C}_{/S} \to \mathcal{C}^T_{/S}$ is the “constant family functor”, and its left and right adjoints $\boldsymbol{\varphi}_! =: \coprod_{T/S}$ and $\boldsymbol{\varphi}_* =: \prod_{T/S} : \mathcal{C}^T_{/S} \to \mathcal{C}_{/S}$ should be thought of as “(local) coproducts and products of families of objects of category $\mathcal{C}_{/S}$ indexed by T”. Then (MS1+) assures us that these local coproducts and products are indeed “local” or “universal”, i.e. compatible with pullbacks.

9.3.12. (Local sums and products in $\mathcal{C}$.) When we have $T \xrightarrow{\varphi} S$ in $\mathcal{S}$, and an object $X \in \operatorname{Ob}\mathcal{C}(T)$ “of $\mathcal{C}$ over T”, we denote $\varphi_!X$ by $\coprod_{T/S} X$ or $\bigoplus_{T/S} X$, and φ_*X by $\prod_{T/S} X$. When S is the final object $e_{\mathcal{S}}$, we write simply $\coprod_T X$ or $\bigoplus_T X$, and $\prod_T X$, respectively. These notations are motivated by usual notations for sheaves and presheaves (cf. SGA 3), as well as the case $\mathcal{S} = \mathit{Sets}$: then $\mathcal{C}(I) = \mathcal{C}(\mathbf{1})^I$, and $\prod_I$, $\coprod_I : \mathcal{C}(\mathbf{1})^I \to \mathcal{C}(\mathbf{1})$ are just the usual product and coproduct functors.

Furthermore, we denote $\varphi_!\varphi^*X$ by $X \times_S T$ or $X \otimes_S T$ (omitting S if $S = e_{\mathcal{S}}$), and $\varphi_*\varphi^*X$ by $\mathbf{Hom}_S(T, X)$ ($\mathbf{Hom}(T, X)$ or even X^T for $S = e_{\mathcal{S}}$). Again, the motivation comes from the case $\mathcal{S} = \mathit{Sets}$. In any case $X \mapsto X \otimes_S T$ and $Y \mapsto \mathbf{Hom}_S(T, Y)$ are adjoint: $\operatorname{Hom}_S(X \otimes_S T, Y) \cong \operatorname{Hom}_T(X|_T, Y|_T) \cong \operatorname{Hom}_S(X, \mathbf{Hom}_S(T, Y))$.

9.3.13. (Local limits.) Suppose we are given inner category C in a topos $\mathcal{E}$, given by its objects of morphisms C_1 and of objects $C_0 \in \mathrm{Ob}\,\mathcal{E}$, together with the source and target morphisms $s, t : C_1 \to C_0$, the identity morphism $i : C_0 \to C_1$, and the composition morphism $\mu : C_2 := C_1 \times_{s,t} C_1 \to C_1$, subject to usual identity and associativity conditions. Suppose that we have an inner functor F from C to a model stack $\mathcal{D} \xrightarrow{p} \mathcal{E}$, i.e. a inner category F in $\mathcal{D}$, such that $p(F) = C$, and $s_F : F_1 \to F_0$ is cartesian. Then we can compute the corresponding inner or local limits $\varinjlim_C F$ and $\varprojlim_C F$ in the classical way: $\varinjlim_C F := \mathrm{Coker}(\coprod_{C_1} F_1 \rightrightarrows \coprod_{C_0} F_0)$, and $\varprojlim_C F := \mathrm{Ker}(\prod_{C_0} F_0 \rightrightarrows \prod_{C_1} F_1)$.

When we have the flatness condition (MS1+), all these local sums, products and limits commute with arbitrary base change, so these notions fit nicely into the Kripke–Joyal philosophy: they might be thought of as "limits in an intuitionistic category $\mathcal{D}$ along a small intuitionistic index category C".

9.3.14. (Model stacks over sites and topoi.) Recall that we have a correspondence (a 2-equivalence of 2-categories, actually) between stacks $\mathcal{C}$ over a site $\mathcal{S}$, and stacks $\tilde{\mathcal{C}}$ over the corresponding topos $\tilde{\mathcal{S}}$. Under this correspondence $\mathcal{C}$ is simply the "restriction" of $\tilde{\mathcal{C}}$ to $\mathcal{S}$, i.e. it is $\tilde{\mathcal{C}} \times_{\tilde{\mathcal{S}}} \mathcal{S}$, and conversely, $\tilde{\mathcal{C}}$ is recovered from $\mathcal{C}$ by restricting $\mathcal{C}^+$ from $\hat{\mathcal{S}}$ to $\tilde{\mathcal{S}}$: $\tilde{\mathcal{C}} := \mathcal{C}^+ \times_{\hat{\mathcal{S}}} \tilde{\mathcal{S}}$ (cf. **9.1.9**). Now, if we have for example a local class of morphisms $\tilde{P}$ in the fibers of $\tilde{\mathcal{C}} \to \tilde{\mathcal{S}}$, we obtain a local class of morphisms P in the fibers of $\mathcal{C} \to \mathcal{S}$ simply by "restriction" since $\mathcal{C}(S) = \tilde{\mathcal{C}}(\tilde{S})$, where $\tilde{S}$ is the sheafification of the presheaf represented by S, and conversely, starting from $P \subset \mathrm{Ar}\,\mathcal{C}$, we can extend it uniquely to a local class $\tilde{P} \subset \mathrm{Ar}\,\tilde{\mathcal{C}}$ as follows: a morphism $f : X \to Y$ in $\tilde{\mathcal{C}}(X)$, $X \in \mathrm{Ob}\,\tilde{\mathcal{S}}$, belongs to $\tilde{P}$ iff its pullbacks $\varphi^*(f) \in \mathrm{Ar}\,\tilde{\mathcal{C}}(\tilde{S}) \cong \mathrm{Ar}\,\mathcal{C}(S)$ with respect to all $\varphi : \tilde{S} \to X$ belong to P. It is immediate that these constructions are inverse to each other, once we take into account that local properties can be also transferred along any $\mathcal{S}$-equivalence of stacks (cf. **9.2.17**).

In particular, we can transfer the three distinguished classes entering into the structure of a model stack from stacks $\mathcal{C} \to \mathcal{S}$ to stacks $\tilde{\mathcal{C}} \to \tilde{\mathcal{S}}$, and conversely. Now it is easy to see that the axioms of a model stack hold for $\mathcal{C}$ iff they hold for $\tilde{\mathcal{C}}$, with the following two exceptions. Firstly, (MS5) cannot be transfered from $\mathcal{C}$ to $\tilde{\mathcal{C}}$, but both (MS5l) and (MS5f) can, and since all model stacks we consider satisfy the stronger condition (MS5f), this is not really a big problem for our considerations. Secondly, it is not immediate that (MS1) extends from $\mathcal{C}$ to $\tilde{\mathcal{C}}$ (the other direction is clear). We'll check this in a moment; for now, let us state

Proposition 9.3.15 *The 2-category of model stacks over a site $\mathcal{S}$, satisfying (MS5f), is 2-equivalent to the 2-category of model stacks over the corresponding topos $\tilde{\mathcal{S}}$, satisfying (MS5f). The same applies if we replace in* *the definition of model stacks (MS5) with a weaker local condition (MS5l). Moreover, any model stack over topos $\tilde{\mathcal{S}}$ defines by restriction a model stack over $\mathcal{S}$ without any additional requirements.*

In this way it is essentially the same thing to consider model stacks over a site or over the corresponding topos.

Proof. The only problem is to transfer (MS1) from $\mathcal{C}$ to $\tilde{\mathcal{C}}$. (a) Let's prove that all $\tilde{\mathcal{C}}(X)$, $X \in \operatorname{Ob}\tilde{\mathcal{S}}$, are closed under arbitrary projective limits. By definition $\tilde{\mathcal{C}}(X) = \mathcal{C}^+(X) = \mathit{Cart}_{\mathcal{S}}(\mathcal{S}_{/X}, \mathcal{C})$, so suppose we want to compute a projective limit $F = \varprojlim F_i$ of cartesian functors $F_i : \mathcal{S}_{/X} \to \mathcal{C}$. To this end we define F fiberwise, i.e. for any $S \xrightarrow{\xi} X$ in $\mathcal{S}_{/X}$ we compute $F(\xi) := \varprojlim F_i(\xi)$ in the fiber $\mathcal{C}(S)$. Now we observe that for all $\varphi : T \to S$ the functor $\varphi^* : \mathcal{C}(S) \to \mathcal{C}(T)$ commutes with arbitrary projective limits, having a left adjoint $\varphi_!$ by (MS1), hence $\varphi^* F(\xi) \cong F(\xi\varphi)$, i.e. we've indeed constructed a cartesian functor $F : \mathcal{S}_{/X} \to \mathcal{C}$, clearly having the universal properly required from $\varprojlim F_i$. The case of inductive limits is dealt with similarly, using that φ^* has a right adjoint φ_*, hence commutes with arbitrary inductive limits as well.

(b) Now we have to show that for any morphism of sheaves $f : X \to Y$ the pullback functor $f^* : \tilde{\mathcal{C}}(Y) \to \tilde{\mathcal{C}}(X)$ has both left and right adjoints $f_!$ and f_*. We can replace $\mathcal{S}$ by any its small generating category $\mathcal{S}'$ since in this case $\tilde{\mathcal{S}'} \cong \tilde{\mathcal{S}}$, and $\tilde{\mathcal{C}}$ is equivalent to the extension to $\tilde{\mathcal{S}'}$ of $\mathcal{C}' := \mathcal{C} \times_{\mathcal{S}} \mathcal{S}'$. Therefore, we can assume $\mathcal{S}$ to be small.

In this case one can actually show the statement for the pullback functor $f^* : \mathcal{C}^+(Y) \to \mathcal{C}^+(X)$ for any morphism of presheaves $f : X \to Y$. The main idea here is that $\mathcal{C}^+(X) = \mathit{Cart}_{\mathcal{S}}(\mathcal{S}_{/X}, \mathcal{C})$, and similarly for $\mathcal{C}^+(Y)$, and under this indentification f^* is the precomposition functor $\bar{f}^*$ with respect to $\bar{f} : \mathcal{S}_{/X} \to \mathcal{S}_{/Y}$. We want to show that $\bar{f}^*$ admits, say, a left adjoint $\bar{f}_!$ (existence of $\bar{f}_*$ can be obtained then by fiberwise duality). We can construct first a "left Kan extension functor" $\bar{f}_!^0 : \mathit{Funct}_{\mathcal{S}}(\mathcal{S}_{/X}, \mathcal{C}) \to \mathit{Funct}_{\mathcal{S}}(\mathcal{S}_{/Y}, \mathcal{C})$ by putting $(\bar{f}_!^0 F)(T \xrightarrow{\eta} Y) := \varinjlim_{\mathcal{S}_{/X \times_Y T}} \varphi_! F(\xi)$, where the limit is computed in $\mathcal{C}(T)$ along all diagrams

$$\begin{tikzcd} S \arrow[r, dashed, "\xi"] \arrow[d, dashed, "\varphi"'] & X \arrow[d, "f"] \\ T \arrow[r, "\eta"] & Y \end{tikzcd} \tag{9.3.15.1}$$

The problem with $\bar{f}_!^0$ is that $\bar{f}_!^0 F$ needn't be cartesian for a cartesian F, at least if we don't have additional flatness conditions like (MS1+). However, if we show that the inclusion $J : \mathit{Cart}_{\mathcal{S}}(\mathcal{S}_{/Y}, \mathcal{C}) \to \mathit{Funct}_{\mathcal{S}}(\mathcal{S}_{/Y}, \mathcal{C})$ admits a left adjoint $J_!$, we can put $f_! = \bar{f}_! := J_! \circ \bar{f}_!^0$. To show existence of $J_!$ we define $(J_!F)(S)$ for any $\mathcal{S}$-functor $F : \mathcal{S}_{/Y} \to \mathcal{C}$ by putting it equal to the inductive limit in $\mathcal{C}(S)$ of $u_{1,!}v_1^* u_{2,!}v_2^* \cdots v_n^* F(S_n)$, taken along the category of all diagrams $S = S_0 \stackrel{u_1}{\leftarrow} T_0 \stackrel{v_1}{\to} S_1 \stackrel{u_2}{\leftarrow} T_2 \stackrel{v_2}{\to} \cdots \stackrel{v_n}{\to} S_n$, all integer $n \geq 0$, where we allow to split any arrow in two of the same direction, and insert couples of identity morphisms at any point. When we have flatness (MS1+), the construction of $J_!F$ is considerably simplified: it suffices to take the inductive limit along all diagrams with $n = 1$.

We don't provide more details because in all our applications the model stack can be constructed directly over the topos $\tilde{\mathcal{S}}$ and then restricted to $\mathcal{S}$ if necessary, so we'll need to apply the above proposition only in the obvious direction.

9.3.16. (Arbitrary limits in a model stack over a topos.) One can easily show that whenever $p : \mathcal{C} \to \mathcal{S}$ is a fibered category satisfying (MS1), and certain types of (say, projective) limits exist both in $\mathcal{S}$ and all fibers $\mathcal{C}(S)$, then these types of limits exist in $\mathcal{C}$ as well, and p commutes with these limits. In particular, *arbitrary (small) projective and inductive limits exist in a model stack $\mathcal{C}$ over a topos $\mathcal{E}$, and the projection $\mathcal{C} \stackrel{p}{\to} \mathcal{E}$ commutes with these limits.* The proof goes as follows. Suppose we want to compute $X = \varprojlim_{\mathcal{I}} X_i$ in $\mathcal{C}$. We put $S_i := p(X_i)$ and compute $S := \varprojlim_{\mathcal{I}} S_i$ in $\mathcal{S}$. Then we put $X := \varprojlim_{\mathcal{I}} \pi_i^* X_i$ in $\mathcal{C}(S)$, where $\pi_i : S \to S_i$ are the natural morphisms, and easily check (using the existence of left adjoints $\varphi_!$) that X is indeed $\varprojlim_{\mathcal{I}} X_i$ in $\mathcal{C}$. To compute inductive limits we put similarly $S := \varinjlim_{\mathcal{I}} S_i$ and $X := \varinjlim_{\mathcal{I}} \lambda_{i,!} X_i$ in $\mathcal{C}(S)$, where $\lambda_i : S_i \to S$ are the natural morphisms.

9.3.17. (Final section of $\mathcal{C} \to \mathcal{S}$.) Let $\mathcal{S}$ be a category with finite projective limits, $\mathcal{C} \stackrel{p}{\to} \mathcal{S}$ be a fibered category with finite projective limits in each fiber $\mathcal{C}(S)$. Suppose all pullback functors $\varphi^* : \mathcal{C}(S) \to \mathcal{C}(T)$ to be left exact. Then the above reasoning shows that finite projective limits exist in $\mathcal{C}$ itself, and p is left exact. Now let us choose a final object $[S]$ in each fiber $\mathcal{C}(S)$. Clearly, $\varphi^*[S] \cong [T]$ for any $\varphi : T \to S$ in $\mathcal{S}$; moreover, $\mathrm{Hom}_{\mathcal{C}}(X, [S]) \cong \mathrm{Hom}_{\mathcal{S}}(p(X), S)$ for any $X \in \mathrm{Ob}\,\mathcal{C}$, and in particular $\mathrm{Hom}_{\mathcal{C}}([T], [S]) \cong \mathrm{Hom}_{\mathcal{S}}(T, S)$. This means that the restriction of p to the fibered subcategory $[\mathcal{S}]$ of $\mathcal{C}$ consisting of objects $[S]$ is an *isomorphism* of categories; taking its inverse we obtain a cartesian section $\sigma : \mathcal{S} \to \mathcal{C}$, clearly the final object of $\varprojlim_{\mathcal{S}} \mathcal{C}$. Since $[S]$ is a final object in $\mathcal{C}(S)$, we obtain a left exact faithful functor $\mathcal{C}(S) \to \mathcal{C}_{/[S]}$, so we can write $\mathrm{Hom}_{[S]}(X, Y)$ or $\mathrm{Hom}_S(X, Y)$ instead of $\mathrm{Hom}_{\mathcal{C}(S)}(X, Y)$, for any X, $Y \in \mathrm{Ob}\,\mathcal{C}(S)$. Moreover, given any $\varphi : T \to S$ in $\mathcal{S}$ and any $X \to [S]$ in $\mathcal{C}(S)$, the pullback $\varphi^* X$ is easily seen to be the fibered product $X \times_{[S]} [T]$ in $\mathcal{C}$. This explains our alternative notations $X \times_{[S]} [T]$ and $X \times_S T$ for the pullback $\varphi^* X$, and allows us to mix fibered products in fibers of $\mathcal{C}$ with fibered products corresponding to pullbacks $\varphi^* X$ without risk of confusion. Notice, however, that the product $X \times Y$ of two objects in $\mathcal{C}(S)$ corresponds to $X \times_{[S]} Y$ in $\mathcal{C}$, so it might be convenient to denote this product by $X \times_S Y$.

Definition 9.3.18 *(Homotopic stack of a model stack.) Given a homotopic stack $\mathcal{C}$ over a site $\mathcal{S}$, or, more generally, a stack $\mathcal{C}$ with a local class of weak equivalences, and a cartesian functor $\gamma : \mathcal{C} \to \mathrm{HO}\,\mathcal{C}$ into another stack $\mathrm{HO}\,\mathcal{C}$, we say that* $\mathrm{HO}\,\mathcal{C}$ is the homotopic stack of $\mathcal{C}$ *if for any stack $\mathcal{D}$ over $\mathcal{S}$ the induced functor $\gamma^* : \mathit{Cart}_{\mathcal{S}}(\mathrm{HO}\,\mathcal{C}, \mathcal{D}) \to \mathit{Cart}_{\mathcal{S}}(\mathcal{C}, \mathcal{D})$ is fully faithful and its essential image consists of all cartesian functors $F : \mathcal{C} \to \mathcal{D}$ that transform weak equivalences into isomorphisms. We define the* homotopic prestack $\mathrm{HO}_p\,\mathcal{C}$ *and* homotopic fibered category $\mathrm{HO}_0\,\mathcal{C}$ *by similar requirements, where $\mathcal{D}$ runs through all prestacks (resp. fibered categories) over $\mathcal{S}$.*

Notice that the above requirements determine $\mathrm{HO}\,\mathcal{C}$, $\mathrm{HO}_p\,\mathcal{C}$ and $\mathrm{HO}_0\,\mathcal{C}$ up to an $\mathcal{S}$-equivalence. Moreover, in the definition of $\mathrm{HO}_0\,\mathcal{C}$ and $\mathrm{HO}_p\,\mathcal{C}$ we can require γ^* to induce an *isomorphism* between $\mathit{Cart}_{\mathcal{S}}(\mathrm{HO}_?\,\mathcal{C}, \mathcal{D})$ and the full subcategory of $\mathit{Cart}_{\mathcal{S}}(\mathcal{C}, \mathcal{D})$ described above. Then $\mathrm{HO}_0\,\mathcal{C}$ and $\mathrm{HO}_p\,\mathcal{C}$ become determined up to an $\mathcal{S}$-isomorphism; in this case we speak about the *strict* homotopic fibered category or prestack of $\mathcal{C}$.

9.3.19. (Homotopic category of a model stack.) We denote by $\mathrm{Ho}\,\mathcal{C}$ the category $\varprojlim_{\mathcal{S}} \mathrm{HO}\,\mathcal{C}$ of global (cartesian) sections of $\mathrm{HO}\,\mathcal{C}$. If $\mathcal{S}$ has a final object e, we usually replace $\varprojlim_{\mathcal{S}} \mathrm{HO}\,\mathcal{C}$ by equivalent category $(\mathrm{HO}\,\mathcal{C})(e)$. Categories $\mathrm{Ho}_p\,\mathcal{C}$ and $\mathrm{Ho}_0\,\mathcal{C}$ are defined similarly. Notice, however, that $\mathrm{Ho}\,\mathcal{C}$ and $\mathrm{Ho}_p\,\mathcal{C}$ depend on the whole stack $\mathcal{C}$, not just on the category $\varprojlim_{\mathcal{S}} \mathcal{C}$ or $\mathcal{C}(e)$ of its global sections.

9.3.20. (Existence of homotopic stacks.) We want to show that $\mathrm{HO}_0\,\mathcal{C}$, $\mathrm{HO}_p\,\mathcal{C}$ and $\mathrm{HO}\,\mathcal{C}$ always exist for any fibered category $\mathcal{C} \to \mathcal{S}$ with a local class of weak equivalences, at least if we don't mind enlarging the universe. Indeed, $\mathrm{HO}_0\,\mathcal{C}$ can be constructed fiberwise by putting $(\mathrm{HO}_0\,\mathcal{C})(S)$ equal to the localization of $\mathcal{C}(S)$ with respect to weak equivalences lying in this category; since all pullback functors $\varphi^* : \mathcal{C}(S) \to \mathcal{C}(T)$ transform weak equivalences into weak equivalences, they induce functors $\varphi^* : (\mathrm{HO}_0\,\mathcal{C})(S) \to (\mathrm{HO}_0\,\mathcal{C})(T)$, thus defining a fibered category $\mathrm{HO}_0\,\mathcal{C} \to \mathcal{S}$ and a cartesian functor $\gamma : \mathcal{C} \to \mathrm{HO}_0\,\mathcal{C}$, having required universal property in its strict form.

Next, we can construct $\mathrm{HO}_p\,\mathcal{C}$ as the prestack associated to $\mathrm{HO}_0\,\mathcal{C}$; recall that this can be done by leaving all objects intact but replacing the Hom-sets in fibers by the global sections of sheafifications of corresponding **Hom**-presheaves (cf. **9.1.12**). If we combine this construction of associated prestack together with the previous construction of $\mathrm{HO}_0\,\mathcal{C}$, we see that the prestack thus obtained satisfies the universal property of $\mathrm{HO}_p\,\mathcal{C}$ in its *strict* form.

Finally, we can construct the homotopic stack $\mathrm{HO}\,\mathcal{C}$ as the stack associated to $\mathrm{HO}_p\,\mathcal{C}$; according to the construction given in **9.1.13**, the natural embedding $\mathcal{S}$-functor $\mathrm{HO}_p\,\mathcal{C} \to \mathrm{HO}\,\mathcal{C}$ is fully faithful. Therefore, we can recover $\mathrm{HO}_p\,\mathcal{C}$ from $\mathrm{HO}\,\mathcal{C}$ by taking the essential image of $J : \mathcal{C} \to \mathrm{HO}\,\mathcal{C}$. We can even recover the strict form of $\mathrm{HO}_p\,\mathcal{C}$ by putting $\mathrm{Ob}\,\mathrm{HO}_p\,\mathcal{C} := \mathrm{Ob}\,\mathcal{C}$, $\mathrm{Hom}_{\mathrm{HO}_p\,\mathcal{C}}(X,Y) := \mathrm{Hom}_{\mathrm{HO}\,\mathcal{C}}(J(X),J(Y))$.

9.3.21. (Homotopic stacks over sites and topoi.) Let $\mathcal{C}$ be a model stack over a site $\mathcal{S}$. Let us denote by $\tilde{\mathcal{C}}$ its extension to the corresponding topos $\tilde{\mathcal{S}}$; then $\mathcal{C}$ is equivalent to the "restriction" $\tilde{\mathcal{C}}|_{\mathcal{S}} = \tilde{\mathcal{C}} \times_{\tilde{\mathcal{S}}} \mathcal{S}$ of $\tilde{\mathcal{C}}$ to $\mathcal{S}$ (cf. **9.1.9**). Now we can construct $\mathrm{HO}_0\,\tilde{\mathcal{C}}$, $\mathrm{HO}_p\,\tilde{\mathcal{C}}$ and $\mathrm{HO}\,\tilde{\mathcal{C}}$ over $\tilde{\mathcal{S}}$, and compare their "restrictions" to $\mathcal{S}$ with $\mathrm{HO}_0\,\mathcal{C}$, $\mathrm{HO}_p\,\mathcal{C}$ and $\mathrm{HO}\,\mathcal{C}$. We claim that *in each of these cases we obtain $\mathcal{S}$-equivalent categories.* Indeed, this is clear for HO_0, because it can be computed fiberwise, hence $\mathcal{C} \mapsto \mathrm{HO}_0\,\mathcal{C}$ commutes with base change $\mathcal{S} \to \tilde{\mathcal{S}}$. As for $(\mathrm{HO}\,\tilde{\mathcal{C}})|_{\mathcal{S}}$ and $\mathrm{HO}\,\mathcal{C}$, their equivalence follows from the 2-equivalence of categories of $\mathcal{S}$-stacks and $\tilde{\mathcal{S}}$-stacks (cf. **9.1.9**), once we take into account that for any stack $\mathcal{D}/\mathcal{S}$ the natural "restriction" functor $\mathit{Cart}_{\tilde{\mathcal{S}}}(\tilde{\mathcal{C}},\tilde{\mathcal{D}}) \to \mathit{Cart}_{\mathcal{S}}(\mathcal{C},\mathcal{D})$ induces an equivalence between the subcategories of cartesian functors that transform weak equivalences into isomorphisms, the class of weak equivalences being *local.*

Now only the case of $\mathrm{HO}_p\,\mathcal{C}$ remains. However, $\mathrm{HO}_p\,\tilde{\mathcal{C}}$ can be computed fiberwise as the essential image of $\tilde{\mathcal{C}} \to \mathrm{HO}\,\tilde{\mathcal{C}}$, so this case follows from that of $\mathrm{HO}\,\mathcal{C}$ already considered.

Therefore, we can always work over a topos if we want to, restricting to the original site at the end if necessary. Notice that *a priori* one would rather expect only the stack $\mathrm{HO}\,\mathcal{C}$ to depend essentially only on the topos $\tilde{\mathcal{S}}$ but not the particular site $\mathcal{S}$ chosen to represent this topos, but, surprisingly, this independence extends to $\mathrm{HO}_p\,\mathcal{C}$ and $\mathrm{HO}_0\,\mathcal{C}$. However, $\mathrm{HO}_p\,\tilde{\mathcal{C}}/\tilde{\mathcal{S}}$ cannot be recovered from $\mathrm{HO}_p\,\mathcal{C}/\mathcal{S}$ alone: we need the cartesian functor $\mathcal{C} \to \mathrm{HO}_p\,\mathcal{C}$ as well, and the same applies to $\mathrm{HO}_0\,\tilde{\mathcal{C}}$.

9.3.22. (Derived cartesian functors between model stacks.) Not surprisingly, the definition of derived cartesian functors between model stacks is completely similar to the classical one (cf. **8.1.14**). Namely, given a cartesian functor $F : \mathcal{C} \to \mathcal{D}$ of model categories over the same site $\mathcal{S}$, its *left derived* $\mathbb{L}F : \mathrm{HO}\,\mathcal{C} \to \mathrm{HO}\,\mathcal{D}$ is a cartesian functor $\mathbb{L}F$ between corresponding homotopy categories, together with a natural transformation $\varepsilon : \mathbb{L}F \circ \gamma_{\mathcal{C}} \to \gamma_{\mathcal{D}} \circ F$ (over $\mathcal{S}$), such that for any other cartesian functor $G : \mathrm{HO}\,\mathcal{C} \to \mathrm{HO}\,\mathcal{D}$ and natural transformation $\zeta : G \circ \gamma_{\mathcal{C}} \to \gamma_{\mathcal{D}} \circ F$ there is a unique natural transformation $\theta : G \to \mathbb{L}F$, for which $\zeta = \varepsilon \circ (\theta \star \gamma_{\mathcal{C}})$. The *right derived functor* $\mathbb{R}F : \mathrm{HO}\,\mathcal{C} \to \mathrm{HO}\,\mathcal{D}$, $\eta : \gamma_{\mathcal{D}} \circ F \to \mathbb{R}F \circ \gamma_{\mathcal{C}}$ is defined similarly.

When $\mathbb{L}F$ maps the full subcategory $\mathrm{HO}_p\,\mathcal{C}$ of $\mathrm{HO}\,\mathcal{C}$ into the full subcategory $\mathrm{HO}_p\,\mathcal{D}$ of $\mathrm{HO}\,\mathcal{D}$, the induced functor $\mathrm{HO}_p\,\mathcal{C} \to \mathrm{HO}_p\,\mathcal{D}$ will be denoted by $\mathbb{L}_pF$ or even $\mathbb{L}F$. Furthermore, the induced functors between the corresponding categories of global sections will be also denoted by $\mathbb{L}F$, e.g. $\mathbb{L}F : \mathrm{Ho}\,\mathcal{C} \to \mathrm{Ho}\,\mathcal{D}$.

Of course, the above definitions and constructions are essentially invariant under extending everything from $\mathcal{S}$ to the corresponding topos $\tilde{\mathcal{S}}$, so we always can work over topoi if we need to.

9.4. (Homotopies in a model stack.) Now we want to construct the homotopic stack $\mathrm{HO}\,\mathcal{C}$ in a manner as close as possible to classical Quillen's approach, based on equivalence $\mathrm{Ho}\,\mathcal{C} \cong \pi\mathcal{C}_{cf}$, and prove or at least state corresponding criteria for the existence and adjointness of derived cartesian functors. Among other things, this will demonstrate that $\mathrm{HO}\,\mathcal{C}$ is actually a $\mathcal{U}$-category, so we don't need to enlarge the universe.

Let us fix a model stack $\mathcal{C}$ over a site $\mathcal{S}$.

9.4.1. (Fibrant and cofibrant objects.) Recall that an object $X \in \mathrm{Ob}\,\mathcal{C}(S)$ is cofibrant (resp. fibrant) iff $\varnothing_{\mathcal{C}(S)} \to X$ is a cofibrant morphism (resp. iff $X \to e_{\mathcal{C}(S)}$ is fibrant). Since the property of a morphism to be a (co)fibration is local, and all pullback functors φ^* preserve the initial and final objects by (MS1), the full subcategories $\mathcal{C}_c$, $\mathcal{C}_f$ and $\mathcal{C}_{cf}$ of $\mathcal{C}$ consisting of cofibrant, fibrant, and fibrant-cofibrant objects are strictly full cartesian subcategories and substacks of $\mathcal{C}/\mathcal{S}$.

9.4.2. (Fibrant and cofibrant replacements.) Given an object $X \in \mathrm{Ob}\,\mathcal{C}(S)$, its *cofibrant replacement* is any acyclic fibration (or in some cases just any weak equivalence) $Q \to X$ with a cofibrant Q. Applying (MS5) to $\varnothing \to X$, we see that cofibrants replacements always exist "globally", i.e. in $\mathcal{C}(S)$ itself. When we have functorial factorizations (MS5f), we can choose the cofibrant replacements functorially, i.e. construct a cartesian functor $Q : \mathcal{C} \to \mathcal{C}_c$ and a natural transformation $\xi : Q \to \mathrm{Id}_{\mathcal{C}}$, such that $\xi_X : Q(X) \to X$ is a cofibrant replacement for all $X \in \mathrm{Ob}\,\mathcal{C}$.

Similar remarks apply of course to *fibrant replacements* $X \to R$, defined as acyclic cofibrations (or just weak equivalences) with a fibrant target.

9.4.3. (Cylinder and path objects.) Cylinder and path objects are defined fiberwise in the classical way (cf. **8.1.10**). For example, a cylinder object $A \times I$ of an object $A \in \mathrm{Ob}\,\mathcal{C}(S)$ is a diagram $A \sqcup A \xrightarrow{\langle \partial_0, \partial_1 \rangle} A \times I \xrightarrow{\sigma} A$ with $\sigma \circ \langle \partial_0, \partial_1 \rangle = \nabla_A = \langle \mathrm{id}_A, \mathrm{id}_A \rangle$, such that $\langle \partial_0, \partial_1 \rangle$ is a cofibration and σ is a weak equivalence. Notice that the global existence of cylinder and path objects is a consequence of (MS5).

9.4.4. (Local, semilocal and global homotopies.) Let us fix two parallel morphisms $f, g : A \rightrightarrows X$ in a fiber $\mathcal{C}(S)$. A *left homotopy h from f to g* is defined in the classical way: it is a morphism $h : A \times I \to X$ from any cylinder object $A \times I \in \operatorname{Ob}\mathcal{C}(S)$ of A as above, such that $f = h\partial_0$ and $g = h\partial_1$. When such a homotopy (globally) exists, we say that *f is globally left homotopic to g* and write $f \overset{\ell}{\approx} g$. We denote by $\pi^\ell_\approx(A, X)$ the quotient of $\operatorname{Hom}_{\mathcal{C}(S)}(A, X)$ with respect to (the equivalence relation generated by) $\overset{\ell}{\approx}$. Next, we say that *f is semilocally left homotopic to g*, if the cylinder object $A \times I$ for A can be still chosen globally, i.e. in $\mathcal{C}(S)$, but the left homotopy $h : A \times I \to X$ from f to g exists only locally, i.e. there is a cover $\{S_\alpha \to S\}$ and some left homotopies $h_\alpha : A \times I|_{S_\alpha} \to X|_{S_\alpha}$ between $f|_{S_\alpha}$ and $g|_{S_\alpha}$. In this case we write $f \overset{\ell}{\simeq} g$, and denote by $\pi^\ell_\simeq(A, X)$ the corresponding quotient of $\operatorname{Hom}_{\mathcal{C}(S)}(A, X)$.

Finally, we say that *f is locally left homotopic to g* and write $f \overset{\ell}{\sim} g$ if both the cylinder object and the homotopy exist only locally, i.e. if there is a cover $\{S_\alpha \to S\}$, such that $f|_{S_\alpha} \overset{\ell}{\approx} g|_{S_\alpha}$. Notice that this is exactly the interpretation of the classical definition in the Kripke–Joyal semantics. The corresponding quotient of $\operatorname{Hom}_S(A, X)$ will be denoted by $\pi^\ell(A, X)$.

Now the formula $(T \xrightarrow{\varphi} S) \mapsto \pi^\ell(\varphi^* A, \varphi^* X)$ defines a presheaf on $\mathcal{S}_{/S}$, denoted by $\boldsymbol{\pi}^\ell(A, X)$. We denote its sheafification by $\tilde{\boldsymbol{\pi}}^\ell(A, X)$, and the sections of this sheafification over S by $\tilde{\pi}^\ell(A, X)$. From the Kripke–Joyal point of view this sheaf $\tilde{\boldsymbol{\pi}}^\ell(A, X)$ is the correct generalization of classical $\pi^\ell(A, X)$.

Of course, all the notions introduced above have their right counterparts. Clearly, $\overset{\ell}{\approx} \Rightarrow \overset{\ell}{\simeq} \Rightarrow \overset{\ell}{\sim}$ and $\overset{r}{\approx} \Rightarrow \overset{r}{\simeq} \Rightarrow \overset{r}{\sim}$.

In the following several lemmas A, B, C, X, Y are objects of $\mathcal{C}(S)$. These lemmas are natural counterparts of Quillen's lemmas 4–8 of [Quillen, 1.1], and are shown essentially in the same way (in the case of $\overset{\ell}{\sim}$ and $\overset{r}{\sim}$ *exactly* in the same way, if we interpret the proofs in Kripke–Joyal semantics).

Lemma 9.4.5 *If A is cofibrant, then $\overset{\ell}{\sim}$ and $\overset{\ell}{\simeq}$ are equivalence relations on $\operatorname{Hom}_{\mathcal{C}(S)}(A, B)$, $\boldsymbol{\pi}^\ell(A, B)$ is a separated presheaf, and $\boldsymbol{\pi}^\ell(A, B) \to \tilde{\boldsymbol{\pi}}^\ell(A, B)$ is injective.*

Proof. Same as in [Quillen, 1.1], Lemma 4. Once we know that $\overset{\ell}{\sim}$ is a *local* equivalence relation, we see that it is representable by a subsheaf

$R \subset \mathbf{Hom}(A,B) \times \mathbf{Hom}(A,B)$, hence the presheaf quotient $\boldsymbol{\pi}^\ell(A,B) = \mathbf{Hom}(A,B)/R$ is separated. The last statement follows from $\tilde{\boldsymbol{\pi}}^\ell(A,B) = a\boldsymbol{\pi}^\ell(A,B)$.

Lemma 9.4.6 *Let A be cofibrant, and $f, g \in \mathrm{Hom}_{\mathcal{C}(S)}(A,B)$. Then:*

(i) $f \overset{\ell}{\simeq} g \Rightarrow f \overset{\ell}{\sim} g \Rightarrow f \overset{r}{\simeq} g \Rightarrow f \overset{r}{\sim} g$.

(ii) If $f \overset{r}{\simeq} g$, there (locally) exists a right homotopy $k : A \to B^I$ from f to g with $s : B \to B^I$ an acyclic cofibration in $\mathcal{C}(S)$. If $f \overset{r}{\sim} g$, same conclusion with B^I existing only locally.

(iii) If $u : B \to C$, then $f \overset{r}{\sim} g \Rightarrow uf \overset{r}{\sim} ug$, and $f \overset{r}{\simeq} g \Rightarrow uf \overset{r}{\simeq} ug$.

Proof. Identical to [Quillen, 1.1], Lemma 5. Notice that to prove the only non-trivial implication $\overset{\ell}{\sim} \Rightarrow \overset{r}{\simeq}$ in (i) we use (MS5) to construct a global path object B^I for B.

Lemma 9.4.7 *If A is cofibrant, then composition in $\mathcal{C}(S)$ induces maps of sets $\pi^r(B,C) \times \pi^r(A,B) \to \pi^r(A,C)$, and similarly for $\pi^r_\simeq$ and $\tilde{\pi}^r$, as well as maps of presheaves $\boldsymbol{\pi}^r(B,C) \times \boldsymbol{\pi}^r(A,B) \to \boldsymbol{\pi}^r(A,C)$ and corresponding sheaves $\tilde{\boldsymbol{\pi}}^r(B,C) \times \tilde{\boldsymbol{\pi}}^r(A,B) \to \tilde{\boldsymbol{\pi}}^r(A,C)$. All these maps are compatible with pullbacks φ^*, for all $\varphi : T \to S$.*

Proof. Same as in [Quillen, 1.1], Lemma 6, for the first two statements, using (iii) of the previous lemma **9.4.6**. The remaining statements follow immediately.

Lemma 9.4.8 *Let A be cofibrant and $p : X \to Y$ be an acyclic fibration. Then the maps of sets $p_* : \pi^\ell(A,X) \to \pi^\ell(A,Y)$, $\pi^\ell_\simeq(A,X) \to \pi^\ell_\simeq(A,X)$ and of presheaves $\boldsymbol{\pi}^\ell(A,X) \to \boldsymbol{\pi}^\ell(A,Y)$ are injective, while the maps of sheaves $\tilde{\boldsymbol{\pi}}^\ell(A,X) \to \tilde{\boldsymbol{\pi}}^\ell(A,Y)$ and of sets $\tilde{\pi}^\ell(A,X) \to \tilde{\pi}^\ell(A,Y)$ are bijective.*

Proof. The proof of injectivity is that of [Quillen, 1.1], Lemma 7, while the proof of surjectivity given in *loc.cit.* actually shows only *local* surjectivity, i.e. that $\boldsymbol{\pi}^\ell(A,p)$ becomes surjective after sheafification. The remaining statements are now immediate.

9.4.9. Notice that in the above notations (after appropriate identifications) we obtain canonical inclusions $\pi^\ell(A,X) \subset \pi^\ell(A,Y) \subset \tilde{\pi}^\ell(A,X) = \tilde{\pi}^\ell(A,Y)$. Therefore, if we fix A and X, but let $p : X \to Y$ run over all acyclic fibrations, we get an embedding of the union of all $\pi^\ell(A,Y)$ into $\tilde{\pi}^\ell(A,X)$. It is an interesting question whether this union equals $\tilde{\pi}^\ell(A, X)$. We'll see later that when this is true, we can compute $\mathrm{HO}_p\mathcal{C}$ by a right fraction calculus in $\pi\mathcal{C}_c$, similar to the constriction of the derived category of an abelian category.

Lemma 9.4.10 *(i) Let $F : \mathcal{C} \to \mathcal{B}$ be a cartesian functor from a model stack $\mathcal{C}$ into a prestack (or, more generally, a fibered category with separated* **Hom***-presheaves) $\mathcal{B}$, transforming weak equivalences into isomorphisms. If $f \overset{\ell}{\sim} g$ or $f \overset{r}{\sim} g$, then $F(f) = F(g)$.*

(ii) Let a cartesian $F : \mathcal{C}_c \to \mathcal{B}$ carry weak equivalences into isomorphisms, with $\mathcal{C}$ and $\mathcal{B}$ as above. Then $f \overset{r}{\sim} g$ implies $F(f) = F(g)$.

(iii) Let $F : \mathcal{C}_{cf} \to \mathcal{B}$ be a cartesian functor as above, then $f \sim g$ implies $F(f) = F(g)$.

Proof. Same as in [Quillen, 1.1], Lemma 8. Indeed, to show (i) assume first that $f \overset{r}{\approx} g$, i.e. we have a right homotopy $k : A \to B^I$ and a path object $B \xrightarrow{s} B^I \xrightarrow{(d_0,d_1)} B \times B$ for B, such that $d_0k = f$, $d_1k = g$. Then $F(d_0)F(s) = \mathrm{id} = F(d_1)F(s)$, hence $F(d_0) = F(d_1)$, $F(s)$ being an isomorphism, hence $F(f) = F(d_0)F(k) = F(d_1)F(k) = F(g)$. Now if we know just $f \overset{r}{\sim} g$, then $f|_{S_\alpha} \overset{r}{\approx} g|_{S_\alpha}$ on some cover $\{S_\alpha \to S\}$, so we obtain $F(f)|_{S_\alpha} = F(g)|_{S_\alpha}$, hence $F(f) = F(g)$, the presheaf $\mathbf{Hom}_{\mathcal{B}_{/S}}(F(A), F(B))$ being separated. The statement for $f \overset{\ell}{\sim} g$ is shown dually.

Now to show (ii) and (iii) for some $f, g : A \rightrightarrows B$ we reason similarly; the only problem is that we need B^I to lie in $\mathcal{C}_c$ (resp. $\mathcal{C}_{cf}$). This is achieved by invoking **9.4.6**, (ii): we can (locally) choose the path object in such a way that $s : B \to B^I$ be an acyclic cofibration; then B^I will lie in $\mathcal{C}_c$ whenever B is itself cofibrant. On the other hand, $B^I \to B \times B$ is always a fibration, hence when B is fibrant, so is B^I.

9.4.11. We denote by $\pi\mathcal{C}_c$ the fibered $\mathcal{S}$-category with the same objects as $\mathcal{C}_c$, but with morphisms given by π^r, and define $\pi\mathcal{C}_f$ dually. According to **9.4.7**, the composition in $\pi\mathcal{C}_c$ is well-defined and compatible with pullbacks, so we've indeed described a fibered category. We define $\tilde{\pi}\mathcal{C}_c$ and $\tilde{\pi}\mathcal{C}_f$ in a similar manner, using $\tilde{\pi}$ instead of π. Clearly, $\tilde{\pi}\mathcal{C}_c$ and $\tilde{\pi}\mathcal{C}_f$ are the prestacks associated to $\pi\mathcal{C}_c$ and $\pi\mathcal{C}_f$ (cf. **9.1.12**).

We construct $\pi\mathcal{C}_{cf}$ and $\tilde{\pi}\mathcal{C}_{cf}$ similarly, taking into account that for a cofibrant A and a fibrant B the equivalence relations $\overset{\ell}{\sim}$, $\overset{\ell}{\simeq}$, $\overset{r}{\sim}$, $\overset{r}{\simeq}$ coincide on $\mathrm{Hom}(A, B)$ by **9.4.6**, so we can write simply $\pi(A, B)$ and $\tilde{\pi}(A, B)$. Notice that by **9.4.8** and its dual both acyclic cofibrations and acyclic fibrations become isomorphisms in $\tilde{\pi}\mathcal{C}_{cf}$; since any weak equivalence in $\mathcal{C}_{cf}$ can be decomposed by (MS5) and (MS3) into an acyclic cofibration followed by an acyclic fibration, with the intermediate object lying automatically in $\mathcal{C}_{cf}$, we conclude that $\mathcal{C}_{cf} \to \tilde{\pi}\mathcal{C}_{cf}$ *transforms weak equivalences into isomorphisms.* Therefore, we obtain a cartesian functor $\bar{\gamma}^+_{cf} : \mathrm{HO}_p\,\mathcal{C}_{cf} \to \tilde{\pi}\mathcal{C}_{cf}$, the target being a prestack.

Finally, **9.4.10** implies existence of cartesian functors $\pi\mathcal{C}_c \to \tilde{\pi}\mathcal{C}_c \xrightarrow{\bar{\gamma}_c} \mathrm{HO}_p\,\mathcal{C}_c \to \mathrm{HO}\,\mathcal{C}_c$, similar functors $\pi\mathcal{C}_f \to \cdots \to \mathrm{HO}\,\mathcal{C}_f$, as well as cartesian functors $\pi\mathcal{C}_{cf} \to \tilde{\pi}\mathcal{C}_{cf} \xrightarrow{\bar{\gamma}} \mathrm{HO}_p\,\mathcal{C} \to \mathrm{HO}\,\mathcal{C}$ and $\bar{\gamma}_{cf} : \tilde{\pi}\mathcal{C}_{cf} \to \mathrm{HO}_p\,\mathcal{C}_{cf}$. Our next goal is to show that $\bar{\gamma} : \tilde{\pi}\mathcal{C}_{cf} \to \mathrm{HO}_p\,\mathcal{C}$ is an $\mathcal{S}$-equivalence.

Lemma 9.4.12 *Cartesian functor $\mathcal{C}_c \to \tilde{\pi}\mathcal{C}_c$ transforms acyclic fibrations into isomorphisms. Dually, $\mathcal{C}_f \to \tilde{\pi}\mathcal{C}_f$ transforms acyclic cofibrations into isomorphisms.*

Proof. Let $p : X \to Y$ be an acyclic fibration of cofibrant objects. Using (MS4) we obtain *local* existence of a section $\sigma : Y \to X$, $p\sigma = \mathrm{id}_Y$. Now if σ' is another (local) section of p (over S or over some $S_\alpha \to S$), then $p\sigma = p\sigma'$ implies $\sigma \overset{\ell}{\sim} \sigma'$ by **9.4.8**, hence $\sigma \overset{r}{\sim} \sigma'$ by **9.4.6**,(i), so the image $\bar{\sigma}$ of σ in $\tilde{\pi}^r(Y, X)$ exists locally and is unique, so it exists globally as well. In other words, $\bar{\sigma} : Y \to X$ is a section of $\bar{p}$ in $\pi\mathcal{C}_c$. On the other hand, consider (locally) σp and id_X; they have equal images under p_*, hence by the same lemmas we obtain $\sigma p \overset{\ell}{\sim} \mathrm{id}_X$ and $\sigma p \overset{r}{\sim} \mathrm{id}_X$, i.e. $\bar{\sigma}\bar{p} = \mathrm{id}_X$ in $\tilde{\pi}\mathcal{C}_c$ (locally, hence also globally, $\tilde{\pi}\mathcal{C}_c$ being a prestack), so $\bar{\sigma}$ is a two-sided inverse to $\bar{p}$.

Theorem 9.4.13 (cf. [Quillen, 1.1], Theorem 1) *The cartesian functor $\bar{\gamma} : \tilde{\pi}\mathcal{C}_{cf} \to \mathrm{HO}_p\,\mathcal{C}$ is an $\mathcal{S}$-equivalence, hence $\mathrm{HO}_p\,\mathcal{C}$ (resp. $\mathrm{HO}\,\mathcal{C}$) is $\mathcal{S}$-equivalent to the prestack (resp. stack) associated to $\mathcal{S}$-fibered category $\pi\mathcal{C}_{cf}$.*

In particular, $\mathrm{HO}_p\,\mathcal{C}$ and $\mathrm{HO}\,\mathcal{C}$ are $\mathcal{U}$-categories.

Proof. (a) First of all, notice that $\bar{\gamma}_{cf}$ and $\bar{\gamma}^+_{cf}$ are adjoint $\mathcal{S}$-equivalences between prestacks $\tilde{\pi}\mathcal{C}_{cf}$ and $\mathrm{HO}_p\,\mathcal{C}_{cf}$, and even $\mathcal{S}$-isomorphisms, inverse to each other, if we choose the *strict* version of $\mathrm{HO}_p\,\mathcal{C}_{cf}$. Indeed, all we have to check is that $\mathcal{C}_{cf} \to \tilde{\pi}\mathcal{C}_{cf}$ transforms weak equivalences into isomorphisms, something that we know already, and that this cartesian functor is universal among all cartesian functors $F : \mathcal{C}_{cf} \to \mathcal{B}$ into $\mathcal{S}$-prestacks $\mathcal{B}$ that transform weak equivalences into isomorphisms, something we know from **9.4.10**,(iii), $\tilde{\pi}\mathcal{C}_{cf}$ being the prestack associated to $\pi\mathcal{C}_{cf}$.

(b) Now let us construct a cartesian functor $\bar{Q} : \mathcal{C} \to \tilde{\pi}\mathcal{C}_c$. Let's choose for this arbitrary *strict* cofibrant replacements $Q(X) \to X$ for all $X \in \mathrm{Ob}\,\mathcal{C}$, strictness understood here as the requirement of $Q(X) \to X$ to be an acyclic fibration, not just a weak equivalence, and put $\bar{Q}(X)$ equal to the image of

$Q(X)$ in $\tilde{\pi}\mathcal{C}_c$. Notice that $\bar{Q}(X)$ *is independent (up to a canonical isomorphism) on the choice of* $Q(X) \to X$. Indeed, whenever we have two such strict cofibrant replacements $p : Q \to X$, $p' : Q' \to X$, we can find another strict replacement $p'' : Q'' \to X$ and acyclic fibrations $\sigma : Q'' \to Q$ and $\sigma' : Q'' \to Q'$, such that $p\sigma = p'' = p'\sigma'$. To achieve this we simply take a strict cofibrant replacement $Q'' \to Q \times_X Q'$, using the fact that acyclic fibrations are stable under base change and composition. Now acyclic fibrations σ and σ' become isomorphisms in $\tilde{\pi}\mathcal{C}_c$ by **9.4.12**, hence an isomorphism $\bar{\sigma}' \circ \bar{\sigma}^{-1} : \bar{Q} \xleftarrow{\sim} \bar{Q}'' \xrightarrow{\sim} \bar{Q}'$. The independence of this isomorphism on the choice of $Q'' \to Q \times_X Q'$ is checked similarly.

(c) So far we have constructed $\bar{Q}$ only on objects. Notice, however, that the independence on the choice of $Q(X) \to X$ implies the compatibility with pullback functors φ^*, since $Q(\varphi^* X) \to \varphi^* X$ and $\varphi^* Q(X) \to \varphi^* X$ are two strict cofibrant replacements of $\varphi^* X$. Thus it remains to define $\bar{Q}$ on morphisms in fibers of $\mathcal{C}$. If $f : X \to Y$ is a morphism in $\mathcal{C}(S)$, and $p_X : Q(X) \to X$, $p_Y : Q(Y) \to Y$ are two strict cofibrant replacements, we can apply (MS4) to lift *locally* $fp_X : Q(X) \to Y$ to a morphism $Q(f) : Q(X) \to Q(Y)$, which is unique up to a left homotopy by **9.4.8**, hence *a fortiori* unique up to a right homotopy by **9.4.6**,(i), hence we obtain locally a well-defined morphism $\bar{Q}(f) : \bar{Q}(X) \to \bar{Q}(Y)$ in $\tilde{\pi}\mathcal{C}_c$, hence $\bar{Q}(f)$ is defined globally as well, $\tilde{\pi}\mathcal{C}_c$ being a prestack. For any $g : Y \to Z$ we get $Q(g)Q(f) \overset{\ell}{\sim} Q(gf)$ and $Q(\mathrm{id}_X) \overset{\ell}{\sim} \mathrm{id}_{Q(X)}$ by **9.4.8**, hence $\bar{Q}(g)\bar{Q}(f) = \bar{Q}(gf)$ and $\bar{Q}(\mathrm{id}_X) = \mathrm{id}_{\bar{Q}(X)}$ by **9.4.6**,(i) again. This finishes the construction of cartesian functor $\bar{Q} : \mathcal{C} \to \tilde{\pi}\mathcal{C}_c$.

(d) We construct a cartesian functor $\bar{R} : \mathcal{C} \to \tilde{\pi}\mathcal{C}_f$ similarly, by putting $\bar{R}(X)$ to be the image of any strict fibrant replacement $i_X : X \to R(X)$. If X is cofibrant, so will be $R(X)$, i_X being an acyclic cofibration. Hence $\bar{R}$ induces a functor $\bar{R}_c : \mathcal{C}_c \to \tilde{\pi}\mathcal{C}_{cf}$. Notice that $f \overset{r}{\sim} g : X \to Y$ in $\mathcal{C}_c$ implies $i_Y f \overset{r}{\sim} i_Y g$ by **9.4.6**,(iii), hence (locally) $R(f)i_X \overset{r}{\sim} R(g)i_X$, i.e. the images of these two elements coincide in $\tilde{\pi}^r(X, R(Y))$; the dual of **9.4.8** now implies $\bar{R}(f) = \bar{R}(g)$ (locally, hence also globally) in $\tilde{\pi}^r(R(X), R(Y)) = \tilde{\pi}(R(X), R(Y))$, i.e. $\bar{R}_c(f) = \bar{R}_c(g)$. This means that $\bar{R}_c$ factorizes through $\pi\mathcal{C}_c$, hence also through $\tilde{\pi}\mathcal{C}_c$, the target category being a prestack. The induced functor will be denoted $\tilde{R}_c : \tilde{\pi}\mathcal{C}_c \to \tilde{\pi}\mathcal{C}_{cf}$.

(e) Now consider the composite cartesian functor $\gamma' : \mathcal{C} \xrightarrow{\bar{Q}} \tilde{\pi}\mathcal{C}_c \xrightarrow{\tilde{R}_c} \tilde{\pi}\mathcal{C}_{cf}$. Its target category is a prestack, and it is immediate from (MS3) that the image under γ' of a weak equivalence $f : X \to Y$ is locally representable by a weak equivalence $RQ(f) : RQ(X) \to RQ(Y)$, hence $\gamma'(f)$ is locally an isomorphism, hence this is true globally. Therefore, γ' induces a cartesian functor $\tilde{\gamma} : \mathrm{HO}_p\,\mathcal{C} \to \tilde{\pi}\mathcal{C}_{cf}$.

(f) Consider cartesian functors between prestacks $\mathrm{HO}_p\,\mathcal{C}_{cf} \xrightarrow{I} \mathrm{HO}_p\,\mathcal{C} \xrightarrow{\tilde{\gamma}} \tilde{\pi}\mathcal{C}_{cf} \xrightarrow{\bar{\gamma}_{cf}} \mathrm{HO}_p\,\mathcal{C}_{cf}$. Clearly, the composite functor is $\mathcal{S}$-isomorphic to the identity, since we can choose $Q(X) = X = R(X)$ for any $X \in \mathrm{Ob}\,\mathcal{C}_{cf}$ in the construction of $\gamma' = \tilde{R}_c\bar{Q}$. The last arrow $\bar{\gamma}_{cf}$ is an $\mathcal{S}$-equivalence by (a), hence $\mathrm{HO}_p\,\mathcal{C}_{cf} \xrightarrow{I} \mathrm{HO}_p\,\mathcal{C} \xrightarrow{\tilde{\gamma}} \tilde{\pi}\mathcal{C}_{cf}$ is an $\mathcal{S}$-equivalence as well; in particular, $\tilde{\gamma}$ is surjective on morphisms. On the other hand, it is faithful, the composite functor $\mathrm{HO}_p\,\mathcal{C} \xrightarrow{\tilde{\gamma}} \tilde{\pi}\mathcal{C}_{cf} \cong \mathrm{HO}_p\,\mathcal{C}_{cf} \xrightarrow{I} \mathrm{HO}_p\,\mathcal{C}$ being isomorphic to the identity functor as well: indeed, it transforms γX into $\gamma RQ(X)$, which is canonically isomorphic to γX via $\gamma(i_{Q(X)})\gamma(p_X)^{-1}$. This proves that I is essentially surjective, and that $\tilde{\gamma}$ is fully faithful; since the same is true for $\bar{\gamma}_{cf}$ and $\bar{\gamma}_{cf}\tilde{\gamma}I \cong \mathrm{Id}$, we see that I is fully faithful as well, hence an $\mathcal{S}$-equivalence. From this and (a) we deduce immediately that $\tilde{\gamma} : \mathrm{HO}_p\,\mathcal{C} \to \tilde{\pi}\mathcal{C}_{cf}$ is an $\mathcal{S}$-equivalence as well, with $\bar{\gamma} = I\bar{\gamma}_{cf}$ its quasi-inverse, q.e.d.

Corollary 9.4.14 (cf. cor. 1 of th. 1, [Quillen, 1.1]) *If A is cofibrant and B fibrant in $\mathcal{C}(S)$, then $[A, B]_S := \mathrm{Hom}_{(\mathrm{HO}\,\mathcal{C})(S)}(\gamma A, \gamma B)$ is canonically isomorphic to $\tilde{\pi}(A, B)$.*

9.4.15. Notice that if we would have (SM5l) instead of (SM5), the whole construction of $\bar{R}$ and $\bar{Q}$ would work only on the level of associated stacks, fibrant and cofibrant replacements being defined only locally.

It is also interesting to note that all the above constructions and statements are still valid when $\mathcal{C}$ is a model *pre*stack over $\mathcal{S}$.

Proposition 9.4.16 *(a) A fibration $p : X \to Y$ in $\mathcal{C}_{cf}(S)$ is acyclic iff $\gamma(p)$ is an isomorphism in $\mathrm{HO}_p\,\mathcal{C}$ iff locally p is a dual of a strong deformation retract, i.e. iff after pulling back to some cover $\{S_\alpha \to S\}$ one can find a section $t : Y \to X$, $pt = \mathrm{id}_X$, and a left homotopy $h : X \times I \to X$ from tp to id_Y with $ph = p\sigma$.*

(b) A morphism $f : X \to Y$ in $\mathcal{C}(S)$ is a weak equivalence iff $\gamma(f)$ is an isomorphism in $\mathrm{HO}\,\mathcal{C}$ (or in its full subcategory $\mathrm{HO}_p\,\mathcal{C}$).

Proof. Same as that of Lemma 1 and Proposition 1 of [Quillen, 1.5].

9.4.17. (Loop and suspension functors.) When $\mathcal{C}$ is a *pointed* model stack over $\mathcal{S}$, i.e. each fiber $\mathcal{C}(S)$ admits a zero object, we can define loop and suspension objects in the usual way. For example, the suspension ΣA of a cofibrant object $A \in \mathrm{Ob}\,\mathcal{C}_c(S)$ can be defined as the cofiber of $A \sqcup A \to A \times I$, for any cylinder object $A \times I$ for A. One checks in the usual manner that $\gamma\Sigma A$ depends (up to a canonical isomorphism in $(\mathrm{HO}_p\,\mathcal{C})(S)$) only on the weak equivalence class of A, so we get well-defined cartesian functors $\Sigma = \mathbb{L}\Sigma$ and $\Omega = \mathbb{R}\Omega : \mathrm{HO}_p\,\mathcal{C} \to \mathrm{HO}_p\,\mathcal{C}$, and corresponding functors $\mathrm{HO}\,\mathcal{C} \to \mathrm{HO}\,\mathcal{C}$, denoted in the same way. When we have (SM5f), we can construct cartesian functors $\Omega : \mathcal{C}_f \to \mathcal{C}_f$ and $\Sigma : \mathcal{C}_c \to \mathcal{C}_c$ as well.

9.4.18. (Fibration and cofibration sequences.) One can define fibration and cofibration sequences in $\mathrm{HO}_p\,\mathcal{C}$ for any pointed model category $\mathcal{C}$ essentially in the same way as in [Quillen, 1.3], and extend these definitions to $\mathrm{HO}\,\mathcal{C}$ "by descent" (cf. **9.1.13**). Then one can construct associated long exact sequences of sheaves of abelian groups/groups/sets on $\mathcal{S}_{/S}$, again by repeating the reasoning of *loc.cit.* for the homotopic prestack $\mathrm{HO}_p\,\mathcal{C}$, and extending to the associated stack $\mathrm{HO}\,\mathcal{C}$ "by descent".

9.4.19. (Criteria for existence of derived functors.) Of course, we have the usual criteria for existence of derived functors. Suppose for example that $F : \mathcal{C} \to \mathcal{D}$ is a cartesian functor between model stacks over $\mathcal{S}$, transforming acyclic cofibrations between fibrant objects (in fibers of $\mathcal{C}$) into weak equivalences in $\mathcal{D}$. Then its left derived functors $\mathbb{L}F = \mathbb{L}_pF : \mathrm{HO}_p\,\mathcal{C} \to \mathrm{HO}_p\,\mathcal{D}$ as well as $\mathbb{L}F : \mathrm{HO}\,\mathcal{C} \to \mathrm{HO}\,\mathcal{D}$ exist, and the natural morphism $\eta_X : \mathbb{L}F(\gamma_{\mathcal{C}}X) \to \gamma_{\mathcal{D}}F(X)$ is an isomorphism for any cofibrant X, i.e. $\mathbb{L}F(X)$ can be computed as $F(P)$ for any cofibrant replacement $P \to X$. The proof is essentially that of **8.1.17** and **8.6.3** (cf. also prop. 1 of [Quillen, 1.4]), at least for $\mathbb{L}_pF : \mathrm{HO}_p\,\mathcal{C} \to \mathrm{HO}_p\,\mathcal{D}$. Then we construct $\mathbb{L}F : \mathrm{HO}\,\mathcal{C} \to \mathrm{HO}\,\mathcal{D}$ by extending $\mathbb{L}_pF$ to associated stacks. In the sequel we'll refer to these criteria as "model stack" or "local" versions of **8.1.17** and **8.6.3**. Notice that the possibility to construct $\mathbb{L}_pF$ on the level of homotopic prestacks is due to the existence of global factorizations (MS5): its weaker version (MS5l) would enable us to construct $\mathbb{L}F$, but not $\mathbb{L}_pF$.

9.4.20. (Adjoint cartesian Quillen functors.) We have a notion of *cartesian Quillen functors* or *pairs*: these are $\mathcal{S}$-adjoint cartesian functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ between model stacks over $\mathcal{S}$, such that F transforms weak equivalences (or just acyclic cofibrations) in fibers of $\mathcal{C}_c$ into weak equivalences in $\mathcal{D}$, and dually G preserves weak equivalences in $\mathcal{D}_f$. In this case derived functors $\mathbb{L}_pF : \mathrm{HO}_p\,\mathcal{C} \to \mathrm{HO}_p\,\mathcal{D}$ and $\mathbb{R}_pG : \mathrm{HO}_p\,\mathcal{D} \to \mathrm{HO}_p\,\mathcal{C}$ exist, can be computed by means of cofibrant (resp. fibrant) replacements, and are $\mathcal{S}$-adjoint. Same applies to their stack extensions $\mathbb{L}F : \mathrm{HO}\,\mathcal{C} \to \mathrm{HO}\,\mathcal{D}$ and $\mathbb{R}G : \mathrm{HO}\,\mathcal{D} \to \mathrm{HO}\,\mathcal{C}$.

9.4.21. (Fibered products of model stacks.) Whenever $\mathcal{C}_1$ and $\mathcal{C}_2$ are fibered categories over $\mathcal{S}$, their fibered product $\mathcal{C}_1 \times_{\mathcal{S}} \mathcal{C}_2$ is another one; this is clearly the *strict* product of $\mathcal{C}_1$ and $\mathcal{C}_2$ in the 2-category of fibered $\mathcal{S}$-categories, i.e. for any fibered $\mathcal{S}$-category $\mathcal{D}$ we have an *isomorphism* of categories $\mathit{Cart}_{\mathcal{S}}(\mathcal{D}, \mathcal{C}_1 \times_{\mathcal{S}} \mathcal{C}_2) \cong \mathit{Cart}_{\mathcal{S}}(\mathcal{D}, \mathcal{C}_1) \times \mathit{Cart}_{\mathcal{S}}(\mathcal{D}, \mathcal{C}_2)$. All this means that this fibered product is a natural replacement of product of categories; in fact, it is the "fiberwise product of two families of categories parametrized by $\mathcal{S}$" since $(\mathcal{C}_1 \times_{\mathcal{S}} \mathcal{C}_2)(S) = \mathcal{C}_1(S) \times \mathcal{C}_2(S)$, and the corresponding pullback functors φ^* can be constructed as $\varphi^*_{\mathcal{C}_1} \times \varphi^*_{\mathcal{C}_2}$. Therefore, a natural replacement for bifunctors is given by cartesian functors like $F : \mathcal{C}_1 \times_{\mathcal{S}} \mathcal{C}_2 \to \mathcal{D}$; if we need, say, a bifunctor contravariant in the first argument and covariant in the second, we use the fiberwise opposite: $G : \mathcal{C}_1^{fop} \times_{\mathcal{S}} \mathcal{C}_2 \to \mathcal{D}$.

Now similarly to what we had in **8.7.2**, $\mathcal{C}_1 \times_{\mathcal{S}} \mathcal{C}_2$ *admits a natural model stack structure for any two model stacks* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *over a site* $\mathcal{S}$. Of course, we declare a morphism $f = (f_1, f_2) \in \operatorname{Ar}(\mathcal{C}_1 \times_{\mathcal{S}} \mathcal{C}_2)(S) = \operatorname{Ar}(\mathcal{C}_1(S) \times \mathcal{C}_2(S))$ to be a (co)fibration or a weak equivalence iff both f_1 and f_2 are. This enables us to derive functors $F : \mathcal{C}_1 \times_{\mathcal{S}} \mathcal{C}_2 \to \mathcal{D}$ and $G : \mathcal{C}_1^{fop} \times_{\mathcal{S}} \mathcal{C}_2 \to \mathcal{D}$ in the obvious manner. We obtain criteria of existence of derived "bifunctors" similar to **8.7.3** as well, where the corresponding requirements (that of $F(P, -)$ to transform acyclic cofibrations between cofibrant objects of $\mathcal{C}_2$ into weak equivalences in $\mathcal{D}$ for any cofibrant object P of $\mathcal{C}_1$, and the symmetric one) have to be understood fiberwise (i.e. $F(P, f)$ has to be a weak equivalence in $\mathcal{D}(S)$ for any $P \in \operatorname{Ob} \mathcal{C}_{1,c}(S)$ and any acyclic cofibration $f : X \to Y$ in $\mathcal{C}_{2,c}$, together with the symmetric condition). In this case $\mathbb{L}_p F(\gamma X_1, \gamma X_2) \cong \gamma F(P_1, P_2)$, where $P_i \to X_i$ are arbitrary cofibrant replacements in $\mathcal{C}_i(S)$.

9.4.22. (Cartesian $\otimes$-structures.) One can define the notion of a $\otimes$-structure and of constraints for such a structure for objects of an arbitrary strictly associative 2-category with strict products, so as to recover the usual definitions in the case of the 2-category of all categories. On the other hand, applying these definitions to the 2-category of fibered categories and cartesian functors over a fixed category $\mathcal{S}$, we obtain the notion of a *cartesian* $\otimes$*-structure* on a fibered category $\mathcal{C}/\mathcal{S}$. By definition, this is just a cartesian functor $\otimes : \mathcal{C} \times_{\mathcal{S}} \mathcal{C} \to \mathcal{C}$; and, for example, an associativity constraint is an $\mathcal{S}$-isomorphism of appropriate cartesian functors $\mathcal{C} \times_{\mathcal{S}} \mathcal{C} \times_{\mathcal{S}} \mathcal{C} \to \mathcal{C}$. A *fibered* $\otimes$*-category over* $\mathcal{S}$ is by definition a fibered category $\mathcal{C} \to \mathcal{S}$ with a cartesian $\otimes$-structure; we speak about AU, ACU etc. fibered $\otimes$-categories depending on the presence and compatibility of appropriate constraints.

If we think of $\mathcal{C} \to \mathcal{S}$ as "a family of categories parametrized by $\mathcal{S}$", then, say, a cartesian ACU $\otimes$-structure on $\mathcal{C}$ is essentially the same thing as an ACU $\otimes$-structure on each fiber $\mathcal{C}(S)$, together with a $\otimes$-functor structure on each pullback functor $\varphi^* : \mathcal{C}(S) \to \mathcal{C}(T)$ (i.e. some functorial isomorphisms $\varphi^*(X \otimes Y) \cong \varphi^* X \otimes \varphi^* Y$, compatible with constraints), the canonical isomorphisms $(\varphi\psi)^* \cong \psi^* \varphi^*$ being required to be isomorphisms of $\otimes$-functors.

Sometimes we denote by $X \otimes_S Y$ the image of (X, Y) under $\otimes$, when both X and Y lie over $S \in \operatorname{Ob} \mathcal{S}$.

9.4.23. (Compatible cartesian $\otimes$-structures on a model stack.) Now let

$\otimes : \mathcal{C} \times_{\mathcal{S}} \mathcal{C} \to \mathcal{C}$ be a cartesian $\otimes$-structure on a model stack $\mathcal{C}$ over a site $\mathcal{S}$. We say that this cartesian $\otimes$-structure is *compatible* with the model structure of $\mathcal{C}$ if the condition (TM) of **8.7.4** holds in each fiber $\mathcal{C}(S)$ of $\mathcal{C}$.

If a compatible cartesian $\otimes$-structure (fiberwise) preserves initial objects in each argument, it admits a left derived $\underline{\underline{\otimes}} : \mathrm{HO}\,\mathcal{C} \times_{\mathcal{S}} \mathrm{HO}\,\mathcal{C} \to \mathrm{HO}\,\mathcal{C}$ or $\underline{\underline{\otimes}} : \mathrm{HO}_p\,\mathcal{C} \times_{\mathcal{S}} \mathrm{HO}_p\,\mathcal{C} \to \mathrm{HO}_p\,\mathcal{C}$, that can be computed by means of cofibrant replacements in each argument, by the same reasoning as in **8.7.6**. We denote also by $\underline{\underline{\otimes}}$ the induced functors between corresponding categories of global (cartesian) sections $\mathrm{Ho}\,\mathcal{C}$ and $\mathrm{Ho}_p\,\mathcal{C}$, thus obtaining $\otimes$-structures on each of these categories, with the same constraints as the original one.

9.4.24. (Compatible external cartesian $\otimes$-actions.) Of course, we can define a *right external cartesian $\otimes$-action* of a fibered AU $\otimes$-category $\mathcal{C}$ on another fibered category $\mathcal{D}$ over the same base category $\mathcal{S}$ as a cartesian functor $\oslash : \mathcal{D} \times_{\mathcal{S}} \mathcal{C} \to \mathcal{D}$, together with external associativity and unity constraints compatible with those of $\mathcal{C}$. A left external cartesian $\otimes$-action $\obslash : \mathcal{C} \times_{\mathcal{S}} \mathcal{D} \to \mathcal{D}$ is defined similarly. Again, this is essentially equivalent to giving a family of external $\otimes$-actions $\oslash_S : \mathcal{D}(S) \times \mathcal{C}(S) \to \mathcal{C}(S)$, compatible with the pullback functors.

When in addition $\mathcal{C}$ and $\mathcal{D}$ are model stacks over a site $\mathcal{S}$, we can define a *compatible* (say, right) cartesian $\otimes$-action $\oslash : \mathcal{D} \times_{\mathcal{S}} \mathcal{C} \to \mathcal{D}$, by requiring (TMe) of **8.7.4** to hold fiberwise. Of course, if $\oslash$ (fiberwise) preserves initial objects in each argument, it can be derived with the aid of cofibrant replacements in each position, thus defining an external cartesian $\otimes$-action of $\mathrm{HO}_p\,\mathcal{C}$ on $\mathrm{HO}_p\,\mathcal{D}$ and of $\mathrm{HO}\,\mathcal{C}$ on $\mathrm{HO}\,\mathcal{D}$, as well as external $\otimes$-actions between corresponding global section categories.

9.4.25. (Equivalent conditions in terms of inner Homs.) Compatibility conditions (TM) and (TMe) have equivalent formulations similar to (TMh) and (SM7a) in terms of appropriate inner Homs, whenever these exist and are cartesian. More precisely, let us consider the case of a cartesian external $\otimes$-action $\oslash : \mathcal{D} \times_{\mathcal{S}} \mathcal{C} \to \mathcal{D}$, with $\mathcal{C}$ and $\mathcal{D}$ model stacks over a site $\mathcal{S}$. We say that a *cartesian* functor $\mathbf{Hom}_{\mathcal{C}} : \mathcal{D}^{fop} \times_{\mathcal{S}} \mathcal{D} \to \mathcal{C}$ is an inner Hom for $\oslash$ if we have functorial isomorphisms

$$\mathrm{Hom}_{\mathcal{D}(S)}(X \oslash_S K, Y) \cong \mathrm{Hom}_{\mathcal{C}(S)}(K, \mathbf{Hom}_{\mathcal{C}}(X, Y)) \\ \text{for all } S \in \mathrm{Ob}\,\mathcal{S},\ X,\ Y \in \mathrm{Ob}\,\mathcal{D}(S),\ K \in \mathrm{Ob}\,\mathcal{C}(S) \quad (9.4.25.1)$$

compatible with all pullback functors, i.e. an $\mathcal{S}$-isomorphism of cartesian functors $\mathcal{D}^{fop} \times_{\mathcal{S}} \mathcal{C}^{fop} \times_{\mathcal{S}} \mathcal{D} \to \mathit{Sets} \times \mathcal{S}$. These functorial isomorphisms can be rewritten in terms of corresponding local Hom-sheaves, thus yielding an

$\mathcal{S}$-isomorphism of cartesian functors $\mathcal{D}^{fop} \times_{\mathcal{S}} \mathcal{C}^{fop} \times_{\mathcal{S}} \mathcal{D} \to \mathbf{SETS}_{\mathcal{S}}$:

$$\mathbf{Hom}_{\mathcal{D}_{/S}}(X \oslash_S K, Y) \cong \mathbf{Hom}_{\mathcal{C}_{/S}}(K, \mathbf{Hom}_{\mathcal{C}}(X, Y)) \quad \text{for all } S \in \operatorname{Ob}\mathcal{S},\ X,\ Y \in \operatorname{Ob}\mathcal{D}(S),\ K \in \operatorname{Ob}\mathcal{C}(S) \tag{9.4.25.2}$$

Now $\oslash$ is compatible with model structures involved, i.e. $\oslash$ satisfies (TMe) fiberwise, iff $\mathbf{Hom}_{\mathcal{C}}$ satisfies (TMh) fiberwise. The proof is still the same classical interplay of adjointness and lifting properties as before in **8.7.7** or in the classical proof of (SM7)⇔(SM7b). Of course, now we have only local lifting properties, so we have to reason locally; Kripke–Joyal semantics provides a formal way to transfer this proof to the local case as usual.

On the other hand, we might also have another flavor of inner Hom, namely, a cartesian functor $\mathbf{Hom}_{\mathcal{D}} : \mathcal{C}^{fop} \times_{\mathcal{S}} \mathcal{D} \to \mathcal{D}$, characterized by

$$\mathbf{Hom}_{\mathcal{D}_{/S}}(X \oslash_S K, Y) \cong \mathbf{Hom}_{\mathcal{D}_{/S}}(X, \mathbf{Hom}_{\mathcal{D}}(K, Y)) \quad \text{for all } S \in \operatorname{Ob}\mathcal{S},\ X,\ Y \in \operatorname{Ob}\mathcal{D}(S),\ K \in \operatorname{Ob}\mathcal{C}(S) \tag{9.4.25.3}$$

We usually denote $\mathbf{Hom}_{\mathcal{D}}(K, X)$ by X^K; when such inner Homs exist, (fiberwise) condition (TMe) for $\oslash$ is equivalent to (fiberwise) condition (SM7a) for $\mathbf{Hom}_{\mathcal{D}}$.

9.4.26. (Deriving inner Homs.) Suppose that $\oslash$ is a compatible external cartesian $\otimes$-action as above, (fiberwise) preserving initial objects in each variable, and that one or both of the above inner Homs exist. Then it is easy to see that inner Homs $\mathbf{Hom}_{\mathcal{C}}$ and $\mathbf{Hom}_{\mathcal{D}}$ (fiberwise) transform initial objects in first variable and final objects in second variable into final objects, hence the same criterion used to derive $\oslash$ is applicable to these inner Homs, once we replace appropriate stacks by their fiberwise duals and use (TMh) or (SM7a) instead of (TMe). In this way we obtain right derived functors $\mathbb{R}\,\mathbf{Hom}_{\mathcal{C}} : \operatorname{HO}\mathcal{D}^{fop} \times_{\mathcal{S}} \operatorname{HO}\mathcal{D} \to \operatorname{HO}\mathcal{C}$ and $\mathbb{R}\,\mathbf{Hom}_{\mathcal{D}} : \operatorname{HO}\mathcal{C}^{fop} \times_{\mathcal{S}} \operatorname{HO}\mathcal{D} \to \operatorname{HO}\mathcal{D}$, and their homotopic prestack counterparts, that can be computed by taking cofibrant replacements of the first argument and fibrant replacements of the second one. Now observe that for any cofibrant $Q \in \operatorname{Ob}\mathcal{D}(S)$ cartesian functors $K \mapsto \varphi^*Q \oslash K$ and $X \mapsto \mathbf{Hom}_{\mathcal{C}}(\varphi^*Q, X)$, where $T \xrightarrow{\varphi} S$, $K \in \operatorname{Ob}\mathcal{C}(T)$, $X \in \operatorname{Ob}\mathcal{D}(T)$, constitute a Quillen $\mathcal{S}_{/S}$-adjoint pair of cartesian functors $\mathcal{C}_{/S} \leftrightarrows \mathcal{D}_{/S}$, hence their derived are $\mathcal{S}_{/S}$-adjoint as well. This implies that $\mathbb{R}\,\mathbf{Hom}_{\mathcal{C}}$ *and* $\mathbb{R}\,\mathbf{Hom}_{\mathcal{D}}$ *are inner Homs for* $\underline{\underline{\oslash}} : \operatorname{HO}\mathcal{D} \times_{\mathcal{S}} \operatorname{HO}\mathcal{C} \to \operatorname{HO}\mathcal{D}$. Of course, we obtain a similar statement on the level of homotopic prestacks, and for corresponding categories of global (cartesian) sections $\operatorname{Ho}_p\mathcal{D}$, $\operatorname{Ho}\mathcal{D}$ etc. as well.

9.4.27. (Possible applications.) The above results would immediately enable us to derive tensor products and inner Homs on a generalized (commutatively) ringed topos $(\mathcal{E}, \mathscr{O})$, and base change with respect to any morphism of generalized ringed topoi, provided we manage to construct appropriate model structures on stacks $\mathfrak{s}\mathbf{SETS}_{\mathcal{E}}$ and $\mathfrak{s}\mathscr{O}$-**MOD** over $\mathcal{E}$.

9.5. (Pseudomodel stack structure on simplicial sheaves.) Throughout this subsection we fix a topos $\mathcal{E}$ and consider the stack $\mathbf{SETS} = \mathbf{SETS}_{\mathcal{E}} \to \mathcal{E}$ of "sets" or "sheaves" over $\mathcal{E}$, characterized by $\mathbf{SETS}_{\mathcal{E}} = \operatorname{Ar}\mathcal{E}$, $\mathbf{SETS}_{\mathcal{E}}(S) = \mathcal{E}_{/S}$, and the stack of simplicial sets or sheaves $\mathfrak{s}\mathbf{SETS}_{\mathcal{E}} \cong \mathbf{CART}_{\mathcal{E}}(\mathbf{\Delta}^0 \times \mathcal{E}, \mathbf{SETS}_{\mathcal{E}})$, characterized by $(\mathfrak{s}\mathbf{SETS}_{\mathcal{E}})(S) = s(\mathcal{E}_{/S})$. We would like to define a reasonable model stack structure on $\mathfrak{s}\mathbf{SETS}_{\mathcal{E}}$. In particular, this would enable us to speak about (co)fibrations and weak equivalences in its fiber $(\mathfrak{s}\mathbf{SETS}_{\mathcal{E}})(e) = s\mathcal{E}$. When $\mathcal{E} = \mathit{Sets}$, we must recover the classical model category structure on sSets; but in general we won't obtain a model category structure on $s\mathcal{E}$, and not even a (local) model stack structure, lifting properties being fulfilled only locally and under some additional restrictions.

Unfortunately, classical construction of a model category structure on sSets is not intuitionistic, so we cannot transfer it immediately to the topos case, i.e. to stack $\mathfrak{s}\mathbf{SETS}_{\mathcal{E}}$ over $\mathcal{E}$. It turns out that we have to weaken either the (local) lifting axiom or the factorization axiom of a model stack. We chose to sacrifice part of the (local) lifting axiom, thus obtaining the notion of a *pseudomodel stack*. Then we can obtain a *pseudo*model structure on $\mathfrak{s}\mathbf{SETS}_{\mathcal{E}}$.

Some of our statements will be proved only for a topos $\mathcal{E}$ with sufficiently many points; while this additional condition shouldn't affect the validity of our statements, it considerably shortens the proofs. The general case might be treated either by transferring results from $\hat{\mathcal{S}}$ to $\tilde{\mathcal{S}}$ in the usual manner, or by using suitable local limits instead of points. This will be done elsewhere.

Notice that we usually don't consider the site case here; of course, we don't lose much, since we can always recover corresponding constructions over a site $\mathcal{S}$ by doing everything over the topos $\tilde{\mathcal{S}}$ first and then pulling back resulting (pre)stacks with respect to $\mathcal{S} \to \tilde{\mathcal{S}}$ (cf. **9.1.9**, **9.3.14** and **9.3.21**).

9.5.1. (Simplicial objects in a fibered category.) Given a fibered category $p_{\mathcal{C}} : \mathcal{C} \to \mathcal{S}$, we define the corresponding *fibered category of simplicial objects* $\mathfrak{s}\mathcal{C}$ by means of the following cartesian square of categories

$$\begin{array}{ccc} \mathfrak{s}\mathcal{C} & \longrightarrow & s\mathcal{C} \\ \downarrow{\scriptstyle p_{\mathfrak{s}\mathcal{C}}} & & \downarrow{\scriptstyle s(p_{\mathcal{C}})} \\ \mathcal{S} & \xrightarrow{I} & s\mathcal{S} \end{array} \tag{9.5.1.1}$$

Here $s\mathcal{C} = Funct(\mathbf{\Delta}^0, \mathcal{C})$ and $s\mathcal{S}$ are the usual categories of simplicial objects, and $I : \mathcal{S} \to s\mathcal{S}$ is the constant simplicial object embedding. By definition $(\mathfrak{s}\mathcal{C})(S) = s(\mathcal{C}(S))$ for any $S \in \operatorname{Ob}\mathcal{S}$, i.e. the fibers of $\mathfrak{s}\mathcal{C}$ consist of simplicial objects over the corresponding fibers of $\mathcal{C}$, and the pullback functors φ^* of $\mathfrak{s}\mathcal{C}$ are just the simplicial extensions $s(\varphi^*)$.

Now if $\mathcal{C} \to \mathcal{S}$ is a (pre)stack over a site $\mathcal{S}$, then $\mathfrak{s}\mathcal{C}$ is obviously another one. In particular, for any topos $\mathcal{E}$ we obtain a well-defined stack $\mathfrak{s}\mathbf{SETS}_{\mathcal{E}}$ over $\mathcal{E}$, characterized by $(\mathfrak{s}\mathbf{SETS}_{\mathcal{E}})(X) = s(\mathcal{E}_{/X})$.

One can extend cartesian functors $F : \mathcal{C} \to \mathcal{D}$ over $\mathcal{S}$ to cartesian functors $\mathfrak{s}F : \mathfrak{s}\mathcal{C} \to \mathfrak{s}\mathcal{D}$ in the usual manner; when no confusion can arise, we denote this simplicial extension by the same letter F. Moreover, this construction is compatible with fibered products: $\mathfrak{s}(\mathcal{C}_1 \times_{\mathcal{S}} \mathcal{C}_2) = \mathfrak{s}\mathcal{C}_1 \times_{\mathcal{S}} \mathfrak{s}\mathcal{C}_2$, so for example a "cartesian bifunctor" $F : \mathcal{C}_1 \times_{\mathcal{S}} \mathcal{C}_2 \to \mathcal{D}$ admits a simplicial extension $F = \mathfrak{s}F : \mathfrak{s}\mathcal{C}_1 \times_{\mathcal{S}} \mathfrak{s}\mathcal{C}_2 \to \mathfrak{s}\mathcal{D}$.

Notice that in these constructions $\mathbf{\Delta}^0$ can be replaced by an arbitrary small index category $\mathcal{I}$. For example, $\mathcal{I} = \mathbf{\Delta}$ yields the stack of cosimplicial objects $\mathfrak{c}\mathcal{C}$.

9.5.2. (Constant simplicial objects.) Let us denote by $q : \mathcal{E} \to Sets$ the canonical morphism of $\mathcal{E}$ into the point topos *Sets*. Then $q_*X = \Gamma(X) = \operatorname{Hom}_{\mathcal{E}}(e_{\mathcal{E}}, X)$, and q^*K is just the constant sheaf or object of $\mathcal{E}$, usually denoted by $\underline{K}_{\mathcal{E}}$. Moreover, q^* extends to a functor $q^* = s(q^*) : sSets \to s\mathcal{E}$; the image under this functor of a simplicial set K will be also denoted by $\underline{K}_{\mathcal{E}}$, or even by K when no confusion can arise.

In particular, we obtain the standard simplices $\underline{\Delta(n)}_{\mathcal{E}}$, for all $n \geq 0$; they still have their characteristic properties $\operatorname{Hom}_{s\mathcal{E}}(\underline{\Delta(n)}_{\mathcal{E}}, X) = \Gamma(X_n)$ for any $X \in \operatorname{Ob} s\mathcal{E}$, and $\mathbf{Hom}_{\mathfrak{s}\mathcal{E}|S}(\underline{\Delta(n)}_{\mathcal{E}}, X) = X_n$ for any $X \in \operatorname{Ob}(\mathfrak{s}\mathbf{SETS}_{\mathcal{E}})(S) = \operatorname{Ob} s(\mathcal{E}_{/S})$.

9.5.3. (Cartesian ACU $\otimes$-structure on **SETS**.) Clearly, the fibered products $\times_S : \mathcal{E}_{/S} \times \mathcal{E}_{/S} \to \mathcal{E}_{/S}$ are compatible with arbitrary pullbacks, thus defining a cartesian "bifunctor" $\otimes = \times_{\mathcal{E}} : \mathbf{SETS}_{\mathcal{E}} \times_{\mathcal{E}} \mathbf{SETS}_{\mathcal{E}} \to \mathbf{SETS}_{\mathcal{E}}$, easily seen to be an ACU cartesian $\otimes$-structure on $\mathbf{SETS}/\mathcal{E}$ (cf. **9.4.22**). Of course, this cartesian $\otimes$-structure admits (cartesian) inner Homs $\mathbf{Hom} = \mathbf{Hom}_{\mathcal{E}} : \mathbf{SETS}_{\mathcal{E}}^{fop} \times_{\mathcal{E}} \mathbf{SETS}_{\mathcal{E}} \to \mathbf{SETS}_{\mathcal{E}}$, given by usual local Hom-objects: $\mathbf{Hom}_{\mathcal{E}|S}(X, Y) = \mathbf{Hom}_S(X, Y) = \mathbf{Hom}_{\mathcal{E}_{/S}}(X, Y)$ is just the object of $\mathcal{E}_{/S}$ representing the Hom-sheaf $\mathbf{Hom}_{\mathcal{E}_{/S}}(X, Y)$ over $\mathcal{E}_{/S}$, for any $X, Y \in \operatorname{Ob}\mathcal{E}_{/S}$.

9.5.4. (Cartesian $\otimes$-action of **SETS** on any flat stack $\mathcal{C}$.) Let $\mathcal{C}$ be any *flat* stack over $\mathcal{E}$, i.e. we require (MS1) and (MS1+). Then $(X, T) \mapsto X \times_S T = \varphi_! \varphi_* X$, where $(T \xrightarrow{\varphi} S) \in \operatorname{Ob}\mathcal{E}_{/S} = \operatorname{Ob}\mathbf{SETS}_{\mathcal{E}}(S)$, $X \in \operatorname{Ob}\mathcal{C}(S)$, defines a cartesian "bifunctor" $\oslash : \mathcal{C} \times_{\mathcal{E}} \mathbf{SETS}_{\mathcal{E}} \to \mathbf{SETS}_{\mathcal{E}}$, easily seen to be a cartesian $\otimes$-action of **SETS** on $\mathcal{C}$ (cf. **9.3.11** and **9.3.12**). Moreover, both flavors of inner Homs exist in this situation: $\mathbf{Hom}_{\mathbf{SETS}} : \mathcal{C}^{fop} \times_{\mathcal{E}} \mathcal{C} \to \mathbf{SETS}_{\mathcal{E}}$ is given simply by the local **Hom**-objects, i.e. $\mathbf{Hom}_{\mathbf{SETS}|S}(X, Y)$ represents the Hom-sheaf $\mathbf{Hom}_{\mathcal{C}_{/S}}(X, Y)$, while $\mathbf{Hom}_{\mathcal{C}} : \mathbf{SETS}_{\mathcal{E}}^{fop} \times_{\mathcal{E}} \mathcal{C} \to \mathcal{C}$ is given by $(T, X) \mapsto X^{T/S} = \varphi_* \varphi^* X$, for any $(T \xrightarrow{\varphi} S) \in \operatorname{Ob} \mathcal{E}_{/S}$, $X \in \operatorname{Ob} \mathcal{C}(S)$ (cf. **9.3.12**).

In the sequel the **SETS**-valued inner Homs will be called *local*, just to distinguish them from all other sorts of inner Homs that will appear. We'll usually denote local Homs simply by **Hom** or $\mathbf{Hom}_S$, or by $\mathbf{Hom}_{\mathcal{C}|S}$ when we want to indicate the stack $\mathcal{C}$ as well.

9.5.5. (Cartesian ACU $\otimes$-structure on $\mathfrak{s}$**SETS**.) Of course, the above cartesian ACU $\otimes$-structure $\otimes = \times : \mathbf{SETS} \times_{\mathcal{E}} \mathbf{SETS} \to \mathbf{SETS}$ extends to a cartesian ACU $\otimes$-structure $\otimes := s(\otimes)$ on $\mathfrak{s}$**SETS**. Whenever we have two simplicial objects $X, Y \in \operatorname{Ob} s(\mathcal{E}_{/S})$, their "tensor product" $X \otimes_S Y$ is simply the componentwise fibered product $X \times_S Y$, given by $(X \times_S Y)_n = X_n \times_S Y_n$.

We claim that *inner Homs* $\mathbf{Hom}_{\mathfrak{s}\mathbf{SETS}} : \mathfrak{s}\mathbf{SETS}^{fop} \times_{\mathcal{E}} \mathfrak{s}\mathbf{SETS} \to \mathfrak{s}\mathbf{SETS}$ *do exist for this cartesian $\otimes$-structure.* Indeed, the requirement for this inner Hom $\mathbf{Hom}_{\mathfrak{s}\mathbf{SETS}|S}(X, Y)$, where X, Y are two simplicial objects over $\mathcal{E}_{/S}$, is

$$\mathbf{Hom}_S(K \otimes_S X, Y) \cong \mathbf{Hom}_S(K, \mathbf{Hom}_{\mathfrak{s}\mathbf{SETS}|S}(X, Y)) \qquad (9.5.5.1)$$

Of course, we are free to choose a simplicial object K after any pullback $S' \to S$ as well. Putting here $K = \underline{\Delta(n)}$, we see that $\mathbf{Hom}_{\mathfrak{s}\mathbf{SETS}|S}(X, Y)$ has to be computed by the classical formula

$$\mathbf{Hom}_{\mathfrak{s}\mathbf{SETS}|S}(X, Y)_n := \mathbf{Hom}_S(\underline{\Delta(n)} \otimes_S X, Y) \qquad (9.5.5.2)$$

Now the verification of (9.5.5.1) for all $K \in \operatorname{Ob} s(\mathcal{E}_{/T})$ can be done directly in the classical manner, the classical proof being intuitionistic, or by using an obvious "local Yoneda lemma".

9.5.6. ($\otimes$-action of $\mathfrak{s}$**SETS** on $\mathfrak{s}\mathcal{C}$.) Now let $\mathcal{C}$ be a *flat* stack over $\mathcal{E}$. The external cartesian $\otimes$-action $\oslash : \mathcal{C} \times_{\mathcal{E}} \mathbf{SETS} \to \mathcal{C}$ of **9.5.4** extends to an external cartesian $\otimes$-action of $\mathfrak{s}$**SETS** on $\mathfrak{s}\mathcal{C}$, usually denoted by $\oslash$ or $\otimes$. Existence of both flavors of inner Homs with respect to this action can be shown again in more or less classical fashion (cf. [Quillen, 2.1]); in particular, $\mathbf{Hom}_{\mathfrak{s}\mathbf{SETS}} : (\mathfrak{s}\mathcal{C})^{fop} \times_{\mathcal{E}} \mathfrak{s}\mathcal{C} \to \mathfrak{s}\mathbf{SETS}$ is still given by (9.5.5.2).

9.5.7. (Standard cofibrant generators of *sSets* and $\mathfrak{s}\mathbf{SETS}_{\mathcal{E}}$.) Recall that *sSets* admits two standard sets of small cofibrant generators (cf. **8.2.10**):

$$I := \{\dot{\Delta}(n) \to \Delta(n) \,|\, n \geq 0\} \qquad (9.5.7.1)$$

$$J := \{\Lambda_k(n) \to \Delta(n) \,|\, 0 \leq k \leq n > 0\} \qquad (9.5.7.2)$$

Here, of course, I generates the cofibrations, and J the acyclic cofibrations.

Now it is reasonable to expect that for any morphism of topoi $f : \mathcal{E}' \to \mathcal{E}$ the pullback functor $f^* : s\mathcal{E} \to s\mathcal{E}'$ will preserve cofibrations and acyclic cofibrations. Let's apply this to $q : \mathcal{E} \to \mathit{Sets}$; we obtain two following sets of maps of constant simplicial sheaves:

$$\underline{I} = \underline{I}_{\mathcal{E}} := \{\underline{\dot{\Delta}(n)} \to \underline{\Delta(n)} \,|\, n \geq 0\} \tag{9.5.7.3}$$

$$\underline{J} = \underline{J}_{\mathcal{E}} := \{\underline{\Lambda_k(n)} \to \underline{\Delta(n)} \,|\, 0 \leq k \leq n > 0\} \tag{9.5.7.4}$$

Clearly, for any reasonable model stack structure on $\mathbf{SETS}_{\mathcal{E}}$ these sets of morphisms $\underline{I}$, $\underline{J} \subset \mathrm{Ar}(\mathfrak{s}\mathbf{SETS}_{\mathcal{E}})(e)$, as well as all their pullbacks, have to be cofibrations, resp. acyclic cofibrations. Moreover, we are tempted to choose these sets $\underline{I}$ and $\underline{J}$ as cofibrant generators for this model stack structure. Unfortunately, in general this doesn't define a model stack structure on $\mathfrak{s}\mathbf{SETS}_{\mathcal{E}}$ unless topos $\mathcal{E}$ is *sequential*, i.e. $\varprojlim_n Z_n \to Z_0$ is an epimorphism for any projective system $\cdots \to Z_2 \to Z_1 \to Z_0$ in $\mathcal{E}$ with epimorphic transition morphisms. This condition is almost never fulfilled: for example, a metrizable topological space X defines a sequential topos iff X is discrete.

9.5.8. (Cofibrant generators of a model stack.) Given a model stack $\mathcal{C} \to \mathcal{E}$ and two sets of morphisms I, $J \subset \mathrm{Ar}\,\mathcal{C}(e)$ in its final fiber, we say that I *and* J *are cofibrant generators for* $\mathcal{C}$ if the acyclic fibrations (resp. fibrations) of $\mathcal{C}$ are exactly the morphisms with the local RLP with respect to (all pullbacks of) all morphisms from I (resp. J). Notice that this condition automatically implies that I (resp. J) consists of cofibrations (resp. acyclic cofibrations).

We say that $\mathcal{C}$ is *cofibrantly generated* if it admits sets of cofibrant generators I, J as above. In this case the model stack structure of $\mathcal{C}$ is completely determined by these two sets. Indeed, we can define acyclic fibrations (resp. fibrations) as morphisms with the local RLP with respect to all pullbacks of morphisms from $\underline{I}$ (resp. $\underline{J}$), then define acyclic cofibrations (resp. cofibrations) as the morphisms with local LLP with respect to all fibrations (resp. acyclic fibrations), and finally define weak equivalences as morphisms which can be decomposed into an acyclic cofibration followed by an acyclic fibration.

If we start from two arbitrary sets I, $J \subset \mathrm{Ar}\,\mathcal{C}(e)$ and define cofibrations, fibrations and weak equivalences as above, we'll still have to check some of the axioms (MS1)–(MS5) to show that we've indeed obtained a model stack structure; in particular, we need to prove (MS5) (factorization), (MS2) (localness and stability of weak equivalences under retracts) and (MS3) (2-out-of-3 for weak equivalences). In addition, we have to check that e.g. acyclic fibrations are indeed the morphisms that are both fibrations and acyclic.

Definition 9.5.9 (Distinguished classes of morphisms in $\mathfrak{s}\mathbf{SETS}_{\mathcal{E}}$.) *We define subsets of fibrations, acyclic fibrations,* **strong** *cofibrations and* **strong** *acyclic cofibrations in sets of morphisms in fibers of $\mathfrak{s}\boldsymbol{SETS}_{\mathcal{E}}$ by means of the above construction, applied to standard cofibrant generators $\underline{I}$, $\underline{J}$ of* **9.5.7**.

This doesn't define a model stack structure on $\mathfrak{s}\mathbf{SETS}_{\mathcal{E}}$ unless $\mathcal{E}$ is sequential: in general we have to sacrifice either the local lifting axiom (MS4) or the factorization axiom (MS5). This is due to the fact that the class of *strong* cofibrations defined above by the local LLP with respect to all acyclic fibrations is too small, and in particular it is not closed under infinite sums and sequential inductive limits.

We deal with this problem by embedding the strong cofibrations into a larger class of cofibrations, stable under small sums, composition, pushouts, sequential inductive limits, and pullbacks with respect to arbitrary morphisms of topoi. In other words, we drop the local lifting axiom (MS4) to be able to obtain factorization (MS5) and even (MS5f).

Definition 9.5.10 *We say that a class of morphisms $\mathscr{P}$ in fibers of a* flat *stack $\mathcal{C}$ over $\mathcal{S}$ is* closed *if:*

1. *$\mathscr{P}$ is local, and in particular stable under all pullbacks φ^*;*
2. *$\mathscr{P}$ contains all isomorphisms in fibers of $\mathcal{C}$;*
3. *$\mathscr{P}$ is stable under pushouts;*
4. *$\mathscr{P}$ is stable under composition;*
5. *$\mathscr{P}$ is stable under "sequential composition", i.e. whenever $A_0 \xrightarrow{i_0} A_1 \xrightarrow{i_1} A_2 \to \cdots$ is a composable infinite sequence of morphisms from $\mathscr{P}$, its "composition" $A_0 \xrightarrow{i} A_\infty = \varinjlim_n A_n$ is also in $\mathscr{P}$.*
6. *$\mathscr{P}$ is stable under retracts (hence also under local retracts);*
7. *$\mathscr{P}$ is stable under arbitrary (small) direct sums;*
8. *$\mathscr{P}$ is stable under all $\varphi_!$, hence under all $-\otimes_S T = \varphi_!\varphi^*$; notice that this condition makes sense only for a flat $\mathcal{C}$. (In fact, stability under $-\otimes_S T$ would suffice for our applications.)*

Similarly, a class of morphisms $\mathscr{P}$ in a category $\mathcal{C}$ is (globally) closed *if it satisfies those of the above conditions that are applicable, i.e. all with exception of the first and the last. If we omit condition 6) from the above list, we obtain the notions of a* semiclosed *and* (globally) semiclosed *class of morphisms.*

For example, the classes of cofibrations and acyclic cofibrations in a model category $\mathcal{C}$ are (globally) closed. Unfortunately, this is not true for model stacks $\mathcal{C}/\mathcal{S}$: in general we don't have 5), 7) and 8).

Definition 9.5.11 *Given a flat stack $\mathcal{C}/\mathcal{S}$ and any set of morphisms $I \subset \operatorname{Ar}\mathcal{C}(e)$ in the final fiber of $\mathcal{C}$, we denote by $\operatorname{Cl} I$ the* closure *of I in $\mathcal{C}$, i.e. the smallest closed class of morphisms in fibers of $\mathcal{C}$ that contains I. We define the* semiclosure $\operatorname{SCl} I$ *similarly; and if $\mathcal{C}$ is just a category, we obtain corresponding global notions* $\operatorname{GCl} I$ *and* $\operatorname{GSCl} I$*, which will be usually denoted simply by* $\operatorname{Cl} I$ *and* $\operatorname{SCl} I$.

Clearly, any intersection of closed sets of morphisms is still closed, so the closure $\operatorname{Cl} I$ always exists and can be constructed as the intersection of all closed sets of morphisms containing I.

If $\mathcal{C}$ is a model category cofibrantly generated by sets I and J of morphisms with small sources, then Quillen's small object argument shows that $\operatorname{Cl} I$ is the set of cofibrations, and $\operatorname{Cl} J$ the set of acyclic cofibrations of $\mathcal{C}$. Unfortunately, this is not true for model stacks.

Definition 9.5.12 *(Acyclic cofibrations of simplicial sheaves.) The set of cofibrations (resp. acyclic cofibrations) in $\mathfrak{s}\boldsymbol{SETS}_{\mathcal{E}}/\mathcal{E}$ is by definition the closure in $\mathfrak{s}\boldsymbol{SETS}_{\mathcal{E}}$ of $\underline{I}_{\mathcal{E}}$ (resp. $\underline{J}_{\mathcal{E}}$) of* **9.5.7**.

Now we are going to check the factorization axiom (MS5f) for these sets of (acyclic) cofibrations and (acyclic) fibrations, and to prove that all strong (acyclic) cofibrations in the sense of **9.5.9** are (acyclic) cofibrations in the sense of above definition. For this we'll need a version of Quillen's small object argument.

Definition 9.5.13 *(Sequentially small and finitely presented objects.) An object X of a category $\mathcal{C}$ is* (sequentially) small *(resp.* finitely presented*) if $\operatorname{Hom}_{\mathcal{C}}(X,-) : \mathcal{C} \to Sets$ commutes with sequential inductive limits (resp. filtered inductive limits). Similarly, an object $X \in \mathcal{C}(S)$ of a fiber of a stack $\mathcal{C}/\mathcal{S}$ is* locally (sequentially) small *(resp.* locally finitely presented*) if $\mathbf{Hom}_{\mathcal{C}_{/T}}(X|_T,-) : \mathcal{C}(T) \to \boldsymbol{SETS}_{\mathcal{S}}(T) = \widetilde{\mathcal{S}_{/T}}$ commutes with sequential (resp. filtered) inductive limits in $\mathcal{C}(T)$, for any $T \to S$ in $\mathcal{S}$.*

Evidently, these properties are indeed local in the sense of **9.2.17**. When $\mathcal{S} = \mathbf{1}$ or *Sets*, we recover the corresponding "global" notions. Notice that the substacks of locally small and locally finitely presented objects are stable under finite inductive limits in fibers of $\mathcal{C}$.

9.5.14. (Small objects in $\mathfrak{s}\mathbf{SETS}_{\mathcal{E}}$.) First of all, notice that the *standard simplices* $\underline{\Delta(n)}_{\mathcal{E}}$ are locally finitely presented, hence also locally small, in $s\mathcal{E}$ (or, more precisely, in $(\mathfrak{s}\mathbf{SETS}_{\mathcal{E}})(e)$). Indeed, this follows from the formula $\mathbf{Hom}(\underline{\Delta(n)}, X) = X_n$, valid for any $X \in \operatorname{Ob} s\mathcal{E}$, and the fact that arbitrary inductive limits in $s\mathcal{E}$ are computed componentwise, hence $\mathbf{Hom}(\underline{\Delta(n)}, -) : X \mapsto X_n$ commutes with arbitrary inductive limits.

Since any finite inductive limit of locally finitely presented objects is again finitely presented, and the constant simplicial sheaf functor $sq^* : sSets \to s\mathcal{E}$ commutes with arbitrary inductive limits, we see that $\underline{A}_{\mathcal{E}}$ is locally finitely presented for any *finite* simplicial set A, such simplicial sets being exactly the finite inductive limits of standard simplices.

Now all simplicial sets involved in standard cofibrant generators I and J of $sSets$ are finite, hence the sources and targets of all morphisms of $\underline{I}_{\mathcal{E}}$ and $\underline{J}_{\mathcal{E}}$ are locally finitely presented and locally small.

9.5.15. ($\mathbf{Hom}(\underline{A}, -)$ for a finite simplicial set A.) Before going on, let's consider the functors $\mathbf{Hom}(\underline{A}, -) : s\mathcal{E} \to \mathcal{E}$, where A is any finite simplicial set. We can represent A as a cokernel of a couple of morphisms between finite sums of standard simplices: $A = \operatorname{Coker}\bigl(\coprod_{j=1}^{t} \Delta(m_j) \rightrightarrows \coprod_{i=1}^{s} \Delta(n_i)\bigr)$ in $sSets$. Applying exact functor $sq^* : A \mapsto \underline{A}$ we obtain a similar relation in $s\mathcal{E}$. Now $\mathbf{Hom}$ is clearly left exact in both arguments, hence

$$\mathbf{Hom}(\underline{A}, X) \cong \operatorname{Ker}\bigl(\prod_{i=1}^{s} X_{n_i} \rightrightarrows \prod_{j=1}^{t} X_{m_j}\bigr) \qquad (9.5.15.1)$$

In other words, $\mathbf{Hom}(\underline{A}, X)$ is a finite projective limit of the components of X.

9.5.16. (Fibrations and acyclic fibrations.) This is applicable in particular to $A = \dot{\Delta}(n) = \operatorname{sk}_{n-1} \Delta(n)$, and we obtain $\mathbf{Hom}(\underline{\dot{\Delta}(n)}, X) \cong (\operatorname{cosk}_{n-1} X)_n$ for any $X \in \operatorname{Ob} s\mathcal{E}$, similarly to the classical case **8.2.10**. On the other hand, $p : X \to Y$ has the local RLP with respect to some $i : A \to B$ in $\mathcal{C}(S)$, $\mathcal{C}/\mathcal{E}$ any stack, iff $\mathbf{Hom}(B, X) \to \mathbf{Hom}(A, X) \times_{\mathbf{Hom}(A,Y)} \mathbf{Hom}(B, Y)$ is an epimorphism in $\mathcal{E}_{/S}$. Applying this to all i from $\underline{I}_{\mathcal{E}}$, we see that $p : X \to Y$ *is an acyclic fibration iff the natural morphisms* $X_n \to (\operatorname{cosk}_{n-1} X)_n \times_{(\operatorname{cosk}_{n-1} Y)_n} Y_n$ *are epimorphic in* $\mathcal{E}_{/S}$ *for all* $n \geq 0$, i.e. a condition completely similar to its classical counterpart **8.2.10**.

Similarly, applying this to all $i : A \to B$ from $\underline{J}_{\mathcal{E}}$, we obtain a description of fibrations $p : X \to Y$ in $s\mathcal{E}$ in terms of a countable set of conditions each requiring a certain morphism between finite projective limits of components of X and Y to be an epimorphism in $\mathcal{E}_{/S}$.

9.5.17. (Stability of fibrations under pullbacks with respect to any morphism of topoi.) An immediate consequence is that *for any finite simplicial set A functors $\mathbf{Hom}(\underline{A}, -)$ commute with pullbacks f^* with respect to any morphism of topoi $f : \mathcal{E}' \to \mathcal{E}$*, i.e. $f^* \mathbf{Hom}(\underline{A}_{\mathcal{E}}, X) \cong \mathbf{Hom}(\underline{A}_{\mathcal{E}'}, f^*X)$ for any $X \in \operatorname{Ob} s\mathcal{E}$. Indeed, this follows from the fact that $\mathbf{Hom}(\underline{A}, X)$ is just some finite projective limit of components of X, and the exactness of f^*. Therefore, the morphisms between finite projective limits of components of X and Y, the epimorphicity of which is equivalent to $p : X \to Y$ being a fibration (resp. an acyclic fibration), are preserved by f^*. Now f^* is exact, and in particular preserves epimorphisms. We conclude that *fibrations and acyclic fibrations in $s\mathcal{E}$ are preserved by pullbacks with respect to arbitrary morphisms of topoi $f : \mathcal{E}' \to \mathcal{E}$.*

9.5.18. (One step of Quillen's construction.) Let $I = \{i_\alpha : A_\alpha \to B_\alpha\}_{\alpha \in I}$ be a (small) set of morphisms in a fiber $\mathcal{C}(S)$ of a flat stack $\mathcal{C}/\mathcal{E}$. Then for any morphism $X \xrightarrow{f} Y$ in $\mathcal{C}(S)$ we construct a functorial decomposition $X \xrightarrow{\xi_f} R_{I,Y}(X) \xrightarrow{R(f)} Y$ as follows. For any $\alpha \in I$ denote by $H_\alpha = H_\alpha(f) \in \operatorname{Ob} \mathcal{E}_{/S}$ the *sheaf* of diagrams

$$\begin{tikzcd} A_\alpha \arrow[r, "i_\alpha"] \arrow[d, dashed, "u_\alpha"] & B_\alpha \arrow[d, dashed, "v_\alpha"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{9.5.18.1}$$

In other words, H_α is just the fibered product $\mathbf{Hom}(A_\alpha, X) \times_{\mathbf{Hom}(A_\alpha, Y)} \mathbf{Hom}(B_\alpha, X)$ in $\mathcal{E}_{/S}$. Now $R_{I,Y}(X)$ is constructed by means of the following cocartesian square:

$$\begin{tikzcd} \coprod_\alpha A_\alpha \otimes H_\alpha \arrow[r, "(i_\alpha)"] \arrow[d, "(u_\alpha)"'] & \coprod_\alpha B_\alpha \otimes H_\alpha \arrow[d, dashed] \arrow[ddr, "(v_\alpha)"] & \\ X \arrow[r, dashed, "\xi_f"] \arrow[drr, "f"'] & R_{I,Y}(X) \arrow[dr, dashed, "R(f)"] & \\ & & Y \end{tikzcd} \tag{9.5.18.2}$$

The top horizontal arrow (i_α) is simply $\coprod_\alpha i_\alpha \otimes \mathrm{id}_{H_\alpha}$. As to the left vertical arrow $(u_\alpha) : \coprod_\alpha A_\alpha \otimes H_\alpha \to X$, its components $u'_\alpha : A_\alpha \otimes H_\alpha \to X$ are constructed by applying $\mathrm{id}_{A_\alpha} \otimes -$ to the natural projection $H_\alpha \to \mathbf{Hom}(A_\alpha, X)$ and composing the result with the canonical "evaluation" morphism $\mathrm{ev}_{A_\alpha, X} : A_\alpha \otimes \mathbf{Hom}(A_\alpha, X) \to X$. The arrow (v_α) is constructed similarly with the aid of the second projection $H_\alpha \to \mathbf{Hom}(B_\alpha, Y)$ and $\mathrm{ev}_{B_\alpha, Y}$, and the commutativity of the outer circuit of the above diagram follows from the definition of H_α and the explicit description of evaluation morphisms involved.

So let us recall the construction of $\mathrm{ev}_{A,X} : A \otimes_S \mathbf{Hom}_S(A, X) \to X$, for any A, $X \in \operatorname{Ob} \mathcal{C}(S)$. Put $H := \mathbf{Hom}_S(A, X)$, and let $\varphi : H \to S$ be the structural morphism of $H \in \operatorname{Ob} \mathcal{E}_{/S}$. Then by definition $A \otimes_S H = \varphi_! \varphi_* A$, hence giving $\mathrm{ev}_{A,X} : \varphi_! \varphi^* A \to X$ is equivalent to giving some $\mathrm{ev}^\flat_{A,X} : A|_H = \varphi^* A \to X|_H$, i.e. an element $\mathrm{ev}^\flat_{A,X} \in \operatorname{Hom}_{\mathcal{C}(H)}(A|_H, X|_H) = (\mathbf{Hom}_S(A, X))(H)$, i.e. an S-morphism from H into $\mathbf{Hom}_S(A, X) = H$. Of course, we put $\mathrm{ev}^\flat_{A,X} := \mathrm{id}_H$.

9.5.19. (Sequential closure of Quillen's construction.) Notice that $R_{I,Y}$ can be treated as a functor $\mathcal{C}(S)_{/Y} \to \mathcal{C}(S)_{/Y}$, and $\xi_f : X \to R_{I,Y}(X)$ is a functorial morphism $\xi : \mathrm{Id}_{\mathcal{C}(S)_{/Y}} \to R_{I,Y}$. Therefore, we can iterate the construction: $X \to R_{I,Y}(X) \to R^2_{I,Y}(X) \to \cdots$. Put $R^\infty_{I,Y}(X) := \varinjlim_n R^n_{I,Y}(X)$; when no confusion can arise, we denote $R^\infty_{I,Y}(X)$ simply by $R_I(X)$ or $R_I(f)$. By construction, we obtain a functorial decomposition $X \to R_I(f) \to Y$ of any $X \xrightarrow{f} Y$. Furthermore, the construction of $R_{I,Y}(X)$ commutes with arbitrary pullbacks $\varphi^* : \mathcal{C}(S) \to \mathcal{C}(T)$, hence the same is true for $R_I(X)$, the stack $\mathcal{C}$ being flat. We conclude that *functors $R_I : \operatorname{Ar} \mathcal{C}(S) \to \mathcal{C}(S)$ define together a cartesian functor $R_I : \mathfrak{Ar}\, \mathcal{C} \to \mathcal{C}$ of stacks over $\mathcal{E}$.*

Proposition 9.5.20 *(Quillen's factorization.) If $I \subset \operatorname{Ar} \mathcal{C}(e)$ is a small set of morphisms in the final fiber of a flat stack $\mathcal{C}$ over a topos $\mathcal{E}$, and all morphisms from I have locally small sources, then $X \to R_I(f)$ belongs to the closure* $\operatorname{Cl} I$ *of I in $\mathcal{C}$, and $R_I(f) \to Y$ has the local RLP with respect to all morphisms from I, for any $f : X \to Y$ in $\mathcal{C}(S)$. In other words, $R_I : \mathfrak{Ar}\, \mathcal{C} \to \mathcal{C}$ together with appropriate natural transformations provides a functorial factorization for all $f : X \to Y$, similar to that of (MS5f).*

Proof. The fact that $X \to R_{I,Y}(X)$ belongs to $\operatorname{Cl} I$ is immediate from (9.5.18.2), **9.5.10** and **9.5.11**. Since $X \to R_I(f) = R^\infty_{I,Y}(X)$ is a sequential composition of morphisms of the above form, it lies in $\operatorname{Cl} I$ as well. Now let us prove that $R_I(f) \to Y$ has the local RLP with respect to any $i_\alpha : A_\alpha \to B_\alpha$ from I. Since the construction of $R_I(f)$ is compatible with all pullbacks φ^*, we may assume that our lifting problem $u : A_\alpha \to R_I(f)$, $v : B_\alpha \to Y$ is given inside $\mathcal{C}(S)$ itself, and, moreover, may even assume $S = e$, replacing $\mathcal{C}$ and $\mathcal{E}$ with $\mathcal{C}_{/S}$ and $\mathcal{E}_{/S}$ if necessary. Now A_α is locally small by assumption, hence $\mathbf{Hom}(A_\alpha, R_I(f)) = \mathbf{Hom}(A_\alpha, \varinjlim_n R^n_{I,Y}(X)) \cong \varinjlim_n \mathbf{Hom}(A_\alpha, R^n_{I,Y}(X))$, so we can find (locally, i.e. on some cover) some $n \geq 0$ and a morphism $u' : A_\alpha \to R^n_{I,Y}(X)$ that represents u. Notice that u' and v define a commutative diagram (9.5.18.1) with $f : X \to Y$ replaced by $f_n : R^n_{I,Y}(X) \to Y$, i.e. $w := (u', v)$ is a section of $H_\alpha = H_\alpha(f_n)$. Now $\mathrm{id}_{B_\alpha} \otimes w$ defines a morphism $B_\alpha \to B_\alpha \otimes H_\alpha$; composing it with the natural embedding of $B_\alpha \otimes H_\alpha$ into the coproduct $\coprod_{\beta\in I} B_\beta \otimes H_\beta$ and the right vertical arrow of (9.5.18.2), we obtain a morphism $h' : B_\alpha \to R^{n+1}_{I,Y}(X)$, making the following diagram commutative:

$$\begin{tikzcd} A_\alpha \arrow[r, "i_\alpha"] \arrow[d, "u'"] & B_\alpha \arrow[d, "h'"] \\ R^n_{I,Y}(X) \arrow[r, "\xi_{f_n}"] & R^{n+1}_{I,Y}(X) \end{tikzcd} \tag{9.5.20.1}$$

Composing h' with the natural embedding of $R^{n+1}_{I,Y}(X)$ into the inductive limit $R_I(f)$, we obtain a (local) solution $h : B_\alpha \to R_I(f)$ of the lifting problem (u, v), q.e.d.

Corollary 9.5.21 *Any morphism $f : X \to Y$ in $s\mathcal{E}$ can be functorially decomposed both into a cofibration $X \to R_{\underline{I}}(X)$ followed by an acyclic fibration $R_{\underline{I}}(X) \to Y$, and into an acyclic cofibration $X \to R_{\underline{J}}(X)$ followed by a fibration $R_{\underline{J}}(X) \to Y$. In other words, (MS5f) holds in $\mathfrak{s}\boldsymbol{SETS}_{\mathcal{E}}/\mathcal{E}$ for this choice of (acyclic) fibrations and cofibrations.*

Proof. Immediate from **9.5.20** and definitions **9.5.12** and **9.5.9**.

9.5.22. Unfortunately, in our case the cofibrations (resp. acyclic cofibrations) are not necessarily strong, i.e. they needn't have the local LLP with respect to all acyclic fibrations (resp. fibrations). The reason for this is that, say, the class of strong cofibrations still contains $\underline{I}$, but is not closed (e.g. it is not closed under infinite direct sums). On the other hand, the opposite inclusion is still true: *any strong cofibration is a cofibration, and similarly for strong acyclic cofibrations.* Indeed, we can apply the classical reasoning of [Quillen]: any strong cofibration $f : X \to Y$ can be factorized into a cofibration $i : X \to Z$ and an acyclic fibration $p : Z \to Y$, and f has the local LLP with respect to p, hence we can locally find a section $\sigma : Y \to Z$ of p, hence f is a local retract of i, hence a cofibration, the class of cofibrations being closed.

Definition 9.5.23 *We say that a morphism $f : X \to Y$ in $s\mathcal{E}$, or in $s\mathcal{E}_{/S} = (\mathfrak{s}\boldsymbol{SETS}_{\mathcal{E}})(S)$ is a* **weak equivalence** *if it can be factorized into an acyclic cofibration followed by an acyclic fibration.*

It is not evident from this definition that being a weak equivalence is a local property, or that weak equivalences satisfy the 2-out-of-3 axiom (MS3). We are going to check this in a moment, at least for topoi with enough points.

Proposition 9.5.24 *Any acyclic cofibration in $s\mathcal{E}$ is both a cofibration and a weak equivalence, and any acyclic fibration is both a fibration and a weak equivalence.*

Proof. (a) Acyclic cofibrations and acyclic fibrations are weak equivalences by definition. (b) Let us show that any acyclic fibration in $s\mathcal{E}$ is a fibration. Indeed, any morphism $\Lambda_k(n) \to \Delta(n)$ of J can be decomposed into $\Lambda_k(n) \to \dot{\Delta}(n) \to \Delta(n)$, and both morphisms of this decomposition are pushouts of morphisms from I. Clearly, this is still true for $\underline{I}_{\mathcal{E}}$ and $\underline{J}_{\mathcal{E}}$. Since strong cofibrations in $s\mathcal{E}$ contain $\underline{I}_{\mathcal{E}}$ and are stable under pushouts and composition, we see that $\underline{J}_{\mathcal{E}}$ consists of strong cofibrations, i.e. any acyclic fibration $p : X \to Y$ in $s\mathcal{E}$ has the local RLP with respect to all morphisms from $\underline{J}_{\mathcal{E}}$, i.e. is a fibration.

(c) Now let's prove that acyclic cofibrations are cofibrations, i.e. $\mathrm{Cl}\,\underline{J}_{\mathcal{E}} \subset \mathrm{Cl}\,\underline{I}_{\mathcal{E}}$. It is enough to show for this that $\underline{J}_{\mathcal{E}} \subset \mathrm{Cl}\,\underline{I}_{\mathcal{E}}$, i.e. that $\underline{J}_{\mathcal{E}}$ consists of cofibrations. But this is shown by the same reasoning as above: any morphism of $\underline{J}_{\mathcal{E}}$ is a composition of pushouts of morphisms from $\underline{I}_{\mathcal{E}}$, hence lies in $\mathrm{Cl}\,\underline{I}_{\mathcal{E}}$.

Proposition 9.5.25 *Let $f : \mathcal{E}' \to \mathcal{E}$ be any morphism of topoi. Then $f^* = sf^* : s\mathcal{E} \to s\mathcal{E}'$ preserves cofibrations, acyclic cofibrations, fibrations, acyclic fibrations and weak equivalences.*

Proof. (a) The stability of fibrations and acyclic fibrations under f^* has been already shown in **9.5.17**, since the property of $p : X \to Y$ to be a fibration or an acyclic fibration is equivalent to epimorphicity of certain morphisms between finite projective limits of components X_n and Y_n. (b) The stability of weak equivalences will immediately follow from that of acyclic cofibrations and acyclic fibrations. (c) Let's deal with the case of, say, cofibrations. We want to show that f^* maps $\mathrm{Cl}\,\underline{I}_{\mathcal{E}}$ into $\mathrm{Cl}\,\underline{I}_{\mathcal{E}'}$. Consider for this the class of morphisms $\mathscr{P}$ in fibers of $\mathfrak{s}\mathbf{SETS}_{\mathcal{E}}/\mathcal{E}$ defined as follows: a morphism $i : A \to B$ in $(\mathfrak{s}\mathbf{SETS}_{\mathcal{E}})(S) = s(\mathcal{E}_{/S})$ belongs to $\mathscr{P}$ iff $f_S^*(i)$ is a cofibration in $s(\mathcal{E}'_{/f^*S})$, where $f_S : \mathcal{E}'_{/f^*S} \to \mathcal{E}_{/S}$ denotes the induced morphism of topoi. One checks immediately that $\mathscr{P}$ is closed and contains $I_{\mathcal{E}}$, hence it has to contain $\mathrm{Cl}\,\underline{I}_{\mathcal{E}}$, i.e. all cofibrations of $\mathfrak{s}\mathbf{SETS}_{\mathcal{E}}$. This proves the required stability for cofibrations; the case of acyclic cofibrations is dealt with similarly, with $\underline{I}$ replaced by $\underline{J}$.

We have already remarked that when $\mathcal{E} = \mathit{Sets}$, we recover the usual model category structure on *Sets*, i.e. the (acyclic) cofibrations, fibrations and weak equivalences in *sSets* obtained from our definitions coincide with their classical counterparts.

Proposition 9.5.26 *Let $f : X \to Y$ be a morphism in $s\mathcal{E}$. If f is an acyclic cofibration, cofibration, acyclic fibration, fibration or weak equivalence in $s\mathcal{E}$, then $f_p = p^*(f) : X_p \to Y_p$ has the same property in sSets for any point $p : Sets \to \mathcal{E}$ of topos $\mathcal{E}$. Conversely, if $\mathcal{E}$ has enough points, and all $f_p : X_p \to Y_p$ are fibrations, acyclic fibrations or weak equivalences in sSets, then the same is true for f, i.e. these properties can be checked pointwise.*

Proof. (a) The first statement is a special case of **9.5.25**, once we take into account the remark made just before this proposition.

(b) Let's prove the second statement. The case of fibrations and acyclic fibrations is evident: indeed, $f : X \to Y$ is a fibration (resp. an acyclic fibration) iff certain morphisms between finite projective limits of components of X and Y are epimorphic (cf. **9.5.17**). Now if $\mathcal{E}$ has enough points, then $u : Z \to W$ is an epimorphism in $\mathcal{E}$ iff all $u_p : Z_p \to W_p$ are surjective; since p^* is exact, we obtain our statement for fibrations and acyclic fibrations.

(c) Suppose that all $f_p : X_p \to Y_p$ are weak equivalences in *sSets*. Consider the decomposition $X \xrightarrow{i} Z \xrightarrow{s} Y$ of f into an acyclic cofibration followed by a fibration, existing by **9.5.21**. For any point p the composition $s_p \circ i_p = f_p$ is a weak equivalence by assumption, as well as i_p, hence s_p is a weak equivalence as well by the 2-out-of-3 axiom in *sSets*, hence s_p is an acyclic fibration for all points p of $\mathcal{E}$, and we can conclude that s is an acyclic fibration in $\mathcal{E}$ by (b). This proves that $f = s \circ i$ is a weak equivalence, q.e.d.

Corollary 9.5.27 *If topos $\mathcal{E}$ has enough points, then weak equivalences in $\mathfrak{s}\boldsymbol{SETS}_{\mathcal{E}}$ constitute a local class, stable under retracts and composition, and satisfying the 2-out-of-3 axiom (MS3). Moreover, in this case $f : X \to Y$ is an acyclic fibration in $s\mathcal{E}$ iff it is both a fibration and a weak equivalence.*

Of course, these statements must be true without any assumption on $\mathcal{E}$. However, when $\mathcal{E}$ has enough points, we are able to deduce the above statements easily from the classical (highly non-trivial) fact that *sSets* is indeed a model category.

9.5.28. Question. Can cofibrations and acyclic cofibrations be checked pointwise? We believe that the answer is negative, and even that not all objects of $s\mathcal{E}$ are cofibrant. The reason for our disbelief is the following. The classical proof of this fact (in *sSets*) goes as follows: we write $A = \varinjlim_n \operatorname{sk}_n A$, and then write each $\operatorname{sk}_{n-1} A \to \operatorname{sk}_n A$ as a pushout of cofibration $\dot{\Delta}(n) \otimes \Sigma_n \to \Delta(n) \otimes \Sigma_n$, where $\Sigma_n \subset A_n$ is the subset of *non-degenerate simplices*, i.e. the complement of the union of images of all degeneracy maps $s_n^i : A_{n-1} \to A_n$. Now we observe that this proof is not intuitionistic since it uses the fact that any simplex is either degenerate or non-degenerate, so we see that this proof doesn't work in $s\mathcal{E}$, i.e. "there is no reason" for A to be cofibrant. Of course, this doesn't prove anything from the formal point of view. We have good candidates for such a non-cofibrant A over topos $\mathcal{E} = \hat{\mathcal{S}}$, $\mathcal{S} = [1] = \{0 \to 1\}$: consider for example the pushout of $\Delta(1) \otimes U \to \Delta(1)$ and $\Delta(1) \otimes U \to \Delta(0) \otimes U$, where U is the only non-trivial open object of $\mathcal{E}$, but the problem here is that it is quite complicated to show that something is *not* a cofibration…

9.5.29. Question. Is it true that if $f : X \to Y$ in $s\mathcal{E}$ is both a weak equivalence and a cofibration, then it is an acyclic cofibration? We expect the answer to this question to be positive, due to the fact that we have postulated stability of cofibrations and acyclic cofibrations under all "local sums" $\varphi_!$, not just under all $- \otimes_S T = \varphi_! \varphi^*$.

9.6. (Pseudomodel stacks.) Let us summarize the properties of distinguished classes of morphisms in $s\mathcal{E}$ and $\mathfrak{s}\mathbf{SETS}_{\mathcal{E}}$. We see that these classes do not satisfy the axioms for a model category or a model stack. The following definition describes a self-dual subset of properties that we have in $s\mathcal{E}$ and $\mathfrak{s}\mathbf{SETS}_{\mathcal{E}}$:

Definition 9.6.1 *(Pseudomodel categories.) We say that a category $\mathcal{C}$ with five distinguished local classes of morphisms (localness understood here as closedness under isomorphisms), called* cofibrations, acyclic cofibrations, fibrations, acyclic fibrations *and* weak equivalences *is a* **pseudomodel category** *if the following conditions hold:*

(PM1) The category $\mathcal{C}$ is closed under finite inductive and projective limits.

(PM2) Each of the five distinguished classes of morphisms in $\mathcal{C}$ is closed under composition and retracts. Acyclic cofibrations and cofibrations are stable under pushouts and finite direct sums. Acyclic fibrations and fibrations are stable under pullbacks and finite products.

(PM3) ("2 out of 3") Given two morphisms $f : X \to Y$ and $g : Y \to Z$, such that any two of f, g and $g \circ f$ are weak equivalences, then so is the third.

(PM5) (Factorization) Any morphism $f : X \to Y$ can be factored both into an acyclic cofibration followed by a fibration, or into a cofibration followed by an acyclic fibration: $f = p \circ j = q \circ i$, with i a cofibration, j an acyclic cofibration, p a fibration, and q an acyclic fibration.

(PM8) Any acyclic cofibration is both a weak equivalence and a cofibration. Any acyclic fibration is both a weak equivalence and a fibration. Any weak equivalence can be factorized into an acyclic cofibration followed by an acyclic fibration.

9.6.2. Notice the absence of the lifting axiom (MS4). All other axioms but (PM8) are more or less similar to those of a closed model category; however, we have to include some stabilities in (PM2), which are consequences of other axioms for model categories; and we mention *five* distinguished classes in this axiom. As to (PM8), notice that we don't require any of the opposite implications, summarized in the following stronger condition:

(PM8+) Acyclic cofibrations are exactly the morphisms which are both cofibrations and weak equivalences. Acyclic fibrations are exactly the morphisms which are both fibrations and weak equivalences.

The reason for omitting this stronger condition is that we have shown only its second half for $s\mathcal{E}$, and we want the axioms of a pseudomodel category to be self-dual.

Definition 9.6.3 *A* **pseudomodel stack** $\mathcal{C}$ *over a topos* $\mathcal{E}$ *is a flat stack* $\mathcal{C}/\mathcal{E}$ *together with five local classes of morphisms in its fibers, called as in* **9.6.1***, satisfying the above axioms (PM1)–(PM3), (PM5), (PM8) in each fiber* $\mathcal{C}(S)$.

Notice that each fiber $\mathcal{C}(S)$ of a pseudomodel stack $\mathcal{C}$ is a pseudomodel category.

9.6.4. We know that $s\mathcal{E}$ has a natural pseudomodel category structure, and $\mathfrak{s}\mathbf{SETS}_{\mathcal{E}}$ a natural pseudomodel stack structure, at least if $\mathcal{E}$ has enough points. These pseudomodel structures enjoy some additional properties, e.g. closedness of sets of cofibrations and acyclic cofibrations, cartesian functorial factorization (MS5f), and the second half of (PM8+).

9.6.5. (Homotopic category of a pseudomodel category.) Once we have a class of weak equivalences in a pseudomodel category $\mathcal{C}$, we can define the corresponding *homotopic category* $\operatorname{Ho}\mathcal{C} = \operatorname{Ho}_0\mathcal{C}$ as the localization of $\mathcal{C}$ with respect to the weak equivalences, at least if we don't mind enlarging the universe $\mathcal{U}$. In most cases one can show *a posteriori* that the homotopic category will be still a $\mathcal{U}$-category, so this enlargement of universe turns out to be unnecessary. Therefore, one can define left and right derived functors $\mathbb{L}F$, $\mathbb{R}F : \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{D}$ of any functor $F : \mathcal{C} \to \mathcal{D}$ between pseudomodel categories (cf. **8.1.14**). When $\mathcal{D}$ is an arbitrary category, without any pseudomodel structure, we apply this definition considering isomorphisms in $\mathcal{D}$ as weak equivalences. For example, $\mathbb{L}F : \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{D}$ is also the left derived functor of $\gamma_{\mathcal{D}} \circ F : \mathcal{C} \to \operatorname{Ho}\mathcal{D}$.

When it comes to considering homotopic fibered categories, (pre)stacks etc., we'll consider only the fibered category $\mathrm{HO}_0\,\mathcal{C}$, given by $(\mathrm{HO}_0\,\mathcal{C})(S) := \operatorname{Ho}\mathcal{C}(S)$.

9.6.6. (Cofibrant objects, replacements,...) We define cofibrant and fibrant objects, (strict) (co)fibrant replacements inside a pseudomodel category in the same way we did it before for model categories. Notations like $\mathcal{C}_c$, $\mathcal{C}_f$ and $\mathcal{C}_{cf}$ will be used to denote the full subcategories consisting of cofibrant objects etc. as before. Notice that $\mathcal{C}_c$, $\mathcal{C}_f$ and $\mathcal{C}_{cf}$ satisfy all the axioms of pseudomodel categories except (PM1), if we agree to consider in (PM2) only those direct sums, pushouts etc. which lie inside these subcategories.

Theorem 9.6.7 *Let $\mathcal{C}$ be a pseudomodel category, $F : \mathcal{C} \to \mathcal{B}$ any functor transforming weak equivalences between cofibrant objects into isomorphisms. Then F admits a left derived $\mathbb{L}F : \operatorname{Ho}\mathcal{C} \to \mathcal{B}$, which can be computed with the aid of cofibrant replacements, i.e. $\eta_P : \mathbb{L}F(\gamma P) \to F(P)$ is an isomorphism for all $P \in \operatorname{Ob}\mathcal{C}_c$, and $\mathbb{L}F(\gamma X) \cong F(P)$ for any cofibrant replacement $P \to X$ of X.*

Applying this theorem to $\mathcal{B} = \operatorname{Ho}\mathcal{D}$, we obtain

Corollary 9.6.8 *If $F : \mathcal{C} \to \mathcal{D}$ is a functor between pseudomodel categories, transforming weak equivalences between cofibrant objects of $\mathcal{C}$ into weak equivalences in $\mathcal{D}$, then F admits a left derived $\mathbb{L}F$, which can be computed by means of cofibrant replacements.*

Proof. (of **9.6.7**) The proof is somewhat similar to that of **9.4.13**.

(a) First of all, we construct a functor $\tilde{F} : \mathcal{C} \to \mathcal{B}$ as follows. For any $X \in \operatorname{Ob}\mathcal{C}$ we choose any its strict cofibrant replacement, i.e. any acyclic fibration $Q \xrightarrow{p} X$ with a cofibrant Q; such cofibrant replacements exist by (PM5). Now we put $\tilde{F}(X) := F(Q)$. We have to check that $\tilde{F}(X)$ doesn't depend on the choice of $Q \xrightarrow{p} X$ up to a canonical isomorphism. We reason as in the proof of **9.4.13**: if $Q' \xrightarrow{p'} X$ is another strict cofibrant replacement, we consider any strict cofibrant replacement $Q'' \xrightarrow{r} Q \times_X Q'$; composing r with the projections of $Q \times_X Q'$ and taking (PM2) into account, we obtain two acyclic fibrations $\sigma : Q'' \to Q$ and $\sigma' : Q'' \to Q'$ between cofibrant objects, such that $p\sigma = p'\sigma'$. Hence $F(\sigma)$ and $F(\sigma')$ are isomorphisms in $\mathcal{B}$, so we obtain an isomorphism $\varphi_{Q''} := F(\sigma') \circ F(\sigma)^{-1} : F(Q) \xrightarrow{\sim} F(Q')$, compatible with $F(p)$ and $F(p')$. Next, we need to check independence of this isomorphism of the choice of $Q'' \xrightarrow{r} Q \times_X Q'$; this is done similarly: if $Q_1'' \xrightarrow{r_1} Q \times_X Q'$ is another such choice, we choose any strict cofibrant replacement Q_2'' of the fibered product of Q'' and Q_1'' over $Q \times_X Q'$, thus obtaining acyclic fibrations $Q_2'' \xrightarrow{u} Q''$ and $Q_2'' \xrightarrow{v} Q_1''$. Now $\varphi_{Q_2''} = F(\sigma' \circ u) \circ F(\sigma \circ u)^{-1} = F(\sigma') \circ F(u) \circ F(u)^{-1} \circ F(\sigma)^{-1} = \varphi_{Q''}$, and $\varphi_{Q_2''} = \varphi_{Q_1''}$ for similar reasons.

(b) Once we have constructed $\tilde{F}$ on objects and have shown independence of $\tilde{F}(X)$ on the choice of strict cofibrant replacements, we can easily define

$\tilde{F}$ on morphisms. Indeed, let $f : X \to Y$ be any morphism in $\mathcal{C}$. Choose any strict cofibrant replacement $Q \xrightarrow{p} Y$; since acyclic fibrations are stable under base change, $Q \times_Y X \to X$ is still an acyclic fibration, so if we choose any strict cofibrant replacement $Q' \to Q \times_Y X$, the composite map $p' : Q' \to Q \times_Y X \to X$ will be still a cofibrant replacement, and the other composition $\bar{f} : Q' \to Q \times_Y X \to Q$ has the property $p \circ \bar{f} = f \circ p'$. Now $F(\bar{f}) : F(Q') \cong \tilde{F}(X) \to F(Q) \cong \tilde{F}(Y)$ is a natural candidate for $\tilde{F}(f)$. Its independence on the choice of $Q' \to Q \times_Y X$ and $Q \to Y$ is shown as in (a). After this $\tilde{F}(\mathrm{id}_X) = \mathrm{id}$ is checked immediately (put $Q' = Q$), and $\tilde{F}(g \circ f) = \tilde{F}(g) \circ \tilde{F}(f)$ for some $X \xrightarrow{f} Y \xrightarrow{g} Z$ is checked as follows: we first choose some strict cofibrant replacement $Q \to Z$, then a strict cofibrant replacement $Q' \to Q \times_Z Y$, and finally a strict cofibrant replacement $Q'' \to Q' \times_Y X$. Then we get morphisms $\bar{g} : Q' \to Q$, $\bar{f} : Q'' \to Q'$, such that $\tilde{F}(g \circ f)$, $\tilde{F}(g)$ and $\tilde{F}(f)$ are canonically identified with $F(\bar{g} \circ \bar{f})$, $F(\bar{g})$ and $F(\bar{f})$:

$$\begin{tikzcd} Q'' \arrow[r, "\bar{f}"] \arrow[d] & Q' \arrow[r, "\bar{g}"] \arrow[d] & Q \arrow[d] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \end{tikzcd} \tag{9.6.8.1}$$

(c) Notice that the 2-out-of-3 axiom (PM3) together with (PM8) imply that $\bar{f} : Q' \to Q$ is a weak equivalence whenever $f : X \to Y$ is one. Since Q' and Q are cofibrant, we conclude that $\tilde{F}(f) = F(\bar{f})$ is an isomorphism in $\mathcal{B}$, i.e. $\tilde{F}$ *transforms weak equivalences into isomorphisms.* By definition, this means that $\tilde{F}$ can be uniquely factorized through $\mathcal{C} \xrightarrow{\gamma} \mathrm{Ho}\,\mathcal{C}$. We denote the arising functor $\mathrm{Ho}\,\mathcal{C} \to \mathcal{B}$ by $\mathbb{L}F$. Thus $(\mathbb{L}F)(\gamma X) = \tilde{F}(X) \cong F(Q)$ for any strict cofibrant replacement $Q \to X$.

(d) We need a natural transformation $\eta : \mathbb{L}F \circ \gamma = \tilde{F} \to F$. It can be defined as follows: if $Q \xrightarrow{p} X$ is any strict cofibrant replacement of X, so that $\tilde{F}(X) \cong F(Q)$, we put $\eta_X := F(p)$. The independence of η_X on the choice of $Q \to X$ and its compatibility with morphisms follows immediately from (a) and (b). Notice that η_X is an isomorphism for any cofibrant X since we can take $Q = X$. We conclude that $(\mathbb{L}F)(\gamma X) \cong (\mathbb{L}F)(\gamma Q) \cong F(Q)$ for any cofibrant replacement $Q \to X$ (not necessarily strict).

(e) Now we want to show that $(\mathbb{L}F, \eta)$ is the left derived functor of F. Let (G, ξ), $G : \mathrm{Ho}\,\mathcal{C} \to \mathcal{B}$, $\xi : G \circ \gamma \to F$ be another such pair. We have to show existence and uniqueness of $\zeta : G \to \mathbb{L}F$, such that $\xi = \eta \circ (\zeta \star \gamma)$. By the universal property of localizations $\mathrm{Hom}(G, \mathbb{L}F) \cong \mathrm{Hom}(G \circ \gamma, \mathbb{L}F \circ \gamma)$, so we have to show existence and uniqueness of $\tilde{\zeta} := \zeta \star \gamma : G \circ \gamma \to \tilde{F} = \mathbb{L}F \circ \gamma$, such that $\xi = \eta \circ \tilde{\zeta}$.

(f) Uniqueness of $\tilde{\zeta}_X : G(\gamma X) \to \tilde{F}(X)$ is shown as follows. Choose any strict cofibrant replacement $Q \xrightarrow{p} X$ and consider the following commutative diagram:

$$\begin{tikzcd} G(\gamma Q) \arrow[r, "\tilde{\zeta}_Q"'] \arrow[rr, bend left, "\xi_Q"] \arrow[d, "G(\gamma(p))"', "\sim"] & \tilde{F}(Q) \arrow[r, "\sim", "\eta_Q"'] \arrow[d, "\tilde{F}(p)"', "\sim"] & F(Q) \arrow[d, "F(p)"] \\ G(\gamma X) \arrow[r, "\tilde{\zeta}_X"] \arrow[rr, bend right, "\xi_X"'] & \tilde{F}(X) \arrow[r, "\eta_X"] & F(X) \end{tikzcd} \tag{9.6.8.2}$$

We see that $\tilde{\zeta}_X = \tilde{F}(p) \circ \eta_Q^{-1} \circ \xi_Q \circ G(\gamma(p))^{-1}$, i.e. it is completely determined by ξ.

(g) Now it remains to show existence of $\tilde{\zeta}$, i.e. we need to show that the morphisms $\tilde{\zeta}_X$ defined as above do not depend on the choice of $Q \xrightarrow{p} X$ and are compatible with morphisms in $\mathcal{C}$. Independence on choice of Q is shown as in (a): we reduce to the case of $Q'' \xrightarrow{\sigma} Q \xrightarrow{p} X$, with σ an acyclic fibration between cofibrant objects, and a 3×3-diagram similar to (9.6.8.2), with one extra horizontal line corresponding to Q'' shows the required independence. As to the compatibility of $\tilde{\zeta}$ with morphisms $f : X \to Y$ in $\mathcal{C}$, it is checked by choosing $Q \to Y$, $Q' \to X$ and $\bar{f} : Q' \to Q$ as in (b), q.e.d.

9.6.9. (Construction of $\bar{Q}$.) Notice that the first part of the above proof, namely, construction of $\tilde{F} : \mathcal{C} \to \mathcal{B}$ and $\mathbb{L}F : \operatorname{Ho}\mathcal{C} \to \mathcal{B}$ in (a)–(c), works for any functor $F : \mathcal{C}_c \to \mathcal{B}$ transforming weak equivalences into isomorphisms. In particular, we can apply it to $\gamma_c : \mathcal{C}_c \to \operatorname{Ho}\mathcal{C}_c$, thus obtaining a functor $\bar{Q} : \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{C}_c$, such that $\bar{Q}(\gamma X)$ is canonically isomorphic to $\gamma_c Q$ for any strict cofibrant replacement $Q \xrightarrow{p} X$. On the other hand, we have a natural functor $I : \operatorname{Ho}\mathcal{C}_c \to \operatorname{Ho}\mathcal{C}$ induced by the embedding $\mathcal{C}_c \to \mathcal{C}$. Clearly, $\bar{Q} \circ I$ is canonically isomorphic to $\mathrm{Id}_{\operatorname{Ho}\mathcal{C}_c}$, since one can always take $Q = X$ while computing $\bar{Q}I(\gamma_c X) = \bar{Q}(\gamma X)$ for a cofibrant X. On the other hand, we have a canonical natural transformation $\eta : I \circ \bar{Q} \to \mathrm{Id}_{\operatorname{Ho}\mathcal{C}}$, given by $\eta_{\gamma X} := \gamma(p) : \gamma Q = I\bar{Q}(\gamma X) \to \gamma X$ as in part (d) of the above proof. Now it is immediate that this $\gamma(p)$ is an isomorphism in $\operatorname{Ho}\mathcal{C}$, p being an acyclic fibration, hence $I \circ \bar{Q}$ is also canonically isomorphic to the identity functor, i.e. *I and $\bar{Q}$ are adjoint equivalences between* $\operatorname{Ho}\mathcal{C}$ *and* $\operatorname{Ho}\mathcal{C}_c$. By duality $\operatorname{Ho}\mathcal{C}_f$ is also equivalent to $\operatorname{Ho}\mathcal{C}$.

9.6.10. (Construction of $\bar{R}_c$.) Notice that the above construction of equivalence $\bar{Q} : \operatorname{Ho}\mathcal{C}_c \to \operatorname{Ho}\mathcal{C}$ still works if we weaken the pseudomodel category axioms for $\mathcal{C}$ by requiring just existence of pullbacks of acyclic fibrations instead of (PM1). Since $\mathcal{C}_c$ satisfies the dual of this condition (it is stable under pushouts of acyclic cofibrations), we can apply the above reasoning to

$\mathcal{C}_c^0$, thus obtaining an equivalence of categories $\bar{R}_c : \operatorname{Ho}\mathcal{C}_c \to \operatorname{Ho}\mathcal{C}_{cf}$, adjoint to the natural "embedding" $\operatorname{Ho}\mathcal{C}_{cf} \to \operatorname{Ho}\mathcal{C}_c$. We conclude that *similarly to the classical case of model categories,* $\operatorname{Ho}\mathcal{C} \cong \operatorname{Ho}\mathcal{C}_c \cong \operatorname{Ho}\mathcal{C}_{cf} \cong \operatorname{Ho}\mathcal{C}_f$ *for any pseudomodel category* $\mathcal{C}$.

Lemma 9.6.11 *Let $\mathcal{C}$ be pseudomodel category satisfying the first half of (PM8+), i.e. such that the acyclic cofibrations are exactly the cofibrations which are also weak equivalences. Suppose that $F : \mathcal{C}_c \to \mathcal{B}$ transforms acyclic cofibrations between objects of $\mathcal{C}_c$ into isomorphisms in $\mathcal{B}$. Then F transforms weak equivalences in $\mathcal{C}_c$ into isomorphisms in $\mathcal{B}$. In particular, if $F : \mathcal{C} \to \mathcal{B}$ transforms acyclic cofibrations between cofibrant objects into isomorphisms, then* **9.6.7** *is applicable.*

Proof. The statement is essentially that of **8.6.3**, but we need to replace our original argument by another one, attributed in [DwyerSpalinski] to K. Brown. This argument goes as follows. Let $f : X \to Y$ be a weak equivalence in $\mathcal{C}_c$. Consider the factorization of the "cograph" $\langle f, \mathrm{id}_Y \rangle : X \sqcup Y \to Y$ into a cofibration $q : X \sqcup Y \to Z$ followed by an acyclic fibration $p : Z \to Y$. The natural embeddings $i : X \to X \sqcup Y$, $j : Y \to X \sqcup Y$ are also cofibrant, X and Y being cofibrant, and $pqi = f$, $pqj = \mathrm{id}_Y$ and p are weak equivalences, hence qi and qj are both weak equivalences and cofibrations, hence acyclic cofibrations. By assumption $F(qi)$ and $F(qj)$ are isomorphisms in $\mathcal{B}$, as well as $F(pqj) = F(\mathrm{id}_Y)$, hence $F(p)$ and $F(f) = F(pqi)$ are also isomorphisms, q.e.d.

9.6.12. (Computation of $\operatorname{Ho}\mathcal{C}$ and $\operatorname{Ho}\mathcal{C}_{cf}$ by means of mixed fraction calculus.) Quillen remarks in [Quillen, 1.1] that $\operatorname{Ho}\mathcal{C}$ and $\operatorname{Ho}\mathcal{C}_{cf}$ do not admit left or right fraction calculus, but can be computed by a mixture of both. We claim that a similar statement is true in any of $\operatorname{Ho}\mathcal{C}$, $\operatorname{Ho}\mathcal{C}_c$, $\operatorname{Ho}\mathcal{C}_f$ and $\operatorname{Ho}\mathcal{C}_{cf}$. Namely, any morphism $\varphi \in \operatorname{Hom}_{\operatorname{Ho}\mathcal{C}}(\gamma X, \gamma Y)$ can be written in form $\gamma(i)^{-1} \circ \gamma(f) \circ \gamma(p)^{-1}$, where $X' \xrightarrow{p} X$ is a strict cofibrant replacement, $Y \xrightarrow{i} Y'$ is a strict fibrant replacement, and $f : X' \to Y'$ a morphism in $\mathcal{C}$, and similarly for any of the smaller categories. In order to show this one simply checks that the set of all such φ's is closed under composition and contains all morphisms from $\mathcal{C}$ and the inverses of all weak equivalences of $\mathcal{C}$. Next, two such expressions $\gamma(i_1)^{-1}\gamma(f_1)\gamma(p_1)^{-1}$ and $\gamma(i_2)^{-1}\gamma(f_2)\gamma(p_2)^{-1}$ are equal, where $X \xleftarrow{p_k} X'_k \xrightarrow{f_k} Y'_k \xleftarrow{i_k} Y$ are as above, iff one can find a strict cofibrant replacement $X' \to X'_1 \times_X X'_2$, a strict fibrant replacement $Y'_1 \coprod_Y Y'_2 \to Y'$ and a morphism $f : X' \to Y'$ making the obvious diagram commutative.

This is almost all we can obtain without the lifting properties, which in the case of model categories enable us to show that any morphism in

$\operatorname{Ho}\mathcal{C}_{cf} \cong \operatorname{Ho}\mathcal{C}$ comes from some morphism in $\mathcal{C}_{cf}$.

9.6.13. (Pointwise pseudomodel structure on $s\mathcal{E}$ and $\mathfrak{s}\mathbf{SETS}_\mathcal{E}$.) Notice that, while the five distinguished sets of morphisms in $s\mathcal{E}$ and $\mathfrak{s}\mathbf{SETS}_\mathcal{E}$ have been defined without any reference to points, we have checked that we do obtain a pseudomodel structure in this way only when $\mathcal{E}$ has enough points. But in this case we can replace acyclic cofibrations and cofibrations by (potentially larger) classes of *pointwise acyclic cofibrations* and *pointwise cofibrations* (of course, a morphism $f : X \to Y$ in $s\mathcal{E}$ is a pointwise cofibration iff $f_p : X_p \to Y_p$ is a cofibration in *sSets* for all points p of $\mathcal{E}$). Enlarging the three other classes in this manner wouldn't produce anything new because of **9.5.26**. In this way we obtain another pseudomodel structure on $s\mathcal{E}$ and $\mathfrak{s}\mathbf{SETS}_\mathcal{E}$, which will be called the *pointwise pseudomodel structure*. In fact, all axioms of pseudomodel structure but (PM5) can be checked now pointwise, and factorization (PM5), and even the functorial factorization (MS5f), follows from our previous results **9.5.21**. Furthermore, the classes of pointwise (acyclic) cofibrations are closed, and we even have (PM8+).

We see that in some respects this pointwise pseudomodel structure is even better than the one we had before; it would be nice to obtain a description of pointwise (acyclic) cofibrations that would be valid in topoi without enough points. For example, a positive answer to **9.5.28** would yield such a description, thus actually showing that our original pseudomodel structure on $s\mathcal{E}$ is even better than we thought before.

In any case, the weak equivalences for these two structures are the same, hence the corresponding homotopic categories $\operatorname{Ho} s\mathcal{E}$ and derived functors also coincide, i.e. *we can freely use pointwise pseudomodel structures while computing derived functors.*

9.7. (Pseudomodel structure on $\mathfrak{s}\mathscr{O}$-$\mathbf{MOD}_\mathcal{E}$ and $s\mathscr{O}$-*Mod*.) Let $(\mathcal{E}, \mathscr{O})$ be a generalized ringed topos, i.e. $\mathscr{O}$ be an arbitrary (inner) algebraic monad over a topos $\mathcal{E}$ (cf. **5.2.3** and **4.9**; notice that we don't need commutativity of $\mathscr{O}$ here). We want to construct a pseudomodel category structure on the category $s\mathscr{O}$-*Mod* of simplicial $\mathscr{O}$-modules in $\mathcal{E}$, as well as a pseudomodel stack structure on the flat stack $\mathfrak{s}\mathscr{O}$-$\mathbf{MOD}_\mathcal{E}$.

9.7.1. (Basic definitions.) Since $L_\Sigma(I)$ and $L_\Sigma(J)$ are cofibrant generators of our standard model category structure on $s\Sigma$-*Mod*, for any algebraic monad Σ, we are tempted to take the sets $L_\mathscr{O}(\underline{I}_\mathcal{E})$ and $L_\mathscr{O}(\underline{J}_\mathcal{E})$ of morphisms inside $(\mathfrak{s}\mathscr{O}\text{-}\mathbf{MOD}_\mathcal{E})(e) = s\mathscr{O}$-*Mod* as the "cofibrant generators" for a pseudomodel structure on $\mathfrak{s}\mathscr{O}$-$\mathbf{MOD}_\mathcal{E}$. Here $L_\mathscr{O}$ denotes both the free $\mathscr{O}$-module functor $\mathcal{E} \to \mathscr{O}$-*Mod* and its simplicial extension $s\mathcal{E} \to s\mathscr{O}$-*Mod*.

More precisely:

- A morphism $f : X \to Y$ in $(\mathfrak{s}\mathscr{O}\text{-}\mathbf{MOD}_{\mathcal{E}})(S) = s\mathscr{O}|_S\text{-}Mod$ is a *fibration* (resp. *acyclic fibration*) iff it has the local RLP with respect to all morphisms from $L_{\mathscr{O}}(\underline{J})$ (resp. $L_{\mathscr{O}}(\underline{I})$).

- A morphism $f : X \to Y$ in $s\mathscr{O}|_S\text{-}Mod$ is a *cofibration* (resp. *acyclic cofibration*) iff it belongs to the closure $\operatorname{Cl} L_{\mathscr{O}}(\underline{I})$ (resp. $\operatorname{Cl} L_{\mathscr{O}}(\underline{J})$) in the sense of **9.5.11**.

- Finally, a morphism $f : X \to Y$ in $s\mathscr{O}|_S\text{-}Mod$ is a *weak equivalence* iff it can be factorized into an acyclic cofibration followed by an acyclic fibration.

9.7.2. Notice that adjoint functors $L_{\mathscr{O}}$ and $\Gamma_{\mathscr{O}}$ between $\mathscr{O}$-*Mod* and $\mathcal{E}$ extend to adjoint cartesian functors $\boldsymbol{L}_{\mathscr{O}}$ and $\boldsymbol{\Gamma}_{\mathscr{O}}$ between corresponding stacks $\mathscr{O}\text{-}\mathbf{MOD}_{\mathcal{E}}$ and $\mathbf{SETS}_{\mathcal{E}}$, and the same applies to their simplicial extensions $\mathfrak{s}\mathscr{O}\text{-}\mathbf{MOD}_{\mathcal{E}} \leftrightarrows \mathfrak{s}\mathbf{SETS}_{\mathcal{E}}$; resulting adjoint cartesian functors will be also denoted by $\boldsymbol{L}_{\mathscr{O}}$ and $\boldsymbol{\Gamma}_{\mathscr{O}}$, or even by $L_{\mathscr{O}}$ and $\Gamma_{\mathscr{O}}$, when no confusion can arise.

In particular, we obtain the following formulas:

$$\operatorname{Hom}_{s\mathscr{O}\text{-}Mod}(L_{\mathscr{O}}(A), X) \cong \operatorname{Hom}_{s\mathcal{E}}(A, \Gamma_{\mathscr{O}}(X)) \tag{9.7.2.1}$$

$$\mathbf{Hom}_{\mathfrak{s}\mathscr{O}\text{-}\mathbf{MOD}_{\mathcal{E}}}(L_{\mathscr{O}}(A), X) \cong \mathbf{Hom}_{\mathfrak{s}\mathbf{SETS}_{\mathcal{E}}}(A, \Gamma_{\mathscr{O}}(X)) \tag{9.7.2.2}$$

9.7.3. (Fibrations and acyclic fibrations in $\mathscr{O}$-*Mod*.) The second of the above formulas immediately implies that some $p : X \to Y$ in $s\mathscr{O}$-*Mod* has the local RLP with respect to $L_{\mathscr{O}}(i)$ for some $i : A \to B$ in $s\mathcal{E}$ iff $\Gamma_{\mathscr{O}}(p)$ has the local RLP with respect to i. Considering here all i from $\underline{I}_{\mathcal{E}}$ (resp. $\underline{J}_{\mathcal{E}}$), we conclude that *$p : X \to Y$ is an acyclic fibration (resp. fibration) in $s\mathscr{O}$-Mod iff $\Gamma_{\mathscr{O}}(p)$ is an acyclic fibration (resp. fibration) in $s\mathcal{E}$*. Furthermore, the latter condition can be expressed in terms of epimorphicity of certain morphisms between finite projective limits of components of $\Gamma_{\mathscr{O}}(X)$ and $\Gamma_{\mathscr{O}}(Y)$. Now notice that $\Gamma_{\mathscr{O}} : \mathscr{O}\text{-}Mod \to \mathcal{E}$ is left exact, and $\Gamma_{\mathscr{O}}(f)$ is an epimorphism in $\mathcal{E}$ iff f is a strict epimorphism in $\mathscr{O}$-*Mod*. We conclude that *$p : X \to Y$ is an acyclic fibration (resp. fibration) in $s\mathscr{O}$-Mod iff certain morphisms between finite projective limits of components of X and Y are strictly epimorphic*, and that *$p : X \to Y$ is an acyclic fibration in $s\mathscr{O}$-Mod iff all $X_n \to Y_n \times_{(\operatorname{cosk}_{n-1} Y)_n} (\operatorname{cosk}_{n-1} X)_n$ are strict epimorphisms in $\mathscr{O}$-Mod.*

Proposition 9.7.4 *Let $f : \mathcal{E}' \to \mathcal{E}$ be a morphism of topoi, $\mathscr{O}$ a generalized ring in $\mathcal{E}$, $\mathscr{O}' := f^*\mathscr{O}$ its pullback to $\mathcal{E}'$. Then all five distinguished classes in $s\mathscr{O}$-Mod are stable under the induced pullback functor $f^* : s\mathscr{O}\text{-}Mod \to s\mathscr{O}'\text{-}Mod$.*

Proof. The proof is completely similar to that of **9.5.25**. (a) Stability of acyclic cofibrations and cofibrations is shown as in *loc.cit.*: we consider the class $\mathscr{P}$ of morphisms in fibers of $\mathfrak{s}\mathscr{O}$-$\mathbf{MOD}_{\mathcal{E}}$, consisting of those $\xi : X \to Y$ in $s\mathscr{O}|_S$-*Mod* which become cofibrations (resp. acyclic cofibrations) in $s\mathscr{O}'|_{f^*S}$-*Mod* after application of f_S^*, easily check that $\mathscr{P}$ is closed and that it contains $L_{\mathscr{O}}(\underline{I_{\mathcal{E}}})$ (resp. $L_{\mathscr{O}}(\underline{J_{\mathcal{E}}})$) since $f^* \circ L_{\mathscr{O}} = L_{\mathscr{O}'} \circ f^*$, hence $\mathscr{P}$ has to contain $\mathrm{Cl}\, L_{\mathscr{O}}(\underline{I_{\mathcal{E}}})$ (resp....), i.e. all cofibrations (resp. acyclic cofibrations) of $\mathfrak{s}\mathscr{O}$-$\mathbf{MOD}_{\mathcal{E}}$. (b) Stability of acyclic fibrations and fibrations immediately follows from *loc.cit.* and **9.7.3**, since $\Gamma_{\mathscr{O}'} \circ f^* = f^* \circ \Gamma_{\mathscr{O}}$. (c) Stability of weak equivalences follows from that of acyclic cofibrations and acyclic fibrations.

9.7.5. (Simple cases.) (a) When $\mathcal{E} = Sets$, hence $\mathscr{O}$ is just a (constant) algebraic monad, then the five distinguished classes in $s\mathscr{O}$-*Mod* defined in **9.7.1** coincide with those given by the model category structure on $s\mathscr{O}$-*Mod* defined in **8.4.8**. Indeed, this is evident for fibrations and acyclic fibrations; as to cofibrations and acyclic cofibrations with respect to this model structure, they clearly constitute closed sets of morphisms containing $L_{\mathscr{O}}(I)$ (resp. $L_{\mathscr{O}}(J)$), hence also the closures of these sets. The opposite inclusion is shown by Quillen's small object argument, which establishes that any cofibration (resp. acyclic cofibration) is a retract of a morphism of $L_{\mathscr{O}}(I)$ (resp. $L_{\mathscr{O}}(J)$), (CM4) being fulfilled in $s\mathscr{O}$-*Mod*, hence lies itself in $L_{\mathscr{O}}(I)$ (resp....). Finally, the case of weak equivalences follows from those of acyclic cofibrations and acyclic fibrations.

(b) On the other hand, we can take $\mathscr{O} = \mathbb{F}_{\varnothing}$, considered as a constant algebraic monad over any topos $\mathcal{E}$. Then $s\mathscr{O}$-*Mod* $= s\mathcal{E}$, $L_{\mathscr{O}}(\underline{I_{\mathcal{E}}}) = \underline{I_{\mathcal{E}}}$, $L_{\mathscr{O}}(\underline{J_{\mathcal{E}}}) = \underline{J_{\mathcal{E}}}$, and we recover the pseudomodel structure on $s\mathcal{E}$ considered before in **9.7**.

Proposition 9.7.6 *If $f : X \to Y$ belongs to any of five distinguished classes of morphisms in $s\mathscr{O}$-Mod, then $f_p : X_p \to Y_p$ belongs to the same class in $s\mathscr{O}_p$-Mod for any point p of $\mathcal{E}$. Conversely, if $\mathcal{E}$ has enough points, and if all $f_p : X_p \to Y_p$ are fibrations (resp. acyclic fibrations, weak equivalences) in $s\mathscr{O}_p$-Mod, then $f : X \to Y$ is itself a fibration (resp....) in $s\mathscr{O}$-Mod.*

Proof. Completely similar to that of **9.5.26**.

Corollary 9.7.7 *If $\mathcal{E}$ has enough points, then* **9.7.1** *defines a pseudomodel structure on $s\mathscr{O}$-Mod and $\mathfrak{s}\mathscr{O}$-$\boldsymbol{MOD}_{\mathcal{E}}$, satisfying the second half of (PM8+), the functorial factorization (MS5f), and having closed classes of cofibrations and acyclic cofibrations.*

Proof. All statements are deduced pointwise from corresponding statements for $s\mathscr{O}$-*Mod*$_p$, valid since $s\mathscr{O}$-*Mod*$_p$ is known to be a model category by **8.4.8**. The only exception is the (functorial) factorization (MS5f), which can be again shown by "local" Quillen's small object argument **9.5.20**, applied to $L_{\mathscr{O}}(\underline{I})$ and $L_{\mathscr{O}}(\underline{J})$, provided we show that these morphisms have locally small sources in $s\mathscr{O}$-*Mod*. But this is immediate from the fact that $\Gamma_{\mathscr{O}}$ commutes with filtered inductive limits, combined with (9.7.2.2), which actually implies that $L_{\mathscr{O}}$ transforms locally small objects of $s\mathcal{E}$ into locally small objects of $s\mathscr{O}$-*Mod*.

9.7.8. (Pointwise pseudomodel structure on $s\mathscr{O}$-*Mod*.) When $\mathcal{E}$ has enough points, we can replace cofibrations and acyclic cofibrations with (potentially larger) classes of pointwise (acyclic) cofibrations, similarly to what we did in **9.6.13**. In this way we obtain a "better" pseudomodel structure on $s\mathscr{O}$-*Mod*, which has closed classes of cofibrations and acyclic cofibrations, and satisfies (MS5f) and (PM8+). We call it the *pointwise pseudomodel structure on* $s\mathscr{O}$-*Mod*. Of course, this pseudomodel structure has the same weak equivalences as before, hence it gives rise to the same homotopic category $\mathcal{D}^{\leq 0}(\mathcal{E},\mathscr{O}) = \operatorname{Ho} s\mathscr{O}\text{-}\mathit{Mod}$ and same derived functors.

9.7.9. (Derived category of $(\mathcal{E},\mathscr{O})$.) We denote by $\mathcal{D}^{\leq 0}(\mathcal{E},\mathscr{O})$ or by $\mathcal{D}^{\leq 0}(\mathscr{O})$ the homotopic category $\operatorname{Ho} s\mathscr{O}$-*Mod*, i.e. the localization of $s\mathscr{O}$-*Mod* with respect to weak equivalences. If $\mathscr{O}$ admits a zero (i.e. a central constant), we have well-defined suspension functors Σ on $(s\mathscr{O}\text{-}\mathit{Mod})_c$ and $\mathbb{L}\Sigma$ on $\mathcal{D}^{\leq 0}(\mathcal{E},\mathscr{O})$; formally inverting this functor in the usual manner (cf. **8.6.12**), we obtain the *stable* homotopic category $\mathcal{D}^{-}(\mathcal{E},\mathscr{O})$ and a natural functor $\mathcal{D}^{\leq 0}(\mathcal{E},\mathscr{O}) \to \mathcal{D}^{-}(\mathcal{E},\mathscr{O})$.

9.7.10. (Additive case.) Now suppose that $\mathscr{O}$ is additive, i.e. it admits a central zero and addition. Then $\mathscr{O}$ is nothing else than a classical ring in $\mathcal{E}$, i.e. $(\mathcal{E},\mathscr{O})$ is a ringed topos in the classical sense. In this case $\mathscr{O}$-*Mod* is additive, hence the Dold–Kan correspondence (cf. **8.5.5**) establishes an equivalence between the category $s\mathscr{O}$-*Mod* of simplicial $\mathscr{O}$-modules and the category $\mathrm{Ch}^{\geq 0}(\mathscr{O}\text{-}\mathit{Mod})$ of non-negative chain complexes of $\mathscr{O}$-modules. Furthermore, weak equivalences in $s\mathscr{O}$-*Mod* correspond exactly to quasi-isomorphisms in $\mathrm{Ch}^{\geq 0}(\mathscr{O}\text{-}\mathit{Mod})$, at least if $\mathcal{E}$ has enough points: indeed, in this case both weak equivalences and quasi-isomorphisms can be checked pointwise, and we are reduced to check this statement over *Sets*, where it is classical. We conclude that $\operatorname{Ho} s\mathscr{O}$-*Mod is indeed equivalent to the full subcategory* $\mathcal{D}^{\leq 0}(\mathcal{E},\mathscr{O})$ *of* $\mathcal{D}(\mathcal{E},\mathscr{O}) := \mathcal{D}(\mathscr{O}\text{-Mod})$, hence *the corresponding stable homotopic category* $\mathcal{D}^{-}(\mathcal{E},\mathscr{O})$ *is equivalent to* $\mathcal{D}^{-}(\mathscr{O}\text{-Mod})$.

9.7.11. (Cofibrant replacements in the additive case.) Furthermore, one checks immediately that the class of injective chain maps with flat cokernels is closed in $\mathrm{Ch}^{\geq 0}(\mathscr{O}\text{-}\mathit{Mod})$ (more precisely, in corresponding stack), and it contains complexes corresponding to simplicial objects $L_{\mathscr{O}}(\underline{I}_{\mathcal{E}})$. This implies that *cofibrations in* $s\mathscr{O}$*-Mod correspond to (some set of) injective chain maps in* $\mathrm{Ch}^{\geq 0}(\mathscr{O}\text{-}Mod)$ *with flat cokernels*. In particular, cofibrant replacements $Q \to X$ in $s\mathscr{O}$-*Mod* yield flat resolutions $N(Q) \to N(X)$ in $\mathrm{Ch}(\mathscr{O}\text{-}Mod)$. This means that deriving functors by means of cofibrant replacements corresponds to deriving functors by means of flat resolutions.

We'll use this remark later to compare our derived tensor products and pullbacks with their classical additive counterparts.

Theorem 9.7.12 *(Derived pullbacks.) Let* $f = (\varphi, \theta) : (\mathcal{E}', \mathscr{O}') \to (\mathcal{E}, \mathscr{O})$ *be a morphism of generalized ringed topoi, i.e.* $\varphi : \mathcal{E}' \to \mathcal{E}$ *is a morphism of topoi and* $\theta : \mathscr{O} \to \varphi_* \mathscr{O}'$ *is a morphism of algebraic inner monads on* $\mathcal{E}$*. Suppose that* $\mathcal{E}'$ *and* $\mathcal{E}$ *have enough points. Then the pullback functor* $f^* : s\mathscr{O}\text{-}Mod \to s\mathscr{O}'\text{-}Mod$ *admits a left derived* $\mathbb{L}f^* : \mathcal{D}^{\leq 0}(\mathcal{E}, \mathscr{O}) \to \mathcal{D}^{\leq 0}(\mathcal{E}', \mathscr{O}')$*, which can be computed by means of cofibrant replacements. When both* $\mathscr{O}$ *and* $\mathscr{O}'$ *are additive,* $\mathbb{L}f^*$ *is identified with its classical counterpart via Dold–Kan correspondence.*

Proof. Let's endow $s\mathscr{O}$-*Mod* and $s\mathscr{O}'$-*Mod* with their pointwise pseudomodel structures. According to **9.6.7**, all we have to check is that f^* preserves weak equivalences between pointwise cofibrant objects. But for any such $u : A \to B$ in $s\mathscr{O}$-*Mod* and any point p of $\mathcal{E}'$ we have $f^*(u)_p = (\theta_p^\sharp)^*(u_{f(p)})$, where $(\theta_p^\sharp)^*$ denotes the scalar extension with respect to algebraic monad homomorphism $\theta_p^\sharp : \mathscr{O}_{f(p)} \to \mathscr{O}'_p$. Now everything follows from the fact that weak equivalences can be checked pointwise and that $(\theta_p^\sharp)^*$ preserves weak equivalences between cofibrant objects by **8.6.2**.

9.7.13. (Functoriality of derived pullbacks.) One can check directly that f^* preserves cofibrations and acyclic cofibrations, using the same trick as in the proof of **9.7.4**. In particular, it preserves cofibrant objects and pointwise cofibrant objects. This immediately implies that whenever $g : (\mathcal{E}'', \mathscr{O}'') \to (\mathcal{E}', \mathscr{O}')$ is another morphism of generalized ringed topoi with enough points, we have $\mathbb{L}(fg)^* = \mathbb{L}g^* \circ \mathbb{L}f^*$.

9.7.14. (Pullbacks with respect to flat morphisms.) Suppose that $f = (\varphi, \theta) : (\mathcal{E}', \mathscr{O}') \to (\mathcal{E}, \mathscr{O})$ is *flat*, i.e. $\theta^\sharp : \varphi^* \mathscr{O} \to \mathscr{O}'$ is a flat extension of algebraic monads in $\mathcal{E}'$, i.e. $(\theta^\sharp)^* : \varphi^*\mathscr{O}\text{-}Mod \to \mathscr{O}'\text{-}Mod$ is exact, and in particular preserves finite projective limits and strict epimorphisms. Since the property of some $p : X \to Y$ to be a fibration or an acyclic fibration in $s\mathscr{O}$-*Mod* can be expressed in terms of such limits and strict epimorphisms, we see that f^* *preserves fibrations and acyclic fibrations for any flat* f. We have already the same property for (acyclic) cofibrations, hence for weak equivalences as well. Now a functor f^* that preserves weak equivalences can be derived in the trivial manner, usually expressed by the formula $\mathbb{L}f^* = f^*$.

9.8. (Derived local tensor products.) Henceforth we assume $(\mathcal{E}, \mathscr{O})$ to be a generalized *commutatively* ringed topos. Then we have a tensor product functor $\otimes = \otimes_{\mathscr{O}} : \mathscr{O}\text{-}Mod \times \mathscr{O}\text{-}Mod \to \mathscr{O}\text{-}Mod$, which canonically extends to a cartesian functor $\otimes : \mathscr{O}\text{-}\mathbf{MOD}_{\mathcal{E}} \times_{\mathcal{E}} \mathscr{O}\text{-}\mathbf{MOD}_{\mathcal{E}} \to \mathscr{O}\text{-}\mathbf{MOD}_{\mathcal{E}}$. Moreover, the simplicial extensions of these functors define an ACU $\otimes$-structure on category $s\mathscr{O}\text{-}Mod$ and on stack $\mathfrak{s}\mathscr{O}\text{-}\mathbf{MOD}_{\mathcal{E}}$. We want to prove the following statement:

Theorem 9.8.1 *Suppose that $\mathcal{E}$ has enough points. Then functor $\otimes : s\mathscr{O}\text{-}Mod \times s\mathscr{O}\text{-}Mod \to s\mathscr{O}\text{-}Mod$ admits a left derived functor $\underline{\underline{\otimes}} = \mathbb{L}\otimes$, which can be computed with the aid of cofibrant replacements in each variable, i.e. $X \underline{\underline{\otimes}} X' \cong Q \otimes Q'$, for any cofibrant replacements $Q \to X$ and $Q' \to X'$ in $s\mathscr{O}\text{-}Mod$. Moreover, $\underline{\underline{\otimes}}$ defines an ACU $\otimes$-structure on $\operatorname{Ho} s\mathscr{O}\text{-}Mod = \mathcal{D}^{\leq 0}(\mathcal{E}, \mathscr{O})$.*

Proof. Of course, the "product" pseudomodel structure on a product of pseudomodel categories $\mathcal{C}_1 \times \mathcal{C}_2$ or on a fibered product of pseudomodel stacks $\mathcal{C}_1 \times_{\mathcal{E}} \mathcal{C}_2$ is defined as in **8.7.2**, i.e. $f = (f_1, f_2) : (X_1, X_2) \to (Y_1, Y_2)$ belongs to one of five distinguished classes of morphisms in $\mathcal{C}_1 \times \mathcal{C}_2$ iff both $f_i : X_i \to Y_i$, $i = 1, 2$, belong to the same distinguished class in $\mathcal{C}_i$. Therefore, everything will follow from **9.6.7**, provided we manage to show that $X \otimes Y$ is cofibrant when both X and Y are cofibrant (in $s\mathscr{O}\text{-}Mod$), and that a tensor product of weak equivalences between cofibrant objects is again a weak equivalence. Now this is very easily done for the pointwise pseudomodel structure on $s\mathscr{O}\text{-}Mod$, for which both statements follow immediately from our former results of **8.7**. We know that both pseudomodel structures on $s\mathscr{O}\text{-}Mod$ have same weak equivalences, hence same homotopic categories and derived functors, so we obtain all statements of the theorem for the "smaller" pseudomodel structure as well, especially if we take into account that cofibrant replacements for the "smaller" pseudomodel structure on $s\mathscr{O}\text{-}Mod$ are automatically pointwise cofibrant replacements, so they can be used to compute $\underline{\underline{\otimes}}$.

9.8.2. Notice that we didn't actually show that the tensor product of cofibrant simplicial $\mathscr{O}$-modules is cofibrant, since the corresponding statement for pointwise cofibrant objects suffices for proving **9.8.1**. However, we'll show in a moment that this statement is true, and, moreover, that a tensor product of acyclic cofibrations between cofibrant objects is again an acyclic cofibration.

9.8.3. (Compatible $\otimes$-structures and $\otimes$-actions.) Suppose that $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{D}$ are three pseudomodel categories, and $\otimes : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{D}$ is a bifunctor, commuting with inductive limits in each argument. We say that $\otimes$ is *compatible with given pseudomodel structures* if the axiom (TM) of **8.7.4** holds. This definition is in particular applicable to $\otimes$-structures on pseudomodel categories and external $\otimes$-actions of one pseudomodel category on another. Moreover, it can be easily extended to the case of pseudomodel stacks $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{D}$ over a topos $\mathcal{E}$ and a cartesian bifunctor $\otimes : \mathcal{C}_1 \times_{\mathcal{E}} \mathcal{C}_2 \to \mathcal{D}$: in this situation we just require (TM) to hold fiberwise.

Notice that whenever $\otimes$ is a compatible ACU $\otimes$-structure on a pseudomodel category, we show as in **8.7.6** that $A \otimes X$ is cofibrant whenever A and X are cofibrant, and that a tensor product of acyclic cofibrations between cofibrant objects is an acyclic cofibration.

Proposition 9.8.4 *Tensor product on $s\mathscr{O}$-Mod and on $\mathfrak{s}\mathscr{O}$-**MOD**$_{\mathcal{E}}$ is compatible with their pseudomodel structure. This applies in particular to $\otimes_{\mathbb{F}_\varnothing} = \times$ on $s\mathcal{E}$.*

Proof. When $\mathcal{E}$ has enough points, this compatibility for the pointwise pseudomodel structure is immediate. Let us show the statement for the "smaller" classes of cofibrations and acyclic cofibrations, given by $\mathrm{Cl}\, L_{\mathscr{O}}(\underline{I}_{\mathcal{E}})$, $\mathrm{Cl}\, L_{\mathscr{O}}(\underline{J}_{\mathcal{E}})$, without assuming $\mathcal{E}$ to have enough points (strictly speaking, we haven't shown that we obtain a pseudomodel structure in this case, but this isn't necessary to discuss the validity of (TM)).

(a) Let us denote by $Z(i,s)$ the source of $i \,\square\, s$, i.e. the pushout of $i \otimes \mathrm{id}_K : A \otimes K \to B \otimes K$ and $\mathrm{id}_A \otimes s : A \otimes K \to A \otimes L$. Then Z is a bifunctor $\mathrm{Ar}\,\mathcal{C} \times \mathrm{Ar}\,\mathcal{C} \to \mathcal{C}$, where $\mathcal{C}$ is either $s\mathscr{O}$-*Mod* or a fiber of $\mathfrak{s}\mathscr{O}$-**MOD**$_{\mathcal{E}}$, $B \otimes L$ can be identified with $Z(\mathrm{id}_B, s)$ and $Z(i, \mathrm{id}_L)$, and $i \,\square\, s : Z(i,s) \to Z(\mathrm{id}_B, s)$ equals $Z((i, \mathrm{id}_B), (\mathrm{id}_K, \mathrm{id}_L)) =: Z(i, \mathrm{id}_B; s)$.

(b) Fix any cofibration s and consider the class $\mathscr{P}$, consisting of morphisms i, such that $i \,\square\, s$ is a cofibration (resp. an acyclic cofibration). According to Lemma **9.8.5** below, this class $\mathscr{P}$ is closed, so it contains all cofibrations (resp. acyclic cofibrations) iff it contains $L_{\mathscr{O}}(\underline{I})$ (resp. $L_{\mathscr{O}}(\underline{J})$). In other words, it suffices to check (TM) for i from $L_{\mathscr{O}}(\underline{I})$ (resp. $L_{\mathscr{O}}(\underline{J})$).

(c) Interchanging the arguments in the reasoning of (b), we see that we can assume that s lies in $L_{\mathscr{O}}(\underline{I})$ as well, i.e. we have to check that $i \,\square\, s$ is a cofibration (resp. acyclic cofibration) whenever i lies in $L_{\mathscr{O}}(\underline{I})$ (resp. $L_{\mathscr{O}}(\underline{J})$) and s lies in $L_{\mathscr{O}}(\underline{I})$.

(d) Now notice that $L_{\mathscr{O}} : s\mathcal{E} \to s\mathscr{O}$-*Mod* commutes with $\square$ and preserves cofibrations and acyclic cofibrations, hence we are reduced to proving the statement in $s\mathcal{E}$ for $s \in \underline{I}$, $i \in \underline{I}$ or $\underline{J}$. Furthermore, $q^* : sSets \to s\mathcal{E}$ also commutes with $\square$ and preserves cofibrations and acyclic cofibrations, so we are reduced to proving a statement about I and J in *sSets*, which follows immediately from the compatibility of *sSets* with its tensor product $\otimes = \times$, q.e.d.

Lemma 9.8.5 *Let $\mathscr{P}$ be a closed class in category $\mathcal{C} = s\mathscr{O}$-Mod or in stack $\mathcal{C} = \mathfrak{s}\mathscr{O}$-$\boldsymbol{MOD}_{\mathcal{E}}$, and $s : K \to L$ be any morphism in this category or in the final fiber of this stack. Then the class $\mathscr{Q} := (\mathscr{P} : s)$ of morphisms i, such that $i \square s$ lies in $\mathscr{P}$, is itself closed.*

Proof. We have to check conditions of **9.5.10** for $\mathscr{Q} = (\mathscr{P} : s)$ one by one.

(a) Condition 1) (localness of $\mathscr{Q}$) is evident, as well as condition 2) (all isomorphisms do lie in $\mathscr{Q}$), 3) (if i' is a pushout of i, then $i' \square s$ is easily seen to be a pushout of $i \square s$), 6) (if i' is a retract of i, then $i' \square s$ is a retract of $i \square s$) and 7) (since $- \square s$ commutes with arbitrary direct sums). So only conditions 4), 5) and 8) remain.

(b) Let us check condition 4), i.e. stability under composition. Let $A \xrightarrow{i} B \xrightarrow{j} C$ be a composable sequence of morphisms in $\mathscr{Q}$. Then $ji \square s = Z(ji, \mathrm{id}_C; s) : Z(ji, s) \to Z(\mathrm{id}_C, s)$ can be decomposed into $Z(i, \mathrm{id}_C; s) : Z(ji, s) \to Z(j, s)$, followed by $Z(j, \mathrm{id}_C; s) = j \square s : Z(j, s) \to Z(\mathrm{id}_C, s)$. Now notice that $Z(i, \mathrm{id}_C; s)$ is a pushout of $Z(i, \mathrm{id}_B; s) = i \square s : Z(i, s) \to Z(\mathrm{id}_B, s)$, just because $Z(-, s)$ is right exact and the following square is cocartesian in $\mathrm{Ar}\,\mathcal{C}$:

$$\begin{tikzcd} i \arrow[r, "{(i,\mathrm{id}_B)}"] \arrow[d, "{(\mathrm{id}_A,j)}"'] & \mathrm{id}_B \arrow[d, "{(\mathrm{id}_B,j)}"] \\ ji \arrow[r, "{(i,\mathrm{id}_C)}"] & j \end{tikzcd} \tag{9.8.5.1}$$

We conclude that $ji \square s$ is a composite of $j \square s$ and a pushout of $i \square s$, hence it lies in $\mathscr{P}$ whenever both $j \square s$ and $i \square s$ do.

(c) Now let us check 5), i.e. stability under sequential composition. Let $A_0 \xrightarrow{i_0} A_1 \xrightarrow{i_1} A_2 \to \ldots$ be a composable sequence of morphisms from $\mathscr{Q}$. Put $B := A_\infty := \varinjlim_n A_n$, and denote by $j_n : A_n \to B$ the natural embeddings, and by $k_n : A_0 \to A_n$ the compositions $i_{n-1} \cdots i_1 i_0$, so as to have $j_n k_n = j_0$. We must check that $j_0 : A_0 \to B$ lies in $\mathscr{Q}$, i.e. that $j_0 \square s = Z(j_0, \mathrm{id}_B; s)$ lies in $\mathscr{P}$. Notice that (j_0, id_B) is the inductive limit of (k_n, id_B), i.e. the sequential composition of $j_0 \xrightarrow{(i_0,\mathrm{id}_B)} j_1 \xrightarrow{(i_1,\mathrm{id}_B)} j_2 \to \cdots$. Since $Z(-; s)$ commutes with arbitrary inductive limits, we see that $j_0 \square s$ is the sequential composition of $Z(i_n, \mathrm{id}_B; s) : Z(j_n, s) \to Z(j_{n+1}, s) = Z(i_n j_n, s)$. But we have seen in (b) that each $Z(i_n, \mathrm{id}_B; s)$ is a pushout of $Z(i_n, \mathrm{id}_{A_{n+1}}; s) = i_n \square s$, hence lies in

$\mathscr{P}$, hence the same is true for their sequential composition $j_0 \square s$, $\mathscr{P}$ being closed.

(d) It remains to check 8), i.e. the stability under $\varphi_!$ in the stack case. This stability follows from the canonical isomorphism $(\varphi_! A) \otimes K \cong \varphi_!(A \otimes \varphi^* K)$, which enables us to identify $\varphi_!(i) \square s$ with $\varphi_!(i \square \varphi^*(s))$, q.e.d.

9.8.6. One checks exactly in the same way as above that the external $\otimes$-action of $s\mathcal{E}$ on $s\mathscr{O}$-*Mod* is compatible with pseudomodel structures and pointwise pseudomodel structures, and that this $\otimes$-action can be derived with the aid of cofibrant replacements (or even pointwise cofibrant replacements).

9.8.7. (Fancy description of this method of proof as "devissage".) We have already used several times the following method to prove stability of cofibrations or acyclic cofibrations under some functor F: we show that the class of morphisms which become (acyclic) cofibrations after applying F is closed, checking conditions of **9.5.10** one by one, and then check that the generators $L_{\mathscr{O}}(\underline{I})$ (resp. $L_{\mathscr{O}}(\underline{J})$) belong to this class; this is usually done by observing that functor F essentially "commutes" with $L_{\mathscr{O}} \circ q^* : sSets \to s\mathscr{O}\text{-}Mod_{\mathcal{E}}$, so we are finally reduced to checking the statetement for morphisms from I (resp. J) in *sSets*.

Up to now we have applied this method to prove stability of cofibrations and acyclic cofibrations under topos pullbacks and base change, and we have also applied it to bifunctor $\otimes$ just now in **9.8.4**, where we have been able to use separately the above reasoning first in one variable (fixing the other), and then in the second, thus reducing (TM) to checking that $i \square s$ is a cofibration for any i, $s \in L_{\mathscr{O}}(\underline{I})$, and an acyclic cofibration for $i \in L_{\mathscr{O}}(\underline{I})$, $s \in L_{\mathscr{O}}(\underline{J})$, something that we can check in *sSets* and apply $L_{\mathscr{O}} \circ q^*$ afterwards. However, we applied this reasoning to an "additive" or right exact functor F (right exact in each variable in the case of $\otimes$) so far. Since we are going to apply it shortly to the much more complicated case of (definitely non-additive) symmetric powers, we'd like to give a fancy description of this method of proof.

Namely, one can imagine that cofibrations are just monomorphisms $i : A \to B$ in some abelian category (e.g. modules over some classical commutative ring Λ, or complexes of such modules), having a projective cokernel $P = B/A$ (one can check that such monomorphisms are just the closure of one-element set $\{0 \to \Lambda\}$ in Λ-*Mod*), and acyclic cofibrations are cofibrations with "negligible" cokernel $P \approx 0$ (something like a complex of projectives with trivial cohomology). We might think of $A \xrightarrow{i} B$ as a "presentation" of P in form B/A. If $i' : A' \to B'$ is a pushout of $i : A \to B$, then $B'/A' \cong B/A$, i.e. a pushout just provides another presentation of the same object. Next, when we decompose a cofibration $A \to B$ into a composition of "simpler" cofibrations $A = A_0 \to A_1 \to \cdots \to A_n = B$, this essentially means that we define on $P = B/A$ an increasing filtration $F_iP = A_i/A_0$ with "simpler" quotients $P_i = A_i/A_{i-1}$, and try to deduce some property of P from similar properties of P_i, i.e. do some sort of *devissage.* When we consider direct sums of morphisms, they correspond to direct sums of cokernels, and retracts of $i : A \to B$ correspond to direct factors of $P = B/A$. Therefore, the above method of proving statements about cofibrations or acyclic cofibrations can be thought of as some sort of devissage, where the role of simple objects is played by the "cokernels" of standard generators from $L_{\mathscr{O}}(\underline{I})$ or $L_{\mathscr{O}}(\underline{J})$.

For example, if $i : A \to B$ and $s : K \to L$ are two such "cofibrations", with projective cokernels $A/B = P$ and $K/L = Q$, then $i \square s : \ldots \to B \otimes L$ of (TM) is injective with cokernel $P \otimes Q$, i.e. $\square$ is just "the tensor product of projective modules in terms of their presentations", and (TM) can be interpreted as saying that $P \otimes Q$ is projective whenever P and Q are projective, and that $P \otimes Q \approx 0$ if in addition either $P \approx 0$ or $Q \approx 0$, a very natural statement indeed. Moreover, in classical situation we would probably show such statements by writing both P and Q as direct factors of free modules, and decomposing these free modules further into direct sums of free modules of rank 1; in our situation we reduce to the case where P and Q are "cokernels" of morphisms from $L_{\mathscr{O}}(\underline{I})$ or $L_{\mathscr{O}}(\underline{J})$ instead.

9.8.8. Once we know that derived local tensor product $\underline{\underline{\otimes}} = \underline{\underline{\otimes}}_{\mathscr{O}}$ exists and can be computed by means of cofibrant replacements, we deduce the usual properties (associativity, commutativity, compatibility with pullbacks and base change etc.) in the same way as before. In particular, we have $X \underline{\underline{\otimes}} K \cong X \underline{\underline{\otimes}}_{\mathscr{O}} \mathbb{L}L_{\mathscr{O}}(K)$, for any $X \in \operatorname{Ob} \mathcal{D}^{\leq 0}(\mathcal{E}, \mathscr{O})$ and $K \in \operatorname{Ob} \operatorname{Ho} s\mathcal{E} = \operatorname{Ob} \mathcal{D}^{\leq 0}(\mathcal{E}, \mathbb{F}_{\varnothing})$. Notice, however, that we cannot derive inner Homs as we did in **8.7.7**, because inner Homs don't commute with fiber functors p^*, so we cannot check required properties pointwise. Therefore, we don't obtain any inner Homs for the ACU $\otimes$-structure on $\mathcal{D}^{\leq 0}(\mathcal{E}, \mathscr{O})$ given by $\underline{\underline{\otimes}}$.

9.8.9. If $\mathscr{O}$ admits a zero, then $\underline{\underline{\otimes}}$ commutes with the suspension functor in each variable, hence induces an ACU $\otimes$-structure $\underline{\underline{\otimes}}$ on the corresponding stable category $\mathcal{D}^{-}(\mathcal{E}, \mathscr{O})$. All formulas and properties mentioned above extend to the stable case as well.

9.8.10. When $\mathscr{O}$ is additive, cofibrant replacements in $s\mathscr{O}$*-Mod* correspond via Dold–Kan to some flat resolutions in the category of chain complexes $\mathrm{Ch}^{\geq 0}(\mathscr{O}\text{-}Mod)$ (cf. **9.7.11**), hence our derived local tensor product corresponds to its classical counterpart, once we take into account Eilenberg–Zilber theorem (cf. **8.5.12**).

9.9. (Derived symmetric powers.) This subsection is dedicated to the proof of existence of derived symmetric powers $\mathbb{L}S^n = \mathbb{L}S^n_{\mathscr{O}} : \mathcal{D}^{\leq 0}(\mathcal{E}, \mathscr{O}) \to \mathcal{D}^{\leq 0}(\mathcal{E}, \mathscr{O})$, for any generalized commutatively ringed topos $(\mathcal{E}, \mathscr{O})$ with sufficiently many points (cf. **9.9.2**). Notice that, unlike our previous considerations, the results of this subsection seem to be new even in the additive case.

9.9.0. (Importance of derived symmetric powers.) Existence of derived symmetric powers is crucial for the goal of this work, since we want to define Chow rings by means of the γ-filtration, exactly the way Grothendieck did it in his proof of Riemann–Roch theorem, and a λ-ring structure on K_0 of perfect complexes (or rather perfect simplicial modules) is necessary for this. A λ-structure can be described either in terms of exterior power operations $\{\lambda^n\}_{n\geq 0}$ or in terms of symmetric powers $\{s^n\}_{n\geq 0}$. We choose the second approach for the following reasons:

- Symmetric powers have good properties in all situations, while exterior powers are nice only for an *alternating* $\mathscr{O}$ (cf. **5.5.4**), e.g. $S^n_{\mathscr{O}}(X \oplus Y) = \bigoplus_{p+q=n} S^p_{\mathscr{O}}(X) \otimes_{\mathscr{O}} S^q_{\mathscr{O}}(Y)$ is true unconditionally, while a similar formula for $\bigwedge^n$ requires alternativity of $\mathscr{O}$.

- We don't want to take care of the $\pm$ signs in all our proofs.

- The proof for symmetric powers is ultimately reduced to the basic case of morphisms from I or J in *sSets*, which can be dealt with the aid of topological realization, while the basic case for exterior powers would be $L_{\mathbb{F}_{\pm 1}}(I)$ or $L_{\mathbb{F}_{\pm 1}}(J)$ in $s\mathbb{F}_{\pm 1}$-*Mod*, more difficult to tackle with.

- Symmetric powers have a long and venerable history of being derived, and they actually were the first example considered by Dold and Puppe in their work [DoldPuppe] on derivation of non-additive functors between abelian categories.

9.9.1. (Notations.) Let us fix a generalized commutatively ringed topos $(\mathcal{E}, \mathscr{O})$. We denote by $\mathcal{C}$ the category $s\mathscr{O}$-*Mod*, endowed with its pseudomodel structure discussed in **9.7.1** and **9.7.7** (when $\mathcal{E}$ doesn't have enough points, we cannot speak about a pseudomodel structure, but we still have well-defined cofibrations and acyclic cofibrations in $\mathcal{C}$) and its compatible ACU $\otimes$-structure $\otimes = \otimes_{\mathcal{E}}$ (cf. **9.8.4**). On some occasions $\mathcal{C}$ will denote the corresponding stack $\mathfrak{s}\mathscr{O}$-$\mathbf{MOD}_{\mathcal{E}}$, and we'll deliberately confuse $\mathcal{C}$ with its final fiber $\mathcal{C}(e) = s\mathscr{O}$-*Mod*.

Notice that almost all our considerations will be actually valid for any pseudomodel category or stack with a compatible ACU $\otimes$-structure, if we assume in addition that $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ commutes with arbitrary inductive limits in each variable (condition automatically fulfilled whenever $\otimes$ admits inner Homs).

When a finite group G acts (say, from the right) on an object X of $\mathcal{C}$, we denote by X/G or X_G its *object of coinvariants*, i.e. the largest strict quotient of X, on which G acts trivially. We can also describe X_G as the cokernel of $\mathrm{pr}_1, \alpha : X \times G \rightrightarrows X$, where $\alpha : X \times G = \coprod_{g \in G} X \to X$ is the group action. Right exactness of $\otimes$ implies $(X \otimes Y)_G \cong X_G \otimes Y$ for any Y with a trivial action of G, hence

$$(X \otimes Y)_{G \times H} \cong ((X \otimes Y)_G)_H \cong (X_G \otimes Y)_H \cong X_G \otimes Y_H \tag{9.9.1.1}$$

for any G-object X and H-object Y.

We denote tensor powers by $T^n(A)$ or $A^{\otimes n}$, and the symmetric powers $T^n(A)/\mathfrak{S}_n = T^n(A)_{\mathfrak{S}_n}$ by $S^n_{\mathscr{O}}(A)$ or simply $S^n(A)$. We use these notations both for these functors in $\mathscr{O}$-*Mod* and for their simplicial extensions to $s\mathscr{O}$-*Mod*.

Theorem 9.9.2 *Let $(\mathcal{E}, \mathscr{O})$ be a generalized commutatively ringed topos, $n \geq 0$ be an integer. Then:*

(a) Symmetric power functor $S^n : s\mathscr{O}\text{-}Mod \to s\mathscr{O}\text{-}Mod$ preserves (pointwise) cofibrations and acyclic cofibrations between (pointwise) cofibrant objects.[1]

(b) If $\mathcal{E}$ has enough points, S^n preserves weak equivalences between cofibrant objects, and the same is true even for weak equivalences between pointwise cofibrant objects.

(c) If $\mathcal{E}$ has enough points, the symmetric power functor admits a left derived functor $\mathbb{L}S^n : \mathcal{D}^{\leq 0}(\mathcal{E}, \mathscr{O}) \to \mathcal{D}^{\leq 0}(\mathcal{E}, \mathscr{O})$, which can be computed with the aid of cofibrant replacements or pointwise cofibrant replacements.

Proof. We want to show that everything follows from (a), which will be proved in the remaining part of this subsection by induction in n. (a)⇒(b): Follows immediately from the fact that weak equivalences can be checked

[1] We give a complete proof of (a) only in the pointwise case, since the general case would require a development of certain techniques we want to avoid in this work, and there are some indications one would need to extend generating sets $L_{\mathscr{O}}(\underline{I}_{\mathcal{E}})$ and $L_{\mathscr{O}}(\underline{J}_{\mathcal{E}})$ in order to prove it, thus replacing cofibrations and acyclic cofibrations with something more similar to corresponding pointwise notions anyway. In any case, this doesn't affect validity of (b) and (c), any cofibration being pointwise cofibrant as well.

pointwise (cf. **9.7.6**), Lemma **8.6.3**, and commutativity of symmetric powers with topos pullbacks and fiber functors p^* in particular. (b)$\Rightarrow$(c): Apply **9.6.7**.

Before proving (a), or actually a more precise statement **9.9.9**, we want to introduce some auxilliary constructions.

9.9.3. (Hypercube construction.) Let us fix an ordered set I; usually it will be a subset of $\mathbb{Z}$ with induced order, e.g. $[1] = \{0, 1\}$, or some other $[m] = \{0, 1, \ldots, m\}$, or $\omega = \{m : m \geq 0\}$ – in short, any ordinal $\leq \omega$. We consider ordered sets as categories with at most one morphism between any two objects in the usual manner. Let us also fix some $n \geq 1$, and put $\mathbf{n} := \{1, 2, \ldots, n\}$ as an (unordered) set. Then the symmetric group $\mathfrak{S}_n = \mathrm{Aut}(\mathbf{n})$ acts from the right on $I^n = \mathit{Funct}(\mathbf{n}, I)$, hence it also acts from the left on functors $I^n \to \mathcal{C}$.

More precisely, if $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in I^n$ and $\sigma \in \mathfrak{S}_n$ is a permutation, then $\varepsilon\sigma = (\varepsilon_{\sigma(1)}, \ldots, \varepsilon_{\sigma(n)})$, and if $P : I^n \to \mathcal{C}$ is a functor, then $\sigma P : I^n \to \mathcal{C}$ is given by $(\sigma P)(\varepsilon) := P(\varepsilon\sigma)$, i.e. $(\sigma P)(\varepsilon_1, \ldots, \varepsilon_n) = P(\varepsilon_{\sigma(1)}, \ldots, \varepsilon_{\sigma(n)})$.

Now let $\{X_i : I \to \mathcal{C}\}_{1 \leq i \leq n}$ be any family of functors. For example, if $I = [1] = \{0 \to 1\}$, then each X_i is just a morphism $X_i(0) \to X_i(1)$ in $\mathcal{C}$. Define $\tilde{X} : I^n \to \mathcal{C}$ by taking the tensor product of values of X_i:

$$\tilde{X}(\varepsilon_1, \ldots, \varepsilon_n) = X_1(\varepsilon_1) \otimes \cdots \otimes X_n(\varepsilon_n) \tag{9.9.3.1}$$

When $I = [1]$, we obtain an n-dimensional hypercube of arrows in $\mathcal{C}$, hence the name for our construction.

9.9.4. (Extension of $\tilde{X}$ to subsets of I^n.) Let $R \subset I^n$ be any subset of I^n, considered here as an ordered set with respect to induced order, or as a full subcategory of I^n. We extend $\tilde{X} : I^n \to \mathcal{C}$ from I^n to the set $\mathfrak{P}(I^n)$ of subsets of I^n, ordered by inclusion, as follows:

$$\tilde{X}_!(R) := \varinjlim_{\varepsilon \in R} \tilde{X}(\varepsilon) \tag{9.9.4.1}$$

Clearly, $\tilde{X}_!(\{\varepsilon\}) = \tilde{X}(\varepsilon)$. More generally, if R has a largest element ε, then $X_!(R) = X(\varepsilon)$. For example, if $I = [m]$, then $X_!(I^n) = \tilde{X}(m, \ldots, m) = X_1(m) \otimes \cdots \otimes X_n(m)$.

Notice that the right action of $\mathfrak{S}_n$ on I^n naturally extends to a right action on $\mathfrak{P}(I^n)$, and we have

$$\tilde{X}_!(R\sigma) = \varinjlim_{\varepsilon \in R} \tilde{X}(\varepsilon\sigma) = (\sigma\tilde{X})_!(R) \tag{9.9.4.2}$$

In other words, $\sigma(\tilde{X}_!) = (\sigma\tilde{X})_!$, so we can write simply $\sigma\tilde{X}_!$.

9.9.5. (Application: multiple box products.) Let $I = [1]$, and $\{X_i\}_{1\leq i\leq n}$ be a family of functors $I \to \mathcal{C}$, i.e. a collection of morphisms $\{X_i(0) \xrightarrow{u_i} X_i(1)\}$ in $\mathcal{C}$. Consider ordered subset $R := I^n - \{(1,1,\ldots,1)\} \subset I^n$, obtained by removing the "final" vertex of our hypercube. Then we get a morphism

$$u_1 \,\square\, u_2 \,\square \cdots \square\, u_n : \tilde{X}_!(R) \to \tilde{X}_!(I^n) = X_1(1) \otimes X_2(1) \otimes \cdots \otimes X_n(1) \quad (9.9.5.1)$$

We say that $u_1 \,\square \cdots \square\, u_n$ is the *multiple box product of* $u_1, \ldots, u_n$. When $n = 2$, we recover the usual $\square$-product $u \,\square\, v$ of **8.7.4**, used in particular to state (TM).

Now it is an easy exercise to check that $(u_1 \,\square\, u_2) \,\square\, u_3 = u_1 \,\square\, u_2 \,\square\, u_3 = u_1 \square (u_2 \square u_3)$; the verification is quite formal and uses only the right exactness of $\otimes$. In other words, *the box-product* $\square : \operatorname{Ar}\mathcal{C} \times \operatorname{Ar}\mathcal{C} \to \operatorname{Ar}\mathcal{C}$ *is associative.* It is of course also commutative, $\otimes$ being commutative. (All this would seem not so unexpected if we use the "fancy language" of **9.8.7** and recall that $\square$ "corresponds to tensor product of cokernels".)

We conclude by an easy induction using (TM), valid in $\mathcal{C} = s\mathscr{O}$*-Mod* by **9.8.4**, that $u_1 \square u_2 \square \cdots \square u_n$ *is a cofibration whenever all* u_i *are cofibrations, and it is an acyclic cofibration if in addition at least one of* u_i *is an acyclic cofibration.*

9.9.6. (Symmetric hypercube construction.) Let us fix an ordered set I and an integer $n > 0$ as above, and any functor $X : I \to \mathcal{C}$. Put all $X_i := X$, $1 \leq i \leq n$, and construct $\tilde{X} : I^n \to \mathcal{C}$ with the aid of tensor product in $\mathcal{C}$ as before:

$$\tilde{X}(\varepsilon_1, \ldots, \varepsilon_n) = X(\varepsilon_1) \otimes \cdots \otimes X(\varepsilon_n) \quad (9.9.6.1)$$

The associativity and commutativity constraints for $\otimes$ yield canonical isomorphisms $\eta_{\sigma,\varepsilon} : \tilde{X}(\varepsilon) \to \tilde{X}(\varepsilon\sigma) = (\sigma\tilde{X})(\varepsilon)$, which can be described on the level of elements as the maps $X(\varepsilon_1) \otimes \cdots \otimes X(\varepsilon_n) \to X(\varepsilon_{\sigma(1)}) \otimes \cdots \otimes X(\varepsilon_{\sigma(n)})$, given by $x_1 \otimes \cdots \otimes x_n \mapsto x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$.

Clearly, $\eta_{\sigma,\varepsilon} : \tilde{X}(\varepsilon) \to (\sigma\tilde{X})(\varepsilon)$ are functorial in ε, i.e. they define a natural transformation $\eta_\sigma : \tilde{X} \to \sigma\tilde{X}$, for any $\sigma \in \mathfrak{S}_n$. Moreover, these η_σ *satisfy a cocycle relation* $\eta_{\sigma\tau} = \sigma(\eta_\tau) \circ \eta_\sigma : \tilde{X} \to \sigma\tilde{X} \to \sigma\tau\tilde{X}$.

By taking inductive limits $\varinjlim_R$ along subsets $R \subset I^n$, we get induced morphisms $\eta_{\sigma,R} : \tilde{X}_!(R) \xrightarrow{\sim} \tilde{X}_!(R\sigma) = (\sigma\tilde{X}_!)(R)$, i.e. η_σ extends to a functorial isomorphism $\tilde{X}_! \to \sigma\tilde{X}_!$, where $\tilde{X}_! : \mathfrak{P}(I^n) \to \mathcal{C}$ is the extension of $\tilde{X}$ defined by (9.9.4.1). It is immediate that these extended η_σ still satisfy the cocycle relation $\eta_{\sigma\tau} = \sigma(\eta_\tau) \circ \eta_\sigma$, i.e. $\eta_{\sigma\tau,R} = \eta_{\tau,R\sigma} \circ \eta_{\sigma,R}$.

In particular, for a *symmetric* $R \subset I^n$ (i.e. stable under $\mathfrak{S}_n$: $R\sigma = R$ for all $\sigma \in \mathfrak{S}_n$) we get isomorphisms $\eta_{\sigma,R} : \tilde{X}_!(R) \xrightarrow{\sim} \tilde{X}_!(R\sigma) = \tilde{X}_!(R)$, and the cocycle relation implies that these $\{\eta_{\sigma,R}\}_{\sigma\in\mathfrak{S}_n}$ define a *right* action of $\mathfrak{S}_n$

on $\tilde{X}_!(R)$, hence we can consider the coinvariants of $\tilde{X}_!(R)$, which will be denoted by $\tilde{X}_!^s(R)$:

$$\tilde{X}_!^s(R) := \tilde{X}_!(R)_{\mathfrak{S}_n} \quad \text{for any symmetric } R \subset I^n \tag{9.9.6.2}$$

We say that $X_!^s : \mathfrak{P}(I^n)^{\mathfrak{S}_n} \to \mathcal{C}$ is the *symmetric extension* of X or $\tilde{X}$.

Notice that the above construction applies to any $R \subset I^n$, fixed by some subgroup $G \subset \mathfrak{S}_n$: we still obtain a right action of G on $\tilde{X}_!(R)$, hence can consider the coinvariants $\tilde{X}_!(R)_G$.

9.9.7. (Application: canonical decomposition of symmetric powers.) Let us apply the above construction to any morphism $u : X(0) \to X(1)$, considered here as a functor $X : [1] \to \mathcal{C}$. We obtain some symmetric extension $\tilde{X}_!^s : \mathfrak{P}([1]^n)^{\mathfrak{S}_n} \to \mathcal{C}$. Put $R_k := [1]^n_{\leq k} = \{\varepsilon \in [1]^n : |\varepsilon| \leq k\}$, where $|\varepsilon|$ denotes the sum $\sum_{i=1}^n \varepsilon_i$ of components of $\varepsilon \in [1]^n \subset \mathbb{Z}^n$. Clearly, $\{0\} = R_0 \subset R_1 \subset \cdots \subset R_n = [1]^n$ is an increasing filtration of $[1]^n$ by symmetric subsets. Put $F_k S^n(u) := \tilde{X}_!^s(R_k) = \tilde{X}_!(R_k)_{\mathfrak{S}_n}$; taking into account obvious identifications of $F_0 S^n(u) = \tilde{X}_!^s(\{0\})$ with $(X(0)^{\otimes n})_{\mathfrak{S}_n} = S^n X(0)$ and of $F_n S^n(u) = \tilde{X}_!^s([1]^n)$ with $S^n X(1)$, we obtain a canonical decomposition of $S^n(u)$:

$$\begin{aligned} S^n(u) : S^n X(0) = F_0 S^n(u) \xrightarrow{\rho_1^{(n)}(u)} F_1 S^n(u) \xrightarrow{\rho_2^{(n)}(u)} \cdots \\ \cdots \longrightarrow F_{n-1} S^n(u) \xrightarrow{\rho_n^{(n)}(u)} F_n S^n(u) = S^n X(1) \end{aligned} \tag{9.9.7.1}$$

Sometimes we consider also $R_{-1} = [1]^n_{\leq -1} = \varnothing$, together with $\rho_0^{(n)}(u) : F_{-1} S^n(u) = \varnothing \to F_0 S^n(u) = S^n X(0)$.

We also introduce a shorter notation $\rho_n(u)$ for the "n-fold symmetric box product" $\rho_n^{(n)}(u) : F_{n-1} S^n(u) \to S^n X(1)$; we'll see in a moment that $\rho_n(u)$ "computes the n-th symmetric power of $\operatorname{Coker} u$", hence its importance for our considerations.

9.9.8. (Fancy explanation.) The meaning of the above decomposition of $S^n(u)$ is better explained in the "fancy language" of **9.8.7**. One can check that the above construction, when applied to an injective homomorphism $X(0) \xrightarrow{u} X(1)$ with projective cokernel P (in Λ-*Mod*, Λ a classical commutative ring), yields a decomposition of $S^n(u)$ into n morphisms $\rho_k^{(n)}$, which correspond to the canonical increasing filtration of $\operatorname{Coker} S^n(u) = S^n X(1)/S^n X(0)$ with quotients isomorphic to $S^k P \otimes S^{n-k} X(0)$. In particular, $\rho_n(u) = \rho_n^{(n)}(u)$ corresponds to the last step of this filtration, with quotient $S^n P$, i.e. $\operatorname{Coker} \rho_n(u) \cong S^n(\operatorname{Coker} u)$: ρ_n *computes symmetric powers of the cokernel*, similarly to the relationship between the box-product and the tensor product.

Now the statement (a) of **9.9.2** will be a consequence of the following more precise statement:

Proposition 9.9.9 *Whenever $u : A \to B$ is a cofibration (resp. acyclic cofibration) between cofibrant objects of $\mathcal{C} = s\mathscr{O}$-Mod, the same is true for all $\rho_k^{(n)}(u) : F_{k-1}S^n(u) \to F_kS^n(u)$, $1 \leq k \leq n$, hence also for their composition $S^n(u) = \rho_n^{(n)}\rho_{n-1}^{(n)} \cdots \rho_1^{(n)} : S^n(A) \to S^n(B)$.*

Proof. Induction in $n \geq 0$, cases $n = 0$ and $n = 1$ being trivial. The induction step will occupy the rest of this subsection; its main idea is of course our usual sort of "devissage" (cf. **9.8.7**). However, we want to work with acyclic cofibrations and cofibrations between cofibrant objects only. The possibility of doing this is due to the fact that *acyclic cofibrations (resp. cofibrations) between cofibrant objects coincide with the closure of $L_{\mathscr{O}}(\underline{I})$ (resp. $L_{\mathscr{O}}(\underline{J})$) inside the full subcategory $\mathcal{C}_c$ of $\mathcal{C} = s\mathscr{O}$-Mod, or, more precisely, inside the full substack $\mathcal{C}_c$ of $\mathcal{C} = \mathfrak{s}\mathscr{O}$-$\boldsymbol{MOD}_{\mathcal{E}}$,* shown in the following lemma:

Lemma 9.9.10 *Let $\mathcal{C}$ be a flat stack over a topos $\mathcal{E}$, I, J be (small) sets of morphisms in the final fiber of $\mathcal{C}$, such that $J \subset \operatorname{Cl} I$. Let us call morphisms from $\operatorname{Cl} I$ (resp. $\operatorname{Cl} J$) cofibrations (resp. acyclic cofibrations), and denote by $\mathcal{C}_c$ the full substack of $\mathcal{C}$ consisting of cofibrant objects. Denote by $\operatorname{Cl}_c I$ (resp. $\operatorname{Cl}_c J$) the closure of I (resp. J) in $\mathcal{C}_c$. Then $\operatorname{Cl}_c I = \operatorname{Cl} I \cap \operatorname{Ar} \mathcal{C}_c$ and $\operatorname{Cl}_c J = \operatorname{Cl} J \cap \operatorname{Ar} \mathcal{C}_c$, i.e. $\operatorname{Cl}_c I$ (resp. $\operatorname{Cl}_c J$) consists exactly of cofibrations (resp. acyclic cofibrations) between cofibrant objects.*

Proof. It suffices to show $\operatorname{Cl}_c J = \operatorname{Cl} J \cap \operatorname{Ar} \mathcal{C}_c$, since the other formula can be then shown by invoking this lemma with $J := I$. Clearly, $\operatorname{Cl} J \cap \operatorname{Ar} \mathcal{C}_c$ is a closed class of morphisms in $\mathcal{C}_c$, containing J, hence also $\operatorname{Cl}_c J$, so one inclusion is trivial.

To prove the opposite inclusion we consider the class $\mathscr{P}$ of morphisms $A \xrightarrow{u} B$ in (fibers of) $\mathcal{C}$ having the following property: all pushouts $A' \xrightarrow{u'} B'$ of $\varphi^*(u)$ with respect to morphisms $\varphi^* A \to A'$ in $\mathcal{C}(T)$ with cofibrant target lie in $\operatorname{Cl}_c J$, where $\varphi : T \to S$ runs over all morphisms in $\mathcal{E}$ with target S, and u lies in $\operatorname{Ar} \mathcal{C}(S)$.

Clearly, $J \subset \mathscr{P}$; if we show that $\mathscr{P}$ is closed in $\mathcal{C}$, we would conclude $\operatorname{Cl} J \subset \mathscr{P}$, hence also the required inclusion $\operatorname{Cl} J \cap \operatorname{Ar} \mathcal{C}_c \subset \mathscr{P} \cap \operatorname{Ar} \mathcal{C}_c \subset \operatorname{Cl}_c J$, applying the characteristic property of morphisms $A \xrightarrow{u} B$ from $\mathscr{P}$ to the pushout with respect to id_A, whenever A is cofibrant.

9.9.11. (Closedness of $\mathscr{P}$.) So we are reduced to proving closedness of $\mathscr{P}$, i.e. verifying conditions 1)–8) of **9.5.10**.

(a) Condition 1) (localness of $\mathscr{P}$) is obvious from the definition of $\mathscr{P}$; since all our constructions are also compatible with pullbacks, we need to check the basic property of morphisms from $\mathscr{P}$ only for pushouts computed in the same fiber, i.e. assume $\varphi = \mathrm{id}_S$ while checking 2)–7), but not 8).

(b) Conditions 2) (all isomorphisms do lie in $\mathscr{P}$) and 3) (stability of $\mathscr{P}$ under pushouts) are evident. Conditions 4) and 5) (stability under composition and sequential composition) are also evident, once we take into account that $\mathrm{Cl}\, J \subset \mathrm{Cl}\, I$ implies that any pushout $u' : A' \to B'$ with cofibrant source A' of a morphism $u : A \to B$ from $\mathscr{P}$ has a cofibrant target B' as well.

(c) Condition 6) (stability under retracts) is somewhat more complicated. Let $v : X \to Y$ be a retract of $u : A \to B$ from $\mathscr{P}$, i.e. we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "i"'] \arrow[d, "v"] \arrow[rr, bend left, "\mathrm{id}_X"] & A \arrow[r, "p"'] \arrow[d, "u"] & X \arrow[d, "v"] \\ Y \arrow[r, "j"] \arrow[rr, bend right, "\mathrm{id}_Y"'] & B \arrow[r, "q"] & Y \end{tikzcd} \tag{9.9.11.1}$$

According to Lemma **9.9.12** below, we may assume $A = X$, $i = p = \mathrm{id}_A$, at the cost of replacing u by its pushout $\tilde{u}$ with respect to p. Since $\tilde{u}$ still lies in $\mathscr{P}$, we can safely do this. Now the pushout $v' : A' \to Y'$ of $v : A \to Y$ with respect to any morphism $A \to A'$ with cofibrant target A' is a retract of the corresponding pushout $u' : A' \to B'$ of u, lying in $\mathrm{Cl}_c\, J$ by assumption. In particular $u' \in \mathrm{Cl}_c\, J \subset \mathrm{Cl}\, J \subset \mathrm{Cl}\, I$ is a cofibration in $\mathcal{C}$, hence the same is true for its retract v', hence the target of v' lies in $\mathcal{C}_c$, i.e. v' is a retract of u' inside $\mathcal{C}_c$, hence it lies in $\mathrm{Cl}_c\, J$ as well.

(d) Let us prove condition 7) (stability under direct sums). Let $\{u_\alpha : A_\alpha \to B_\alpha\}_{\alpha \in K}$ be a family of morphisms from $\mathscr{P}$, and put $A := \bigoplus A_\alpha$, $B := \bigoplus B_\alpha$, $u := \bigoplus u_\alpha : A \to B$. We have to check that u lies in $\mathscr{P}$. Let $\sigma : A = \bigoplus A_\alpha \to A'$ be any morphism with cofibrant target. Consider the components $\sigma_\alpha : A_\alpha \to A'$ of σ, together with their direct sum $\tilde{\sigma} := \bigoplus \sigma_\alpha : A \to A' \otimes K = \bigoplus_\alpha A'$. Denote by ∇ the codiagonal map $A' \otimes K = \bigoplus_\alpha A' \to A'$, and consider the following commutative diagram with cocartesian squares:

$$\begin{tikzcd} A \arrow[r, "\tilde{\sigma}"] \arrow[d, "u"] & A' \otimes K \arrow[r, "\nabla"] \arrow[d, "v"] & A' \arrow[d, "u'"] \\ B \arrow[r] & \bigoplus B'_\alpha \arrow[r] & B' \end{tikzcd} \tag{9.9.11.2}$$

Here v is the direct sum of pushouts $v_\alpha : A' \to B'_\alpha$ of morphisms u_α. Each such pushout lies in $\mathrm{Cl}_c\, J$ by assumption, hence the same is true for their direct sum v, and in particular the source and the target of v lie in $\mathcal{C}_c$. On the other hand, the right square of the above diagram is cocartesian in $\mathcal{C}_c$, hence u' lies in $\mathrm{Cl}_c\, J$ as well. Now we just observe that u' is exactly the pushout of u with respect to $\sigma = \nabla\tilde{\sigma}$.

(e) It remains to check 8), i.e. stability under $\varphi_!$ or "local direct sums". So let us fix some $\varphi : T \to S$ in $\mathcal{E}$; we can assume $S = e$, replacing $\mathcal{E}$ and $\mathcal{C}$ by $\mathcal{E}_{/S}$ and $\mathcal{C}_{/S}$ if necessary. Let $u : A \to B$ be a morphism of $\mathcal{C}(T)$ lying in $\mathscr{P}$. We want to show that any pushout of $\varphi_!(u) : \varphi_! A \to \varphi_! B$ with respect to a morphism $\sigma : \varphi_! A \to A'$ in $\mathcal{C}(e)$ with cofibrant target lies in $\mathrm{Cl}_c\, J$. Denote by $\sigma^\flat : A \to \varphi^* A'$ the morphism in $\mathcal{C}(T)$ defined by adjointness of $\varphi_!$ and φ^*. Let $v : \varphi^* A' \to Z$ be the pushout in $\mathcal{C}(T)$ of u with respect to $\sigma^\flat$; since $\varphi^* A'$ is cofibrant and u lies in $\mathscr{P}$, v lies in $\mathrm{Cl}_c\, J$, hence the same is true for $\varphi_!(v)$. Now consider the following commutative diagram in $\mathcal{C}(e)$ with cocartesian squares:

$$\begin{tikzcd}
\varphi_! A \arrow[r, "\varphi_!(\sigma^\flat)"] \arrow[d, "\varphi_!(u)"] & \varphi_! \varphi^* A' \arrow[r, "\nabla"] \arrow[d, "\varphi_!(v)"] & A' \arrow[d, "u'"] \\
\varphi_! B \arrow[r] & \varphi_! Z \arrow[r] & B'
\end{tikzcd} \tag{9.9.11.3}$$

Clearly, $u' : A' \to B'$ is a pushout of $\varphi_!(v)$ in $\mathcal{C}_c(e)$, hence it lies in $\mathrm{Cl}_c\, J$ as well. On the other hand, u' is exactly the pushout in $\mathcal{C}(e)$ of $\varphi_!(u)$ with respect to $\nabla \circ \varphi_!(\sigma^\flat) = \sigma$, q.e.d.

Lemma 9.9.12 *Suppose that $v : X \to Y$ is a retract of $u : A \to B$ in some category $\mathcal{C}$, by means of a retract diagram* (9.9.11.1). *Suppose that the pushout $\tilde{u} : X \to Z$ of u with respect to $p : A \to X$ exists in $\mathcal{C}$. Then v is a retract of $\tilde{u}$, and one can find a retract diagram $(\tilde{\imath}, \tilde{\jmath}) : v \to \tilde{u}$, $(\tilde{p}, \tilde{q}) : \tilde{u} \to v$ with property $\tilde{\imath} = \tilde{p} = \mathrm{id}_X$.*

Proof. Consider the following diagram induced by (9.9.11.1), where the right square is cocartesian:

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "v"] & A \arrow[r, "p"] \arrow[d, "u"] & X \arrow[d, dashed, "\tilde{u}"] \arrow[ddr, "v"] & \\
Y \arrow[r, "j"] & B \arrow[r, dashed, "r"] \arrow[drr, "q"'] & Z \arrow[dr, dashed, "\tilde{q}"] & \\
 & & & Y
\end{tikzcd} \tag{9.9.12.1}$$

Put $\tilde{\imath} := \tilde{p} := \mathrm{id}_X$, $\tilde{\jmath} := rj$. Then these morphisms $(\tilde{\imath}, \tilde{\jmath}) : v \to \tilde{u}$ and $(\tilde{p}, \tilde{q}) : \tilde{u} \to v$ do satisfy the retract conditions, thus proving that v is a retract of $\tilde{u}$ with required property. Indeed, $\tilde{u}\tilde{\imath} = \tilde{u}\,\mathrm{id}_X = \tilde{u}pi = rjv = \tilde{\jmath}v$, $v\tilde{p} = v = \tilde{q}\tilde{u}$, $\tilde{p}\tilde{\imath} = \mathrm{id}_X$, and $\tilde{q}\tilde{\jmath} = \tilde{q}rj = qj = \mathrm{id}_Y$, q.e.d.

Before proving the induction step of **9.9.9**, we need to construct some decompositions of $\rho_k^{(n)}(u)$, where $u = \cdots u_2 u_1$ is a finite or sequential composition of morphisms, into some "simple" morphisms.

9.9.13. (Sieves, symmetric sieves and their extensions.) Let I be an ordered set as before. We say that a subset $R \subset I^n$ is a *sieve* if $x \leq y$ and $y \in R$ implies $x \in R$. We say that R is a *symmetric sieve* if it is fixed by $\mathfrak{S}_n$: $R = R\sigma$ for all $\sigma \in \mathfrak{S}_n$. An *extension of sieves* is simply a couple (R, R') of sieves in I^n, such that $R \subset R'$. Finally, an extension $R \subset R'$ of symmetric sieves will be said to be *simple* if $R \neq R'$, but there are no symmetric sieves R'', such that $R \subsetneq R'' \subsetneq R'$.

9.9.14. (Structure of simple extensions of symmetric sieves.) Let I be an ordinal $\leq \omega$, i.e. either a finite set $[m]$, or the infinite set ω of non-negative integers. Let $R \subsetneq R'$ be any non-trivial extension of symmetric sieves. Choose any minimal element $\gamma = (\gamma_1, \ldots \gamma_n)$ of the complement $R' - R$ of R in R' (e.g. choose γ with minimal value of $|\gamma|$). Clearly, $\gamma \notin R$, but $\gamma - e_k = (\gamma_1, \ldots, \gamma_k - 1, \ldots, \gamma_n) \in R$, for any k, such that $\gamma_k > 0$. One easily checks that whenever we are given a symmetric sieve $R \subset I^n$ and an element $\gamma \in I^n$ with the above property, then $R'' := R \cup \gamma\mathfrak{S}_n$ is a symmetric sieve in I^n, and, moreover, the extension $R \subset R''$ is simple.

We conclude that any non-trivial extension of symmetric sieves $R \subset R'$ contains a simple extension $R \subset R'' = R \cup \gamma\mathfrak{S}_n \subset R'$ of the above form; if $R \subset R'$ is itself simple, then $R' = R''$, i.e. *all simple extensions of symmetric sieves are of form $R \subset R \cup \gamma\mathfrak{S}_n$.* Since we are free to replace γ by any $\gamma\sigma$, *we can assume that $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n$.*

9.9.15. (Relation to the symmetric hypercube construction.) Now let us fix any functor $X : I \to \mathcal{C}$, i.e. a finite or infinite sequence of composable morphisms, and apply the symmetric hypercube construction. Let $R \subset R' = R \cup \gamma\mathfrak{S}_n$ be a simple extension of symmetric sieves as above, $\gamma = (\gamma_1, \ldots, \gamma_n) \in I^n$, and denote by G the stabilizer $\mathrm{Stab}_{\mathfrak{S}_n}(\gamma)$ of γ. Put $W' := \{\gamma_1 - 1, \gamma_1\} \times \cdots \times \{\gamma_n - 1, \gamma_n\} \cap I^n$ – the "small hypercube at γ", and $W := W' \cap R = W' - \{\gamma\}$, the latter equality being true because of the characteristic property of γ. Clearly, $W \subset R$ and $W' \subset R'$, so we get a commutative square

$$\begin{tikzcd} X_!(W) \arrow[r] \arrow[d] & X_!(W') \arrow[d] \\ X_!(R) \arrow[r] & X_!(R') \end{tikzcd} \tag{9.9.15.1}$$

Notice that W and W' are stabilized by G, hence G acts from the right on $X_!(W)$ and $X_!(W')$, as well as on $X_!(R)$ and $X_!(R')$, on which in fact a larger group $\mathfrak{S}_n \supset G$ acts (cf. **9.9.6**), and all arrows are compatible with these group actions, whence a commutative diagram of coinvariants:

$$\begin{tikzcd} X_!(W)_G \arrow[r, "v_\gamma"] \arrow[d] & X_!(W')_G \arrow[d] \\ X_!^s(R) \arrow[r, "u_{R,\gamma}"] & X_!^s(R') \end{tikzcd} \tag{9.9.15.2}$$

We want to prove that this square is cocartesian, and find an expression for v_γ in terms of box-products of morphisms $u_k : X(k-1) \to X(k)$ and their "symmetric box-powers" $\rho_d(u_k)$.

Proposition 9.9.16 *Square* (9.9.15.2) *is cocartesian.*

Proof. (a) First of all, the above commutative square can be constructed componentwise in terms of components $X(i)_k$ of simplicial $\mathscr{O}$-modules $X(i)$, so we can check its couniversality componentwise, i.e. replace all $X(i)$ with $X(i)_k$ and consider the corresponding symmetric hypercube construction in $\mathcal{C} := \mathscr{O}$-*Mod*, not in $s\mathscr{O}$-*Mod*. This reduction step is in fact not indispensable (all we actually need is a functor $\tilde{X} : I^n \to \mathcal{C}$ together with a "cocycle" $\{\eta_\sigma : \tilde{X} \xrightarrow{\sim} \sigma\tilde{X}\}$, so we can work in any $\mathcal{C}$), but it can simplify the understanding of the proof.

(b) We have to check that (9.9.15.2) becomes a cartesian square of sets after an application of contravariant functor $h_V := \mathrm{Hom}_{\mathcal{C}}(-, V)$, for any $V \in \mathrm{Ob}\,\mathcal{C}$. Let us fix such a V and denote by P, $\tilde{P}$ and $\tilde{P}_*$ the compositions of X, $\tilde{X}$ and $\tilde{X}_!$ with h_V. Denote by σ_* the maps $h_V(\eta_\sigma) : \tilde{P}(\varepsilon\sigma) \to \tilde{P}(\varepsilon)$ and $\tilde{P}_*(S\sigma) \to \tilde{P}_*(S)$, for any $S \subset I^n$. Taking into account that h_V transforms inductive limits into projective limits and coinvariants into invariants, we see that we have to prove that the following commutative square of sets is cartesian:

$$\begin{tikzcd} \tilde{P}_*(R')^{\mathfrak{S}_n} \arrow[r, "u^*"] \arrow[d, "q^*"] & \tilde{P}_*(R)^{\mathfrak{S}_n} \arrow[d, "p^*"] \\ \tilde{P}_*(W')^G \arrow[r, "v^*"] & \tilde{P}_*(W)^G \end{tikzcd} \tag{9.9.16.1}$$

where $\tilde{P}_*(S) = \varprojlim_{\varepsilon\in S} \tilde{P}(\varepsilon)$ is a projective limit of sets, i.e. it is just the set of collections $x = \{x_\varepsilon\}_{\varepsilon\in S}$, $x_\varepsilon \in \tilde{P}(\varepsilon)$, compatible in the following sense: $x_\varepsilon|_\delta = x_\delta$ for any $\delta \leq \varepsilon$ in S, where of course the "restriction" $z|_\delta$ denotes the image of $z \in \tilde{P}(\varepsilon)$ under the canonical "restriction" map $\tilde{P}(\varepsilon) \to \tilde{P}(\delta)$.

(c) Let us check that an element $x = \{x_\varepsilon\}_{\varepsilon\in R'}$ of $\tilde{P}_*(R')^{\mathfrak{S}_n}$ is completely determined by its restrictions $u^*(x)$ and $q^*(x)$ to R and W', respectively. Indeed, $u^*(x)$ determines all x_ε for $\varepsilon \in R$, and $q^*(x)$ determines x_γ, hence also all $x_{\gamma\sigma} = \sigma_*^{-1}(x_\gamma)$, x being $\mathfrak{S}_n$-invariant. We see that all components of x are thus determined since $R' = R \cup \gamma\mathfrak{S}_n$.

(d) Now it remains to check that given any two such collections $y = \{y_\varepsilon\}_{\varepsilon\in R} \in \tilde{P}_*(R)^{\mathfrak{S}_n}$ and $z = \{z_\varepsilon\}_{\varepsilon\in W'} \in \tilde{P}_*(W')^G$, such that $p^*(y) = v^*(z)$, i.e. $y_\varepsilon = z_\varepsilon$ for all $\varepsilon \in W = W' \cap R$, one can find a compatible collection $x = \{x_\varepsilon\}_{\varepsilon\in R'}$ in $\tilde{P}_*(R')^{\mathfrak{S}_n}$, which restricts to y on R and to z on W'. Of course, we put $x_\varepsilon := y_\varepsilon$ for $\varepsilon \in R$ and $x_\varepsilon := z_\varepsilon$ for $\varepsilon \in W'$, thus defining x on $R \cup W' = R \cup \{\gamma\}$, these two definitions being compatible on $R \cap W' = W$. Furthermore, the element x being constructed has to be $\mathfrak{S}_n$-invariant, i.e. we must have $x_{\gamma\sigma} = \sigma_*^{-1}(x_\gamma) = \sigma_*^{-1}(z_\gamma)$ for all $\sigma \in \mathfrak{S}_n$. This indeed defines x on $\gamma\mathfrak{S}_n$ since if $\gamma\sigma = \gamma\tau$, then $\tau = g\sigma$ for some $g \in G$, hence $\tau_*^{-1}(z_\gamma) = \sigma_*^{-1}(g_*^{-1}(z_\gamma)) = \sigma_*^{-1}(z_\gamma)$ since z was supposed to be G-invariant. We have thus determined $x = (x_\varepsilon)$ for all $\varepsilon \in R' = R \cup \gamma\mathfrak{S}_n$.

(e) Notice that this collection $x = (x_\varepsilon)$ is $\mathfrak{S}_n$-invariant as required. Indeed, for $\varepsilon \in R$ this is evident since y was supposed to be $\mathfrak{S}_n$-invariant, and for $\varepsilon = \gamma\sigma$ we get $x_{\varepsilon\tau} = x_{\gamma\sigma\tau} = (\sigma\tau)_*^{-1}(z_\gamma) = \tau_*^{-1}(\sigma_*^{-1}(z_\gamma)) = \tau_*^{-1}(x_\varepsilon)$ as required.

(f) Now let us check that this collection $x \in (\prod_{\varepsilon\in R'} \tilde{P}(\varepsilon))^{\mathfrak{S}_n}$ is compatible, i.e. that it lies in $\varprojlim_{R'} \tilde{P}(\varepsilon) = \tilde{P}_*(R')$. This would finish the proof. So let's prove $x_{\varepsilon'}|_\varepsilon = x_\varepsilon$ for any $\varepsilon \leq \varepsilon' \in R' = R \cup \gamma\mathfrak{S}_n$. If ε' lies in R, the same is true for $\varepsilon \leq \varepsilon'$, R being a sieve, and the required compatibility is evident since $x_\varepsilon = y_\varepsilon$, $x_{\varepsilon'} = y_{\varepsilon'}$, and y was supposed to be a compatible family. So assume $\varepsilon' \in \gamma\mathfrak{S}_n$. If ε also lies in $\gamma\mathfrak{S}_n$, then $|\varepsilon'| = |\varepsilon|$, hence $\varepsilon' = \varepsilon$ and the compatibility is trivial, so we assume $\varepsilon \in R$. Now the $\mathfrak{S}_n$-invariance of x shows that $x_{\varepsilon'}|_\varepsilon = x_\varepsilon$ iff the same is true for $\varepsilon\sigma^{-1} \leq \varepsilon'\sigma^{-1}$, i.e. we can assume $\varepsilon' = \gamma$. Since $\varepsilon \leq \gamma$ and $\varepsilon \neq \gamma$, $\varepsilon_i < \gamma_i$ at least for one index i; put $\delta := \gamma - e_i = (\gamma_1, \ldots, \gamma_i - 1, \ldots, \gamma_n) \in W$. Now $\varepsilon \leq \delta \leq \gamma$, $x_\gamma|_\delta = x_\delta$ because both δ and γ lie in W' and z was supposed to be compatible, and $x_\delta|_\varepsilon = x_\varepsilon$ because both δ and ε lie in R, q.e.d.

Proposition 9.9.17 *Let $I \subset \mathbb{Z}$, $n \geq 0$, $X : I \to \mathcal{C}$ and $\gamma = (\gamma_1, \ldots, \gamma_n)$ be as above, and suppose that $\gamma_1 \leq \cdots \leq \gamma_n$. Put $G := \mathrm{Stab}_{\mathfrak{S}_n}(\gamma) \subset \mathfrak{S}_n$, $\tilde{W}' := \{\gamma_1 - 1, \gamma_1\} \times \cdots \times \{\gamma_n - 1, \gamma_n\} \subset \mathbb{Z}^n$, $\tilde{W} := \tilde{W}' - \{\gamma\}$, $W' := \tilde{W}' \cap I^n$,*

$W := \tilde{W} \cap I^n = W' - \{\gamma\}$, and denote by υ_γ the morphism $\tilde{X}_!(W)_G \to \tilde{X}_!(W')_G$ induced by inclusion $W \subset W'$. Then:

(a) We can extend X to a functor $\mathbb{Z} \to \mathcal{C}$, without changing υ_γ. Moreover, we can do it in such a way that $\tilde{X}_!(W) \cong \tilde{X}_!(\tilde{W})$ and $\tilde{X}_!(W') \cong \tilde{X}_!(\tilde{W}')$.

(b) If $\gamma = 0 = (0, 0, \ldots, 0)$, then υ_γ is the morphism $\varnothing \to S^n X(0)$.

(c) If $\gamma_1 = \gamma_2 = \cdots = \gamma_n = k$, then υ_γ coincides with $\rho_n(u_k)$, where $u_k : X(k-1) \to X(k)$ is the transition morphism, provided we put $X(-1) = \varnothing$.

(d) Suppose that $\gamma_d < \gamma_{d+1}$ for some $0 < d < n$, and put $e := n - d$. Write $\gamma = \gamma' \star \gamma''$, where $\star$ denotes concatenation of sequences $\star : \mathbb{Z}^d \times \mathbb{Z}^e \to \mathbb{Z}^n$, i.e. $\gamma' = (\gamma_1, \ldots, \gamma_d)$ and $\gamma'' = (\gamma_{d+1}, \ldots, \gamma_n)$. Then υ_γ can be identified with $\upsilon_{\gamma'} \boxempty \upsilon_{\gamma''}$.

(e) Let $\{k_1, \ldots, k_s\}$ be the set of all distinct components of γ, listed in increasing order, and $d_i > 0$ be the multiplicity of k_i in γ, so that $\sum_i d_i = n$. Then υ_γ can be identified with $\rho_{d_1}(u_{k_1}) \boxempty \cdots \boxempty \rho_{d_s}(u_{k_s})$, where $u_{k_i} : X(k_i - 1) \to X(k_i)$ are the transition morphisms of X, and $X(-1) := \varnothing$.

Proof. Notice that (e) follows from (d) and (c) by induction in s, and (b) is a special case of (c), or just evident by itself. In order to show (a) we first extend $I = [m]$ to $I = \omega$ if I was finite by putting $X(k) := X(m)$ for $k > m$, and $u_k : X(k-1) \to X(k)$ is of course taken equal to $\mathrm{id}_{X(m)}$. Then we extend X to $\mathbb{Z}$ by putting $X(k) := \varnothing$ for $k < 0$. Now (a) is trivial except for statement $\tilde{X}_!(W) = \tilde{X}_!(\tilde{W})$; but if we compare inductive limits involved we see that the diagram of the second inductive limit is obtained from that of the first by adding some initial objects and some morphisms, all of them coming from these new initial objects. Of course, this cannot change the inductive limit. Therefore, we can suppose $I = \mathbb{Z}$ while proving (c) and (d).

Now (c) is immediate since it involves only the restriction of $X : \mathbb{Z} \to \mathcal{C}$ to the subset $\{k-1, k\} \cong [1]$, given by the morphism $u_k : X(k-1) \to X(k)$, and $W \subset W'$ corresponds exactly to $[1]^n_{\leq n-1} \subset [1]^n$, hence $\upsilon_\gamma = \rho_n(u_k)$ by definition **9.9.7** of ρ_n.

Let us show the remaining statement (d). Put $G' := \mathrm{Stab}_{\mathfrak{S}_d}(\gamma')$, $G'' := \mathrm{Stab}_{\mathfrak{S}_e}(\gamma'')$; then clearly $G = G' \times G''$ if we identify $\mathfrak{S}_d \times \mathfrak{S}_e$ with a subgroup of $\mathfrak{S}_n$ in the usual way. Next, consider $W_1 \subset W_1' \subset I^d$ and $W_2 \subset W_2' \subset I^e$, constructed from γ' and γ'' in the same way $W \subset W'$ have been constructed from γ. Then $W' = W_1' \times W_2'$ and $W = W_1' \times W_2 \cup W_1 \times W_2'$. Now $W_1' \times W_2$ and $W_1 \times W_2'$ are two subsieves of $W_1' \times W_2'$ with intersection equal to $W_1 \times W_2$; this implies formally (using only properties of inductive limits) that the following square is cocartesian:

$$\begin{tikzcd} \tilde{X}_!(W_1 \times W_2) \arrow[r] \arrow[d] & \tilde{X}_!(W_1' \times W_2) \arrow[d] \\ \tilde{X}_!(W_2 \times W_2') \arrow[r] & \tilde{X}_!(W) \end{tikzcd} \tag{9.9.17.1}$$

Clearly, $\tilde{X}_!(R_1 \times R_2) = \tilde{X}_!(R_1) \otimes \tilde{X}_!(R_2)$ for any subsets $R_1 \subset I^d$, $R_2 \subset I^e$. Applying this observation to the above diagram we see that $\tilde{X}_!(W) \to \tilde{X}_!(W') = \tilde{X}_!(W_1') \times \tilde{X}_!(W_2') = \tilde{X}(\gamma') \otimes \tilde{X}(\gamma'')$ is exactly the box-product of $\tilde{X}_!(W_1) \to \tilde{X}_1(W_1')$ and $\tilde{X}_!(W_2) \to \tilde{X}_2(W_2')$. (One might have also seen from the very beginning that $\tilde{X}_!(W) \to \tilde{X}_!(W')$ equals the multiple box product $u_{\gamma_1} \square \cdots \square u_{\gamma_n}$ and use the associativity of box products mentioned in **9.9.5**; in fact, we have just proved this associativity.) Taking coinvariants under $G = G' \times G''$ with the aid of (9.9.1.1), we obtain $v_\gamma = v_{\gamma'} \square v_{\gamma''}$ as required, q.e.d.

Corollary 9.9.18 *Let $u : X(0) \to X(1)$ be any morphism in $\mathcal{C}$, $0 < k \leq n$. Then $\rho_k^{(n)}(u) : F_{k-1}S^n(u) \to F_kS^n(u)$ is a pushout of $\mathrm{id}_{S^{n-k}X(0)} \otimes \rho_k(u) = i \square \rho_k(u)$, where i is the only morphism $\varnothing \to S^{n-k}X(0)$.*

Proof. Put $I := [1]^n$, $R := [1]^n_{\leq k-1}$, $R' := [1]^n_{\leq k}$, $\gamma := (0, \ldots, 0, 1, 1, \ldots 1)$, $|\gamma| = k$. Then $R \subset R' = R \cup \gamma\mathfrak{S}_n$ is a simple extension of symmetric sieves, so we are in position to apply **9.9.16**, which tells us that $u_{R,\gamma} = \rho_k^{(n)}(u) : \tilde{X}_!^s(R) \to \tilde{X}_!^s(R')$ is a pushout of $v_\gamma : \tilde{X}_!(W)_G \to \tilde{X}_!(W')_G$ for $G = \mathrm{Stab}_{\mathfrak{S}_n}(\gamma)$. Now **9.9.17** yields $v_\gamma = \mathrm{id}_{S^{n-k}X(0)} \otimes \rho_k(u)$ as required.

Now we are ready to formulate the statement we are really going to prove:

Theorem 9.9.19 *For any $n \geq 1$ functors ρ_n preserve (pointwise) cofibrations and acyclic cofibrations in $\mathcal{C} = s\mathscr{O}$-Mod and $\mathfrak{s}\boldsymbol{SETS}_{\mathcal{E}}$.*[2]

9.9.20. (This statement implies the main theorem.) Before proving the above statement, let us show how it will imply **9.9.9**, hence also **9.9.2**. Indeed, applying **9.9.19** to any cofibration $u : \varnothing \to A$ and taking into account that in this case $F_{n-1}S^n(u) = \varnothing$, and $\rho_n(u) : F_{n-1}S^n(u) \to S^n(A)$ can be identified with $S^n(u) : \varnothing \to S^n(A)$, we see that $S^n(A)$ *is cofibrant for any cofibrant A.* Next, let us take any cofibration (resp. acyclic cofibration)

[2] We provide a complete proof only in the pointwise case

$u : A \to B$ between cofibrant objects A and B. Then all $S^{n-k}A$ are cofibrant, and all $\rho_k(u)$ are cofibrations (resp. acyclic cofibrations) by **9.9.19**, hence the same is true for all $S^{n-k}A \otimes \rho_k(u)$ by (TM), hence also for their pushouts $\rho_k^{(n)}(u)$, $0 < k \leq n$. Since $F_0 S^n(u) = S^n A$ is already known to be cofibrant, we prove by induction in k that all $F_k S^n(u)$ are also cofibrant, i.e. all $\rho_k^{(n)}$ are cofibrations (resp. acyclic cofibrations) between cofibrant objects, exactly as claimed in **9.9.9**.

9.9.21. Proof of **9.9.19**. Of course, we use our usual sort of devissage, explained in **9.8.7**. Let us denote by $\mathscr{P}$ the set of morphisms $u : A \to B$ in $\mathcal{C} = s\mathscr{O}\text{-}\mathit{Mod}$ (or rather in the fibers of $\mathfrak{s}\mathscr{O}$**-MOD**$_\mathcal{E}$, but we can work in a fixed fiber throughout all steps of the proof but one), such that $\rho_n(u)$ is a cofibration (resp. acyclic cofibration) for all $n \geq 1$ (in particular, $u = \rho_1(u)$ is itself a cofibration, resp. acyclic cofibration). We are going to show that $\mathscr{P}$ is closed and that it contains the standard generators $L_\mathscr{O}(\underline{I_\mathcal{E}})$ (resp. $L_\mathscr{O}(\underline{J_\mathcal{E}})$); this would imply the theorem. So let's check conditions 1)–8) of **9.5.10** one by one.

(a) Condition 1) (localness of $\mathscr{P}$) is evident, as well as condition 2), since $\rho_n(u)$ is an isomorphism whenever u is one. Condition 6) is also evident: if u' is a retract of u, then $\rho_n(u')$ is a retract of $\rho_n(u)$, retracts being preserved by arbitrary functors $\operatorname{Ar}\mathcal{C} \to \operatorname{Ar}\mathcal{C}$. Now let's check 3). Suppose that u' is a pushout of u; then $u' \square v$ is easily seen to be a pushout of $u \square v$ for any v, hence by induction the n-fold box power $u'^{\square n} = u' \square \cdots \square u'$ is a pushout of $u^{\square n}$, i.e. we have a cocartesian square relating these two morphisms. Applying the right exact functor of taking $\mathfrak{S}_n$-coinvariants to this square, we see that $\rho_n(u')$ is a pushout of $\rho_n(u)$, thus proving 3).

(b) Let's check condition 5) (stability under sequential composition). Let $X(0) \xrightarrow{u_1} X(1) \xrightarrow{u_2} X(2) \to \cdots$ be a composable sequence of morphisms from $\mathscr{P}$, considered here as a functor $X : I \to \mathcal{C}$, where $I = \omega$. Denote the sequential composition of this sequence by $w : X(0) \to X(\omega) := \varinjlim_k X(k)$. We apply the symmetric hypercube construction to functor X, introducing notations $\tilde{X}$, $\tilde{X}_!$ and $\tilde{X}_!^s$ as before. Next, we construct an infinite sequence of simple extensions of symmetric sieves $R_0 \subset R_1 \subset R_2 \subset \cdots$ in I^n as follows. Put $R_0 := \{\varepsilon \in I^n \,|\, \exists i : \varepsilon_i = 0\}$, and define R_N for $N > 0$ by induction. Namely, we choose any element $\gamma^{(N)} \in I^n - R_{N-1}$ with minimal possible $|\gamma^{(N)}|$; since R_{N-1} is symmetric, we can assume that $\gamma_1^{(N)} \leq \cdots \leq \gamma_n^{(N)}$. Now we put $R_N := R_{N-1} \cup \gamma^{(N)}\mathfrak{S}_n$; according to **9.9.14**, $R_{N-1} \subset R_N$ is indeed a simple extension of symmetric sieves. Notice that $X_!^s(R_0 \cap [m]^n) \to X_!^s([m]^n)$ can be identified with $\rho_n(u_m u_{m-1} \cdots u_1)$; taking $\varinjlim_m$, we see that $X_!^s(R_0) \to X_!^s(I^n)$ can be identified with $\rho_n(w)$. On the other hand, by construction $\bigcup_N R_N = I^n$, hence $\varinjlim_N X_!^s(R_N) = X_!^s(I^n) = S^n X(\omega)$, i.e.

$\rho_n(w) : X_!^s(R_0) \to X_!^s(I^n)$ is the sequential composition of morphisms $w_N : X_!^s(R_{N-1}) \to X_!^s(R_N)$. According to **9.9.16** and **9.9.17**, w_N is a pushout of $v_{\gamma^{(N)}} = \rho_{d_1}(u_{k_1}) \,\square \cdots \square\, \rho_{d_s}(u_{k_n})$, where $0 < k_1 < \cdots < k_s$ are distinct components of $\gamma^{(N)}$ (all necessarily > 0 since $\gamma^{(N)} \notin R_0$) and $d_i > 0$ are their multiplicities, $\sum d_i = n$. (One can even check that each such box product appears exactly for one value of N.) Since all u_k were supposed to be in $\mathscr{P}$, all $\rho_d(u_k)$ are cofibrations (resp. acyclic cofibrations), hence the same is true for their box products $v_{\gamma^{(N)}}$ by (TM), hence for their pushouts w_N and the sequential composion $\rho_n(w)$ of w_N as well. Therefore, $w \in \mathscr{P}$.

(c) Condition 4) (stabilitity under compositions) can be either reduced to 5) by extending $X(0) \xrightarrow{u} X(1) \xrightarrow{v} X(2)$ to the right by infinitely many copies of $\mathrm{id}_{X(2)}$, or proved directly by the same reasoning applied to $I = [2]$. In the latter case the only difference is that the sequence of symmetric sieves R_N stops after a finite number of extensions, i.e. we get $R_N = [2]^n$ for some N (actually for $N = n + 1$), thus obtaining a decomposition of $\rho_n(vu)$ into pushouts of morphisms $\rho_n(u)$, $\rho_{n-k}(u) \,\square\, \rho_k(v)$, where $0 < k < n$, and $\rho_n(v)$.

(d) Let $\{u_i : A_i \to B_i\}_{i\in T}$ be any family of morphisms from $\mathscr{P}$. We want to check that $u := \bigoplus u_i$ also lies in $\mathscr{P}$. Consider the following formula for $S^n(B)$:

$$S^n(B) = S^n\Bigl(\bigoplus_{i\in T} B_i\Bigr) = \bigoplus_{\substack{\alpha\in\mathbb{N}_0(T)\\ |\alpha|=n}} \bigotimes_{\alpha_i>0} S^{\alpha_i} B_i \tag{9.9.21.1}$$

This formula can be deduced for example from a similar formula for tensor powers $B^{\otimes n}$:

$$B^{\otimes n} = \Bigl(\bigoplus_{i\in T} B_i\Bigr)^{\otimes n} = \bigoplus_{\varphi\in T^n} \bigotimes_{1\leq i\leq n} B_i^{\otimes \varphi_i} \tag{9.9.21.2}$$

The latter formula can be shown first for a finite T by induction, and then the case of an arbitrary T is shown by taking a filtered inductive limit along finite subsets $T_0 \subset T$. Then the formula for $S^n(B)$ is obtained by re-grouping the direct summands according to $\mathfrak{S}_n$-orbit decomposition of T^n, and computing the $\mathfrak{S}_n$-coinvariants, taking into account that $T^n/\mathfrak{S}_n = \{\alpha \in \mathbb{N}_0(T) : |\alpha| = n\}$ (another natural notation for $T^n/\mathfrak{S}_n$ is S^nT — symmetric power in *Sets*).

Reasoning in essentially the same way with box products, box powers and "symmetric box powers" ρ_n, we obtain a similar formula for $\rho_n(u)$:

$$\rho_n(u) = \rho_n\Bigl(\bigoplus_{i\in T} u_i\Bigr) = \bigoplus_{\substack{\alpha\in\mathbb{N}_0(T)\\ |\alpha|=n}} \square_{\alpha_i>0}\rho_{\alpha_i}(u_i) \tag{9.9.21.3}$$

This formula shows immediately that $\rho_n(u)$ is an (acyclic) cofibration whenever all $\rho_d(u_i)$ are, i.e. $\mathscr{P}$ is closed under direct sums.

(e) Now it remains to check 8), i.e. stability of $\mathscr{P}$ under "local direct sums" $\varphi_!$, for any $\varphi : T \to S$ in $\mathcal{E}$. We can assume $S = e$ as usual. If we would be able to check cofibrations and acyclic cofibrations pointwise, we would simply apply

$$(\varphi_! X)_p = \bigoplus_{q \in T_p} X_q \tag{9.9.21.4}$$

and reduce everything to (d). Notice, however, that this remark actually proves our theorem for the pointwise cofibrant structure on $s\mathscr{O}$-*Mod*, thus allowing us to derive symmetric powers with the aid of pointwise cofibrant replacements, hence they can be derived with the aid of cofibrant replacements as well. In general case we need to apply some "local variant" of formula (9.9.21.1); this is not so easy to do because the proof of (9.9.21.2)⇒(9.9.21.1) is not intuitionistic, so we postpone this verification for the time being, since we have just explained how to prove **9.9.2**(b) and (c) without this verification, using pointwise pseudomodel structures instead.

(f) Once we have shown that $\mathscr{P}$ is closed, all we need to check is that it contains the standard generators $L_{\mathscr{O}}(\underline{I}_{\mathcal{E}})$ (resp. $L_{\mathscr{O}}(\underline{J}_{\mathcal{E}})$). Since symmetric and tensor powers, cofibrations and acyclic cofibrations, and all our other constructions as well, are preserved by base change functors $L_{\mathscr{O}} : s\mathcal{E} \to s\mathscr{O}$-*Mod* and $q^* : sSets \to s\mathcal{E}$, we are reduced to check the statement for I and J in *sSets*. This is done in Lemma **9.9.22** below, so we can finish the proof of **9.9.19**, which is already known to imply the main theorem **9.9.2**.

Lemma 9.9.22 *(a) Let $u : A \to B$ be any cofibration in sSets (e.g. a generator $\dot{\Delta}(n) \to \Delta(n)$ from I). Then $\rho_m(u)$ is a cofibration in sSets, for any $m > 0$.*

(b) Let $u : \Lambda_k(n) \to \Delta(n)$ be a standard acyclic cofibration from J, $0 \leq k \leq n > 0$. Then $\rho_m(u)$ is an acyclic cofibration in sSets, for any $m > 0$.

Proof. (a) First of all, $u^{\square n}$ is a cofibration in *sSets*, and $\rho_n(u)$ is obtained from $u^{\square n}$ by taking $\mathfrak{S}_n$-coinvariants. Recall that cofibrations in *sSets* are exactly the (componentwise) injective maps of simplicial sets. Now the statement follows from the following fact: if $i : X \to Y$ is an injective map of $\mathfrak{S}_n$-sets, then the induced map of coinvariants (i.e. orbit sets) $i_{\mathfrak{S}_n} : X/\mathfrak{S}_n \to Y/\mathfrak{S}_n$ is also injective.

(b) We prove the statement by induction in $m > 0$, case $m = 1$ being trivial. We know by (a) that $\rho_m(u)$ is a cofibration, and all $\rho_k(u)$, $0 < k < m$, are acyclic cofibrations by induction hypothesis, hence the same is true for all $S^{m-k}A \otimes \rho_k(u)$ and for their pushouts $\rho_k^{(m)}(u) : F_{k-1}S^m(u) \to F_k S^m(u)$ (cf. **9.9.18**), and we conclude that $w := \rho_{m-1}^{(m)} \cdots \rho_2^{(m)} \rho_1^{(m)} : S^m A \to F_{m-1}S^m(u)$ is an acyclic cofibration, and in particular a weak equivalence.

Now $S^m(u) = \rho_m(u)w : S^nA \to S^nB$ by **9.9.7**, so by the 2-out-of-3 axiom it would suffice to show that $S^m(u)$ is a weak equivalence.

(c) Let us show that $S^m(u)$ is a weak equivalence, for any $u : \Lambda_k(n) \to \Delta(n)$ from J. Consider for this the embedding $h : \Delta(0) \to \Lambda_k(n)$, defined by the k-th vertex of $\Lambda_k(n)$, and put $h' := hu$:

$$\begin{tikzcd} \Delta(0) \arrow[d, "h"] \arrow[dr, "h'"] & \\ \Lambda_k(n) \arrow[r, "u"] & \Delta(n) \end{tikzcd} \tag{9.9.22.1}$$

By the 2-out-of-3 axiom it would suffice to show that both $S^n(h)$ and $S^n(h')$ are weak equivalences. Now a morphism in *sSets* is a weak equivalence iff this is true for its geometric realization; using the fact that $|X \times Y| = |X| \times |Y|$ for any two finite simplicial sets X and Y, and that $|\cdot|$ is right exact, we see that $|S^nX| = S^n|X| = |X|^n/\mathfrak{S}_n$ for any finite simplicial set X, so, say, $|S^n(h)|$ can be identified with $S^n(|h|)$.

(d) Notice that both $|h|$ and $|h'|$ are homotopy equivalences. More precisely, if we denote by $\pi : Z := |\Lambda_k(n)| \to \mathrm{pt} = |\Delta(0)|$ the only map from Z into a point, then $\pi \circ |h| = \mathrm{id}_{\mathrm{pt}}$, and $|h| \circ \pi$ is homotopic to id_Z. In fact $Z = |\Lambda_k(n)|$ equals $\{(\lambda_0, \ldots, \lambda_n) : \text{all } \lambda_i \geq 0, \sum_i \lambda_i = 1, \exists i \neq k : \lambda_i = 0\}$, and $|\Delta(n)|$ is described similarly without the last condition. The k-th vertex of $\Delta(n)$, i.e. the image of $|h|$ or $|h'|$, is just the k-th basis vector $(0, \ldots, 1, \ldots, 0)$. Now we define maps $H_t : |Z| \to |Z|$ for all real $0 \leq t \leq 1$ by

$$H_t(\lambda_0, \lambda_1, \ldots, \lambda_k, \ldots, \lambda_n) := (t\lambda_0, t\lambda_1, \ldots, 1 - t(1 - \lambda_k), \ldots, t\lambda_n) \tag{9.9.22.2}$$

Clearly, this formula defines a continuous family of maps $H_t : |\Delta(n)| \to |\Delta(n)|$, respecting $|\Lambda_k(n)| \subset |\Delta(n)|$, such that H_0 is just the constant map onto the k-th vertex, and $H_1 = \mathrm{id}$, i.e. $H = \{H_t\}_{0 \leq t \leq 1}$ yields a homotopy from id to the constant map both for $|\Lambda_k(n)|$ and for $|\Delta(n)|$.

(e) Now we put $\tilde{H}_t := S^n(H_t) : S^nZ \to S^nZ$ for all $t \in [0, 1]$, where Z is either $|\Lambda_k(n)|$ or $|\Delta(n)|$. Clearly, $\tilde{H}_t$ depends continuously on t, hence it defines a homotopy $\tilde{H}$ between id_{S^nZ} and a constant map, hence $S^nZ \to \mathrm{pt}$ and $\mathrm{pt} \to S^nZ$ are weak equivalences, both for $Z = |\Lambda_k(n)|$ and $Z = |\Delta(n)|$, q.e.d.

10 Perfect cofibrations and Chow rings

The aim of this chapter is to define perfect cofibrations and perfect simplicial objects in $s\mathscr{O}$-*Mod*, where $\mathcal{E} = (\mathcal{E}, \mathscr{O})$ is an arbitrary generalized (commutatively) ringed topos, usually assumed to have enough points. Then we define K_0 of the set of isomorphism classes of perfect objects in $\mathcal{D}^{\leq 0}(\mathcal{E}, \mathscr{O}) = \operatorname{Ho} s\mathscr{O}\text{-}Mod$ and introduce on this $K^0(\mathcal{E}) := K_0(\operatorname{Perf}(\mathcal{E}, \mathscr{O}))$ a pre-λ-ring structure. We postpone the verification of the fact that this is a λ-ring structure; instead, we provisionally replace $K^0(\mathcal{E})$ by its largest quotient which is a λ-ring, and use it to construct $\operatorname{Ch}\mathcal{E}$ as the associated graded with respect to the γ-filtration. In this way we construct a reasonable intersection and Chern class theory on any generalized ringed topos, and in particular any generalized scheme.

We finish this chapter by a computation of $K^0(\widehat{\operatorname{Spec}\mathbb{Z}})$ and $\operatorname{Ch}(\widehat{\operatorname{Spec}\mathbb{Z}})$, obtaining "correct" answers. The case of a projective bundle will be treated elsewhere.

10.0. (Notations.) We fix a generalized (commutatively) ringed topos $\mathcal{E} = (\mathcal{E}, \mathscr{O})$, usually supposed to have enough points. We denote by $\mathcal{C}$ either the pseudomodel category $s\mathscr{O}$-*Mod* of simplicial $\mathscr{O}$-modules, or the corresponding stack $\mathfrak{s}\mathscr{O}\text{-}\mathbf{MOD}_{\mathcal{E}}$. We denote standard finite ordered sets by $[n] = \{0, 1, \ldots, n\}$, and the category of all such finite ordered sets by $\mathbf{\Delta}$; thus a simplicial object over $\mathcal{D}$ is a functor $X : \mathbf{\Delta}^0 \to \mathcal{D}$. The full subcategory of $\mathbf{\Delta}$ consisting of $[n]$ with $n \leq m$ will be denoted by $\mathbf{\Delta}_{\leq m}$ (cf. **8.2.4**, where it has been denoted $\mathbf{\Delta}_m$), and $X_{\leq m} : \mathbf{\Delta}^0_{\leq m} \to \mathcal{D}$ will denote the m-th truncation of X.

The standard generating sets for cofibrations and acyclic cofibrations in *sSets* will be still denoted by I and J. Thus $I = \{\dot{\Delta}(n) \to \Delta(n) : n \geq 0\}$ and $J = \{\Lambda_k(n) \to \Delta(n) : 0 \leq k \leq n > 0\}$, and $L_{\mathscr{O}}(\underline{I}_{\mathcal{E}})$ and $L_{\mathscr{O}}(\underline{J}_{\mathcal{E}})$ are our generators for the pseudomodel structure of $s\mathscr{O}$-*Mod*.

10.1. (Simplicial dimension theory.) Up to now we didn't use much the fact that $\mathcal{C} = s\mathscr{O}$-*Mod*, usually only using the fact that we can reason componentwise, and sometimes reducing the statements being proved to basic case *sSets*, where the needed statements are usually quite well known. Now we'd like to construct a *simplicial* or *homotopic dimension theory* for objects and morphisms in $\mathcal{C}$, together with canonical dimension decompositions, similar to the theory of brute truncations of a complex of modules over an additive sheaf of rings.

Several first definitions will be in fact valid in any category $\mathcal{C} = s\mathcal{D}$ of simplicial objects over a category $\mathcal{D}$ with finite inductive limits:

10.1.1. (Morphisms concentrated in dimension $> n$ and $\leq n$.) Let $f : X \to$

Y be any morphism in $\mathcal{C} = s\mathcal{D}$. We say that f *is concentrated in dimension* $> n$, if f induces an isomorphism $f_{\leq n} : X_{\leq n} \to Y_{\leq n}$ of n-truncated objects. This is equivalent to saying that all $f_i : X_i \to Y_i$ are isomorphisms for $0 \leq i \leq n$, or that $\mathrm{sk}_n(f) : \mathrm{sk}_n X \to \mathrm{sk}_n Y$ is an isomorphism in $\mathcal{C}$.

We say that f *is concentrated in dimension* $\leq n$, or that f *has (relative) dimension* $\leq n$, if the following square is cocartesian, where $\mathrm{sk}_n X \to X$ and $\mathrm{sk}_n Y \to Y$ are the canonical morphisms (cf. **8.2.4**):

$$\begin{tikzcd} \mathrm{sk}_n X \arrow[r, "\mathrm{sk}_n f"] \arrow[d] & \mathrm{sk}_n Y \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{10.1.1.1}$$

Since $\mathrm{sk}_n(f)$ is completely determined by the truncation $f_{\leq n}$, we see that *any* $f : X \to Y$ *of relative dimension* $\leq n$ *is completely determined (up to a unique isomorphism) by* X *and* $f_{\leq n}$*, i.e. by the collection* $f_i : X_i \to Y_i$*,* $i \leq n$*, if* X *and* Y *are already fixed.*

We extend the above notions to any $n \in \mathbb{Z}$, possibly negative, by putting $\mathbf{\Delta}_n := \varnothing$, $\mathrm{sk}_n X := \varnothing_{\mathcal{C}}$ for all $n < 0$. When f is concentrated in dimension $> n$, we also say that it is concentrated in dimension $\geq n + 1$, and so on. If f is concentrated in dimension $\leq b$ and $\geq a$, we say that f *is concentrated in dimensions* $[a, b]$. When $a = b$, we say that f *is pure(ly) of dimension* a instead. We allow here $a = -\infty$ and $b = +\infty$ as well.

When $n < 0$, any f is concentrated in dimension $> n$, but only isomorphisms are concentrated in dimension $\leq n$.

One checks very easily using formula $\mathrm{sk}_n \circ \mathrm{sk}_m = \mathrm{sk}_{\min(m,n)}$ for any m, $n \in \mathbb{Z}$, itself an immediate consequence of the universal property of $\mathrm{sk}_n X$, that whenever $m \leq n$, condition "f is concentrated in dimension $\leq m$" implies that "f is concentrated in dimension $\leq n$", and "f is concentrated in dimension $> n$" implies "f is concentrated in dimension $> m$". In particular, we can define the *(relative) dimension* $\dim f$ *of* f as the infimum of all $n \in \mathbb{Z}$, such that f is concentrated in dimension $\leq n$. Then f is concentrated in dimension $\leq n$ iff $\dim f \leq n$. Another immediate consequence is that *if* f *is concentrated in dimensions* $[a, b]$ *and* $[a, b] \subset [a', b']$*, then* f *is concentrated in dimensions* $[a', b']$.

Finally, we extend all the above notions to objects $X \in \mathrm{Ob}\,\mathcal{C}$ by considering $\nu_X : \varnothing \to X$, e.g. $\dim X := \dim \nu_X$ and so on. The only object of dimension < 0 is the initial object $\varnothing$, which has $\dim \varnothing = -\infty$, and the simplicial objects of dimension ≤ 0 (hence pure of dimension 0) are the constant simplicial objects, since $\mathrm{sk}_0 X$ is the constant simplicial object X_0 for any X.

10.1.2. (Compatibility with composition and inductive limits.) One checks immediately from definitions that the above properties are stable under composition, and in one case even satisfy the 2-out-of-3 property: *if two of f, g, $g \circ f$ are concentrated in dimension $> n$, then so is the third. If $g \circ f$ and f are of relative dimension $\leq n$ (resp. are concentrated in $[a, b]$), then the same holds for g.* In particular, if X and Y are of dimension $\leq n$, the same is true for any $f : X \to Y$.

Since sk_n commutes with arbitrary inductive limits (and functors φ^* and $\varphi_!$ in stack $\mathfrak{s}\mathscr{O}$**-MOD**$_\mathcal{E}$ for any $\varphi : T \to S$, if we want to consider stacks here), we see that the above properties of simplicial objects are stable under all sorts of inductive limits, and the property of a morphism f to be concentrated in $[a, b]$ is stable under pushouts, composition, sequential composition, retracts, direct sums (and local direct sums $\varphi_!$ in the stack situation), hence *morphisms concentrated in dimension $[a, b]$ constitute a closed class in $\mathcal{C} = s\mathscr{O}$-Mod or $\mathcal{C} = \mathfrak{s}\mathscr{O}$**-MOD**$_\mathcal{E}$* in the sense of **9.5.10**.

10.1.3. (Example: standard generators of *sSets*.) It is easy to see that the standard simplex $\Delta(n) \in \mathrm{Ob}\, sSets$ is of dimension n (i.e. $\dim X = n$), hence any finite simplicial set has finite dimension, equal in fact to the highest dimension in which it has non-degenerate simplices.

Consider a standard cofibration $i_n : \dot{\Delta}(n) \to \Delta(n)$ from I. It is of dimension $\leq n$ because both sides are. On the other hand, $\mathrm{sk}_{n-1} \Delta(n) = \dot{\Delta}(n)$, i.e. i_n is of dimension $> n-1$, hence i_n *is pure of dimension* n. One checks similarly that the standard acyclic cofibrations $\Lambda_k(n) \to \Delta(n)$, $0 \leq k \leq n > 0$, are concentrated in dimension $[n-1, n]$.

Notation. We denote by $I_{\leq n}$, $I_{\geq n}$, $I_{[a,b]}$, I_S (where $S \subset \mathbb{Z}$) and $I_{=n} = I_{[n,n]}$ the subsets of I consisting of generators of specified dimensions.

10.1.4. (Functorial dimensional decompositions.) We claim that *any morphism $f : X \to Y$ can be uniquely (up to a unique isomorphism) factorized into $X \xrightarrow{g} Z \xrightarrow{h} Y$, where g is of dimension $\leq n$ and h is concentrated in dimension $> n$.* Indeed, $\mathrm{sk}_n h$ has to be an isomorphism, i.e. all $h_i : Z_i \to Y_i$, $i \leq n$, are isomorphisms; we can assume, replacing Z_i by isomorphic objects Y_i if necessary, that $h_i = \mathrm{id}_{Y_i}$, for all $i \leq n$. Then $h_{\leq n} = \mathrm{id}$, hence $\mathrm{sk}_n h = \mathrm{id}$ and $\mathrm{sk}_n f = \mathrm{sk}_n h \, \mathrm{sk}_n g = \mathrm{sk}_n g$, i.e. $\mathrm{sk}_n g$ is known; since g is of dimension $\leq n$, this completely determines g, which can be identified with the pushout of $\mathrm{sk}_n g = \mathrm{sk}_n f$ with respect to $\mathrm{sk}_n X \to X$. In this way we obtain uniqueness; conversely, we can always define $g = \mathrm{sk}_{\leq n}(f)$ and $h = \mathrm{sk}_{>n}(f)$ with required properties by means of the following cocartesian square:

$$\begin{tikzcd} \mathrm{sk}_n X \arrow[r, "\mathrm{sk}_n f"] \arrow[d, dashed] & \mathrm{sk}_n Y \arrow[d] \arrow[ddr, bend left] & \\ X \arrow[r, dashed, "\mathrm{sk}_{\leq n} f"] \arrow[drr, "f"'] & Z \arrow[dr, dashed, "\mathrm{sk}_{>n} f"] & \\ & & Y \end{tikzcd} \tag{10.1.4.1}$$

Therefore, the above factorization $f = \mathrm{sk}_{>n}(f) \circ \mathrm{sk}_{\leq n}(f)$ is functorial in f, and commutes with arbitrary inductive limits.

Now suppose that $n < m$. We can construct a decomposition of $f : X \to Y$ into $X \xrightarrow{u} W \xrightarrow{v} Z \xrightarrow{w} Y$, where u is concentrated in $\leq n$, v in $[n+1, m]$ and w in $> m$, for example by putting $u := \mathrm{sk}_{\leq n}(f)$, $v := \mathrm{sk}_{\leq m} \mathrm{sk}_{>n}(f)$, $w := \mathrm{sk}_{>m} \mathrm{sk}_{>n}(f) = \mathrm{sk}_{>m}(f)$. Such decompositions are also unique (apply previous result first to $f = (wv)u$, then to wv), and since $u = \mathrm{sk}_{\leq n} \mathrm{sk}_{\leq m}(f) = \mathrm{sk}_{\leq n}(f)$, $v = \mathrm{sk}_{>n} \mathrm{sk}_{\leq m}(f)$, $w = \mathrm{sk}_{>m}(f)$ constitute another such decomposition, we have $\mathrm{sk}_{\leq m} \mathrm{sk}_{>n} = \mathrm{sk}_{>n} \mathrm{sk}_{\leq m}$ whenever $n < m$. We put

$$\mathrm{sk}_{[n,m]}(f) := \mathrm{sk}_{\leq m} \mathrm{sk}_{\geq n}(f) = \mathrm{sk}_{\geq n} \mathrm{sk}_{\leq m}(f), \quad \text{for any } n \leq m, \tag{10.1.4.2}$$

$$\mathrm{sk}_{=n}(f) := \mathrm{sk}_{[n,n]}(f) \quad \text{for any } n. \tag{10.1.4.3}$$

Since $\mathrm{sk}_{\leq n}(f) = \mathrm{sk}_{=n}(f) \circ \mathrm{sk}_{\leq n-1}(f)$, $\mathrm{sk}_{<0}(f) = \mathrm{id}$, and $\varinjlim_n \mathrm{sk}_{\leq n}(f) = f$, we obtain a canonical decomposition of f into a sequential composition of morphisms $\mathrm{sk}_{=n}(f)$, each of pure dimension n:

$$X = F_0 X \xrightarrow{\mathrm{sk}_{=0} f} F_1 X \xrightarrow{\mathrm{sk}_{=1} f} F_2 X \to \cdots F_\infty X = Y \tag{10.1.4.4}$$

The above decomposition is functorial in f, unique up to an isomorphism, and we have e.g. $\mathrm{sk}_{[a,b]}(f) = \mathrm{sk}_{=b}(f) \circ \mathrm{sk}_{=b-1}(f) \circ \cdots \circ \mathrm{sk}_{=a}(f)$. In the additive case this decomposition corresponds to an increasing filtration $F_k P$ on a chain complex P, "cokernel" of f, namely, the brute filtration: $(F_k P)_n = 0$ for $n \geq k$, $(F_k P)_n = P_n$ for $n < k$.

10.1.5. (Extension of terminology.) Given any subset $S \subset \mathbb{Z}$, we say that f *is concentrated in dimension(s)* S if $\mathrm{sk}_{=n}(f)$ are isomorphisms for all $n \notin S$, i.e. if the corresponding steps of (10.1.4.4) are trivial. In this case f is a (finite or sequential) composition of morphisms isomorphic to $\mathrm{sk}_{=n}(f)$, for elements $n \in S$ listed in increasing order.

Clearly, this definition is compatible with our previous terminology when $S = [a, b]$.

Proposition 10.1.6 *(a) If a morphism $f : X \to Y$ belongs to $\mathrm{Cl}\, L_{\mathscr{O}}(\underline{I}_S)$, where $S \subset \mathbb{Z}$ is any subset, then $\mathrm{sk}_n(f)$ and its pushout $\mathrm{sk}_{\leq n}(f)$ belong to $\mathrm{Cl}\, L_{\mathscr{O}}(\underline{I}_{S\cap[0,n]})$, and $\mathrm{sk}_{>n}(f)$ belongs to $\mathrm{Cl}\, L_{\mathscr{O}}(\underline{I}_{S\cap[n+1,\infty)})$, hence $\mathrm{sk}_{[a,b]}(f)$ belongs to $\mathrm{Cl}\, L_{\mathscr{O}}(\underline{I}_{S\cap[a,b]})$.*

(b) If $f : X \to Y$ is a cofibration, then $\mathrm{sk}_n f$, $\mathrm{sk}_{\leq n} f$, $\mathrm{sk}_{>n} f$, $\mathrm{sk}_{[a,b]} f$ and $\mathrm{sk}_{=n} f$ are also cofibrations. Moreover, $\mathrm{sk}_{[a,b]} f$ lies in the closure of $L_{\mathscr{O}}(\underline{I}_{[a,b]})$.

*(c) The set of cofibrations concentrated in dimension $[a,b]$ coincides with the closure (in $\mathcal{C} = \mathfrak{s}\mathscr{O}$-**MOD**$_{\mathcal{E}}$) of $L_{\mathscr{O}}(\underline{I}_{[a,b]})$.*

(d) More generally, the set of cofibrations concentrated in dimensions $S \subset \mathbb{Z}$ coincides with the closure of $\underline{I}_S$.

Proof. (a) is reduced by our usual "devissage" (cf. **9.8.7**) to the case of a standard cofibration $i_n : \dot{\Delta}(n) \to \Delta(n)$ in *sSets* with $n \in S$, where it is immediate, i_n being purely of dimension n. The ability to apply devissage to sk_n is due to the fact that sk_n commutes with any inductive limits, composition, sequential composition, local direct sums ($\varphi_!$) and so on. Case of $\mathrm{sk}_{>n}$ is not much more complicated: the only difference is that now $\mathrm{sk}_{>n}(gf) = \mathrm{sk}_{>n}\, g \circ (\text{some pushout of } \mathrm{sk}_{>n} f)$, similarly to what we had for $u \mapsto u \Box i$ in **9.8.5**. We can emphasize this similarity by means of a statement that contains both these situations, cf. **10.1.7** below.

(b) is now immediate from (a), and (c) follows from (b): one inclusion is due to the fact that morphisms concentrated in dimensions $[a,b]$ constitute a closed class, and the other inclusion follows from (b) and the fact that f is concentrated in dimensions $[a,b]$ iff f is isomorphic to $\mathrm{sk}_{[a,b]} f$.

Let's show (d). If $f : X \to Y$ lies in $\mathrm{Cl}\, L_{\mathscr{O}}(I_S)$, then by (a) $\mathrm{sk}_{=n}(f)$ lies in $\mathrm{Cl}\, L_{\mathscr{O}}(I_{S\cap\{n\}})$; for $n \notin S$ this means that $\mathrm{sk}_{=n}(f)$ is an isomorphism, i.e. the n-th step of dimensional decomposition is trivial, hence f is concentrated in dimensions from S by definition. Conversely, if f is a cofibration concentrated in dimensions from S, then it is a (finite or sequential) composition of morphisms $\mathrm{sk}_{=n}(f)$ for $n \in S$; each of these morphisms lies in $\mathrm{Cl}\, L_{\mathscr{O}}(I_{=n}) \subset \mathrm{Cl}\, L_{\mathscr{O}}(I_S)$ by (b), hence the same is true for their composition f.

Lemma 10.1.7 *Let $\mathcal{C}$ and $\mathcal{D}$ be any two categories closed under (small) inductive limits, F, $G : \mathcal{C} \to \mathcal{D}$ be two functors commuting with arbitrary inductive limits, and $\eta : F \to G$ be a natural transformation. For any $u : X \to Y$ in $\mathcal{C}$ denote by $Z(u)$ the fourth vertex of the cocartesian square built on $F(u)$ and η_X, and by $\eta[u] : Z(u) \to G(B)$ the natural morphism, thus defining functors $Z : \mathrm{Ar}\,\mathcal{C} \to \mathcal{D}$ and $\eta[\cdot] : \mathrm{Ar}\,\mathcal{C} \to \mathrm{Ar}\,\mathcal{D}$. Then:*

(a) $Z(\mathrm{id}_Y) = F(Y)$, and $Z((u, \mathrm{id}_Y)) : Z(u) \to Z(\mathrm{id}_Y)$ can be identified with $\eta[u]$.

(b) *For any composable* $X \xrightarrow{u} Y \xrightarrow{v} W$ *the morphism* $Z(u, \mathrm{id}_W) : Z(vu) \to Z(v)$ *is a pushout of* $Z(u, \mathrm{id}_Y) = \eta[u]$*, and* $\eta[vu] = \eta[v] \circ Z(u, \mathrm{id}_W)$*.*

(c) *If* $u' : X' \to Y'$ *is a pushout of* $u : X \to Y$*, then* $\eta[u']$ *is a pushout of* $\eta[u]$*.*

(d) $\eta\left[\bigoplus_\alpha u_\alpha\right] = \bigoplus_\alpha \eta[u_\alpha]$*.*

(e) *If* u *is a sequential composition* $\cdots u_2 u_1 u_0$*, then* $\eta[u]$ *is sequential composition of pushouts of* $\eta[u_n]$*.*

Proof. Statement (a) is immediate, and (b) is shown by a simple diagram chase, not involving any properties of F or G. Statement (c) is shown by another diagram chase, using that F and G preserve cocartesian squares. Remaining statements are then shown in the same way as in **9.8.5**.

Proposition 10.1.8 *(a) If* $i : A \to B$ *and* $s : X \to Y$ *are cofibrations of dimension* $\leq n$ *and* $\leq m$*, respectively, then* $i \square s$ *is a cofibration of dimension* $\leq n + m$*. In particular,* $\dim(i \square s) \leq \dim i + \dim s$ *for any two cofibrations* i *and* s*.*

(b) If X *and* Y *are cofibrant, then* $\dim(X \otimes Y) \leq \dim X + \dim Y$*.*

Proof. (b) is immediate from (a) since $\nu_{X\otimes Y} = \nu_X \square \nu_Y$. Let's prove (a). By **10.1.6**,(c) we see that i lies in the closure of $L_{\mathscr{O}}(\underline{I}_{\leq n})$, and s in that of $L_{\mathscr{O}}(\underline{I}_{\leq m})$. Applying devissage for the box product, first in one variable and then in the other (application of devissage is made possible by lemma **9.8.5**), we are finally reduced to checking that $w := i_n \square i_m$ is a cofibration of dimension $\leq n + m$ in *sSets*, for any n, $m \geq 0$. First of all, we see that the target $\Delta(n) \otimes \Delta(m) = \Delta(n) \times \Delta(m)$ of w is of dimension $n+m$: its simplices are order-preserving maps $[k] \to [n] \times [m]$, and non-degenerate simplices correspond to injective such maps, which can exist only for $k \leq m+n$. Next, w is a cofibration, hence componentwise injective; this immediately implies that it preserves non-degenerate simplices, hence its source can have non-degenerate simplices only in dimension $\leq n+m$, i.e. both the source and the target of w are of dimension $\leq n + m$, hence the same holds for w itself.

10.1.9. One shows even simpler that *for any morphism of generalized ringed topoi* $f : (\mathcal{E}', \mathscr{O}') \to (\mathcal{E}, \mathscr{O})$*, functor* f^* *preserves cofibrations and cofibrant objects concentrated in dimension* $[a, b]$. This is actually true for arbitrary morphisms or objects of $s\mathcal{E}$*-Mod* (not just cofibrant), just because the components of sk_n, $\mathrm{sk}_{\leq n}$ and $\mathrm{sk}_{>n}$ are computed with the aid of finite inductive limits only.

10.1.10. (Constant cofibrations. Components of cofibrations.) We say that a morphism $f_0 : X_0 \to Y_0$ in $\mathscr{O}$*-Mod* is a *(constant) cofibration* if it becomes a cofibration in $s\mathscr{O}$*-Mod* when considered as a morphism of constant simplicial objects. Since constant objects and morphisms are exactly those of dimension ≤ 0, we see that f_0 is a cofibration iff it lies in $\operatorname{Cl} L_{\mathscr{O}}(\underline{I}_{=0})$. Since $Z \mapsto Z_0$ commutes with any limits, our usual devissage argument shows that (constant) cofibrations in $\mathscr{O}$*-Mod* are *exactly* morphisms from the closure in $\mathscr{O}$*-Mod* of one-element set $\varnothing_{\mathscr{O}} \to L_{\mathscr{O}}(1)$. For example, cofibrations in *Sets* are just the injective maps.

Now functor $P_n : s\mathscr{O}\text{-}Mod \to \mathscr{O}\text{-}Mod$, $Z \mapsto Z_n$, also preserves any limits, so we can apply devissage again and prove that *whenever $f : X \to Y$ is a cofibration in $s\mathscr{O}$-Mod, all its components $f_n : X_n \to Y_n$ are cofibrations in $\mathscr{O}$-Mod, i.e. they belong to* $\operatorname{Cl}\{\varnothing_{\mathscr{O}} \to L_{\mathscr{O}}(1)\}$.

Proposition 10.1.11 *Let $(\mathcal{E}, \mathscr{O})$ be a generalized ringed topos, not necessarily commutative. Then:*

(a) All components $f_n : X_n \to Y_n$ of a cofibration $f : X \to Y$ in $s\mathscr{O}$-Mod are monomorphisms.

(b) Any constant cofibration $f_0 : X_0 \to Y_0$ in $\mathscr{O}$-Mod is a monomorphism.

Proof. Statement (a) follows immediately from (b), since we have just shown that all components of a cofibration are constant cofibrations. Let's prove (b). We assume for simplicity that $\mathcal{E}$ has enough points, so it is enough to prove (b) over $\mathcal{E} = Sets$. Then we can assume that $\mathscr{O}$ has at least one constant, i.e. $\mathscr{O}(0) = \mathscr{O}^{(0)} = \varnothing_{\mathscr{O}}$ is non-empty: otherwise we can do scalar extension with respect to $r : \mathscr{O} \mapsto \mathscr{O} \boxtimes \mathbb{F}_1 = \mathscr{O}\langle c^{[0]}\rangle$ (here we use the non-commutative tensor product $\boxtimes$ of **4.5.17**, i.e. c is not required to commute with any operations of $\mathscr{O}$). Then $M \to r_* r^* M$ is injective for any $\mathscr{O}$-module M, since it can be identified with embedding $M \to M \oplus L_{\mathscr{O}}(1)$, which admits a section when M is non-empty, and is injective for trivial reasons if M is empty (this reasoning wouldn't be valid over an arbitrary topos $\mathcal{E}$, where we would have to decompose $\mathcal{E}$ into the open subtopos over which M locally admits a section, and its closed complement). Therefore, injectivity of $r^*(i)$ would imply injectivity of i, and r^* preserves cofibrations, so the reduction step is done.

So let us assume that $\mathscr{O}$ admits a constant c. We claim that then *any constant cofibration $f_0 : X_0 \to Y_0$ admits a left inverse*, hence is injective. This statement is shown again by "devissage", using that constant cofibrations coincide with the closure of $i_0 : L_{\mathscr{O}}(0) \to L_{\mathscr{O}}(1)$ in $\mathscr{O}$*-Mod*, and that i_0 admits a left inverse π_0, defined e.g. by the constant $c \in L_{\mathscr{O}}(0)$ (elements of

M are in one-to-one correspondence with $\mathscr{O}$-homomorphisms $L_{\mathscr{O}}(1) \to M$, and condition $\pi_0 i_0 = \mathrm{id}$ is automatic, $L_{\mathscr{O}}(0) = \varnothing_{\mathscr{O}}$ being initial).

Proposition 10.1.12 *Morphism $f : X \to Y$ is a cofibration iff all $(\mathrm{sk}_{\geq n} f)_n$ are constant cofibrations iff all $(\mathrm{sk}_{=n} f)_n$ are constant cofibrations, where $n \geq 0$ runs through all non-negative integers.*

Proof. We know that $\mathrm{sk}_{\geq n} f$ and $\mathrm{sk}_{=n} f$ are cofibrations whenever f is one (cf. **10.1.6**), and that all components of a cofibration are constant cofibrations (cf. **10.1.10**), hence the first condition implies the other two. On the other hand, $\mathrm{sk}_{=n} f = \mathrm{sk}_{\leq n} \mathrm{sk}_{\geq n} f$ is a pushout of $\mathrm{sk}_n \mathrm{sk}_{\geq n} f$, hence $(\mathrm{sk}_{=n} f)_n$ is a pushout of $(\mathrm{sk}_{\geq n} f)_n$, i.e. the second condition implies the third.

Now suppose that $(\mathrm{sk}_{=n} f)_n$ is a constant cofibration. Since f is a sequential composition of the $\mathrm{sk}_{=n} f$, it would suffice to show that $\mathrm{sk}_{=n} f$ is a cofibration, for all $n \geq 0$. Putting $g := \mathrm{sk}_{=n} f$, we are reduced to checking the following statement: *if $g : X' \to Y'$ is purely of dimension n, and $g_n : X'_n \to Y'_n$ is a constant cofibration, then g is itself a cofibration.* According to Lemma **10.1.14** below, any $g : X' \to Y'$ purely of dimension n is a pushout of $g_n \square i_n = g_n \square L_{\mathscr{O}}(i_n)$, where $i_n : \dot{\Delta}(n) \to \Delta(n)$ is the standard cofibrant generator. We know already that $\square$ respects cofibrations (cf. **9.8.4**), whence the statement.

During the proof we have obtained the following interesting statement:

Corollary 10.1.13 *A morphism $f : X \to Y$ is a cofibration purely of dimension n iff it is a pushout of $u \square i_n$, for some constant cofibration u.*

Lemma 10.1.14 *If $f : X \to Y$ is purely of dimension $n \geq 0$, then it can be canonically identified with a pushout of $f_n \square i_n = f_n \square L_{\mathscr{O}}(i_n)$, where $f_n : X_n \to Y_n$ is the n-th component of f, considered here as a morphism of constant simplicial objects, and $i_n : \dot{\Delta}(n) \to \Delta(n)$ is the standard cofibrant generator of sSets.*

Proof. First of all, f is a pushout of $\mathrm{sk}_n f$, being of dimension $\leq n$, $f' := \mathrm{sk}_n f$ is still purely of dimension n, and $f_n = (\mathrm{sk}_n f)_n$, so we can replace f with $\mathrm{sk}_n f$ and assume that X and Y are of dimension $\leq n$. Now consider the following commutative diagram:

$$
\begin{tikzcd}
X_n \otimes \dot{\Delta}(n) \arrow[rr, "f_n \otimes \mathrm{id}_{\dot{\Delta}(n)}"] \arrow[d, "\mathrm{id}_{X_n} \otimes i_n"] & & Y_n \otimes \dot{\Delta}(n) \arrow[d, dashed, "\tilde{\imath}"] \arrow[ddr, "\mathrm{id}_{Y_n} \otimes i_n"] \arrow[dddl, two heads, "w"'] & \\
X_n \otimes \Delta(n) \arrow[rr, dashed, "\tilde{f}"] \arrow[drrr, "f_n \otimes \mathrm{id}_{\Delta(n)}"'] \arrow[ddr, "\mathrm{ev}_{\Delta(n),X}"'] & & Z \arrow[dr, dashed, "f_n \square i_n"] \arrow[ddl, dashed, two heads, "h"] & \\
 & & & Y_n \otimes \Delta(n) \arrow[d, "\mathrm{ev}_{\Delta(n),Y}"] \\
 & X \arrow[rr, "f"] & & Y
\end{tikzcd}
\tag{10.1.14.1}
$$

Here $\mathrm{ev}_{\Delta(n),X} : X_n \otimes \Delta(n) \to X$ is the "evaluation map", defined by adjointness from the canonical isomorphism $\mathbf{Hom}(\Delta(n), X) \cong X_n$, and $\mathrm{ev}_{\Delta(n),Y}$ is defined similarly, so the only solid arrow left unexplained is $w : Y_n \otimes \dot{\Delta}(n) \to X$. By adjointness it suffices to construct $w^\flat : Y_n \to \mathbf{Hom}(\dot{\Delta}(n), X)$. First of all, we have a canonical morphism $f_* : \mathbf{Hom}(\dot{\Delta}(n), X) \to \mathbf{Hom}(\dot{\Delta}(n), Y)$; it is even an *iso*morphism since $\mathbf{Hom}(\dot{\Delta}(n), X)$ is a certain finite projective limit of components X_k of X with $k \leq n-1$, and all $f_k : X_k \to Y_k$, $0 \leq k \leq n-1$, are isomorphisms, f being purely of dimension n by assumption. Composing the inverse of this isomorphism f_* with $i_n^* : Y_n = \mathbf{Hom}(\Delta(n), Y) \to \mathbf{Hom}(\dot{\Delta}(n), Y)$, we obtain our $w^\flat$.

Now it is immediate that the subdiagram of the above diagram consisting of solid arrows is commutative, so we can construct Z by means of a cocartesian square, and define remaining arrows $\tilde{\imath}$, $\tilde{f}$, $f_n \,\square\, i_n$ and h, thus completing the construction of the diagram.

We want to show that the square of this diagram with vertices in Z, $Y_n \otimes \Delta(n)$, X and Y is cocartesian. Since all simplicial objects involved in the diagram are of dimension $\leq n$, it would suffice to show that this square becomes cocartesian after truncation to dimension $\leq n$, i.e. that its components in dimensions $k \leq n$ constitute a cocartesian square. When we truncate to dimensions $\leq n-1$, i_n becomes an isomorphism, hence the same is true for $\mathrm{id}_{X_n} \otimes i_n$, $\mathrm{id}_{Y_n} \otimes i_n$, $\tilde{\imath}$, hence $(f_n \,\square\, i_n)_{\leq n-1}$ is also an isomorphism, and $f_{\leq n-1}$ is an isomorphism as well, f being concentrated in dimension $> n-1$, so our square is cocartesian in dimensions $< n$ for trivial reasons.

Now it remains to compute the components in dimension n. First of all, $(\Delta(n))_n$ consists of all order-preserving maps $\varphi : [n] \to [n]$, and $(\dot{\Delta}(n))_n$ consists of all such non-surjective maps, i.e. $(\Delta(n))_n = (\dot{\Delta}(n))_n \sqcup \{\sigma\}$, where $\sigma = \mathrm{id}_{\mathbf{n}}$ is the only non-degenerate n-dimensional simplex of $\Delta(n)$. This immediately implies that $(\mathrm{id}_{X_n} \otimes i_n)_n$ can be identified with the canonical embedding $X_n \otimes \dot{\Delta}(n)_n \to \big(X_n \otimes \dot{\Delta}(n)_n\big) \oplus X_n$, hence $\tilde{\imath}$ can be identified with $Q := Y_n \otimes \dot{\Delta}(n)_n \to Q \oplus X_n \cong Z_n$, and a similar description is valid for $(\mathrm{id}_{Y_n} \otimes i_n)_n$, with the second summand equal to Y_n instead of X_n. This means that we can identify $(f_n \,\square\, i_n)_n$ with $\mathrm{id}_Q \oplus f_n$, i.e. we get the following commutative diagram:

$$\begin{tikzcd} X_n \arrow[r, "\otimes\sigma"] \arrow[d, "f_n"] & Q \oplus X_n \arrow[r, "h_n"] \arrow[d, "(f_n \square i_n)_n"] & X_n \arrow[d, "f_n"] \\ Y_n \arrow[r, "\otimes\sigma"] & Q \oplus Y_n \arrow[r, "(\mathrm{ev}_{\Delta(n),Y})_n"] & Y_n \end{tikzcd} \tag{10.1.14.2}$$

The composites of horizontal arrows are equal to identity (because $\sigma = \mathrm{id}_{\mathbf{n}}$), hence the outer circuit is cocartesian, and the left square is cocartesian because $(f_n \,\square\, i_n)_n = \mathrm{id}_Q \oplus f_n$, hence the right square is also cocartesian, q.e.d.

10.2. (Finitary closures and perfect cofibrations.) Now we are going to define and study the basic properties of *perfect cofibrations* and *perfect (simplicial) objects*, over any generalized commutatively ringed topos $\mathcal{X} = (\mathcal{X}, \mathscr{O})$, sometimes supposed to have enough points. The basic idea behind these definitions is the following. We define *finitarily closed* sets of morphisms in a category or in fibers of a stack by imposing "finitary" versions of conditions 1)–8) of **9.5.10**, e.g. we allow only finite direct sums and finite compositions. Then we can define *finitary closure* of any set of morphisms; and the *perfect cofibrations* will be exactly morphisms from the finitary closure of the set $L_{\mathscr{O}}(\underline{I}_{\mathcal{X}})$ of standard cofibrant generators for $\mathfrak{s}\mathscr{O}$-$\mathbf{MOD}_{\mathcal{X}}$, and *perfect* or *perfectly cofibrant objects* X will be the objects for which $\nu_X : \varnothing \to X$ is a perfect cofibration.

Then we are going to prove some basic properties of perfect cofibrations and perfect objects, essentially applying the same (or even simpler) "devissage" as before (cf. **9.8.7**). Since most of the proofs would just repeat some of the arguments applied before to prove properties of cofibrations, we are going to give references to our former proofs instead of re-proving the statements for the perfect case.

10.2.0. (Retract and retract-free versions of the theory.) The theory developped below can be actually constructed in two different fashions, depending on whether we allow retracts in the definition of finitary closed sets (*"retract version"* of the theory) or not (*"retract-free version"*). We'll usually develop only the retract version, leaving the retract-free version to the reader. Usually the retract condition in the definition of finitarily closed sets will be used only to propagate itself, so the proofs for the retract-free theory are just sub-proofs of those considered by us.

However, when there will be some subtle points where the retract and the retract-free variants differ, we'll point this out explicitly. One of such examples will be encountered during the consideration of affine generalized schemes, since a locally free $\mathscr{O}$-module needn't be globally free, while a locally projective $\mathscr{O}$-module is globally projective. There is another important point affected by the choice of theory:

Terminology. A *vector bundle* $\mathscr{E}$ over $(\mathcal{X}, \mathscr{O})$ will be just any (locally) finitely presented locally projective $\mathscr{O}$-module (resp. locally free in the retract-free version of the theory).

Definition 10.2.1 *Let $\mathcal{C}$ be a stack over a site $\mathcal{S}$, such that finite inductive limits exist in each fiber of $\mathcal{C}/\mathcal{S}$ and all pullback functors $\varphi^* : \mathcal{C}(S) \to \mathcal{C}(T)$ are right exact, and $\mathscr{P} \subset \operatorname{Ar}\mathcal{C}$ be any set of morphisms lying in the fibers of $\mathcal{C}$. We say that $\mathscr{P}$ is* finitarily closed *if the following conditions hold (cf.* **9.5.10***):*

1) *$\mathscr{P}$ is local, and in particular stable under base change $\varphi^* : \mathcal{C}(S) \to \mathcal{C}(T)$ with respect to any $\varphi : T \to S$ in $\mathcal{S}$.*

2) *$\mathscr{P}$ contains all isomorphisms in fibers of $\mathcal{C}/\mathcal{S}$.*

3) *$\mathscr{P}$ is stable under pushouts.*

4) *$\mathscr{P}$ is stable under composition (of two morphisms in a fiber of $\mathcal{C}$).*

6) *$\mathscr{P}$ is stable under retracts (in the retract-free version of the theory this condition is omitted).*

Similarly, a set $\mathscr{P} \subset \mathcal{C}$ of morphisms in an arbitrary category $\mathcal{C}$ with finite inductive limits is said to be (globally) finitarily closed *if it fulfills conditions 2), 3), 4) and 6) (in the retract-free version the latter condition is omitted).*

When we want to emphasize that we work in the retract-free version, we speak about finitarily semiclosed *sets.*

10.2.2. (Stability under finite direct sums.) Notice that conditions 2), 3) and 4) actually imply a weakened version of 7):

7^w) $\mathscr{P}$ is stable under finite direct sums.

Indeed, an empty direct sum is an isomorphism $\varnothing \to \varnothing$, hence lies in $\mathscr{P}$ by 2), and $u \oplus v = (u \oplus \mathrm{id}) \circ (\mathrm{id} \oplus v)$ is a composition of a pushout of u and a pushout of v, hence belongs to $\mathscr{P}$ by 3) and 4).

10.2.3. (Examples.) (a) Notice that any closed class of morphisms is also finitarily closed, e.g. the sets of cofibrations or acyclic cofibrations in pseudomodel stack $\mathfrak{s}\mathscr{O}$-$\mathbf{MOD}_{\mathcal{X}}$ are finitarily closed.

(b) Another example is given by *strong cofibrations* and *strong acyclic cofibrations*, i.e. morphisms with the local LLP with respect to all acyclic fibrations (resp. all fibrations) in a pseudomodel stack. More generally, any class of morphisms characterized by local LLP with respect to another local class of morphisms is finitarily closed.

Definition 10.2.4 *Let I be any set of morphisms in a category $\mathcal{C}$ with finite inductive limits. Its* (global) finitary closure*, denoted by* $\mathrm{GFinCl}_{\mathcal{C}}\, I$, $\mathrm{GFinCl}\, I$ *or just* $\mathrm{FinCl}\, I$, *is the smallest finitarily closed subset of* $\mathrm{Ar}\,\mathcal{C}$ *containing I. Similarly, suppose $\mathcal{C}/\mathcal{S}$ be a stack, such that finite inductive limits exist in each fiber of $\mathcal{C}/\mathcal{S}$, all $\varphi^* : \mathcal{C}(S) \to \mathcal{C}(T)$ are right exact, and that $\mathcal{S}$ admits a final object e. Then for any set $I \subset \mathrm{Ar}\,\mathcal{C}(e)$ we define its* finitary closure $\mathrm{FinCl}\, I$ *or* $\mathrm{FinCl}_{\mathcal{C}}\, I$ *as the smallest finitarily closed subset $\mathscr{P} \subset \mathrm{Ar}\,\mathcal{C}$ containing I.*

When we want to emphasize that we work in the retract-free theory, we speak about finitary semiclosures*, and denote them by* $\mathrm{FinSCl}\, I$.

Any intersection of finitarily closed sets is again finitarily closed, hence $\mathrm{FinCl}_{\mathcal{C}}\, I$ exist for any I: we just have to take the intersection of all finitarily closed sets containing I.

Lemma 10.2.5 *Let $\mathcal{C}$ be a category with finite inductive limits, $I \subset \mathrm{Ar}\,\mathcal{C}$. Then:*

(a) The global finitary semiclosure $\mathrm{GFinSCl}\, I$ of I consists of the morphisms $f \in \mathrm{Ar}\,\mathcal{C}$ that are (isomorphic to) finite (possibly empty) compositions of pushouts of morphisms from I.

(b) The global finitary closure $\mathrm{GFinCl}\, I$ consists of retracts of morphisms from the global finitary semiclosure $\mathrm{GFinSCl}\, I$. Moreover, by **9.9.12** *it is enough to consider here only retracts with fixed source.*

Proof. (a) Clearly, all such morphisms have to belong to $\mathrm{GFinSCl}\, I$; conversely, the set of all such morphisms is easily seen to be stable under pushouts and composition, and to contain all isomorphisms and I, whence the opposite inclusion.

(b) We may replace I with $\mathrm{GFinSCl}\, I$ and assume I to be already finitarily semiclosed. Denote by $\tilde{I}$ the set of all fixed source retracts of morphisms from I; since I is closed under pushouts, lemma **9.9.12** implies that $\tilde{I}$ contains *all* retracts of morphisms from I. All we need to check is that $\tilde{I} \supset I$ is finitarily closed. It is obviously stable under pushouts and retracts, so all we need to check is that $\tilde{I}$ is closed under composition.

(c) Recall that f' is a fixed source retract of f iff there exist morphisms i and σ, such that $if' = f$, $\sigma f = f'$, and $\sigma i = \mathrm{id}$. Now suppose that composable morphisms f' and g' are fixed source retracts of morphisms f and g from I. First of all, if (j, τ) are the retract morphisms for g', then the same morphisms satisfy the retract property for gf' and $g'f'$, i.e. $g'f'$ is a retract of gf', so it would suffice to prove $gf' \in \tilde{I}$, i.e. we can assume $g' = g \in I$. Now let (i, σ) be the retract morphisms for f', and denote by $\tilde{g}$ the pushout of g with respect to i; since $\sigma i = \mathrm{id}$, the pushout of $\tilde{g}$ with respect to σ can be identified with g, i.e. we get the following commutative diagram with *two* cocartesian squares, one built on i and $\tilde{\imath}$, and the other built on σ and $\tilde{\sigma}$:

$$\begin{tikzcd} & B \arrow[r, dashed, "\tilde{g}"] \arrow[d, bend right, "\sigma"'] & \tilde{C} \arrow[d, dashed, bend right, "\tilde{\sigma}"'] \\ A \arrow[ur, "f"] \arrow[r, "f'"] & B' \arrow[r, "g"] \arrow[u, bend right, "i"'] & C \arrow[u, dashed, bend right, "\tilde{\imath}"'] \end{tikzcd} \tag{10.2.5.1}$$

Now $(\tilde{\imath}, \tilde{\sigma})$ satisfy the retract relations for gf' and $\tilde{g}f$, i.e. gf' is a retract of $\tilde{g}f$, and the latter morphism belongs to I since $\tilde{g}$ is a pushout of g.

Lemma 10.2.6 *Let $\mathcal{S}$ be a site with final object e, $\mathcal{C}/\mathcal{S}$ be a stack with right exact pullback functors, and $I \subset \mathrm{Ar}\,\mathcal{C}(e)$ be any set of morphisms. Then a morphism $f : X \to Y$ in $\mathcal{C}(S)$, $S \in \mathrm{Ob}\,\mathcal{S}$, belongs to the finitary closure $\mathrm{Cl}_{\mathcal{C}}\, I$ iff it locally belongs to the global finitary closure of $\varphi^* I$, i.e. iff one can find a cover $\{S_\alpha \to S\}$, such that each $f|_{S_\alpha}$ belongs to the global finitary closure in $\mathcal{C}(S_\alpha)$ of the pullback $I|_{S_\alpha}$ of I to S_α.*

Proof. Clearly, any morphism from $\mathrm{GFinCl}_{\mathcal{C}(S_\alpha)}\, I|_{S_\alpha}$ lies in $\mathrm{FinCl}_{\mathcal{C}}\, I$, and the latter class of morphisms is local, hence the second condition ("f locally belongs to the global finitary closure") implies the first. To show the opposite inclusion we have to check that the set of morphisms f having this property is finitarily closed. This fact is immediate, once we take into account that all φ^* are right exact: the only non-trivial case is that of compositions, but if f belongs to the global finitary closure on one cover $\{S_\alpha \to S\}$, and g on another $\{S'_\beta \to S\}$, then all we have to do is to choose a common refinement of these two covers. (Notice that such an argument wouldn't work for sequential compositions and infinite direct sums, i.e. finitarity is essential here.)

Definition 10.2.7 *Let $\mathcal{X} = (\mathcal{X}, \mathscr{O})$ be a generalized ringed topos. We say that a morphism $f : X \to Y$ in $\mathfrak{s}\mathscr{O}$-$\boldsymbol{MOD}_{\mathcal{X}}(S) = s\mathscr{O}|_S$-Mod is* perfectly cofibrant*, or a* perfect cofibration*, or just* perfect *if it belongs to the finitary closure* $\mathrm{FinCl}\, L_{\mathscr{O}}(\underline{I}_{\mathcal{X}})$ *of the set of standard cofibrant generators of* $\mathfrak{s}\mathscr{O}$*-**MOD***$_{\mathcal{X}}$. *We say that* f *is a* globally perfect cofibration *or just* globally perfect *if it belongs to the global finitary closure* $\mathrm{GFinCl}\, L_{\mathscr{O}|_S}(\underline{I}_{\mathcal{X}_{/S}})$ *of the set of standard cofibrant generators in* $s\mathscr{O}|_S$*-Mod.*

An object $X \in \mathrm{Ob}\, \mathfrak{s}\mathscr{O}\text{-}\boldsymbol{MOD}_{\mathcal{X}}(S) = \mathrm{Ob}\, s\mathscr{O}|_S\text{-}Mod$ *is said to be* perfect *or* perfectly cofibrant *(resp.* globally perfect*) if* $\varnothing \to X$ *is a perfect cofibration (resp. globally perfect cofibration).*

Definition 10.2.8 *An object* $\bar{X}$ *of the homotopic category* $\mathcal{D}^{\leq 0}(\mathcal{X}, \mathscr{O}) = \mathrm{Ho}\, s\mathscr{O}\text{-}Mod$ *is* perfect *if it is isomorphic to an object of the form* γX*, for some perfect* X *in* $s\mathscr{O}$*-Mod. Similarly, a morphism in* $\mathcal{D}^{\leq 0}(\mathcal{X}, \mathscr{O})$ *is* perfect *if it is isomorphic to* $\gamma(f)$ *for some perfect cofibration in* $s\mathscr{O}$*-Mod.*

If $\mathscr{O}$ *admits a zero, we extend these definitions to the stable homotopic category* $\mathcal{D}^{-}(\mathcal{X}, \mathscr{O})$ *in the natural way, e.g. an object* $(\bar{X}, n)$ *of* $\mathcal{D}^{-}(\mathcal{X}, \mathscr{O})$ *is perfect iff for some* $N \geq 0$ *the object* $\Sigma^N \bar{X}$ *is perfect in* $\mathcal{D}^{\leq 0}(\mathcal{X}, \mathscr{O})$*.*

Notice that these homotopic category notions are not local, i.e. if $\bar{X} \in \mathrm{Ob}\, \mathcal{D}^{\leq 0}$ becomes perfect on some cover, this doesn't necessarily imply perfectness of $\bar{X}$ itself. This is one of the reasons we prefer to work with perfect objects and morphisms inside $s\mathscr{O}$*-Mod* and not $\mathcal{D}^{\leq 0}(\mathcal{X}, \mathscr{O})$ in almost all situations.

10.2.9. (Direct description of perfect objects and cofibrations.) According to **10.2.6**, an object or a morphism in $s\mathscr{O}$*-Mod* is perfect iff locally it is globally perfect, i.e. iff it becomes globally perfect on some cover. According to **10.2.5**, a morphism is globally perfect iff it is a (fixed source) retract of a finite composition of pushouts of morphisms from $L_{\mathscr{O}}(\underline{I}_{\mathcal{X}})$; in the retract-free theory, of course, we don't even have to consider retracts.

This explicit description, together with the fact that strong cofibrations constitute a finitarily closed set of morphisms containing $L_{\mathscr{O}}(\underline{I})$, immediately implies the properties of perfect objects and cofibrations summarized in the following proposition:

Proposition 10.2.10 *(a) All perfect morphisms are strong cofibrations (i.e. have the local LLP with respect to all acyclic fibrations) and cofibrations (i.e. lie in* $\mathrm{Cl}\, L_{\mathscr{O}}(\underline{I})$*). All perfect objects are cofibrant and strongly cofibrant.*

(b) Any globally perfect cofibration has bounded dimension. Any perfect cofibration or perfect object has locally bounded dimension; if $\mathcal{X}$ *is quasi-compact, any perfect cofibration or object has bounded dimension.*

10.2.11. (Dimensional decomposition of perfect cofibrations.) Reasoning in the same way as in **10.1.6**, with "finitary devissage" used instead of "devissage", we obtain the following statement:

Proposition. *(a) Functors* sk_n*,* $\mathrm{sk}_{\leq n}$*,* $\mathrm{sk}_{>n}$ *and* $\mathrm{sk}_{[a,b]}$ *of* **10.1** *transform perfect cofibration into perfect cofibrations.*

(b) A perfect cofibration is concentrated in dimensions $S \subset \mathbb{Z}$ *iff it belongs to* $\mathrm{FinCl}\, L_{\mathscr{O}}(\underline{I}_S)$*. In particular, perfect cofibrations purely of dimension* n *coincide with* $\mathrm{FinCl}\, L_{\mathscr{O}}(\underline{I}_{=n})$*.*

(c) A perfect cofibration f *of bounded dimension* $N < +\infty$ *(condition automatically verified for a quasicompact* $\mathscr{X}$*) can be canonically decomposed into a finite composition of perfect cofibrations* $\mathrm{sk}_{=k}(f)$*,* $0 \leq k \leq N$*, each of them purely of dimension* k*.*

10.2.12. (Constant perfect objects and cofibrations.) Similarly to **10.1.10**, we say that a morphism $f_0 : X_0 \to Y_0$ in $\mathscr{O}$*-Mod* is a *(constant) perfect cofibration* if it becomes a perfect cofibration in $s\mathscr{O}$*-Mod* when considered as a morphism of constant simplicial objects. An object X_0 of $\mathscr{O}$*-Mod* is said to be *perfect* if it is perfect as a constant object of $s\mathscr{O}$*-Mod*.

One checks, reasoning as in **10.1.10**, that *the constant perfect cofibrations coincide with the finitary closure* $\mathrm{FinCl}\{\varnothing_{\mathscr{O}} \to L_{\mathscr{O}}(1)\}$. Applying **10.2.6** and **10.2.5**, we see that a constant morphism is a (constant) perfect cofibration iff it can be locally represented as a retract of a standard embedding $X_0 \to X_0 \oplus L_{\mathscr{O}}(n)$. Therefore, constant perfect objects are just local retracts of free $\mathscr{O}$-modules of finite rank, i.e. the *vector bundles* (cf. **10.2.0**).

Clearly, all components $f_n : X_n \to Y_n$ of a perfect cofibration $f : X \to Y$ in $s\mathscr{O}$*-Mod* are constant perfect cofibrations (same reasoning as in **10.1.10**). In particular, *all components* X_n *of a perfect simplicial object* X *are vector bundles.*

10.2.13. (Perfect cofibrations of simplicial sets.) Let's consider the situation $\mathscr{X} = Sets$, $\mathscr{O} = \mathbb{F}_\varnothing$, i.e. $\mathscr{O}$*-Mod* $= Sets$, $s\mathscr{O}$*-Mod* $= sSets$.

(a) First of all, (constant) perfect cofibrations in *Sets* are just injective maps of sets $f_0 : X_0 \to Y_0$, such that $Y_0 - f_0(X_0)$ is finite. In particular, constant perfect sets are just the finite sets ("vector bundles over $\mathbb{F}_\varnothing$").

(b) Therefore, a perfect cofibration $f : X \to Y$ in *sSets* has the following properties: it is a cofibration (i.e. a componentwise injective map) of bounded dimension $N < +\infty$, and all components $f_n : X_n \to Y_n$ are constant perfect cofibrations, i.e. all sets $Y_n - f_n(X_n)$ are finite. Conversely, one easily checks that whenever $f : X \to Y$ is a cofibration in *sSets* of bounded dimension $N < +\infty$, such that $Y_n - f_n(X_n)$ is finite (for all n or just for $n \leq N$), then one can represent $f : X \to Y$ as a finite composition of pushouts of standard generators i_n. In order to do this just choose a simplex $\sigma \in Y_n - f_n(X_n)$ of minimal dimension n; then all its faces already lie in X, so we can "glue in" this simplex to X along its boundary, i.e. decompose f into $X \to X' \to Y$, where $X \to X'$ is the pushout of $i_n : \dot{\Delta}(n) \to \Delta(n)$ with respect to the map $\dot{\Delta}(n) \to X$ defined by the faces of σ; one can check then that $X' \to Y$ is still a cofibration (i.e. componentwise injective map) satisfying the above conditions for the same N but with smaller value of $\sum_{n \leq N} |Y_n - f_n(X_n)|$. The proof is concluded by induction in this number.

(c) In particular, a simplicial set X is perfect iff it has bounded dimension and all its components X_n are finite. This is equivalent to saying that the set of non-degenerate simplices of X is finite, i.e. that X is a finite simplicial set. Hence *perfect simplicial sets are exactly the finite simplicial sets.*

(d) Now let $f : X \to Y$ be any cofibration of simplicial sets, and suppose Y to be perfect, i.e. finite. Then f maps non-degenerate simplices of X into non-degenerate simplices of Y, all components of f being injective, hence X also has finitely many non-degenerate simplices, i.e. it is finite, or perfect. Now it is immediate that f has bounded dimension, and all $Y_n - f_n(X_n) \subset Y_n$ are finite, hence f is a perfect cofibration. We have just shown that *any cofibration of simplicial sets with perfect target is itself perfect, and its source is also perfect.*

(e) This is applicable in particular to cofibrations $i_n \square i_m$ with target $\Delta(n) \otimes \Delta(m) = \Delta(n) \times \Delta(m)$, i.e. *all $i_n \square i_m$ are perfect cofibrations in sSets.*

Proposition 10.2.14 *Let $f : \mathcal{X}' = (\mathcal{X}', \mathscr{O}') \to \mathcal{X} = (\mathcal{X}, \mathscr{O})$ be any morphism of generalized commutatively ringed topoi. Then $f^* : s\mathscr{O}\text{-Mod} \to s\mathscr{O}'\text{-Mod}$ preserves perfect cofibrations and objects. Moreover, $\dim f^*(u) \leq \dim u$ for any (perfect) cofibration u, and $\dim f^*X \leq \dim X$ for any (perfectly) cofibrant object X.*

Proof. The statement about dimensions doesn't need perfectness, and has been already discussed in **10.1.9**. The fact about perfect objects follows immediately from that about perfect cofibrations, and the latter is shown by an obvious finitary devissage.

Proposition 10.2.15 *Let $\mathcal{X} = (\mathcal{X}, \mathscr{O})$ be a generalized (commutatively) ringed topos. Then:*

(a) $i \square s$ is a perfect cofibration in $s\mathscr{O}$-Mod whenever i and s are. Furthermore, $\dim(i \square s) \leq \dim i + \dim s$.

(b) $X \otimes Y = X \otimes_{\mathscr{O}} Y$ is perfect for any two perfect simplicial objects X and Y of $s\mathscr{O}$-Mod. Furthermore, $\dim(X \otimes Y) \leq \dim X + \dim Y$.

Proof. (b) is an obvious consequence of (a) since $\nu_{X \otimes Y} = \nu_X \square \nu_Y$. The second statement of (a) has been shown in **10.1.8**. The first statement is reduced by finitary version of devissage used in **9.8.4** to the case of $i_n \Box i_m$ in *sSets*, already dealt with in **10.2.13**,(e), q.e.d.

Corollary 10.2.16 *(Componentwise criterion of perfectness.) Let $f : X \to Y$ be a morphism in $s\mathscr{O}$-Mod of bounded dimension $N < +\infty$. Then f is a perfect cofibration iff all $(\mathrm{sk}_{\geq n} f)_n$, $0 \leq n \leq N$, are constant perfect cofibrations iff all $(\mathrm{sk}_{=n} f)_n$, $0 \leq n \leq N$, are constant perfect cofibrations.*

Proof. These conditions are necessary by **10.2.11** and **10.2.12**. The proof of the opposite implications goes exactly in the same way as in **10.1.12**, taking into account that in our case the dimensional decomposition of f is finite: $f = \mathrm{sk}_{=N}(f) \circ \cdots \circ \mathrm{sk}_{=1}(f) \circ \mathrm{sk}_{=0}(f)$, and that $u \Box i_n = u \Box L_{\mathscr{O}}(i_n)$ is a perfect cofibration whenever u is a constant perfect cofibration by **10.2.15**.

Proposition 10.2.17 *Let $(\mathcal{X}, \mathscr{O})$ be as above. Denote by $S^n = S^n_{\mathscr{O}}$ the symmetric power functors, and by ρ_n the "n-th symmetric box power" functor of* **9.9.7**. *Then:*

(a) $\rho_n(u)$ is a perfect cofibration in $s\mathscr{O}$-Mod whenever u is one, for any $n \geq 0$. Furthermore, $\dim \rho_n(u) \leq n \dim(u)$.

(b) $S^n(X)$ is a perfect simplicial object whenever X is one. Furthermore, $\dim S^n(X) \leq n \dim X$.

Proof. Statement (b) is a consequence of (a) since $\nu_{S^n X} = \rho_n(\nu_X)$. Statement (a) is shown simultaneously for all values of $n \geq 1$ (case $n = 0$ is trivial since $\rho_0(u) = \nu_{L_{\mathscr{O}}(1)}$) by our usual finitary devissage. The only non-trivial case is that of $\rho_n(vu)$, but according to **9.9.21**(c), this $\rho_n(vu)$ is a finite composition of pushouts of $\rho_k(v) \Box \rho_{n-k}(u)$, $0 \leq k \leq n$. This accomplishes the devissage step once we take **10.2.15** into account. The statement about dimensions is shown similarly, considering only perfect cofibrations of relative dimension $\leq d$, i.e. elements of the finitary closure $\mathrm{FinCl}\, L_{\mathscr{O}}(\underline{I}_{\leq d})$: then from $\dim \rho_k(v) \leq k \dim v \leq kd$, and similarly for $\rho_{n-k}(u)$, we deduce $\dim(\rho_k(v) \Box \rho_{n-k}(u)) \leq nd$, hence the same is true for the composition $\rho_n(vu)$ of pushouts of these morphisms. Therefore, our finitary devissage reduces everything to the case of $\rho_n(i_m)$ in *sSets*; the statement is immediate there by **10.2.13**,(d) and (e). Since the target of $\rho_n(i_m)$ is $S^n\Delta(m)$, itself a quotient of $\Delta(m)^n$, hence nm-dimensional, we see that the same holds for $\rho_n(i_m)$, i.e. $\dim \rho_n(i_m) \leq nm$. This proves the dimension statement.

10.2.18. (Derived category case.) Since derived functors $\underline{\underline{\oplus}}$, $\underline{\underline{\otimes}}$ and $\mathbb{L}S^n$ exist and can be computed by means of cofibrant replacements (cf. **9.8.1** and **9.9.2**), we can extend our previous results to $\mathcal{D}^{\leq 0}(\mathcal{X}, \mathscr{O}) = \mathrm{Ho}\, s\mathscr{O}\text{-}Mod$:

whenever $\bar{X}$ and $\bar{Y}$ are perfect in $\mathcal{D}^{\leq 0}$, the same holds for $\bar{X}\underline{\underline{\oplus}}\bar{Y}$, $\bar{X}\underline{\underline{\otimes}}\bar{Y}$ and $\mathbb{L}S^n\bar{X}$. Furthermore, left derived pullbacks $\mathbb{L}f^*$ with respect to morphisms of generalized ringed topoi preserve perfectness as well (cf. **9.7.12**).

One might try to extend the above results to morphisms in $\mathcal{D}^{\leq 0}$. The idea is to observe that any morphism in $\mathcal{D}^{\leq 0}$ can be represented or replaced by a cofibration between cofibrant objects, and to "derive" $\square$ and ρ_n with the aid of such "doubly cofibrant replacements", using a more sophisticated version of **9.6.7**. We don't want to provide more details for the time being.

10.2.19. (Perfect cofibrations between perfect objects.) Denote by $\mathcal{C}_p$ the full substack of $\mathcal{C} = \mathfrak{s}\mathscr{O}\text{-}\mathbf{MOD}_{\mathcal{X}}$, consisting of perfect objects. Since $L_{\mathscr{O}}(\underline{I}) \subset \operatorname{Ar}\mathcal{C}_p$, we might ask whether the finitary closure of $L_{\mathscr{O}}(\underline{I})$ inside $\mathcal{C}_p$ coincides with the set of perfect cofibrations between perfect objects, i.e. whether it is sufficient to consider pushouts with perfect target while constructing a perfect cofibration between perfect objects. In fact, **10.2.5** and **10.2.6** immediately imply that the answer to this question is positive.

10.2.20. (Perfectness of cones and cylinders.) Suppose $\mathscr{O}$ to have a zero. Then for any map $f : X \to Y$ of simplicial $\mathscr{O}$-modules we can construct its *cylinder* $Cyl(f)$ and *cone* $C(f)$ by means of the following cocartesian squares (cf. **8.6.8**):

$$\begin{tikzcd} X \otimes \{0\} \arrow[r, "f"] \arrow[d] & Y \otimes \{0\} \arrow[d, dashed] \\ X \otimes \Delta(1) \arrow[r, dashed] & Cyl(f) \end{tikzcd} \qquad \begin{tikzcd} X \otimes \{1\} \arrow[r] \arrow[d] & Cyl(f) \arrow[d, dashed] \\ 0 \arrow[r, dashed] & C(f) \end{tikzcd} \tag{10.2.20.1}$$

If X is perfect, then $\mathrm{id}_X \otimes \Delta(\partial^0) : X \otimes \{0\} \to X \otimes \Delta(1)$ is a perfect cofibration, since $\Delta(\partial^0)$ is a perfect cofibration of simplicial sets. Therefore, $Y \cong Y \otimes \{0\} \to Cyl(f)$ is a perfect cofibration whenever X is perfect. If $f : X \to Y$ is a perfect cofibration between perfect objects, then $X \otimes \Delta(1) \to Cyl(f)$ is a perfect cofibration, being a pushout of f, hence $X \otimes \{1\} \to X \otimes \Delta(1) \to Cyl(f)$ is also perfect, hence the same holds for its pushout $0 \to C(f)$, i.e. *the cone of a perfect cofibration between cofibrant objects is perfect.*

10.2.21. (Perfectness of suspensions.) Suppose $\mathscr{O}$ still has a zero, and let X be any perfect simplicial $\mathscr{O}$-module. Then $X \oplus X \to X \otimes \Delta(1)$ is a perfect cofibration, being equal to $\mathrm{id}_X \otimes j = \nu_X \,\square\, j$, since $j : \Delta(0) \sqcup \Delta(0) \to \Delta(1)$ is a perfect cofibration of simplicial sets. Hence the cofiber $0 \to \Sigma X$ of $\mathrm{id}_X \otimes j$, i.e. its pushout with respect to $X \oplus X \to 0$, is a perfect cofibration, i.e. *the suspension ΣX of any perfect object is perfect.*

In particular, this justifies the definition of perfectness in the stable case, given in **10.2.8**.

10.2.22. (Additive case.) Now suppose that $\mathscr{O}$ is additive, i.e. is a classical sheaf of commutative rings. Then:

(a) Constant perfect cofibrations are just injective maps $f : \mathscr{F}' \to \mathscr{F}$ of $\mathscr{O}$-modules with the cokernel $\mathscr{E} := \operatorname{Coker} f$ a vector bundle, i.e. a local retract of a free $\mathscr{O}$-module (resp. a locally free $\mathscr{O}$-module in the retract-free theory). Indeed, it is evident that any constant perfect cofibration has this property; conversely, if this is the case, $\mathscr{F} \to \mathscr{E} = \operatorname{Coker} f$ locally splits, $\mathscr{E}$ being locally projective, hence $\mathscr{F}' \to \mathscr{F}$ is locally isomorphic to $\mathscr{F}' \to \mathscr{F}' \oplus \mathscr{E}$, i.e. to a retract of a map $\mathscr{F}' \to \mathscr{F}' \oplus \mathscr{O}(n)$.

(b) Simplicial $\mathscr{O}$-modules of dimension $\leq n$ correspond via Dold–Kan to chain complexes concentrated in (chain) degrees $[0, n]$. In order to see this we just recall that by definition of dimension and of sk_n the simplicial $\mathscr{O}$-modules of dimension $\leq n$ coincide with the essential image of the left Kan extension $I_{n,!}$ of the functor $I_n : \mathbf{\Delta}_{\leq n} \to \mathbf{\Delta}$ (cf. **8.2.4**), and the associated simplicial object functor $K : \mathrm{Ch}_{\geq 0}(\mathscr{O}\text{-}\mathit{Mod}) \to s\mathscr{O}\text{-}\mathit{Mod}$ is also defined by means of a certain left Kan extension $J_!$ for $J : \mathbf{\Delta}_+ \to \mathbf{\Delta}$, cf. **8.5.5** and **8.5.3**. Then our statement follows almost immediately from transitivity of left Kan extensions and definitions, once we consider the following commutative square of categories

$$\begin{tikzcd} \mathbf{\Delta}_{+,\leq n} \arrow[r, "I_{+,n}"] \arrow[d, "J_n"] & \mathbf{\Delta}_+ \arrow[d, "J"] \\ \mathbf{\Delta}_{\leq n} \arrow[r, "I_n"] & \mathbf{\Delta} \end{tikzcd} \tag{10.2.22.1}$$

and observe that $((I_{+,n})_! X)_m = \varinjlim_{[m] \rightarrowtail [p], p \leq n} X_p$ equals 0 for $m > n$ and X_m for $m \leq n$.

(c) Since $\operatorname{sk}_n X \to X$ can be described as the universal (final) object in the category of morphisms from simplicial objects of dimension $\leq n$ into X, we see that $\operatorname{sk}_n X$ corresponds under Dold–Kan equivalence to the "brute truncation" $\sigma_{\geq -n} P$ of corresponding chain complex P. This enables us to compute the counterparts of $\operatorname{sk}_{\leq n}$, $\operatorname{sk}_{>n}$, $\operatorname{sk}_{=n}$ for chain complexes, since adjoint equivalences $K : \mathrm{Ch}_{\geq 0}(\mathscr{O}\text{-}\mathit{Mod}) \leftrightarrows s\mathscr{O}\text{-}\mathit{Mod} : N$ have to preserve arbitrary inductive limits. For example, a chain map $f' : X \to Y$ corresponds to a morphism of simplicial objects of dimension $\leq n$ iff $f'_k : X_k \to Y_k$ are isomorphisms for all $k > n$.

(d) In particular, if a map of simplicial $\mathscr{O}$-modules f corresponds to a chain map $f' = Nf : X \to Y$, then $\operatorname{sk}_{=n}(f)$ corresponds to $f'_{=n} : (\cdots \to X_{n+1} \to X_n \to Y_{n-1} \to \cdots \to Y_0) \to (\cdots \to X_{n+1} \to Y_n \to Y_{n-1} \to \cdots \to Y_0)$. Since $\operatorname{sk}_{=n}(f)$ can be identified with $K(f'_{=n})$, formula (8.5.3.2) yields an identification of $(\operatorname{sk}_{=n}(f))_n$ with $\mathrm{id}_Q \oplus f_n : Q \oplus X_n \to Q \oplus Y_n$, where Q is a certain direct sum of Y_k with $k < n$. In particular, description given in (a) implies that $(\mathrm{sk}_{=n}(f))_n = \mathrm{id}_Q \oplus f_n$ is a constant perfect cofibration iff $f_n : X_n \to Y_n$ is one.

(e) Combining (b) and (d) with the dimensionwise criterion of perfectness **10.2.16**, we see that *perfect cofibrations of bounded dimension correspond via Dold–Kan to chain maps $f : X \to Y$ with all components $f_n : X_n \to Y_n$ as in (a), such that all f_n are isomorphisms for $n \gg 0$.* Taking (a) into account we see that *perfect cofibrations of bounded dimension correspond via Dold–Kan to injective chain maps $f : \mathscr{F}. \to \mathscr{F}'_.$ with the cokernel a finite complex of vector bundles, i.e. a perfect complex in classical sense.*

(f) Applying the above result to morphisms $0 \to X$, we see that *perfect simplicial objects of bounded dimension correspond via Dold–Kan to bounded chain complexes of vector bundles, i.e. to perfect complexes in the classical sense.* If $\mathcal{X}$ is quasicompact, any perfect simplicial object has bounded dimension, hence perfect simplicial objects correspond exactly to perfect complexes in the classical sense.

10.3. (K_0 of perfect morphisms and objects.) Our nearest goal is to construct K_0 of perfect morphisms and/or objects over a generalized ringed topos $\mathcal{X} = (\mathcal{X}, \mathscr{O})$, sometimes assumed to have enough points. We do this in quite a natural way, suggested by our definition of perfect objects, which may be thought of as a certain modification of Waldhausen's construction to the case when the basic category is not required to have a zero object (cf. **10.3.31** for a more detailed comparison). In fact, we define *three* variants of such K_0:

- K_0 of all perfect cofibrations, denoted by $K^0_{big}(\mathcal{X}, \mathscr{O})$.
- K_0 of perfect cofibrations between perfect objects $K^0_{perf}(\mathcal{X}, \mathscr{O})$ or simply $K^0(\mathcal{X}, \mathscr{O})$. It will be also called "K_0 of perfect simplicial $\mathscr{O}$-modules" or "K_0 of perfect objects".
- K_0 of constant perfect objects, i.e. vector bundles over $\mathcal{X}$, denoted by $K^0_{vect}(\mathcal{X}, \mathscr{O})$ or $\hat{K}^0(\mathcal{X}, \mathscr{O})$.

Besides, we can construct the above notions in the retract and retract-free settings, cf. **10.2.0**, and the six arising abelian groups (four of them actually are commutative pre-λ-rings) are related to each other by some canonical homomorphisms.

Definition 10.3.1 *The* K_0 of perfect cofibrations, *or* "the big K_0" *of* $\mathcal{X} = (\mathcal{X}, \mathscr{O})$*, denoted by* $K^0_{big}(\mathcal{X}, \mathscr{O})$*, is the free abelian group generated by all perfect cofibrations (or just by their isomorphism classes)* $[u]$ *in* $s\mathscr{O}$*-Mod, modulo following relations:*

0) If u is an isomorphism, then $[u] = 0$.

1) If $X \xrightarrow{u} Y \xrightarrow{v} Z$ are composable perfect cofibrations, then $[vu] = [v] + [u]$.

2) If $X' \xrightarrow{u'} Y'$ is a pushout of a perfect cofibration $X \xrightarrow{u} Y$, then $[u'] = [u]$.

3) If $X \xrightarrow{u} Y$ is isomorphic to $Z \xrightarrow{v} W$ in the derived category $\mathcal{D}^{\leq 0}(\mathcal{X}) = \operatorname{Ho} s\mathscr{O}$-Mod, then $[v] = [u]$.

Notice that condition 0) is actually superfluous, being a special case of 3), since 1) already implies $[\mathrm{id}_X] + [\mathrm{id}_X] = [\mathrm{id}_X]$, hence $[\mathrm{id}_X] = 0$. On the other hand, 0) and 1) already imply that $[u]$ actually depends on the isomorphism class of u in $s\mathscr{O}$-*Mod*, something that is also immediate from 2). Finally, the last condition 3) implies that $[u]$ depends only on the isomorphism class of u in $\mathcal{D}^{\leq 0}(\mathcal{X}, \mathscr{O})$, i.e. we might say that $K^0_{big}(\mathcal{X}, \mathscr{O})$ is generated by isomorphism classes of perfect morphisms in $\mathcal{D}^{\leq 0}(\mathcal{X}, \mathscr{O})$ (cf. **10.2.8**), and use only morphisms in derived category in the above definition (actually condition 2) is not so easy to state in the derived category). However, we prefer to work in $s\mathscr{O}$-*Mod* for technical reasons.

Notice that it is not clear whether $K^0_{big}(\mathcal{X}, \mathscr{O})$ admits a small set of generators, i.e. whether $K^0_{big}(\mathcal{X}, \mathscr{O})$ is a ($\mathcal{U}$-)small abelian group. That's why we introduce a "smaller" version of K^0, which can be shown to coincide with the previous one in "good situations".

Definition 10.3.2 *The* K_0 of perfect cofibrations between perfect objects, *or simply* K_0 of perfect objects, *denoted by $K^0_{perf}(\mathcal{X}, \mathscr{O})$ or $K^0(\mathcal{X}, \mathscr{O})$, is the free abelian group generated by isomorphism classes $[u]$ of perfect cofibrations $u : X \to Y$ between perfect objects in $s\mathscr{O}$-Mod, subject to the same relations 0)–3) as above, where of course we consider only perfect cofibrations between perfect objects (e.g. X' in 2) has to be perfect).*

Clearly, $[u] \mapsto [u]_{big}$ defines a homomorphism $K^0(\mathcal{X}, \mathscr{O}) \to K^0_{big}(\mathcal{X}, \mathscr{O})$, which can be shown to be an isomorphism in all "good" situations (e.g. if $\mathcal{X}$ is quasicompact and $\mathscr{O}$ additive, but not only in this case). On the other hand, it is easy to deduce from **10.2.5** and **10.2.6** that the set of isomorphism classes of perfect objects of $s\mathscr{O}$-*Mod* is small, hence $K^0(\mathcal{X}, \mathscr{O})$ admits a small set of generators and in particular is itself small. Therefore, we usually consider this $K^0(\mathcal{X}, \mathscr{O})$, for example to avoid all possible set-theoretical complications.

10.3.3. (Image of a perfect object in K^0.) Given any perfect object X of $s\mathscr{O}$-*Mod*, we denote by $[X]$ the element $[\nu_X]$ of $K^0(\mathcal{X}, \mathscr{O})$ or $K^0_{big}(\mathcal{X}, \mathscr{O})$

(if we want to distinguish these two situations, we write $[X]$ and $[X]_{big}$, respectively), where $\nu_X : \varnothing \to X$ is the only morphism from the initial object as before.

Now if $X \xrightarrow{u} Y$ is any perfect cofibration between perfect objects, then $\nu_Y = u \circ \nu_X$, hence by 1) we get $[Y] = [\nu_Y] = [u] + [\nu_X] = [u] + [X]$, i.e. $[u] = [Y] - [X]$. In particular, $K^0(\mathcal{X}, \mathscr{O})$ *is generated by elements* $[X]$ *corresponding to isomorphism classes of perfect objects.* (Of course, this is not necessarily true for $K^0_{big}(\mathcal{X}, \mathscr{O})$.) That's why we call $K^0(\mathcal{X}, \mathscr{O})$ the K_0 of perfect objects.

Since $[u] = [Y] - [X]$, we might define $K^0(\mathcal{X}, \mathscr{O})$ as the free abelian group generated by isomorphism classes $[X]$ (either in $s\mathscr{O}$-*Mod* or $\mathcal{D}^{\leq 0} = \operatorname{Ho} s\mathscr{O}$-*Mod*) of perfect objects, modulo following relations:

0′) $[\varnothing] = 0$, where $\varnothing$ denotes the initial object of $s\mathscr{O}$-*Mod*.

2′) Whenever we have a cocartesian square (10.3.3.1) of perfect objects in $s\mathscr{O}$-*Mod* with perfect cofibrations for horizontal arrows, $[Y] - [X] = [Y'] - [X']$.

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d] & Y \arrow[d] \\ X' \arrow[r, "u'"] & Y' \end{tikzcd} \tag{10.3.3.1}$$

3′) If two perfect objects X and Y become isomorphic in $\mathcal{D}^{\leq 0}(\mathcal{X}, \mathscr{O})$, then $[X] = [Y]$.

Notice that we don't need to write down relations corresponding to 1), since $[vu] = [Z] - [X] = ([Z] - [Y]) + ([Y] - [X]) = [v] + [u]$ is automatic for any composable couple of perfect cofibrations $X \xrightarrow{u} Y \xrightarrow{v} Z$ between perfect objects. On the other hand, condition 3′) implies following weaker conditions:

3^w) If $f : X \to Y$ is a weak equivalence between perfect objects, then $[X] = [Y]$.

3^a) If $f : X \to Y$ is an acyclic cofibration between perfect objects, then $[X] = [Y]$.

Notice that 3^w) doesn't imply 3′), since under the condition of 3′) we know only that X and Y can be connected by a path of weak equivalences, with intermediate nodes not necessarily perfect.

10.3.4. (Image of direct sums in K_0.) Recall that any finitarily closed set of morphisms, e.g. the set of perfect cofibrations, is closed under finite direct sums since $u \oplus v = (u \oplus \mathrm{id}) \circ (\mathrm{id} \oplus v)$ (cf. **10.2.2**). Relations 1) and 2) immediately imply

4) $[u \oplus v] = [u] + [v]$

Similarly, relations 0′) and 2′), where we put $X := \varnothing$, $Y' = X' \oplus Y$, imply $[X \oplus Y'] = [X] + [Y']$. We can also deduce this formula from 4) applied to ν_X and $\nu_{Y'}$.

10.3.5. (K_0 of any finitarily closed set of morphisms.) More generally, let $\mathcal{C}$ be any category and $\mathscr{P} \subset \operatorname{Ar}\mathcal{C}$ be any (globally) finitarily (semi)closed set of morphisms in $\mathcal{C}$. Notice that in particular this includes existence in $\mathcal{C}$ of all pushouts of morphisms from $\mathscr{P}$. Then we can define $K_0(\mathscr{P})$ as the free abelian group generated by (isomorphism classes of) morphisms from $\mathscr{P}$, modulo relations 1) and 2), hence also automatically 0). Moreover, this K_0 is functorial in the following sense: if $\mathscr{P}' \subset \operatorname{Ar}\mathcal{C}'$ is another finitarily semiclosed set as above, and $F : \mathcal{C} \to \mathcal{C}'$ is any right exact functor such that $F(\mathscr{P}) \subset \mathscr{P}'$, then F induces a homomorphism $F_* : K_0(\mathscr{P}) \to K_0(\mathscr{P}')$.

However, this construction is insufficient to describe our K^0_{perf} and K^0_{big} because of relations 3), which involve isomorphisms in another (namely, derived) category. In order to cover these cases we consider *triples* $(\mathcal{C}, \mathscr{P}, \gamma)$, where $\mathscr{P} \subset \operatorname{Ar}\mathcal{C}$ is as above, and $\gamma : \mathcal{C} \to \tilde{\mathcal{C}}$ is an arbitrary functor. Then we can define $K_0(\mathscr{P}, \gamma)$ by taking the quotient of $K_0(\mathscr{P})$ modulo relations 3), i.e. $[u] = [u']$ whenever $\gamma(u)$ is isomorphic to $\gamma(u')$ in $\tilde{\mathcal{C}}$. This construction is also functorial, if we define a morphism $(F, \tilde{F}, \eta) : (\mathcal{C}, \mathscr{P}, \gamma) \to (\mathcal{C}', \mathscr{P}', \gamma')$ as a triple consisting of a right exact functor $F : \mathcal{C} \to \mathcal{C}'$, such that $F(\mathscr{P}) \subset \mathscr{P}'$, its “derived functor” $\tilde{F} : \tilde{\mathcal{C}} \to \tilde{\mathcal{C}}'$, and a functorial isomorphism $\eta : \tilde{F} \circ \gamma \xrightarrow{\sim} \gamma' \circ F$ (usually we just write $\tilde{F} \circ \gamma = \gamma' \circ F$).

For example, taking $\mathcal{C} := s\mathscr{O}\text{-}\mathcal{M}od$, $\mathscr{P} := \{\text{perfect cofibrations in } \mathcal{C}\}$, $\tilde{\mathcal{C}} := \operatorname{Ho}\mathcal{C} = \mathcal{D}^{\leq 0}(\mathcal{X}, \mathscr{O})$, we recover $K^0_{big}(\mathcal{X}, \mathscr{O})$. Abelian group $K^0_{perf}(\mathcal{X}, \mathscr{O})$ can be also constructed in this manner if we take the full subcategory $\mathcal{C}_{perf} \subset \mathcal{C}$ instead of $\mathcal{C}$, but keep same $\tilde{\mathcal{C}} = \mathcal{D}^{\leq 0}(\mathcal{X}, \mathscr{O})$. Finally, group homomorphism $K^0_{perf}(\mathcal{X}, \mathscr{O}) \to K^0_{big}(\mathcal{X}, \mathscr{O})$ is a special case of functoriality just discussed.

10.3.6. (K_0 of constant perfect objects, i.e. vector bundles.) In particular, we can consider K_0 of (constant) perfect cofibrations between constant perfect objects, i.e. vector bundles. The resulting abelian group will be denoted by $K^0_{vect}(\mathcal{X}, \mathscr{O})$ or $\hat{K}^0(\mathcal{X}, \mathscr{O})$.

Reasoning as in **10.3.3**, we see that $\hat{K}^0(\mathcal{X}, \mathscr{O})$ is generated by isomorphism classes $[X]$ of vector bundles over $\mathcal{X}$ modulo relations 0′) and 2′), where of course we consider only *constant* cocartesian squares (10.3.3.1), i.e. all four objects involved must be vector bundles, and $X \xrightarrow{u} Y$ must be a constant perfect cofibration between vector bundles, i.e. it must lie in the finitary closure of $\{\varnothing_{\mathscr{O}} \to L_{\mathscr{O}}(1)\}$ (cf. **10.2.12**).

Notice the absence of relations 3′). Such relations are actually unnecessary due to the following fact: *if two constant simplicial objects become* *isomorphic in* $\operatorname{Ho} s\mathscr{O}\text{-}\mathit{Mod}$*, then they are already isomorphic in* $\mathscr{O}\text{-}\mathit{Mod}$*.* We are going to check this in a moment.

10.3.7. ($\boldsymbol{\pi}_0$ of a simplicial $\mathscr{O}$-module.) (a) Given any $X \in \operatorname{Ob} s\mathscr{O}\text{-}\mathit{Mod}$, we define $\boldsymbol{\pi}_0 X = \boldsymbol{\pi}_0(X) \in \operatorname{Ob}\mathscr{O}\text{-}\mathit{Mod}$ as the cokernel of $d_0^X, d_1^X : X_1 \rightrightarrows X_0$ in $\mathscr{O}\text{-}\mathit{Mod}$. Clearly, $\boldsymbol{\pi}_0$ is a functor $s\mathscr{O}\text{-}\mathit{Mod} \to \mathscr{O}\text{-}\mathit{Mod}$, and $\boldsymbol{\pi}_0 X_0 = X_0$ for any constant simplicial object X_0.

(b) It is immediate that the set $\mathscr{P}$ of morphisms f in fibers of $\mathfrak{s}\mathscr{O}\text{-}\mathbf{MOD}_{\mathcal{X}}$, such that $\boldsymbol{\pi}_0(f)$ is an isomorphism, is closed (cf. **9.5.10**), $\boldsymbol{\pi}_0 X$ being defined with the aid of a finite inductive limit of components of X. It is equally obvious that $\boldsymbol{\pi}_0$ commutes with generalized ringed topos pullbacks. Now the standard acyclic cofibrations $\Lambda_k(n) \to \Delta(n)$, $0 \leq k \leq n > 0$ from J clearly induce isomorphisms on π_0, both $\Lambda_k(n)$ and $\Delta(n)$ being connected and non-empty, hence $L_{\mathscr{O}}(\underline{J})$ lies in $\mathscr{P}$, hence $\mathscr{P}$ contains all acyclic cofibrations, i.e. $\boldsymbol{\pi}_0(f)$ *is an isomorphism for any acyclic cofibration* f*.*

(c) Now let $f : X \to Y$ be an acyclic fibration in $s\mathscr{O}\text{-}\mathit{Mod}$. According to **9.7.3**, this means that $X_n \to Y_n \times_{(\operatorname{cosk}_{n-1} Y)_n} (\operatorname{cosk}_{n-1} X)_n$ are strict epimorphisms for all $n \geq 0$; in particular, $f_0 : X_0 \to Y_0$ and $X_1 \to Y_1 \times_{(Y_0 \times Y_0)} (X_0 \times X_0)$ have to be strict epimorphisms, being just special cases of the above morphisms for $n = 0$ and $n = 1$. Notice that $\boldsymbol{\pi}_0 X = \operatorname{Coker}(X_1 \rightrightarrows X_0)$ actually depends only on $X_R \subset X_0 \times X_0$, the image of $X_1 \to X_0 \times X_0$, and in our case X_R is the preimage of Y_R under strict epimorphism $X_0 \times X_0 \to Y_0 \times Y_0$. This immediately implies that $X_0 \twoheadrightarrow Y_0 \twoheadrightarrow Y_0/Y_R = \boldsymbol{\pi}_0 Y$ is the cokernel of $X_R \rightrightarrows X_0$ as well (indeed, X_R contains $X_0 \times_{Y_0} X_0$, hence X_0/X_R has to factorize through Y_0, $X_0 \to Y_0$ being a strict epimorphism, and the kernel of $Y_0 \to X_0/X_R$ obviously contains the image of X_R in $Y_0 \times Y_0$, i.e. Y_R; the rest is trivial). In other words, $\boldsymbol{\pi}_0 X \cong \boldsymbol{\pi}_0 Y$, i.e. $\boldsymbol{\pi}_0(f)$ *is an isomorphism for any acyclic fibration* f*.*

(d) Combining (b) and (c) together, we see that $\boldsymbol{\pi}_0$ transforms weak equivalences into isomorphisms, hence it can be derived in a trivial manner, yielding a functor $\boldsymbol{\pi}_0 : \mathcal{D}^{\leq 0}(\mathcal{X}, \mathscr{O}) \to \mathscr{O}\text{-}\mathit{Mod}$.

(e) An interesting consequence is that $\boldsymbol{\pi}_0$ *commutes with the forgetful functor* $\Gamma_{\mathscr{O}} : \mathscr{O}\text{-}\mathit{Mod} \to \mathcal{X}$, i.e. $\boldsymbol{\pi}_0 X = \operatorname{Coker}(X_1 \rightrightarrows X_0)$ in $\mathscr{O}\text{-}\mathit{Mod}$ coincides with the same cokernel computed in topos $\mathcal{X}$. Indeed, one can find a weak equivalence (even an acyclic cofibration) $f : X \to X'$, and $\Gamma_{\mathscr{O}}(f)$ *is a weak equivalence iff* f *is one* by **9.7.6**, **9.5.26** and **8.4.8**, hence both $\boldsymbol{\pi}_0(f)$ and $\boldsymbol{\pi}_0\Gamma_{\mathscr{O}}(f)$ are isomorphisms, and we are reduced to the case of a fibrant X (fibrant in $s\mathscr{O}\text{-}\mathit{Mod}$ or $s\mathcal{X}$ – this is the same thing). Consider $\boldsymbol{\pi}_0\Gamma_{\mathscr{O}} X := \operatorname{Coker}(X_1 \rightrightarrows X_0)$ in $\mathcal{X}$; all we have to check is that $X_0 \to \boldsymbol{\pi}_0\Gamma_{\mathscr{O}} X$ is compatible with the $\mathscr{O}$-module structure on X_0, i.e. that it is compatible with the structure maps $\mathscr{O}(n) \times X_0^n \to X_0$. This is immediate, once we take into account that $\boldsymbol{\pi}_0(X \times Y) \cong \boldsymbol{\pi}_0 X \times \boldsymbol{\pi}_0 Y$ for any fibrant sheaves of sets X and Y.

(f) In any case, we have constructed a functor $\boldsymbol{\pi}_0 : \mathcal{D}^{\leq 0}(\mathcal{X}, \mathscr{O}) \to \mathscr{O}\text{-}\mathit{Mod}$, having property $\boldsymbol{\pi}_0 \gamma A_0 \cong A_0$ for any constant simplicial $\mathscr{O}$-module A_0, hence $\gamma A_0 \cong \gamma B_0$ implies $A_0 \cong B_0$ in $\mathscr{O}\text{-}\mathit{Mod}$ for any two $\mathscr{O}$-modules A_0 and B_0, exactly as claimed in **10.3.6**.

10.3.8. (Morphisms between different versions of K_0.) Of course, we get canonical homomorphisms of abelian groups

$$K^0_{vect}(\mathcal{X}, \mathscr{O}) = \hat{K}^0(\mathcal{X}) \longrightarrow K^0_{perf}(\mathcal{X}, \mathscr{O}) = K^0(\mathcal{X}) \longrightarrow K^0_{big}(\mathcal{X}, \mathscr{O}) \tag{10.3.8.1}$$

which can be deduced from general functoriality of **10.3.5**. Furthermore, we obtain similar homomorphisms for the retract-free versions of the above groups, as well as canonical homomorphisms from the retract-free into the retract versions, fitting together into a commutative 3×2-diagram.

10.3.9. (K^0_{vect} in the additive case.) Now suppose $\mathscr{O}$ to be additive, so that $\mathcal{X} = (\mathcal{X}, \mathscr{O})$ is just a classical ringed topos. In this case constant perfect cofibrations are just injective maps of $\mathscr{O}$-modules $\mathscr{F}' \to \mathscr{F}$ with the cokernel a vector bundle (cf. **10.2.22**,(a)). Consider any short exact sequence of vector bundles on $\mathcal{X}$:

$$0 \longrightarrow \mathscr{F}' \xrightarrow{u} \mathscr{F} \xrightarrow{v} \mathscr{E} \longrightarrow 0 \tag{10.3.9.1}$$

According to the description just recalled, u is a perfect cofibration between constant perfect objects, and we get a cocartesian square as in (10.3.3.1):

$$\begin{tikzcd} \mathscr{F}' \arrow[r, "u"] \arrow[d] & \mathscr{F} \arrow[d, "v"] \\ 0 \arrow[r, "u'"] & \mathscr{E} \end{tikzcd} \tag{10.3.9.2}$$

Relations 0′) and 2′) yield $[\mathscr{F}] - [\mathscr{F}'] = [\mathscr{E}] - [0] = [\mathscr{E}]$, i.e. we have following relations in $K^0_{vect}(\mathcal{X}, \mathscr{O})$:

5$^+$) $[\mathscr{F}] = [\mathscr{F}'] + [\mathscr{E}]$ for any short exact sequence of vector bundles $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{E} \to 0$.

Conversely, relations 5$^+$) imply 0′) and 2′). As to 0′), putting $\mathscr{F}' = \mathscr{F} = \mathscr{E} = 0$ immediately yields $[0] = 0$. In order to show 2′) consider any square (10.3.3.1) and put $E := \operatorname{Coker} u$. Since u' is a pushout of u, $\operatorname{Coker} u' \cong \operatorname{Coker} u = E$; on the other hand, E has to be a vector bundle, u being a perfect cofibration. Relations 5$^+$) for $0 \to X \xrightarrow{u} Y \to E \to 0$ and $0 \to X' \xrightarrow{u'} Y' \to E \to 0$ yield $[Y'] - [X'] = [E] = [Y] - [X]$ as required in 2′).

We conclude that *in the additive case $K^0_{vect}(\mathcal{X}, \mathscr{O})$ coincides with the free abelian group generated by isomorphism classes of vector bundles modulo relations* 5^+)*, i.e. with the classical (Grothendieck's) K_0 of vector bundles over $\mathcal{X}$.*

10.3.10. (K^0_{perf} in the additive case.) Now let $\mathscr{O}$ still be additive, and consider $K^0 = K^0_{perf}(\mathcal{X})$. Let's suppose in addition all perfect objects over $\mathcal{X}$ to have bounded dimension (quasicompactness of $\mathcal{X}$ would suffice for this). In this case perfect cofibrations between perfect objects correspond via Dold–Kan equivalence to monomorphic chain maps $f : X \to Y$ of perfect non-negatively graded chain complexes (i.e. finite chain complexes consisting of vector bundle) with $E := \operatorname{Coker} f$ also a perfect complex, cf. **10.2.22**,(e). Reasoning as above we see that in this case $K^0_{perf}(\mathcal{X})$ is the free abelian group generated by isomorphism classes $[E]$ of perfect non-negatively graded chain complexes, modulo relations 5^+), where we consider all short exact sequences of perfect complexes, and $3'$). This is *almost* the classical K_0 of the triangulated category of perfect complexes (since all distinguished triangles are isomorphic to triangles defined by short exact sequences of complexes) up to some minor points.

10.3.11. (Contravariance of K^0.) Let $f : (\mathcal{X}', \mathscr{O}') \to (\mathcal{X}, \mathscr{O})$ be any morphism of generalized ringed topoi. Recall that according to **10.2.14**), the pullback functor $f^* : s\mathscr{O}\text{-}Mod \to s\mathscr{O}'\text{-}Mod$ preserves perfect cofibrations and perfect objects hence also perfect cofibrations between perfect objects. Furthermore, f^* admits a left derived $\mathbb{L}f^* : \mathcal{D}^{\leq 0}(\mathcal{X}, \mathscr{O}) \to \mathcal{D}^{\leq 0}(\mathcal{X}', \mathscr{O}')$, which can be computed with the aid of cofibrant replacements (cf. **9.7.12**), i.e. $\mathbb{L}f^*\gamma X \cong \gamma f^* X$ for any cofibrant and in particular for any perfect X. This means that we are in position to apply general functoriality statements of **10.3.5**, provided we consider only perfect cofibrations between perfect objects, thus obtaining the following statement:

Proposition. *For any morphism of generalized ringed topoi $f : \mathcal{X}' \to \mathcal{X}$ we have well-defined canonical homomorphisms $f^* : K^0_{perf}(\mathcal{X}) \to K^0_{perf}(\mathcal{X}')$ and $K^0_{vect}(\mathcal{X}) \to K^0_{vect}(\mathcal{X}')$, uniquely determined by rule*

$$f^*[X] = [f^*X], \quad f^*[u] = [f^*(u)] \tag{10.3.11.1}$$

Furthermore, these homomorphisms f^* are functorial in f, i.e. $(f \circ g)^* = g^* \circ f^*$.

Notice that we don't obtain a similar statement for K^0_{big}; this is probably due to the fact that a more "correct" definition of K^0_{big} should involve only perfect cofibrations between cofibrant objects.

Theorem 10.3.12 (Multiplication on K^0.) *Let $\mathcal{X} = (\mathcal{X}, \mathscr{O})$ be a generalized commutatively ringed topos. Then the tensor product $\otimes = \otimes_{\mathscr{O}}$ induces a commutative multiplication law (i.e. symmetric biadditive map) on $K^0_{perf}(\mathcal{X})$ and $K^0_{vect}(\mathcal{X})$ by the rules*

$$[X] \cdot [Y] = [X \otimes Y], \quad [u] \cdot [v] = [u \mathbin{\square} v] \tag{10.3.12.1}$$

where X and Y are perfect objects, and u and v are perfect cofibrations between perfect objects (resp. constant perfect objects and perfect cofibrations between such for $K^0_{vect}(\mathcal{X})$), hence the same is true for $X \otimes Y$ and $u \mathbin{\square} v$ by **10.2.15**. *Furthermore, $K^0_{perf}(\mathcal{X})$ and $K^0_{vect}(\mathcal{X})$ become commutative rings under this multiplication, and canonical maps $K^0_{vect}(\mathcal{X}) \to K^0_{perf}(\mathcal{X})$ are ring homomorphisms, as well as the maps f^* induced by generalized ringed topos pullbacks.*

Proof. (a) All we have to check is that (10.3.12.1) really defines a product on K^0_{perf} or K^0_{vect}; all remaining statements will follow immediately from associativity and commutativity of tensor products and their compatibility with topos pullbacks/scalar extension. We'll discuss only the case of K^0_{perf}; K^0_{vect} is treated similarly if we consider only constant perfect objects. Notice that the formula $[X] \cdot [Y] = [X \otimes Y]$ is a special case of $[u] \cdot [v] = [u \mathbin{\square} v]$ for $u = \nu_X$, $v = \nu_Y$, so we'll concern ourselves only with the latter formula.

(b) We have to check that the map $(u, v) \mapsto [u \mathbin{\square} v]$ is compatible with the relations 0)–3) in each argument. By symmetry it suffices to check this compatibility only in first argument, while assuming v to be fixed. Now relation 0) is trivial: if u is an isomorphism, so is $u \mathbin{\square} v$, hence $[u \mathbin{\square} v] = 0$. Relation 2) is also simple since $u' \mathbin{\square} v$ is a pushout of $u \mathbin{\square} v$ whenever u' is a pushout of u, hence $[u' \mathbin{\square} v] = [u \mathbin{\square} v]$ as expected. Relation 1) follows from the fact that $u'u \mathbin{\square} v$ can be decomposed into $u' \mathbin{\square} v$ and a pushout of $u \mathbin{\square} v$ (cf. **10.1.7** or **9.8.5**), hence $[u'u \mathbin{\square} v]$ is indeed equal to $[u' \mathbin{\square} v] + [u \mathbin{\square} v]$ by 1) and 2). So only relations 3) remain.

(c) Before checking compatibility with relations 3) let's show the following formula: *if $X \xrightarrow{u} Y$ and $Z \xrightarrow{v} W$ are perfect cofibrations between perfect objects, then $[u \mathbin{\square} v] = [Y \otimes W] + [X \otimes Z] - [X \otimes W] - [Y \otimes Z]$.* Indeed, consider the diagram used to define $u \mathbin{\square} v$:

$$\begin{tikzcd}
X \otimes Z \arrow[r, "u \otimes \mathrm{id}_Z"] \arrow[d] & Y \otimes Z \arrow[d, dashed] \arrow[ddr] & \\
X \otimes W \arrow[r, dashed, "u'"] \arrow[drr] & T \arrow[dr, dashed, "u \square v"] & \\
 & & Y \otimes W
\end{tikzcd} \tag{10.3.12.2}$$

All objects involved in this diagram are perfect, and all morphisms are perfect cofibrations by **10.2.15**; furthermore, 2) or 2′) is applicable to this cocartesian square, yielding $[u'] = [u \otimes \mathrm{id}_Z] = [Y \otimes Z] - [X \otimes Z]$. On the other hand, $[X \otimes W] + [u'] + [u \,\square\, v] = [Y \otimes W]$ by 1); combining these formulas together, we obtain the announced formula for $[u \,\square\, v]$.

(d) Now suppose that perfect cofibrations between perfect objects $X \xrightarrow{u} Y$ and $X' \xrightarrow{u'} Y'$ become isomorphic in $\mathcal{D}^{\leq 0}(\mathcal{X})$, i.e. $\gamma(u') \cong \gamma(u)$, and in particular $\gamma X \cong \gamma X'$ and $\gamma Y \cong \gamma Y'$. Applying the formula of (c) to $[u \,\square\, v]$ and $[u' \,\square\, v]$, we see that it would suffice to show the following statement: *if $\gamma X \cong \gamma X'$ and $\gamma Y \cong \gamma Y'$ in $\mathcal{D}^{\leq 0}(\mathcal{X})$, then $[X \otimes Y] = [X' \otimes Y']$.*

(e) This statement follows from existence of derived tensor products $\underline{\underline{\otimes}}$ and the fact that they can be computed with the aid of cofibrant replacements (cf. **9.8.1**): indeed, we have $\gamma(X \otimes Y) \cong \gamma X \underline{\underline{\otimes}} \gamma Y \cong \gamma X' \underline{\underline{\otimes}} \gamma Y' \cong \gamma(X' \otimes Y')$, hence $[X \otimes Y] = [X' \otimes Y']$ in $K^0(\mathcal{X})$ by 3).

(f) The above reasoning shows that for any fixed v the rule $[u] \mapsto [u \,\square\, v]$ determines a well-defined map $h_v : K^0(\mathcal{X}) \to K^0(\mathcal{X})$; interchanging arguments, we obtain compatibility with relations 0)–3) in v as well, i.e. show that $[v] \mapsto h_v$ is a well-defined homomorphism $K^0(\mathcal{X}) \to \mathrm{End}(K^0(\mathcal{X}))$, which corresponds by adjointness to a $\mathbb{Z}$-linear map $K^0(\mathcal{X}) \otimes_{\mathbb{Z}} K^0(\mathcal{X}) \to K^0(\mathcal{X})$, i.e. to a $\mathbb{Z}$-bilinear map $K^0(\mathcal{X}) \times K^0(\mathcal{X}) \to K^0(\mathcal{X})$ with required property $([u], [v]) \mapsto [u \,\square\, v]$, q.e.d.

10.3.13. (Identity of $K^0(\mathcal{X})$.) It is worth mentioning that the identity of $K^0(\mathcal{X})$ under the multiplication just defined is $1 := [L_{\mathscr{O}}(1)] = [\nu_{L_{\mathscr{O}}(1)}]$. This is due to the fact that $L_{\mathscr{O}}(1) \otimes X \cong X$ (i.e. $L_{\mathscr{O}}(1)$ is the unit for $\otimes$) and $K^0(\mathcal{X})$ is generated by $[X]$, or can be equally easy deduced directly from $\nu_{L_{\mathscr{O}}(1)} \,\square\, u \cong u$ (i.e. $\nu_{L_{\mathscr{O}}(1)}$ is a unit for $\square$).

10.3.14. (General remarks on pre-λ-rings.) Given any commutative ring K, we denote by $\hat{G}^t(K)$ or by $1 + K[[t]]^+$ the set of formal power series over K in one indeterminate t with constant term equal to one, considered as an abelian group under multiplication. We write the group law of $\hat{G}^t(K)$ additively; when we consider a formal series $f(t)$ of $1 + K[[t]]^+$ as an element of this group, we denote it by $\{f(t)\}$; thus $\{f(t)\} + \{g(t)\} = \{f(t)g(t)\}$. In this way we obtain a functor $\hat{G}^t$ from commutative rings into abelian groups.

Of course, t can be replaced here by any other letter, e.g. $\hat{G}^u(K) = 1 + K[[u]]^+$.

Now recall the following definition (cf. SGA 6, V.2.1):

Definition 10.3.15 *A pre-λ-ring is a commutative ring K, endowed with an abelian group homomorphism $\lambda_t : K \to \hat{G}^t(K) = 1 + K[[t]]^+$, such that $\lambda_t(x) = 1 + xt + \cdots$ for any $x \in K$.*

Recall that the coefficient at t^n of $\lambda_t(x)$ is usually denoted by $\lambda^n(x)$ or $\lambda^n x$; thus a pre-λ-ring is a commutative ring K together with a family of unary operations $\lambda^n : K \to K$, $n \geq 0$, called *exterior power operations,* such that $\lambda^0(x) = 1$ and $\lambda^1(x) = x$ for any $x \in K$, and

$$\lambda^n(x+y) = \sum_{p+q=n} \lambda^p(x)\lambda^q(y) \tag{10.3.15.1}$$

10.3.16. (Symmetric operations on a pre-λ-ring.) Given any pre-λ-ring K, one defines *symmetric (power) operations* s^n on K by means of the following generating series:

$$s_t(x) = \sum_{n\geq 0} s^n(x)t^n := \lambda_{-t}(x)^{-1} \tag{10.3.16.1}$$

Of course, one can write arising relations explicitly:

$$\sum_{p+q=n} (-1)^p \lambda^p(x) s^q(x) = 0 \tag{10.3.16.2}$$

Since $f(t) \mapsto f(-t)^{-1}$ is an automorphism of $1 + K[[t]]^+$, $s_t : K \to 1+K[[t]]^+$ is an abelian group homomorphism as well, and $s_t(x) = \lambda_{-t}(x)^{-1} = (1 - xt + \cdots)^{-1} = 1 + xt + \cdots$, i.e. we still have $s^0(x) = 1$, $s^1(x) = x$, and

$$s^n(x+y) = \sum_{p+q=n} s^p(x)s^q(y) \tag{10.3.16.3}$$

In other words, symmetric power operations define another pre-λ-structure on K.

Conversely, given any abelian group homomorphism $s_t : K \to 1+K[[t]]^+$, such that $s_t(x) = 1+xt+\cdots$, or equivalently, a collection of symmetric power operations $\{s^n : K \to K\}_{n\geq 0}$ satisfying the above relations, we can recover λ_t from (10.3.16.1), thus obtaining a pre-λ-ring structure on K. In other words, *pre-λ-rings admit an equivalent description in terms of symmetric operations s^n.*

Theorem 10.3.17 *Let $\mathcal{X} = (\mathcal{X}, \mathscr{O})$ be as above. There are unique $\mathbb{Z}$-linear maps $s^n : K^0(\mathcal{X}) \to K^0(\mathcal{X})$, $n \geq 0$, where $K^0(\mathcal{X})$ denotes either $K^0_{perf}(\mathcal{X})$ or $K^0_{vect}(\mathcal{X})$, such that*

$$s^n[X] = [S^n(X)], \quad s^n[u] = [\rho_n(u)] \tag{10.3.17.1}$$

where $S^n = S^n_{\mathscr{O}}$ is the n-th symmetric power functor, and $\rho_n(u)$ is defined in **9.9.7**. *Furthermore, these maps s^n are compatible with ring homomorphisms f^* induced by generalized ringed topos morphisms as well as with ring* *homomorphism $K^0_{vect}(\mathcal{X}) \to K^0_{perf}(\mathcal{X})$, and they satisfy following relations:*

$$s^0(\xi) = 1, \tag{10.3.17.2}$$
$$s^1(\xi) = \xi, \tag{10.3.17.3}$$
$$s^n(\xi+\eta) = \sum_{p+q=n} s^p(\xi)\cdot s^q(\eta) \tag{10.3.17.4}$$

In other words, s^n are the symmetric power operations for a unique pre-λ-ring structure on $K^0(\mathcal{X})$.

Proof. We deal again only with the case $K^0(\mathcal{X}) = K^0_{perf}(\mathcal{X})$; the case of K^0_{vect} is treated similarly, restricting all considerations to constant perfect objects and cofibrations. Also notice that $\rho_n(\nu_X) = \nu_{S^n X}$, hence formula $s^n[u] = [\rho_n(u)]$ implies $s^n[X] = [S^n(X)]$, i.e. it is enough to consider only the first formula.

(a) First of all, notice that $[S^0(X)] = [L_{\mathscr{O}}(1)] = 1$ for any perfect X, and $[\rho_0(u)] = [\nu_{L_{\mathscr{O}}(1)}] = 1$ for any u by definition, hence s^0 is well-defined and $s^0(\xi) = 1$ for all $\xi \in K^0(\mathcal{X})$. Similarly, $[S^1(X)] = [X]$ and $[\rho_1(u)] = [u]$ just because $\rho_1(u) = u$, hence s_1 is also well-defined and $s^1(\xi) = \xi$ for all ξ.

(b) For any perfect cofibration u between perfect objects we put

$$s_t(u) := \sum_{n\geq 0}[\rho_n(u)] \in 1 + K^0(\mathcal{X})[[t]]^+ \tag{10.3.17.5}$$

In order for this expression to make sense we need to know that all $\rho_n(u)$ are perfect cofibrations between perfect objects; this is true by **10.2.17**, **9.9.7** and **9.9.18** (in fact, we obtain by induction in k that all $\rho^{(n)}_k(u) : F_{k-1}S^n(u) \to F_k S^n(u)$ are perfect cofibrations between perfect objects). Moreover, the free term $s^0(u)$ of $s_t(u)$ equals one by (a), hence $s_t(u)$ has indeed free term equal to one. We write $s_t(X)$ instead of $s_t(\nu_X)$ for any perfect X as usual.

(c) Now all we have to check is that $[u] \to s_t(u)$ is a well-defined map $K^0(\mathcal{X}) \to 1 + K^0(\mathcal{X})[[t]]^+$, i.e. that the $s_t(u)$ satisfy relations 0)–3) of **10.3.1**. This is evident for 0); for 2) we just use the fact that $\rho_n(u')$ is a pushout of $\rho_n(u)$ whenever u' is a pushout of u (cf. **9.9.21**,(a)). As to 1), we use that $\rho_n(vu)$ can be decomposed into a composition of pushouts of $\rho_k(v) \,\square\, \rho_{n-k}(u)$, $0 \leq k \leq n$ (cf. **9.9.21**,(c)), hence $[\rho_n(vu)] = \sum_{p+q=n}[\rho_p(v)][\rho_q(u)]$, i.e. $s_t(vu) = s_t(v)s_t(u)$ as required by 1).

(d) So only condition 3) remains. Since for any perfect cofibration between perfect objects $u : X \to Y$ we have $s_t(u) = s_t(Y)\cdot s_t(X)^{-1}$, it would suffice to show that $\gamma X \cong \gamma X'$ in $\mathcal{D}^{\leq 0}(\mathcal{X})$ implies $s_t(X) = s_t(X')$. This is immediate from existence of derived symmetric powers and the fact that they can be computed by means of cofibrant replacements (cf. **9.9.2**): we obtain $\gamma(S^n X) \cong \mathbb{L}S^n(\gamma X) \cong \mathbb{L}S^n(\gamma X') \cong \gamma(S^n X')$, hence $[S^n X] = [S^n X']$ by 3). This completes the proof of existence of symmetric power operations s^n with required property, and the construction of the pre-λ-structure on $K^0(\mathcal{X})$.

(e) We still have to check that the symmetric power operations s^n are compatible with maps f^* and $K^0_{vect}(\mathcal{X}) \to K^0_{perf}(\mathcal{X})$; the latter statement is evident, and the former is an immediate consequence of commutativity of generalized ringed topos pullbacks with symmetric powers of $\mathscr{O}$-modules, q.e.d.

10.3.18. (K_0 of an algebraic monad Λ.) All of the above constructions are applicable to any algebraic monad Λ, since it can be considered as an algebraic monad over the point topos *Sets*. We'll usually write $K^0(\Lambda)$ and $K^0_{vect}(\Lambda) = \hat{K}^0(\Lambda)$ instead of $K^0(Sets, \Lambda)$ and $K^0_{vect}(Sets, \Lambda)$. These are abelian groups for an arbitrary Λ, and commutative pre-λ-rings for a commutative Λ. Furthermore, any algebraic monad homomorphism $\rho : \Lambda \to \Lambda'$ defines a generalized ringed topos morphism $(Sets, \Lambda') \to (Sets, \Lambda)$, whence canonical homomorphisms (even pre-λ-rings homomorphisms for commutative Λ and Λ') $\rho_* : K^0(\Lambda) \to K^0(\Lambda')$. Of course, ρ_* is induced by scalar extension, e.g. $\rho_*[P] = [P_{(\Lambda')}]$ for any projective Λ-module P of finite type. When Λ is additive, we recover classical K^0 of a ring according to **10.3.9**.

10.3.19. (Computation of $\hat{K}^0(\mathbb{F}_\varnothing)$.) In particular, the pre-λ-rings $\hat{K}^0(\mathbb{F}_\varnothing) = K^0_{vect}(Sets, \mathbb{F}_\varnothing)$ and $K^0(\mathbb{F}_\varnothing)$ are quite important since $(Sets, \mathbb{F}_\varnothing)$ is the 2-final object in the 2-category of generalized ringed topoi, and therefore we obtain pre-λ-ring homomorphisms $\hat{K}^0(\mathbb{F}_\varnothing) \to \hat{K}^0(\mathcal{X})$ and $\hat{K}^0(\mathbb{F}_\varnothing) \to K^0(\mathbb{F}_\varnothing) \to K^0(\mathcal{X})$ for any generalized ringed topos $\mathcal{X} = (\mathcal{X}, \mathscr{O})$.

We'll show in several steps that $\hat{K}^0(\mathbb{F}_\varnothing) \cong K^0(\mathbb{F}_\varnothing) \cong \mathbb{Z}$ considered as a pre-λ-ring with respect to its only λ-ring structure given by $\lambda^k(n) = \binom{n}{k}$, $\lambda_t(n) = (1+t)^n$, $s_t(n) = (1-t)^{-n}$, $s^k(n) = \binom{n+k-1}{k}$ (cf. SGA 6 V 2.5).

Let's show the statement about $\hat{K}^0(\mathbb{F}_\varnothing) = K^0_{vect}(\mathbb{F}_\varnothing)$. A perfect (constant) $\mathbb{F}_\varnothing$-module is just a finite set X, hence $\hat{K}^0(\mathbb{F}_\varnothing)$ is generated by $[X]$, for all finite sets X. Since $[X \oplus Y] = [X] + [Y]$ by **10.3.4**, and any finite set X is a direct sum (i.e. disjoint union) of $|X|$ copies of $\mathbf{1}$, we get $[X] = |X| \cdot [\mathbf{1}] = |X| \cdot 1$ since $[\mathbf{1}]$ is the identity of $\hat{K}^0(\mathbb{F}_\varnothing)$. In particular, the canonical ring homomorphism $\mathbb{Z} \to \hat{K}^0(\mathbb{F}_\varnothing)$ is surjective. On the other hand, the map $X \mapsto |X|$ obviously satisfies relations $0')$ and $2')$ since a constant perfect cofibration is just an injective map $u : X \to Y$ with finite $Y - u(X)$, and the number of elements in the complement of the image of u is obviously preserved under any pushouts. This yields a map $\hat{K}^0(\mathbb{F}_\varnothing) \to \mathbb{Z}$, $[X] \mapsto |X|$ in the opposite direction, clearly inverse to the previous one, hence

$\hat{K}^0(\mathbb{F}_\varnothing) \cong \mathbb{Z}$. This isomorphism is obviously compatible with multiplication; as to the λ-structures, $S^n X$ is just the n-th symmetric power $X^{(n)} := X^n/\mathfrak{S}_n$ of a set X, hence $|S^n X| = \binom{|X|+n-1}{n}$ as claimed.

10.3.20. (Computation of $[i_n]$ in $K^0(\mathbb{F}_\varnothing)$ and $K^0(\mathcal{X})$.) Now let's compute $[i_n]$ in $K^0(\mathbb{F}_\varnothing)$, where $i_n : \dot{\Delta}(n) \to \Delta(n)$ denotes a standard cofibrant generator of *sSets*. Consider for this the standard acyclic cofibration $j : \Lambda_n(n) \to \Delta(n)$. On one hand, it can be decomposed into $\Lambda_n(n) \xrightarrow{u} \dot{\Delta}(n) \xrightarrow{i_n} \Delta(n)$, where u is a pushout of i_{n-1}, hence $[j] = [i_n] + [u] = [i_n] + [i_{n-1}]$. On the other hand, $[j] = 0$ by 3) and 0) since $\gamma(j)$ is an isomorphism in the derived category, j being an acyclic cofibration. We conclude that $[i_n] + [i_{n-1}] = 0$ for any $n > 0$, and clearly $[i_0] = 1$, hence

$$[i_n] = (-1)^n \quad \text{for any } n \geq 0. \tag{10.3.20.1}$$

Notice that the above formula holds in any $K^0(\mathcal{X})$ as well, $K^0(\mathbb{F}_\varnothing) \to K^0(\mathcal{X})$ being a ring homomorphism, where of course $[i_n]$ is understood as $L_{\mathscr{O}}(\underline{i_n})$.

Now we are going to deduce from (10.3.20.1) some sort of Euler characteristic formula:

Proposition 10.3.21 *Let $\mathcal{X} = (\mathcal{X}, \mathscr{O})$ be a generalized (commutatively) ringed topos, $u : X \to Y$ be a perfect cofibration of bounded dimension $\dim u \leq N < +\infty$ between perfect objects. Then all $\operatorname{sk}_{=n} u$ are also perfect cofibrations between perfect objects, all $(\operatorname{sk}_{=n} u)_n$ are constant perfect cofibrations between vector bundles, and*

$$[u] = \sum_{n=0}^{N} [\operatorname{sk}_{=n} u] = \sum_{n=0}^{N} (-1)^n [(\operatorname{sk}_{=n} u)_n] \tag{10.3.21.1}$$

In particular, $[u]$ lies in the image of $\hat{K}^0(\mathcal{X}) \to K^0(\mathcal{X})$.

Proof. Indeed, the dimensional decomposition (10.1.4.4) of u into $\operatorname{sk}_{=n} u$ consists of finitely many steps, all $\operatorname{sk}_{=n} u$ being isomorphisms for $n > N$, and all $\operatorname{sk}_{=n} u : F_n(u) \to F_{n+1}(u)$ are perfect cofibrations by **10.2.11**, hence by induction all $F_n(u)$ are also perfect, $F_0(u) = X$ being perfect by assumption, hence $[\operatorname{sk}_{=n} u]$ is an element of $K^0(\mathcal{X})$, and the first equality of (10.3.21.1) follows from relations 1). Furthermore, all components of $\operatorname{sk}_{=n} u$, and in particular $w_n := (\operatorname{sk}_{=n} u)_n$, are constant perfect cofibrations between vector bundles by **10.2.12**, hence $[w_n]$ makes sense both in $\hat{K}^0(\mathcal{X})$ and $K^0(\mathcal{X})$. Finally, $\operatorname{sk}_{=n} u$ is a pushout of $w_n \square i_n$ by **10.1.14**, hence $[\operatorname{sk}_{=n} u] = [w_n][i_n] = (-1)^n [w_n]$ by (10.3.20.1). Computing the sum over all $n \leq N$ we obtain the second half of (10.3.21.1), q.e.d.

In particular, we obtain the following interesting statement:

Corollary 10.3.22 *If all perfect simplicial objects over $\mathcal{X} = (\mathcal{X}, \mathscr{O})$ have bounded dimension, e.g. if $\mathcal{X}$ is quasicompact, then the canonical pre-λ-ring homomorphism $\hat{K}^0(\mathcal{X}) \to K^0(\mathcal{X})$ is surjective, i.e. $K^0(\mathcal{X})$ is generated by elements $[P]$, where P runs over isomorphism classes of vector bundles over $\mathcal{X}$.*

For example, $\hat{K}^0(\Lambda) \to K^0(\Lambda)$ is surjective for any generalized ring Λ.

10.3.23. ($\hat{K}^0$ and K^0 of a classical field k.) Now let k be a classical (i.e. additive) field. Then $\hat{K}^0(k) \to K^0(k)$ is surjective, and $\mathbb{Z} \to \hat{K}^0(k)$ is an isomorphism, the inverse being given by $[V] \mapsto \dim_k V$ (cf. **10.3.9**). We claim that $\hat{K}^0(k) \to K^0(k)$ is also an isomorphism, i.e. $K^0(k) = K^0_{perf}(K)$ is also isomorphic to $\mathbb{Z}$ with its standard λ-structure. Indeed, **10.3.10** immediately implies that $\chi : [X] \mapsto \sum_{n\geq 0}(-1)^n \dim_k H^n(NX)$ is a well-defined homomorphism $K^0(k) \to \mathbb{Z}$, such that $\chi(1) = 1$, where NX denotes the normalized chain complex associated to a simplical k-vector space. This χ is clearly an inverse to the surjective map $\mathbb{Z} = \hat{K}^0(k) \to K^0(k)$.

10.3.24. ($K^0(\mathbb{F}_\varnothing) = \mathbb{Z}$.) We already know that $\hat{K}^0(\mathbb{F}_\varnothing) \cong \mathbb{Z}$ and that the homomorphism $\hat{K}^0(\mathbb{F}_\varnothing) \to K^0(\mathbb{F}_\varnothing)$ is surjective. We claim that it is in fact an isomorphism, i.e. $K^0(\mathbb{F}_\varnothing) \cong \mathbb{Z}$ with the standard λ-structure. Indeed, since $\mathbb{Q}$ is an extension of $\mathbb{F}_\varnothing$, we get a map $K^0(\mathbb{F}_\varnothing) \to K^0(\mathbb{Q})$, $[X] \mapsto [L_{\mathbb{Q}}(X)]$, and $K^0(\mathbb{Q})$ is isomorphic to $\mathbb{Z}$ via the map χ, hence the composite map $\chi \circ L_{\mathbb{Q}}$ is an additive map $K^0(\mathbb{F}_\varnothing) \to \mathbb{Z}$ mapping 1 into 1, hence the surjection $\mathbb{Z} \to K^0(\mathbb{F}_\varnothing)$ must be an isomorphism.

10.3.25. ($K^0(\mathbb{F}_{1^n}) = \hat{K}^0(\mathbb{F}_{1^n}) = \mathbb{Z}$.) Now let us compute $\hat{K}^0(\mathbb{F}_{1^n})$ and $K^0(\mathbb{F}_{1^n})$, where $\mathbb{F}_{1^n}$ is the generalized ring defined in **5.1.16**. Recall that $\mathbb{F}_{1^n} = \mathbb{F}_1[\zeta^{[1]} \,|\, \zeta^n = \boldsymbol{e}\}$ and $\mathbb{F}_1 = \mathbb{F}_\varnothing[0^{[0]}]$, and $\mathbb{F}_{1^n}$-*Mod* consists of sets X together with a marked point $0 = 0_X \in X$ and a bijection $\zeta = \zeta_X : X \to X$ respecting 0_X, such that $\zeta_X^n = \mathrm{id}_X$, i.e. an action of the cyclic group $C_n := \mathbb{Z}/n\mathbb{Z}$. We claim that *the canonical λ-homomorphisms $\mathbb{Z} = \hat{K}^0(\mathbb{F}_\varnothing) \to \hat{K}^0(\mathbb{F}_{1^n}) \to K^0(\mathbb{F}_{1^n})$ are actually isomorphisms*, i.e. *both $\hat{K}^0(\mathbb{F}_{1^n})$ and $K^0(\mathbb{F}_{1^n})$ are canonically isomorphic to $\mathbb{Z}$.*

(a) First of all, a free $\mathbb{F}_{1^n}$-module of rank r is just the $nr+1$-element set $0 \sqcup \mathbf{r} \times C_n$, where $C_n = \mathbb{Z}/n\mathbb{Z}$ is the cyclic group of order n with the obvious action of ζ. Conversely, if X is a finite set with a marked point 0, such that the stabilizer in C_n of any non-zero $x \in X$ is trivial, then the C_n-orbit decomposition of X is of the above form, hence X is a free $\mathbb{F}_{1^n}$-module of finite rank.

(b) An immediate consequence is that *any projective $\mathbb{F}_{1^n}$-module of finite type is free*. Indeed, any projective module is a retract, i.e. both a quotient and a submodule of a free module, and (a) shows that any submodule of a free $\mathbb{F}_{1^n}$-module is free. Furthermore, all constant perfect cofibrations are easily seen to be of the form $X \to X \oplus L_{\mathbb{F}_{1^n}}(r)$, i.e. $X \to X \sqcup \mathbf{r} \times C_n$. This means that $\hat{K}^0(\mathbb{F}_{1^n})$ is generated by $[L_{\mathbb{F}_{1^n}}(1)] = 1$, i.e. $\mathbb{Z} = \hat{K}^0(\mathbb{F}_\varnothing) \to \hat{K}^0(\mathbb{F}_{1^n})$ is surjective, and $\hat{K}^0(\mathbb{F}_{1^n}) \to K^0(\mathbb{F}_{1^n})$ is surjective by **10.3.22**.

(c) It remains to check that these two surjections have trivial kernel. It would suffice for this to construct a $\mathbb{Z}$-linear map $\chi : K^0(\mathbb{F}_{1^n}) \to \mathbb{Z}$, such that $\chi(1) = 1$. We define such a map as follows: put $A := \mathbb{Z} \otimes_{\mathbb{F}_1} \mathbb{F}_{1^n} = \mathbb{Z}[\zeta]/(\zeta^n - 1)$, choose any maximal ideal $\mathfrak{m} \subset A$ (such an ideal exists since A is a free $\mathbb{Z}$-module of rank $n > 0$, hence a non-trivial commutative ring), and put $k := A/\mathfrak{m}$. Then k is a classical field, and we have a canonical generalized ring homomorphism $\rho : \mathbb{F}_{1^n} \to A \to k$, which induces a map $\rho_* : K^0(\mathbb{F}_{1^n}) \to K^0(k) \cong \mathbb{Z}$ with the required property.

10.3.26. (Polynomial functions.) Let A be any abelian group. We denote by $A^{\mathbb{N}_0}$ the set of all A-valued functions $f : \mathbb{N}_0 \to A$ defined on the set of non-negative integers. Of course, $A^{\mathbb{N}_0}$ has a canonical abelian group structure given by pointwise addition: $(f+g)(n) = f(n) + g(n)$.

For any $f \in A^{\mathbb{N}_0}$ we denote by $\Delta f \in A^{\mathbb{N}_0}$ the A-valued function given by

$$(\Delta f)(n) := f(n+1) - f(n) \quad \text{for all } n \geq 0. \tag{10.3.26.1}$$

We say that f is a *A-valued polynomial function* if $(\Delta^n f)(0) = 0$ for all $n > N$, where $N \geq 0$ is some integer. (The smallest N with this property is called the *degree* of f.) An easy induction shows that this condition is equivalent to $\Delta^n f = 0$ for all $n > N$, or to $\Delta^{N+1} f = 0$. Furthermore, another induction in $x \geq 0$ yields

$$f(x) = \sum_{n \geq 0} \binom{x}{n} \cdot (\Delta^n f)(0) \tag{10.3.26.2}$$

for any $x \geq 0$ and any $f \in A^{\mathbb{N}_0}$. Notice that this sum is finite for any $x \geq 0$ since $\binom{x}{n} = 0$ for $n > x$.

Now if $f \in A^{\mathbb{N}_0}$ is an A-valued polynomial function, all terms corresponding to $n > N$ in the above sum are automatically zero, hence

$$f(x) = \sum_{n=0}^{N} \binom{x}{n} \cdot c_n \quad \text{for some } c_n \in A. \tag{10.3.26.3}$$

Conversely, if $f(x)$ is given by the above formula for some $c_n \in A$, $0 \leq n \leq N$, then $(\Delta f)(x)$ is easily seen to be equal to $\sum_{n=0}^{N-1} \binom{x}{n} \cdot c_{n+1}$, and an obvious induction yields $(\Delta^n f)(0) = c_n$, where of course $c_n := 0$ for $n > N$, thus proving that f is an A-valued polynomial function, as well as showing the uniqueness of coefficients c_n.

In other words, we see that the abelian group $A[\binom{x}{n}] \subset A^{\mathbb{N}_0}$ of A-valued polynomials is canonically isomorphic to $A \otimes \mathbb{Z}[\binom{x}{n}]$, where $\mathbb{Z}[\binom{x}{n}] \subset \mathbb{Q}[x]$ is the set of all integer-valued polynomials, a free $\mathbb{Z}$-module with basis $\binom{x}{n}$, $n \geq 0$.

Another immediate consequence of the above constructions is that any polynomial function $f : \mathbb{N}_0 \to A$ can be uniquely extended by (10.3.26.3) to a polynomial function $f : \mathbb{Z} \to A$, where the coefficients c_n are necessarily equal to $(\Delta^n f)(0)$, i.e. the extension of f is given by the Newton extrapolation formula (10.3.26.2). In particular, we can consider

$$f(-1) = \sum_{n=0}^{N} (-1)^n c_n = \sum_{n \geq 0} (-1)^n (\Delta^n f)(0) \tag{10.3.26.4}$$

10.3.27. (Hilbert function of a perfect cofibration.) Let $u : X \to Y$ be a perfect cofibration in $s\mathscr{O}$-*Mod*. We denote by $P_u : \mathbb{N}_0 \to \hat{K}^0(\mathcal{X}, \mathscr{O})$ the *Hilbert function* of u, given by

$$P_u(n) := [u_n], \quad \text{for any } n \geq 0. \tag{10.3.27.1}$$

Here $u_n : X_n \to Y_n$ denotes the corresponding component of u; according to **10.2.12**, all u_n are indeed constant cofibrations between vector bundles, hence $[u_n]$ indeed makes sense in $\hat{K}^0(\mathcal{X})$. Furthermore, we have $P_{u \square v} = P_u P_v$ (i.e. $P_{u \square v}(n) = P_u(n) \cdot P_v(n)$ for all $n \geq 0$) since $(u \square v)_n$ obviously equals $u_n \square v_n$. Similarly, $P_{\rho_n(u)} = s^n P_u$.

If X is a perfect simplicial object, we define its *Hilbert function* P_X by $P_X := P_{\nu_X}$, i.e. $P_X(n) = [X_n]$. Then $X \mapsto P_X$ satisfies $0'$) and $2'$), $P_{X \otimes Y} = P_X P_Y$, and $P_{S^n X} = s^n P_X$, i.e. all natural relations of $K^0(\mathcal{X})$ except $3'$) are fulfilled.

Notice that $u \mapsto P_u$ satisfies relations 0)–2), since for example if u' is a pushout of u, then all u'_n are pushouts of u_n, hence $P_{u'}(n) = [u'_n] = [u_n] = P_u(n)$ by 2) in $\hat{K}^0(\mathcal{X})$. However, relations 3) are not satisfied, since components of a weak equivalence needn't be isomorphisms.

Proposition 10.3.28 (Hilbert polynomials and Euler characteristic.) *Let $\mathcal{X} = (\mathcal{X}, \mathscr{O})$ be a generalized ringed topos as usual, $u : X \to Y$ be a perfect cofibration of bounded dimension $\dim u \leq N < +\infty$ between perfect objects (e.g. if $\mathcal{X}$ is quasicompact, any such u has bounded dimension). Then $P_u \in \hat{K}^0(\mathcal{X})^{\mathbb{N}_0}$ is a polynomial function of degree $\leq N$; more precisely,*

$$P_u(n) = \sum_{k=0}^{N} c_k \binom{n}{k}, \quad \text{where } c_k := [(\mathrm{sk}_{=k}\, u)_k] \tag{10.3.28.1}$$

Furthermore, the image of $P_u(-1) = \sum_{k=0}^{N}(-1)^k c_k$ under the canonical λ-homomorphism $\hat{K}^0(\mathcal{X}) \to K^0(\mathcal{X})$ equals $[u]$:

$$[u] = P_u(-1) = \sum_{k=0}^{N}(-1)^k(\Delta^k P_u)(0) = \sum_{k=0}^{N}(-1)^k c_k = \sum_{k=0}^{N}(-1)^k[(\mathrm{sk}_{=k}\, u)_k] \tag{10.3.28.2}$$

Similar formulas are valid for a perfect object X of dimension $\leq N$, e.g. P_X is a polynomial function of degree $\leq N$, and $P_X(-1) = [X]$.

Proof. Of course, the statements about P_X follow from those about P_u applied to $u = \nu_X$, so we'll treat only the case of P_u. Notice that it would suffice to show (10.3.28.1): the remaining statements would then follow from (10.3.26.2), (10.3.26.4) and (10.3.21.1). Since $u \mapsto P_u$ satisfies relations 0)–2) and the dimensional decomposition (10.1.4.4) of u is finite, it is enough to show (10.3.28.1) for u purely of dimension k. In this case u is a pushout of $w \,\square\, i_k$, where $w := u_k = (\mathrm{sk}_{=k}\, u)_k$ (cf. **10.1.14**), hence $P_u = P_{w\square i_k} = P_w \cdot P_{i_k} = wP_{i_k}$, w being constant, i.e. we are reduced to proving

$$P_{i_k}(n) = \binom{n}{k} \quad \text{in } \hat{K}^0(\mathbb{F}_\varnothing) = \mathbb{Z}. \tag{10.3.28.3}$$

This equality is shown by direct computation: by definition, $P_{i_k}(n) = [i_{k,n}] = |\Delta(k)_n| - |\dot{\Delta}(k)_n|$, $\Delta(k)_n$ consists of all non-decreasing maps $\varphi : [n] \to [k]$, and $\dot{\Delta}(k)_n$ of all non-surjective such maps, hence $P_{i_k}(n)$ equals to the number of surjective non-decreasing maps $\varphi : [n] \twoheadrightarrow [k]$, easily seen to be equal to $\binom{n}{k}$, such maps being in one-to-one correspondence to k-element subsets $I = \{n_1, \ldots, n_k\} \subset \{1, 2, \ldots, n\}$ by the following rule: $I = \{j \in [n] : j > 0, \varphi(j-1) < \varphi(j)\}$. This finishes the proof of **10.3.28**.

10.3.29. (Relation to Euler characteristic.) Recall that in the additive case $P_{KA}(-1) = \sum_{k\geq 0}(-1)^k[A_k]$ for any perfect complex A, according to the computation already mentioned in **8.0.11**. Therefore, the above formula $[u] = P_u(-1)$ should be thought of as a counterpart of the Euler characteristic formula.

10.3.30. (Suspensions, cones and cylinders.) Suppose that $\mathscr{O}$ admits a zero, so that we can construct suspensions, cones and cylinders.

(a) Notice that $\Delta(0) \to \Delta(1)$ is an acyclic cofibration of finite simplicial sets, hence $[\Delta(1)] = [\Delta(0)] = 1$ in $K^0(\mathbb{F}_\varnothing)$. We can use this to conclude $[i_1] = -1$ for $i_1 : \Delta(0) \sqcup \Delta(0) = \dot{\Delta}(1) \to \Delta(1)$, or just apply (10.3.20.1).

(b) Let X be a perfect simplicial object, hence its suspension ΣX is also perfect by **10.2.21**. Recall that ΣX is the cofiber of $\mathrm{id}_X \otimes i_1 = \nu_X \,\square\, i_1$:

$X \oplus X \to X \otimes \Delta(1)$, i.e. $0 \to \Sigma X$ is a pushout of this morphism, hence $[\Sigma X] = [\nu_X] \cdot [i_1] = -[X]$. If $u : X \to Y$ is a perfect cofibration between perfect objects, the same holds for $\Sigma(u)$, this morphism being a pushout of $u \,\square\, i_1$, and $[\Sigma(u)] = [u] \cdot [i_1] = -[u]$.

(c) Now let $f : X \to Y$ be a perfect cofibration between perfect objects. Applying 2) to cocommutative squares of (10.2.20.1), we obtain $[Y \otimes \{0\} \to Cyl(f)] = [f] \cdot [\Delta(0) \to \Delta(1)] = [f] \cdot 0 = 0$, hence $[Cyl(f)] = [Y]$. The second square yields $[C(f)] = [Cyl(f)] - [X] = [Y] - [X]$, exactly what one usually has in the classical (additive) situation.

10.3.31. (Relation to Waldhausen's construction.) Our definition of K_0 might be thought of as a modification of Waldhausen's construction of K_0 presented in [Waldhausen]. Since Waldhausen defines higher algebraic K-theory as well, this observation might be used to define higher K-functors in our situation by modifying Waldhausen's construction in a suitable way.

Recall that Waldhausen considers *categories* $\mathcal{C}$ *with cofibrations and weak equivalences*, sometimes called *Waldhausen categories*. A Waldhausen category is a pointed category $\mathcal{C}$ (i.e. it has a zero object 0) with two classes of morphisms $\mathscr{P}$ and $\mathscr{W} \subset \operatorname{Ar}\mathcal{C}$, called *cofibrations* and *weak equivalences*, satisfying certain conditions. Namely, both classes are supposed to contain all isomorphisms and be closed under composition. Secondly, the following conditions have to be fulfilled:

Co2) Morphisms $0 \to X$, for all $X \in \operatorname{Ob}\mathcal{C}$, are cofibrations.

Co3) Cofibrations are stable under pushouts (in particular, pushouts of cofibrations are required to exist).

We2) (Gluing lemma) If the left horizontal arrows u and v in the diagram

$$\begin{tikzcd} B \arrow[d, "\sim"] & A \arrow[l, "u"'] \arrow[r] \arrow[d, "\sim"] & C \arrow[d, "\sim"] \\ B' & A' \arrow[l, "v"] \arrow[r] & C' \end{tikzcd} \tag{10.3.31.1}$$

are cofibrations, and the vertical arrows are weak equivalences, then the induced map of pushouts $B \vee_A C \to B' \vee_{A'} C'$ is also a weak equivalence.

For any cofibration $X \xrightarrow{u} Y$ we define its cofiber E as $Y \vee_X 0$. Clearly, if u' is any pushout of u, then the cofiber of u' is also equal to E. Now Waldhausen's $K_0(\mathcal{C}) = K_0(\mathcal{C}, \mathscr{P}, \mathscr{W})$ can be defined as the free abelian group generated by objects $[X]$ of $\mathcal{C}$ modulo relations (W_1) $[Y] = [X] + [E]$ for any cofibration $X \xrightarrow{u} Y$ with cofiber E, and (W_2) $[X] = [X']$ for any weak equivalence $X \to X'$.

Now we can see a certain similarity to our construction, where we consider the category $\mathcal{C}$ of all perfect objects, $\mathscr{P}$ is the set of perfect cofibrations, and $\mathscr{W}$ is the set of weak equivalences in $\mathcal{C}$, at least in the case when $\mathscr{O}$ admits a zero. Reasoning as in **10.3.9** (with cofibers instead of cokernels), we can show that relations (W_1) are in fact equivalent to relations $0')$ and $2')$, and (W_2) correspond to our $3^w)$, but not $3')$ (i.e. we impose more relations than Waldhausen).

It is an interesting question whether the axiom We2), which makes sense for $\mathscr{O}$ without a zero as well, holds for our choice of $\mathcal{C}$, $\mathscr{P}$ and $\mathscr{W}$. The answer is extremely likely to be positive (for example, it is positive for the classical case of pointed simplicial sets, i.e. simplicial $\mathbb{F}_1$-modules). If the answer *is* positive, we might try to apply Waldhausen's construction directly to our situation, thus immediately defining higher algebraic K-groups as well.

10.4. (Projective modules over $\mathbb{Z}_\infty$.) Our next aim is to compute $\hat{K}^0(\widehat{\operatorname{Spec}\mathbb{Z}})$. However, we need to know more about projective bundles $\mathscr{E}$ over $\widehat{\operatorname{Spec}\mathbb{Z}}$ to do this. Since $\mathscr{E}|_{\operatorname{Spec}\mathbb{Z}}$ is given by a finitely generated projective (i.e. free) $\mathbb{Z}$-module, we just have to study finitely generated projective modules over $\mathbb{Z}_{(\infty)}$ or $\mathbb{Z}_\infty$. In fact, we are going to show that such projective modules are free as well, and that any constant perfect cofibration between vector bundles over $\mathbb{Z}_\infty$ is of the form $\mathbb{Z}_\infty^{(n)} \to \mathbb{Z}_\infty^{(n)} \oplus \mathbb{Z}_\infty^{(m)} \cong \mathbb{Z}_\infty^{(n+m)}$, and similarly for $\mathbb{Z}_{(\infty)}$ and $\bar{\mathbb{Z}}_\infty$. This would imply $\hat{K}^0(\mathbb{Z}_\infty) = K^0(\mathbb{Z}_\infty) = \mathbb{Z}$, and similarly for $\mathbb{Z}_{(\infty)}$ and $\bar{\mathbb{Z}}_\infty$.

10.4.1. (Strictly convex archimedian valuation rings.) We are going to treat our three cases $\mathbb{Z}_\infty$, $\mathbb{Z}_{(\infty)}$ and $\bar{\mathbb{Z}}_\infty$ simultaneously. In order to do this we fix following data:

- A (classical) field K together with a non-trivial archimedian valuation $|\cdot|$. Notice that necessarily $\operatorname{char} K = 0$, i.e. $K \supset \mathbb{Q}$. We consider K as a topological field with respect to the topology given by its norm $|\cdot|$.

- $V \subset K$ denotes the (generalized) valuation ring of $|\cdot|$ in K, defined as in **5.7.13**. In other words, $V(n) \subset K^{(n)} = K^n$ consists of $\lambda = (\lambda_1, \dots, \lambda_n) \in K^n$ with $\sum_i |\lambda_i| \leq 1$.

- $\tilde{\mathbb{Q}}$ denotes the closure of $\mathbb{Q} \subset K$ in K, and $\tilde{\mathbb{Q}}_+ \subset \tilde{\mathbb{Q}}$ denotes the closure of $\mathbb{Q}_+$. Since any element of $\tilde{\mathbb{Q}}$ is a limit of a Cauchy sequence in $\mathbb{Q}$, we obtain a natural embedding $\tilde{\mathbb{Q}} \subset \mathbb{R}$, i.e. we can treat elements of $\tilde{\mathbb{Q}} \subset K$ as real numbers.

- We require $(K, |\cdot|)$ to be *strictly convex*, meaning the following:

$$|x + y| = |x| + |y| \quad \text{for } x,\, y \in K \;\Rightarrow\; y = 0 \text{ or } xy^{-1} \in \tilde{\mathbb{Q}}_+. \qquad (10.4.1.1)$$

- Finally, we require $|x| \in \tilde{\mathbb{Q}}_+$ for any $x \in K$.

The above conditions are easily seen to be satisfied by $K = \mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ with their standard archimedian norms, given by absolute value. In these cases we have $V = \mathbb{Z}_{(\infty)}$, $\mathbb{Z}_\infty$ and $\bar{\mathbb{Z}}_\infty$, respectively.

Theorem 10.4.2 (Projective modules over strictly convex archimedian valuation rings.) *Let (K, V) be as above, i.e. V be a strictly convex archimedian valuation ring, e.g. $V = \mathbb{Z}_\infty$, $\mathbb{Z}_{(\infty)}$ or $\bar{\mathbb{Z}}_\infty$. Denote by $V^{(n)}$ or $L_V(n)$ the free V-module of rank n. Then:*

(a) Any projective V-module P of finite type is free of finite rank, i.e. isomorphic to some $L_V(n)$, $n \geq 0$.

(b) If $L_V(n)$ is isomorphic to $L_V(m)$, then $n = m$.

(c) Any constant perfect cofibration of V-modules is of the form $M \to M \oplus L_V(n)$. In particular, any constant perfect cofibration between vector bundles over V is of the form $L_V(m) \to L_V(m) \oplus L_V(n) \cong L_V(m+n)$.

Proof. Notice that (b) follows immediately from existence of embedding $\rho : V \to K$: applying scalar extension functor $\rho^* = K \otimes_V -$ to $L_V(n)$ we obtain $L_K(n) = K^n$, a K-vector space of dimension n, and $K^n \simeq K^m$ is possible only for $n = m$.

As to the proof of (a) and (c), it will occupy the rest of this subsection. We are going to obtain matrix descriptions of objects involved in (a) and (c), reduce (c) to (a) and prove (a). Therefore, we fix K and V as above until the end of this subsection.

10.4.3. (Finitely generated projective modules over any Σ.) Let P be a finitely generated projective module over an arbitrary algebraic monad Σ. Fix any finite system $f_1, \ldots, f_n$ of generators of P, or equivalently, a strict epimorphism $\varphi : L_\Sigma(n) \twoheadrightarrow P$, such that $\varphi(e_i) = f_i$, where $e_i = \{i\}$ denotes the i-th basis element of $L_\Sigma(n)$. Since P is projective, φ admits a section $\sigma : P \to L_\Sigma(n)$, and $p := \sigma\varphi \in \mathrm{End}_\Sigma(L_\Sigma(n)) = M(n; \Sigma) = \Sigma(n)^n$ is a projector ($p^2 = p$) with $P = \mathrm{Coker}(p, \mathrm{id}) = \mathrm{Ker}(p, \mathrm{id}) = p(L_\Sigma(n))$. Conversely, if $p \in \mathrm{End}_\Sigma(L_\Sigma(n))$ is a projector, then the corresponding direct factor P of $L_\Sigma(n)$ is projective and finitely generated, being a strict quotient of $L_\Sigma(n)$. Therefore, any finitely generated projective Σ-module P with n generators can be described by means of a projector $p \in M(n, \Sigma) = \Sigma(n)^n$, and any such projector determines such a projective Σ-module with n fixed generators.

10.4.4. (Constant perfect cofibrations over any Σ.) Similarly, let $f : M \to N$ be a constant perfect cofibration of Σ-modules. According to **10.2.12**, such an f is a fixed-source retract of a morphism of the form $i_M : M \to M \oplus L_\Sigma(n)$, i.e. there are Σ-homomorphisms $\sigma : M \oplus L_\Sigma(n) \twoheadrightarrow N$ and $j : N \to M \oplus L_\Sigma(n)$, such that $\sigma j = \mathrm{id}_N$, $jf = i_M$ and $\sigma i_M = j$. Putting $p := j\sigma$, we obtain a projector $p = p^2 \in \mathrm{End}_\Sigma(M \oplus L_\Sigma(n))$, such that $pi_M = i_M$; conversely, any such projector p defines a direct factor N of $M \oplus L_\Sigma(n)$, such that $M \to N$ is a retract of i_M, i.e. a constant perfect cofibration with source M. In this way we obtain a description of constant perfect cofibrations in terms of projectors as well.

Notice that $\sigma : M \oplus L_\Sigma(n) \twoheadrightarrow N$ is necessarily a strict epimorphism, i.e. N is generated by M and elements $u_i := \sigma(e_i)$, $1 \leq i \leq n$. Conversely, let $f : M \to N$ be a constant perfect cofibration, and $u_1, \ldots, u_m \in N$ be any system of elements which generates N together with $f(M)$, thus defining a strict epimorphism $\sigma' : M \oplus L_\Sigma(m) \twoheadrightarrow N$. Recall that any constant perfect cofibration is strongly cofibrant and in particular has the LLP with respect to strict epimorphisms (i.e. surjective maps) of Σ-modules; applying this to f and σ', we see that f is a fixed-source retract of $M \to M \oplus L_\Sigma(m)$.

10.4.5. (Projective modules over V.) Now let P be a finitely generated projective module over V. Choose a system of generators $u_1, \ldots, u_n$ of P with *minimal* $n \geq 0$. According to the above considerations, this yields a description of P as a direct factor (retract) of $L_V(n)$, given by a certain projector $a = a^2 \in \mathrm{End}_V(L_V(n)) = M(n; V)$. We will ultimately show that the minimality of n implies $a = \mathrm{id}$, i.e. $P \cong L_V(n)$, thus proving **10.4.2**,(a).

Since $V \subset K$, $M(n; V) \subset M(n; K) = K^{n\times n}$, i.e. a is given by a certain matrix $(a_{ij})_{1\leq i,j\leq n}$ with coefficients in K, such that $a(e_j) = \sum_i a_{ij} e_i$. Denote by $\|\cdot\|$ the L_1-norm in K^n, i.e. $\|\lambda_1 e_1 + \cdots + \lambda_n e_n\| = |\lambda_1| + \cdots + |\lambda_n|$. Then $V(n)$ can be identified with $\{\lambda \in K^n : \|\lambda\| \leq 1\}$, and a matrix $a \in M(n; K)$ lies in $M(n; V) = V(n)^n$ iff all $a(e_j)$ lie in $V(n)$, i.e. iff

$$\sum_{i=1}^{n} |a_{ij}| \leq 1, \quad \text{for all } 1 \leq j \leq n. \tag{10.4.5.1}$$

We fix P and a matrix $a = a^2$ as above, and put $u_j := a(e_j)$; these elements generate a V-submodule of $L_V(n)$ isomorphic to P, which will be usually identified with P. Notice that elements u_j are identified then with our original system of generators of P.

10.4.6. (Constant perfect cofibrations over V.) Similarly, let $f : M \to N$ be a constant perfect cofibration over V. Choose a system of generators u'_1, $\ldots$, $u'_n \in N$ of N "over M" (i.e. u'_i together with $f(M)$ generate N) with a *minimal* $n \geq 0$. Then f is a retract of $i_M : M \to M \oplus L_V(n)$, given by a certain projector $p = p^2 \in \mathrm{End}_V(M \oplus L_V(n))$, such that $pi_M = i_M$. Denote by a the composite map $L_V(n) \to M \oplus L_V(n) \xrightarrow{p} M \oplus L_V(n) \to 0 \oplus L_V(n) = L_V(n)$, or equivalently, the map induced by p on strict quotient $(M \oplus L_V(n))/M \cong L_V(n)$; clearly, $a = a^2$ is a projector in $\mathrm{End}_V(L_V(n))$ defining projective module P, the cofiber of f.

We will ultimately show that $p = \mathrm{id}_M \oplus a$, hence f will be identified with $M \to M \oplus P$; this will enable us to deduce **10.4.2**,(c) from **10.4.2**,(a).

Let us denote $u'_j := p(e_j) \in M \oplus L_V(n)$; these elements can be still identified with our original generators of N, considered here as a submodule of $M \oplus L_V(n)$. The projections $u_j := a(e_j)$ of u'_j to $L_V(n)$ will then generate P.

10.4.7. (Elements of direct sums over V.) Let M and M' be two arbitrary V-modules. Since $M \oplus M'$ is generated by $M \cup M'$, any element $z \in M \oplus M'$ can be written in form $z = t(x_1, \ldots, x_m, y_1, \ldots, y_n)$ for some $t \in V(m+n)$, $x_i \in M$, $y_j \in M'$, m, $n \geq 0$. Using definition of V we obtain

$$z = \sum_i \lambda_i x_i + \sum_j \mu_j y_j, \quad \text{where } \sum_i |\lambda_i| + \sum_j |\mu_j| \leq 1,\ x_i \in M,\ y_j \in M'. \tag{10.4.7.1}$$

Put $\lambda := \sum_i |\lambda_i|$, $\mu := \sum_j |\mu_j|$, $x := \sum_i (\lambda^{-1}\lambda_i) x_i \in M$, $y := \sum_j (\mu^{-1}\mu_j) y_j \in M'$; if $\lambda = 0$, then all $\lambda_i = 0$, so we put $x := 0$, and similarly $y = 0$ if $\mu = 0$. Then $|\lambda| + |\mu| = \lambda + \mu \leq 1$, and $z = \lambda x + \mu y$, i.e. *any element z of $M \oplus M'$ can be written in form $\lambda x + \mu y$ for $x \in M$, $y \in M'$, $|\lambda| + |\mu| \leq 1$.* This decomposition is not unique; however, elements $\lambda x \in M$ and $\mu y \in M'$ are completely determined by z, being its images under projections $M \oplus M' \to M$ and $M \oplus M' \to M'$.

This is applicable in particular to $z \in M \oplus L_V(n)$. In this case we put $\|z\| := \|\mu y\| \leq 1$. This number is well-defined, $\|z\| = 0$ iff $z \in M$, and $\|z\| = 1$ implies $|\mu| \geq 1$ since $\|y\| \leq 1$, y being an element of $L_V(n) = V(n) \subset K^n$, hence $|\mu| = 1$, $\lambda = 0$ in view of $|\lambda| + |\mu| \leq 1$, i.e. $\|z\| = 1$ *implies* $\lambda = 0$ *and* $z \in L_V(n)$.

10.4.8. (First consequence of minimality of n.) Now let us return back to the situation of **10.4.5**, i.e. we still have a projective V-module P, determined by a projector $a = a^2 \in M(n; V) \subset M(n; K)$ with minimal possible n. Put $I := \{i \in \mathbf{n} : \|u_i\| = 1\}$, where $u_i = a(e_i)$ as before.

Let $x \in K^n$ be any vector. Then $\|a(x)\| = \sum_i |\sum_j a_{ij} x_j| \leq \sum_i \sum_j |a_{ij}| \cdot |x_j| = \sum_j |x_j| \cdot (\sum_i |a_{ij}|) \leq \sum_j |x_j| = \|x\|$ since $\sum_i |a_{ij}| = \|a(e_j)\| \leq 1$ for any j. Now suppose that we have an equality: $\|a(x)\| = \|x\|$. This implies in particular an equality in $\sum_j |x_j| \cdot (\sum_i |a_{ij}|) \leq \sum_j |x_j|$, which is possible only if for any j we have either $x_j = 0$ or $\sum_i |a_{ij}| = 1$, i.e. $j \in I$. Therefore, *if* $\|a(x)\| = \|x\|$, *then* $x_i = 0$ *for* $i \notin I$.

Since $a^2 = a$, we have $a(u_j) = a^2(e_j) = a(e_j) = u_j$ for any j, and in particular $\|a(u_j)\| = \|u_j\|$, so by the above observation $a_{ij} = 0$ for $i \notin I$ and any j. This means that u_j lies in the V-submodule generated by $\{e_i\}_{i\in I}$; applying a we see that any $u_j = a(u_j)$ lies in the V-span of $\{u_i = a(e_i)\}_{i\in I}$, i.e. P is generated already by $\{u_i\}_{i\in I}$. Since n was chosen to be minimal possible, we must have $I = \mathbf{n}$, i.e.

$$\|u_i\| = \|a(e_i)\| = \sum_{j=1}^{n} |a_{ji}| = 1 \quad \text{for all } 1 \leq i \leq n. \tag{10.4.8.1}$$

10.4.9. (Consequence for constant perfect cofibrations.) Now let $f : M \to N$ be a constant perfect cofibration, $u'_1, \ldots, u'_n$ be a minimal system of relative generators of N over M as before, and define $p = p^2 \in \mathrm{End}_V(M \oplus L_V(n))$, $a = a^2 \in \mathrm{End}_V(L_V(n))$, P as in **10.4.6**. Then $u'_i = p(e_i) \in N \subset M \oplus L_V(n)$, hence it can be written in form

$$u'_i = \lambda_i v_i + \mu_i w_i, \quad \text{where } |\lambda_i| + |\mu_i| \leq 1,\ v_i \in M,\ w_i \in L_V(n). \tag{10.4.9.1}$$

Furthermore, $\mu_i \neq 0$, otherwise u'_i would lie in M and we could remove it from our system of relative generators, thus obtaining a contradiction with minimality of n, and $\mu_i w_i$ equals $u_i = a(e_i)$, the projection of $u'_j \in M \oplus L_V(n)$ to $L_V(n)$.

Now put $I := \{i : \|u_i\| = 1\} = \{i : \|u'_i\| = 1\}$ as before. For any i we have $a(u_i) = u_i$ and $u_i = \mu_i w_i$ for $\mu_i \neq 0$, hence $a(w_i) = w_i$ and $\|a(w_i)\| = \|w_i\|$ as well, and by **10.4.8** we obtain $(w_i)_j = 0$ for $j \notin I$, i.e. $w_i = \sum_{j\in I}(w_i)_j e_j$, and $\sum_j |(w_i)_j| \leq 1$ since $w_i \in L_V(n)$. Now $p(w_i) = \sum_{j\in I}(w_i)_j u'_j$ belongs to the V-submodule N' of $M \oplus L_V(n)$ generated by M and $\{u'_j\}_{j\in I}$, hence the same is true for $u'_i = p(u'_i) = p(\lambda_i v_i + \mu_i w_i) = \lambda_i v_i + \mu_i p(w_i)$, taking into account that $p|_M = \mathrm{id}_M$ and $|\lambda_i| + |\mu_i| \leq 1$. In other words, $\{u'_i\}_{i\in I}$ is a smaller system of relative generators of N over M unless $I = \mathbf{n}$. Since n was supposed to be minimal, this means $\|u_i\| = \|u'_i\| = |\mu_i| \cdot \|w_i\| = 1$ for all i, hence $|\mu_i| = 1$ and $\lambda_i = 0$, i.e. *all* u'_i *actually lie in* $L_V(n) \subset M \oplus L_V(n)$. This means exactly that $p = \mathrm{id}_M \oplus a$, i.e. *any constant perfect cofibration over* V *is of the form* $M \to M \oplus P$ *for a finitely generated projective* P.

Therefore, we have reduced **10.4.2**,(c) to **10.4.2**,(a) as promised before, so it remains to show the latter statement.

10.4.10. (Finite Markov chains defined by projective V-modules.) Now let us forget about perfect cofibrations and deal with a projective V-module P with a minimal system of generators $u_1, \ldots, u_n$, given by a matrix $a = a^2 = (a_{ij}) \in M(n;V) \subset M(n;K)$ as in **10.4.5**. We have already seen that minimality of n implies $\|u_j\| = \sum_i |a_{ij}| = 1$ for all j. Put $p_{ij} := |a_{ij}|$, $0 \leq p_{ij} \leq 1$, thus obtaining a matrix $p = (p_{ij}) \in M(n; \tilde{\mathbb{Q}}_+) \subset M(n; \mathbb{R})$. We claim that p is still a projector, i.e. $p^2 = p$. Indeed, $a^2 = a$ means $a_{ik} = \sum_j a_{ij}a_{jk}$, hence $p_{ik} = |a_{ik}| \leq \sum_j |a_{ij}| \cdot |a_{jk}| = \sum_j p_{ij}p_{jk}$. On the other hand, $1 = \sum_i p_{ik} \leq \sum_{i,j} p_{ij}p_{jk} = \sum_j p_{jk} = 1$, hence all inequalities under consideration have to be equalities, meaning that $p_{ik} = \sum_j p_{ij}p_{jk}$, i.e. $p^2 = p$ as claimed.

Notice that one can associate to $p = (p_{ij})$ a finite Markov chain with n states by putting the probability of going from state j into state i in one step equal to p_{ij}. Condition $\sum_i p_{ij} = 1$ now means that the sum of probabilities to end up into some state starting from state j equals one, and $p^2 = p$ means that the probability of going from i into j in two steps equals the probability to go from i into j in one step, so we've got some very special Markov chains here.

We consider also the directed graph Γ with n vertices, such that the edge $i \to j$ belongs to Γ iff $p_{ij} > 0$, i.e. iff $a_{ij} \neq 0$.

10.4.11. (Transitivity of Γ.) We claim that *if* $p_{ij} > 0$ *and* $p_{jk} > 0$*, then* $p_{ik} > 0$. Indeed, we know that all $p_{ij} \geq 0$, and if $p_{ij} > 0$ and $p_{jk} > 0$, then at least one summand in $p_{ik} = \sum_j p_{ij}p_{jk}$ is strictly positive, and the others are non-negative, hence their sum $p_{ik} > 0$.

10.4.12. (Reflexivity of Γ.) Another immediate consequence of minimality of n is that *for all i the edge $i \to i$ belongs to* Γ*, i.e.* $a_{ii} \neq 0$ *and* $p_{ii} > 0$. Indeed, if $a_{ii} = 0$, then u_i is a V-linear combination of elements $\{e_j\}_{j \neq i}$, hence $u_i = a(u_i)$ is a V-linear combination of $\{u_j\}_{j \neq i}$, hence $\{u_j\}_{j \neq i}$ already generate P as a V-module. This contradicts the minimality of n.

10.4.13. (Connectedness of Γ.) We can suppose that Γ is connected. Indeed, otherwise we can split our index set $\mathbf{n}$ into $S \sqcup T$, such that $a_{ij} = 0$ unless $(i, j) \in S \times S \cup T \times T$. Renumbering our generators in such a way that all elements of S are listed before all elements of T, we obtain a decomposition $a = a' \oplus a''$, where a' and a'' are two projectors of smaller size, defining a decomposition $P = P' \oplus P''$; now it would be sufficient to show that both P' and P'' are free to conclude that P is free, so we proceed by an induction in n, and we have to consider only the case of a connected Γ.

10.4.14. (Strong connectedness of Γ.) Now we assume Γ to be connected and show that it can be even supposed to be *strongly connected*, i.e. $p_{ij} > 0$ for any i and j. (This is equivalent to the ergodicity of the corresponding Markov chain.) Indeed, for any $j \in \mathbf{n}$ consider $S_j := \{i : p_{ij} > 0\}$. Reflexivity of Γ means $j \in S_j$, and transitivity means $S_k \subset S_j$ for any $k \in S_j$. Now choose i_0, such that $|S_{i_0}|$ is minimal possible, and put $S := S_{i_0}$. Clearly, $S \neq \varnothing$ since

$i_0 \in S$, and $S_i \subset S$ for any $i \in S$ implies $S_i = S$ for any $i \in S$ in view of the minimality of $|S|$. In other words, for any i and $j \in S$ we have $p_{ij} > 0$, and for $i \notin S$, $j \in S$ we have $p_{ij} = 0$. If $S = \mathbf{n}$, Γ is strongly connected and we are done. Otherwise let's reorder our indices in $\mathbf{n}$ so that $S = \mathbf{k} = \{1, 2, \ldots, k\}$, $0 < k < n$. Since $a_{ij} = 0$ for $i > k$, $j \leq k$, we see that the matrix a is block-diagonal: $a = \left(\begin{smallmatrix} a' & c \\ 0 & a'' \end{smallmatrix}\right)$. Clearly $a' = (a')^2$, and $a' \in M(k; V)$ since the sums of valuations of elements taken along columns of a' coincide with such sums computed along corresponding columns of a. Similarly, $a'' = (a'')^2$ and $a'' \in M(n-k; V)$. Denote by P' and P'' projective V-modules defined by a' and a''. Then (a', a) describe $f : P' \to P$ as a retract of $L_V(k) \to L_V(n)$, hence $f : P' \to P$ is a perfect cofibration between projective V-modules with cofiber equal to P''. Clearly the $(u_i)_{1 \leq i \leq k}$ generate P' as a V-module, and the $(u_i)_{k < i \leq n}$ constitute a system of relative generators of P over P'. Since the union of any system of generators of P' and any system of relative generators of P over P' consititutes a system of generators of P, minimality of n for P implies minimality of k for P' and of $n - k$ for P relatively to P'. We have already shown in **10.4.9** that in this situation we have $P = P' \oplus P''$; since P' is generated by $k < n$ elements and P'' by $n - k < n$ elements, they are free by induction hypothesis, hence P is also free, and we are done.

10.4.15. (All columns of p are equal.) Now we assume Γ to be strongly connected and prove that all columns of p are equal, i.e. $p_{ij} = p_{ii} =: \delta_i > 0$ for all i and j. First of all, denote by $v \in \mathbb{R}^n$ the vector with all components equal to 1. Then $p^t v = v$: indeed, $(p^t v)_j = \sum_i p_{ij} v_i = \sum_i p_{ij} = 1$. Let $x \in \mathbb{R}^n$ be another vector with property $p^t x = x$. We claim that x *is necessarily proportional to v, i.e. all components x_i of x are equal.* Indeed, put $m := \min_i x_i$; replacing x with $x - mv$, we may assume $m = 0$, i.e. all $x_i \geq 0$, and $x_k = 0$ for at least one k. We are going to show that $x = 0$. Indeed, we know that all $x_i \geq 0$, $x_k = 0$, all $p_{ik} > 0$ by strong connectedness of Γ, and $x_k = \sum_i p_{ik} x_i$. This is possible only if all $x_i = 0$, i.e. if $x = 0$.

Now let's fix any index i and put x equal to the transpose of i-th row of p: $x_j := p_{ij}$. Since $p^2 = p$, we have $p^t x = x$, hence all components of x have to be equal, i.e. $p_{ij} = p_{ii}$ for all j and i as claimed.

10.4.16. (Renormalization of a.) Up to now we haven't used strict convexity, but only relation "$|x| \in \tilde{\mathbb{Q}}_+$ for all $x \in K$". Now we are going to use strict convexity. For any x, $y \in K^*$ let's write $x \sim y$ and say that x *and* y *are proportional* if $yx^{-1} \in \tilde{\mathbb{Q}}_+$. Strict convexity now asserts that $|x+y| = |x|+|y|$ implies $x \sim y \sim x + y$. An easy induction shows that $|x_1 + \cdots + x_k| = |x_1| + \cdots + |x_k|$ for $x_i \in K^*$ implies $x_i \sim x_1 + \cdots + x_k$ as well.

Consider now equality $a_{ik} = \sum_j a_{ij} a_{jk}$. Since $p_{ik} = \sum p_{ij} p_{jk}$ as well, and $|a_{ij}| = p_{ij} > 0$ by strong connectedness of Γ, we can apply strict convexity and obtain $a_{ik} \sim a_{ij}a_{jk}$ for any i, j, k. In particular, $a_{ii} \sim a_{ii}^2$, hence $1 \sim a_{ii}$, i.e. $a_{ii} \in \tilde{\mathbb{Q}}_+$; we'll write this as "$a_{ii} > 0$" and "$a_{ii} = |a_{ii}| = p_{ii}$".

Put $\varepsilon_i := a_{1i}/|a_{1i}|$. Clearly, $|\varepsilon_i| = 1$, hence ε_i is invertible in V, so we can replace our original system of generators u_i by $\varepsilon_i u_i$. This amounts to replacing a with its conjugate a' by the diagonal matrix with entries ε_i, i.e. $a'_{ij} = \varepsilon_i a_{ij} \varepsilon_j^{-1}$. This matrix a' describes the same projective module P, and we have $|a'_{ij}| = |a_{ij}| = p_{ij}$. On the other hand, obviously $\varepsilon_1 = 1$ and $a'_{1j} = |a_{1j}| = p_{1j} \in \tilde{\mathbb{Q}}_+$ by construction. Therefore, replacing a with a', we may assume all $a_{1i} \sim 1$. Since $a_{1i}a_{ij} \sim a_{1j}$ for any i and j, we obtain $a_{ij} \sim 1$, i.e. $a_{ij} \in \tilde{\mathbb{Q}}_+$, or, equivalently, $a_{ij} = |a_{ij}| = p_{ij}$, i.e. we can identify a with p. In particular, all columns of a are now equal.

10.4.17. (End of proof.) We have just shown that we can assume all columns of a to be equal, i.e. $u_1 = u_2 = \cdots = u_n$. Since $\{u_i\}_{1 \leq i \leq n}$ was supposed to be a minimal system of generators of P, this is possible only if $n = 1$. Then $a_{11} = 1$ and $P = L_V(1)$ is indeed free as claimed in **10.4.2**.

10.4.18. (Minimal number of generators.) Notice that the number n of elements in a minimal system of generators of a projective V-module P must be actually equal to the rank of P, i.e. it is the only integer $m \geq 0$, for which $P \simeq L_V(m)$. Indeed, uniqueness of m follows from **10.4.2**,(b), $L_V(m)$ obviously admits a system of m generators, and doesn't admit a smaller system of generators just because the K-vector space $L_K(m) = K^m = L_V(m) \otimes_V K$ doesn't.

10.4.19. (Consequence for $K^0(V)$.) An immediate consequence of **10.4.2** is that $\hat{K}^0(V) = K^0(V) = \mathbb{Z}$ for a strictly convex V as above, e.g. $V = \mathbb{Z}_{(\infty)}$, $\mathbb{Z}_\infty$ or $\bar{\mathbb{Z}}_\infty$. Indeed, since any finitely generated projective V-module is free, $\mathbb{Z} = \hat{K}^0(\mathbb{F}_\varnothing) \to \hat{K}^0(V)$ is surjective, and $\hat{K}^0(V) \to K^0(V)$ is surjective for any generalized ring V. On the other hand, we have a map $K^0(V) \to K^0(K) = \mathbb{Z}$ induced by the embedding of V into its "fraction field" K. Since the composite of all these ring homomorphisms equals $\mathrm{id}_\mathbb{Z}$, we see that all of them have to be isomorphisms as claimed.

10.4.20. (Example over $\mathbb{F}_\infty$.) We have just shown that any finitely generated projective $\mathbb{Z}_\infty$-module is free, i.e. $\mathbb{Z}_\infty$ behaves exactly like a local ring in this respect. However, its "residue field" $\mathbb{F}_\infty$ is not so nice, and it actually admits not only non-free modules (cf. **5.7.6**), but also non-free projective modules of finite type! Indeed, consider the endomorphism $p \in \mathrm{End}_{\mathbb{F}_\infty}(L_{\mathbb{F}_\infty}(2))$, given on the basis elements by $p : \{1\} \mapsto \{1\}$, $\{2\} \mapsto ?\{1\}+?\{2\}$ in the notation of **4.8.13**. Then it is easy to see that $p^2 = p$, and the corresponding projective module $P = p(L_{\mathbb{F}_\infty}(2))$ is the $\mathbb{F}_\infty$-submodule of $L_{\mathbb{F}_\infty}(2)$ generated by $\{1\}$ and $?\{1\}+?\{2\}$. It consists of 5 elements, namely, 0, $\{1\}$, $?\{1\}+?\{2\}$, $-\{1\}$ and

$-?\{1\}-?\{2\}$. Since $|\mathbb{F}_\infty(n)| = 3^n$ for any $n \geq 0$ (cf. **4.8.13**), P cannot be a free $\mathbb{F}_\infty$-module. (This example has been communicated to the author by A. Smirnov.)

10.5. (Vector bundles over $\widehat{\operatorname{Spec}\mathbb{Z}}$.) Now we want to obtain a reasonable description of vector bundles $\mathscr{E}$ over the compactified $\widehat{\operatorname{Spec}\mathbb{Z}}$. It will be used in **10.5.22** to compute $\hat{K}^0(\widehat{\operatorname{Spec}\mathbb{Z}})$.

10.5.1. (Compactified $\widehat{\operatorname{Spec}\mathbb{Z}}$.) Recall that $\widehat{\operatorname{Spec}\mathbb{Z}}$, the "compactification of $\operatorname{Spec}\mathbb{Z}$", can be constructed as follows (cf. **7.1**). Fix any integer $N > 1$, put $B_N := \mathbb{Z}[1/N]$ and $A_N := B_N \cap \mathbb{Z}_{(\infty)}$; then B_N is a unary localization both of $\mathbb{Z}$ and A_N since $B_N = \mathbb{Z}[N^{-1}] = A_N[(1/N)^{-1}]$, i.e. $\operatorname{Spec} B_N$ is a principal open subscheme both in $\operatorname{Spec}\mathbb{Z}$ and $\operatorname{Spec} A_N$. Then generalized scheme $\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$ is obtained by gluing $\operatorname{Spec}\mathbb{Z}$ and $\operatorname{Spec} A_N$ along their open subschemes isomorphic to $\operatorname{Spec} B_N$.

This generalized scheme $\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$ does depend on the choice of $N > 1$; however, we get natural "transition morphisms" $f_N^M : \widehat{\operatorname{Spec}\mathbb{Z}}^{(M)} \to \widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$ whenever $N \mid M$ (cf. **7.1.8**), so these $\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$ constitute a projective system of generalized schemes, indexed by the filtered set of integers $N > 1$, ordered by divisibility.

Then we define the "true" compactification $\widehat{\operatorname{Spec}\mathbb{Z}}$ as $\varprojlim_{N>1} \widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$. There are two understandings of this projective limit: we can compute it either in the category of pro-generalized schemes (cf. **7.1.13**), or in the category of generalized ringed spaces (cf. **7.1.14**). Both approaches are equivalent when we consider only finitely presented objects over $\widehat{\operatorname{Spec}\mathbb{Z}}$ (cf. **7.1.17**); since we are now concerned only with perfect simplicial $\mathscr{O}_{\widehat{\operatorname{Spec}\mathbb{Z}}}$-modules, all components of which are vector bundles, hence finitely presented, we may freely choose between these two approaches.

10.5.2. (Finitely presented $\mathscr{O}_{\widehat{\operatorname{Spec}\mathbb{Z}}}$-modules.) Recall that we have an open embedding $\operatorname{Spec}\mathbb{Z} \to \widehat{\operatorname{Spec}\mathbb{Z}}$, as well as "point at infinity" $\hat{\eta} : \operatorname{Spec}\mathbb{Z}_{(\infty)} \to \widehat{\operatorname{Spec}\mathbb{Z}}$ and "generic point" $\hat{\xi} : \operatorname{Spec}\mathbb{Q} \to \widehat{\operatorname{Spec}\mathbb{Z}}$ (cf. **7.1**).

Now let $\mathscr{F}$ be any finitely presented $\mathscr{O}_{\widehat{\operatorname{Spec}\mathbb{Z}}}$-module. Then we put $M_\mathbb{Z} := \Gamma(\operatorname{Spec}\mathbb{Z}, \mathscr{F}|_{\operatorname{Spec}\mathbb{Z}})$, $M_\infty := \mathscr{F}_\infty = \Gamma(\operatorname{Spec}\mathbb{Z}_{(\infty)}, \hat{\eta}^*\mathscr{F})$, and finally $M_\mathbb{Q} := \mathscr{F}_\xi := \Gamma(\operatorname{Spec}\mathbb{Q}, \hat{\xi}^*\mathscr{F})$. Clearly, $M_\mathbb{Z}$ is a finitely presented $\mathbb{Z}$-module, M_∞ is a finitely presented $\mathbb{Z}_{(\infty)}$-module, and $M_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Q} \cong M_\mathbb{Q} \cong M_\infty \otimes_{\mathbb{Z}_{(\infty)}} \mathbb{Q}$, for example because $\hat{\xi} : \operatorname{Spec}\mathbb{Q} \to \widehat{\operatorname{Spec}\mathbb{Z}}$ can be factorized both through $\operatorname{Spec}\mathbb{Z} \subset \widehat{\operatorname{Spec}\mathbb{Z}}$ and $\hat{\eta} : \operatorname{Spec}\mathbb{Z}_{(\infty)} \to \widehat{\operatorname{Spec}\mathbb{Z}}$. Therefore, we obtain a canonical isomorphism of $\mathbb{Q}$-vector spaces $\theta_M : M_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Q} \xrightarrow{\sim} M_\infty \otimes_{\mathbb{Z}_{(\infty)}} \mathbb{Q}$.

In this way we obtain a functor F from the category of finitely presented $\mathscr{O}_{\widehat{\operatorname{Spec}\mathbb{Z}}}$-modules into the category $\mathcal{C}$ of triples $(M_{\mathbb{Z}}, M_{\infty}, \theta_M)$, consisting of a finitely generated $\mathbb{Z}$-module $M_{\mathbb{Z}}$, a finitely presented $\mathbb{Z}_{(\infty)}$-module M_{∞}, and an isomorphism of $\mathbb{Q}$-vector spaces $\theta_M : M_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} M_{\infty} \otimes_{\mathbb{Z}_{(\infty)}} \mathbb{Q}$. Furthermore, we have seen in **7.1.22** that *this functor F is actually an equivalence of categories.* In particular, *the category of finitely presented $\mathscr{O}_{\widehat{\operatorname{Spec}\mathbb{Z}}}$-modules doesn't depend on the localization theory $\mathcal{T}^?$ used to construct spectra involved.*

10.5.3. (Vector bundles over $\widehat{\operatorname{Spec}\mathbb{Z}}$.) Since any vector bundle $\mathscr{E}$ over $\widehat{\operatorname{Spec}\mathbb{Z}}$ is in particular a finitely presented $\mathscr{O}_{\widehat{\operatorname{Spec}\mathbb{Z}}}$-module, we can apply the above considerations, thus obtaining a triple $E = (E_{\mathbb{Z}}, E_{\infty}, \varphi)$, where $E_{\mathbb{Z}}$ is a finitely generated projective (hence free) $\mathbb{Z}$-module, E_{∞} is a finitely generated projective (hence free by **10.4.2**) $\mathbb{Z}_{(\infty)}$-module, and $\varphi : E_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} \to E_{\infty} \otimes_{\mathbb{Z}_{(\infty)}} \mathbb{Q}$ is an isomorphism of their scalar extensions to $\mathbb{Q}$. Conversely, any such triple $(E_{\mathbb{Z}}, E_{\infty}, \varphi)$ defines a finitely presented $\mathscr{O}_{\widehat{\operatorname{Spec}\mathbb{Z}}}$-module, which is locally free over $\operatorname{Spec}\mathbb{Z}$, and free in some neighbourhood of ∞ as well by the inductive limit argument, since $\mathscr{E}_{\infty} = E_{\infty} \xrightarrow{\sim} L_{\mathbb{Z}_{(\infty)}}(r)$ is free. Therefore, $\mathscr{E}$ is a vector bundle. Among other things, we have just shown that *any vector bundle $\mathscr{E}$ over $\widehat{\operatorname{Spec}\mathbb{Z}}$ is locally free*, i.e. *vector bundles over $\widehat{\operatorname{Spec}\mathbb{Z}}$ in the retract and retract-free versions of the theory coincide.* Therefore, we may limit ourselves to the consideration of the retract-free version of vector bundles. Using **10.4.2**,(c), we see that this is equally true for constant perfect cofibrations between vector bundles.

10.5.4. (Rank of a vector bundle over $\widehat{\operatorname{Spec}\mathbb{Z}}$.) Another immediate consequence is that *any vector bundle $\mathscr{E}$ over $\widehat{\operatorname{Spec}\mathbb{Z}}$ has a well-defined rank $r \geq 0$, equal to* $\dim_{\mathbb{Q}} \mathscr{E}_{\xi}$. In other words, $\mathscr{E}$ is locally isomorphic to $\mathscr{O}^{(r)}_{\widehat{\operatorname{Spec}\mathbb{Z}}} = L_{\mathscr{O}_{\widehat{\operatorname{Spec}\mathbb{Z}}}}(r)$. Indeed, this is obvious over $\operatorname{Spec}\mathbb{Z}$, and true over $\mathscr{O}_{\widehat{\operatorname{Spec}\mathbb{Z}},\infty} = \mathbb{Z}_{(\infty)}$ by **10.4.2**, hence true over some neighbourhood of infinity by the inductive limit argument.

10.5.5. (Matrix description of a vector bundle over $\widehat{\operatorname{Spec}\mathbb{Z}}$.) Let $\mathscr{E}$ be a vector bundle over $\widehat{\operatorname{Spec}\mathbb{Z}}$ of rank r. We know that $\mathscr{E}$ is completely determined by $(E_{\mathbb{Z}}, E_{\infty}, \varphi)$, where $E_{\mathbb{Z}} \simeq \mathbb{Z}^r$, $E_{\infty} \simeq \mathbb{Z}^{(r)}_{(\infty)}$, and φ is a $\mathbb{Q}$-isomorphism between $E_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $E_{\infty} \otimes_{\mathbb{Z}_{(\infty)}} \mathbb{Q}$. Another equivalent description: $E = \mathscr{E}_{\xi}$ is a $\mathbb{Q}$-vector space of dimension r, $E_{\mathbb{Z}} \subset E$ is a $\mathbb{Z}$-structure on E (i.e. a lattice in E), and $E_{\infty} \subset E$ is a free $\mathbb{Z}_{(\infty)}$-structure on E (i.e. an octahedron inside E centered at the origin).

Such a situation admits a description in terms of bases and matrices. Namely, let us choose any base $(e_i)_{1\leq i\leq r}$ of $\mathbb{Z}_{(\infty)}$-module E_{∞} (i.e. $\{\pm e_i\}$ are the vertices of octahedron E_{∞}), and any base $(f_i)_{1\leq i\leq r}$ of $\mathbb{Z}$-lattice $E_{\mathbb{Z}} \subset E$.

Since both (e_i) and (f_i) are bases of $\mathbb{Q}$-vector space E, they are related to each other by means of a matrix $A = (a_{ij}) \in GL_r(\mathbb{Q})$:

$$e_i = \sum_{j=1}^{r} a_{ij} f_j, \quad a_{ij} \in \mathbb{Q} \tag{10.5.5.1}$$

10.5.6. (Dependence on the choice of bases.) If we choose another base (f'_i) for $E_{\mathbb{Z}}$, it is related to (f_i) by means of a matrix $B = (b_{ij}) \in GL_r(\mathbb{Z})$, where $f_i = \sum_j b_{ij} f'_j$. Then $e_i = \sum_{j,k} a_{ij} b_{jk} f'_k$, i.e. A is replaced by $A' = AB$. Similarly, if we replace (e_i) by another base (e'_i) of E_∞, these two bases are related to each other by means of $C = (c_{ij}) \in GL_r(\mathbb{Z}_{(\infty)}) = \mathrm{Oct}_r$, where $e'_i = \sum_j c_{ij} e_j$, and we obtain $A' = CA$. Therefore, *multiplying A from the left by matrices from $GL_r(\mathbb{Z}_{(\infty)}) = \mathrm{Oct}_r$ and from the right by matrices from $GL_r(\mathbb{Z})$ doesn't change corresponding vector bundle $\mathscr{E}$.* Conversely, if two matrices A and A' define isomorphic vector bundles, we can assume (by transporting all structure) these vector bundles to coincide, i.e. A and A' would arise from different choices of bases in same modules E_∞ and $E_{\mathbb{Z}}$, hence $A' = CAB$ for some $C \in GL_r(\mathbb{Z}_{(\infty)})$ and $B \in GL_r(\mathbb{Z})$. We have just shown the following statement:

Isomorphism classes of vector bundles $\mathscr{E}$ of rank r over $\widehat{\operatorname{Spec}\mathbb{Z}}$ are in one-to-one correspondence with double cosets $GL_r(\mathbb{Z}_{(\infty)})\backslash GL_r(\mathbb{Q})/GL_r(\mathbb{Z})$.

This description of isomorphism classes of vector bundles in terms of double cosets of matrices is completely similar to the classical description of vector bundles over a projective curve over a field.

10.5.7. (Operations on vector bundles in terms of matrices.) All reasonable operations with vector bundles, such as direct sums, tensor products, symmetric, tensor and exterior powers, can be computed by means of corresponding operations with matrices. For example, let $\mathscr{E}$ and $\mathscr{E}'$ be two vector bundles of rank r and r', respectively, and $(e_i)_{1\leq i\leq r}$, $(f_i)_{1\leq i\leq r}$, $(e'_j)_{1\leq j\leq r'}$, $(f'_j)_{1\leq j\leq r'}$ be the bases used to construct matrices $A \in GL_r(\mathbb{Q})$ and $A' \in GL_{r'}(\mathbb{Q})$ representing $\mathscr{E}$ and $\mathscr{E}'$. Then $\mathscr{E} \otimes_{\mathscr{O}} \mathscr{E}'$ is given by $E_{\mathbb{Z}} \otimes_{\mathbb{Z}} E'_{\mathbb{Z}}$ and $E_\infty \otimes_{\mathbb{Z}_{(\infty)}} E'_\infty$ in $E \otimes_{\mathbb{Q}} E'$, and $(f_i \otimes f'_j)_{i,j}$, $(e_i \otimes e'_j)_{i,j}$ constitute natural bases for these tensor products of free modules. The matrix relating these two bases will be exactly the Kronecker product $A \otimes A'$ of matrices A and A' as claimed.

As to the other operations, they can be dealt with in a completely similar fashion, by considering naturally arising bases. Notice that for the exterior powers we use alternativity of $\mathbb{Z}_{(\infty)}$ (valid since $\mathbb{Z}_{(\infty)} \subset \mathbb{Q}$) to assure that an exterior power of a free module is still free, with the base given in the classical way.

10.5.8. (Picard group of $\widehat{\mathrm{Spec}\,\mathbb{Z}}$.) Let's compute Picard group $\mathrm{Pic}(\widehat{\mathrm{Spec}\,\mathbb{Z}})$ using the description of vector bundles given in **10.5.6**. We get $\mathrm{Pic}(\widehat{\mathrm{Spec}\,\mathbb{Z}}) = GL_1(\mathbb{Z}_{(\infty)})\backslash GL_1(\mathbb{Q})/GL_1(\mathbb{Z}) = \{\pm 1\}\backslash\mathbb{Q}^*/\{\pm 1\} = \mathbb{Q}^*_+$, (cf. also **7.1.36**), i.e. *isomorphism classes of line bundles over* $\widehat{\mathrm{Spec}\,\mathbb{Z}}$ *are parametrized by positive rational numbers.* We have just seen that tensor product of vector bundles corresponds to Kronecker product of matrices; for line bundles this means product of rational numbers, i.e. $\mathrm{Pic}(\widehat{\mathrm{Spec}\,\mathbb{Z}}) \cong \mathbb{Q}^*_+$ *as an abelian group.* Notice that $\mathbb{Q}^*_+ = \mathbb{Z}^{(\mathbb{P})}$, where $\mathbb{P}$ is the set of prime numbers, i.e. $\mathrm{Pic}(\widehat{\mathrm{Spec}\,\mathbb{Z}})$ *is a free abelian group with countably many generators.*

Since $\mathrm{Pic}(\widehat{\mathrm{Spec}\,\mathbb{Z}})$ is often written additively, and $\mathbb{Q}^*_+$ is written multiplicatively, we denote by $\log\mathbb{Q}^*_+$ the group $\mathbb{Q}^*_+$ written in additive form; its elements will be often denoted by $\log\lambda$, $\lambda \in \mathbb{Q}^*_+$ (this is just a formal notation, but we can identify $\log\mathbb{Q}^*_+$ with a subgroup of $\mathbb{R}^+$ by means of the "true" logarithm if we like). Thus $\log\lambda + \log\mu = \log(\lambda\mu)$.

We denote by $\mathscr{O}(\log\lambda) = \mathscr{O}_{\widehat{\mathrm{Spec}\,\mathbb{Z}}}(\log\lambda)$ the line bundle on $\widehat{\mathrm{Spec}\,\mathbb{Z}}$ given by 1×1-matrix (λ). Thus $x \mapsto [\mathscr{O}(x)]$ is an isomorphism $\log\mathbb{Q}^*_+ \xrightarrow{\sim} \mathrm{Pic}(\widehat{\mathrm{Spec}\,\mathbb{Z}})$.

10.5.9. ("Serre twists" on $\widehat{\mathrm{Spec}\,\mathbb{Z}}$.) Given any $\mathscr{O}_{\widehat{\mathrm{Spec}\,\mathbb{Z}}}$-module $\mathscr{F}$ and any $x \in \log\mathbb{Q}^*_+$, we denote by $\mathscr{F}(x)$ the corresponding "Serre twist" $\mathscr{F}\otimes_{\mathscr{O}}\mathscr{O}(x)$. Since $\mathscr{O}(x+y) = \mathscr{O}(x)\otimes_{\mathscr{O}}\mathscr{O}(y)$, we have $\mathscr{F}(x+y) = (\mathscr{F}(x))(y)$. This operation has nearly all the properties one has for Serre twists on a projective line over a field (with the obvious difference that the Picard group of a projective line is $\mathbb{Z}$, not $\log\mathbb{Q}^*_+$).

For example, if $\mathscr{E}$ is a vector bundle of rank r, so is $\mathscr{E}(\log\lambda)$; furthermore, if $\mathscr{E}$ is given by matrix A, then $\mathscr{E}(\log\lambda)$ is given by λA.

10.5.10. (Determinant line bundles.) Let $\mathscr{E}$ be a vector bundle of rank r on $\widehat{\mathrm{Spec}\,\mathbb{Z}}$, given by a matrix A. In the classical case $\mathbb{P}^1_k$ of a projective line over a field we have $c_1(\mathscr{E}) = c_1(\det\mathscr{E})$, and the first Chern class $c_1 : \mathrm{Pic}(\mathbb{P}^1) \to CH^1(\mathbb{P}^1)$ induces an isomorphism, both sides being isomorphic to $\mathbb{Z}$. (In this case $c_1(\mathscr{E})$ is actually the *degree* of $\mathscr{E}$.) Therefore, over $\mathbb{P}^1$ we might identify $c_1(\mathscr{E}) \in CH^1(\mathbb{P}^1)$ with $[\det\mathscr{E}] \in \mathrm{Pic}(\mathbb{P}^1)$. We can try to do the same over $\widehat{\mathrm{Spec}\,\mathbb{Z}}$. In our case $\det\mathscr{E}$ will be given by $\det A$, and its class in $\mathrm{Pic}(\widehat{\mathrm{Spec}\,\mathbb{Z}})$ is given by $\log|\det A| \in \log\mathbb{Q}^*_+$. We'll see later that this is indeed a valid computation of $\deg\mathscr{E} = c_1(\mathscr{E})$ over $\widehat{\mathrm{Spec}\,\mathbb{Z}}$; we'll just remark now that this computation already shows us how the logarithms of volumes appear as arithmetic intersection numbers or arithmetic degrees of vector bundles in Arakelov geometry.

10.5.11. (Homomorphisms of vector bundles.) Since the functor $F : \mathscr{E} \mapsto (\Gamma(\mathrm{Spec}\,\mathbb{Z},\mathscr{E}), \mathscr{E}_\infty, \varphi_{\mathscr{E}})$ of **10.5.2** is an equivalence of categories, we can compute $\mathrm{Hom}_{\widehat{\mathrm{Spec}\,\mathbb{Z}}}(\mathscr{E},\mathscr{E}')$ for any two vector bundles $\mathscr{E}$ and $\mathscr{E}'$ in terms of their matrices. In fact, if $\mathscr{E}$ is given by $E_{\mathbb{Z}}$ and E_∞ inside $\mathbb{Q}$-vector space E, and $\mathscr{E}'$ is given similarly by $E'_{\mathbb{Z}}$ and $E'_\infty \subset E'$, then $\mathrm{Hom}_{\widehat{\mathrm{Spec}\,\mathbb{Z}}}(\mathscr{E},\mathscr{E}')$ can be identified with $\mathbb{Q}$-linear maps $u : E \to E'$, such that $u(E_{\mathbb{Z}}) \subset E'_{\mathbb{Z}}$ and $u(E_\infty) \subset E'_\infty$. If we choose bases as in **10.5.5**, thus obtaining matrices $A \in GL_r(\mathbb{Q})$, $A' \in GL_{r'}(\mathbb{Q})$, then we can consider the (transposed) matrix $U \in M(r,r';\mathbb{Q})$ of u with respect to bases (e_i) and (e'_j), defined by $u(e_i) = \sum_j u_{ij}e'_j$. Then the conditions for U to define a homomorphism $\mathscr{E} \to \mathscr{E}'$ are the following:

- $\sum_j |u_{ij}| \leq 1$ for any $1 \leq i \leq r$ (this is equivalent to $u(E_\infty) \subset E'_\infty$).
- $A^{-1}UA' \in M(r,r';\mathbb{Z})$ (this expresses $u(E_{\mathbb{Z}}) \subset E'_{\mathbb{Z}}$).

10.5.12. (Global sections of $\mathscr{E}$ and Mumford-regular vector bundles.) The above considerations can be applied to $\Gamma(\widehat{\mathrm{Spec}\,\mathbb{Z}},\mathscr{E}) = \mathrm{Hom}_{\mathscr{O}_{\widehat{\mathrm{Spec}\,\mathbb{Z}}}}(\mathscr{O},\mathscr{E})$. We obtain immediately that the $\mathbb{F}_{\pm 1}$-module $\Gamma(\widehat{\mathrm{Spec}\,\mathbb{Z}},\mathscr{E})$ can be identified with $E_{\mathbb{Z}} \cap E_\infty \subset E$. Notice that this intersection is always a finite set, more or less for compactness (or boundedness) reasons.

Similarly, $\Gamma_{\widehat{\mathrm{Spec}\,\mathbb{Z}}}(\mathscr{E}(\log\lambda)) \cong \mathrm{Hom}_{\mathscr{O}}(\mathscr{O}(-\log\lambda),\mathscr{E})$ can be identified with $E_{\mathbb{Z}} \cap \lambda E_\infty$, i.e. $f_{\mathscr{E}} : \log\lambda \mapsto \mathrm{card}(E_{\mathbb{Z}} \cap \lambda E_\infty)$ is the "Hilbert function" of $\mathscr{E}$. We see that Hilbert functions of vector bundles over $\widehat{\mathrm{Spec}\,\mathbb{Z}}$ are closely related to the problem of counting lattice points inside polyhedra (octahedra in this case) with rational vertices. Another interesting thing is that asymptotically $f_{\mathscr{E}}(\log\lambda) = r\log\lambda + \log|\det A| + \log(2^r/r!) + o(1)$ when $\lambda \to +\infty$, i.e. the rank and the "arithmetic degree" $c_1(\mathscr{E}) = \log|\det A|$ determine the asymptotic behavior of $f_{\mathscr{E}}$.

We say that a vector bundle $\mathscr{E}$ is *Mumford-regular* if all the vertices of octahedron $E_\infty \subset E$ lie in lattice $E_{\mathbb{Z}}$, i.e. if $\mathscr{E}$ is given by a matrix A with integer coefficients. (Usually a vector bundle $\mathscr{E}$ over $\mathbb{P}^N$ is said to be Mumford-regular if higher cohomology groups $H^q(\mathbb{P}^N,\mathscr{E}(-q))$ vanish for all $q > 0$; any such $\mathscr{E}$ is known to be generated by its global sections.) However, in our case a Mumford-regular vector bundle needn't be generated by its global sections (i.e. $E_\infty \cap E_{\mathbb{Z}}$ doesn't necessarily generate $E_{\mathbb{Z}}$), as illustrated by the vector bundle of rank 3 defined by

$$A = \begin{pmatrix} 1 & & \\ & 1 & \\ -1 & -1 & 2 \end{pmatrix} \qquad (10.5.12.1)$$

Clearly, for any vector bundle $\mathscr{E}$ we can find a Serre twist $\mathscr{E}(\log\lambda)$, which is Mumford-regular. More precisely, there is a minimal λ_0 with this property (namely, the inverse of the g.c.d. of all entries a_{ij} of A, i.e. the generator of the fractional ideal generated by the a_{ij}), and all other λs are precisely the multiples of λ_0.

10.5.13. (Mumford-antiregularity and homomorphisms into line bundles.) Let's consider the dual problem. Given any vector bundle $\mathscr{E}$ of rank r, we can consider $\operatorname{Hom}_{\mathscr{O}}(\mathscr{E}, \mathscr{O}(\log\lambda))$. It can be identified with the set of linear forms $u \in E_{\mathbb{Z}}^* \subset E^*$, such that $u(E_\infty) \subset [-\lambda, \lambda]$, i.e. $u \in E_{\mathbb{Z}}^* \cap \lambda E_\infty^*$, where E_∞^* is the dual of E_∞, a cube with rational vertices in E^* centered at the origin. In other words, the "dual Hilbert function"

$$p_{\mathscr{E}}(\lambda) := \operatorname{card}\operatorname{Hom}_{\mathscr{O}}(\mathscr{E}, \mathscr{O}(\log\lambda)) = |E_{\mathbb{Z}}^* \cap \lambda E_\infty^*| \tag{10.5.13.1}$$

counts the number of lattice points of $E_{\mathbb{Z}}^*$ inside cubes λE_∞^*. This function has somewhat better properties than the "ordinary Hilbert functions" considered before. For example,

$$p_{\mathscr{E}\oplus\mathscr{E}'}(\lambda) = p_{\mathscr{E}}(\lambda) \cdot p_{\mathscr{E}'}(\lambda) \tag{10.5.13.2}$$

an equality that has no counterpart for "ordinary Hilbert functions". Furthermore, we have $\log p_{\mathscr{E}}(\lambda) = r\log(2\lambda) - \log|\det A| + o(1)$ asymptotically in $\lambda \to +\infty$, without any additional constants, where A is any matrix defining $\mathscr{E}$.

We say that $\mathscr{E}$ is *Mumford-antiregular* if the dual base (e_i^*) of the base (e_i) consisting of vertices of octahedron E_∞, i.e. the centers of faces of cube E_∞^*, lie in dual lattice $E_{\mathbb{Z}}^*$. Clearly, this is equivalent to requiring all coefficients of A^{-1} to be integer.

Of course, there are $\lambda \in \mathbb{Q}_+^*$, such that $\mathscr{E}(-\log\lambda)$ is Mumford-antiregular, and such λ's are multiples of minimal such λ_0, equal to the g.c.d. of coefficients of A^{-1}. In particular, if $\mathscr{E}$ is Mumford-antiregular, so is $\mathscr{E}(-\log\lambda)$ for all "positive" λ, i.e. for all $\lambda \in \mathbb{N}$.

It turns out that for any $\mathscr{E}$ one can find a polynomial $\tilde{p}_{\mathscr{E}} \in \mathbb{Q}[T]$, such that $p_{\mathscr{E}}(\lambda) = \tilde{p}_{\mathscr{E}}(\lambda)$ for all $\lambda \in \mathbb{Q}_+^*$, for which $\mathscr{E}(-\log\lambda)$ is Mumford-antiregular. We'll return to this question later, indicating a method for computation of polynomial $\tilde{p}_{\mathscr{E}}$.

10.5.14. (Matrix description of perfect cofibrations of vector bundles.) Let $q : \mathscr{E}' \to \mathscr{E}$ be a perfect cofibration with cofiber $\mathscr{E}''$, i.e. $0 \to \mathscr{E}' \xrightarrow{q} \mathscr{E} \xrightarrow{\pi} \mathscr{E}'' \to 0$ is a cofibration sequence, and $[\mathscr{E}] = [\mathscr{E}'] + [\mathscr{E}'']$ in $\hat{K}^0(\widehat{\operatorname{Spec}\mathbb{Z}})$. We want to obtain a matrix description of such a situation. Let r be the rank of $\mathscr{E}'$, s be the rank of $\mathscr{E}''$; then the rank of $\mathscr{E}$ equals $r+s$ since we have a short exact sequence of generic fibers. Let us denote $E := \mathscr{E}_\xi$, $E_{\mathbb{Z}} := \Gamma(\operatorname{Spec}\mathbb{Z}, \mathscr{E}) \subset E$ and $E_\infty := \mathscr{E}_\infty$ as before, and similarly for $\mathscr{E}'$ and $\mathscr{E}''$. We identify E' with a $\mathbb{Q}$-vector subspace of E by means of q_ξ;

then E'' can be identified with E/E'. Since $\operatorname{Spec}\mathbb{Z}$ is additive, a cofibration sequence of free $\mathbb{Z}$-modules $0 \to E'_{\mathbb{Z}} \to E_{\mathbb{Z}} \to E''_{\mathbb{Z}} \to 0$ is nothing else than a short exact sequence. Therefore, if $(f'_i)_{1\leq i\leq r}$ is any base of $E'_{\mathbb{Z}}$, $(f''_j)_{1\leq j\leq s}$ any base of $E''_{\mathbb{Z}}$, $f_i := f'_i$ for $1 \leq i \leq r$, and $f_{r+i} \in E_{\mathbb{Z}}$ are any lifts of f''_i to $E_{\mathbb{Z}}$, $1 \leq i \leq s$, then $(f_i)_{1\leq i\leq r+s}$ is a base of $E_{\mathbb{Z}}$. Similarly, $E'_\infty \to E_\infty$ is a perfect cofibration of vector bundles over $\mathbb{Z}_{(\infty)}$, hence by **10.4.2**,(c) it is isomorphic to $L_{\mathbb{Z}_{(\infty)}}(r) \to L_{\mathbb{Z}_{(\infty)}}(r) \oplus L_{\mathbb{Z}_{(\infty)}}(s) = L_{\mathbb{Z}_{(\infty)}}(r+s)$. In other words, we can choose a $\mathbb{Z}_{(\infty)}$-base $(e_i)_{1\leq i\leq r+s}$ of E_∞, such that its first r elements $(e'_i := e_i)_{1\leq i\leq r}$ constitute a base of E'_∞; then the images e''_i of e_{r+i}, $1 \leq i \leq s$, constitute a base of E''_∞.

Now let $A \in GL_{r+s}(\mathbb{Q})$ be the matrix relating (e_i) to (f_i), and define $A' \in GL_r(\mathbb{Q})$ and $A'' \in GL_s(\mathbb{Q})$ similarly. Then A, A', A'' describe vector bundles $\mathscr{E}$, $\mathscr{E}'$ and $\mathscr{E}''$, respectively, and by construction $A = \left(\begin{smallmatrix} A' & 0 \\ * & A'' \end{smallmatrix}\right)$ is block-triangular with diagonal blocks A' and A''. Conversely, if a vector bundle $\mathscr{E}$ admits a description in terms of a block-triangular matrix A of the above form, we obtain morphisms of corresponding vector bundles $\mathscr{E}' \to \mathscr{E} \to \mathscr{E}''$; this sequence is a perfect cofibration sequence over $\operatorname{Spec}\mathbb{Z}$, being just a short exact sequence of free $\mathbb{Z}$-modules there, and a perfect cofibration sequence in some neighbourhood of ∞, since $\mathscr{E}'_\infty \to \mathscr{E}_\infty \to \mathscr{E}''_\infty$ is a perfect cofibration sequence of free $\mathbb{Z}_{(\infty)}$-modules by construction.

Therefore, *perfect cofibrations $\mathscr{E}' \to \mathscr{E}$ between vector bundles and perfect cofibration sequences $\mathscr{E}' \to \mathscr{E} \to \mathscr{E}''$ over $\widehat{\operatorname{Spec}\mathbb{Z}}$ correspond to block-triangular decompositions $A = \left(\begin{smallmatrix} A' & 0 \\ * & A'' \end{smallmatrix}\right)$ of corresponding matrices.* Of course, we obtain such block-triangular decompositions only for special choices of bases for E_∞ and $E_{\mathbb{Z}}$, i.e. they hold only for some special representatives $A \in GL_{r+s}(\mathbb{Q})$ of double coset in $GL_{r+s}(\mathbb{Z}_{(\infty)})\backslash GL_{r+s}(\mathbb{Q})/GL_{r+s}(\mathbb{Z})$. However, A' and A'' might have been chosen arbitrarily.

Proposition 10.5.15 (Canonical representatives for $GL_n(\mathbb{Q})/GL_n(\mathbb{Z})$.) *Any right coset $GL_n(\mathbb{Q})/GL_n(\mathbb{Z})$ contains exactly one matrix $A = (a_{ij}) \in GL_n(\mathbb{Q})$ satisfying following conditions:*

- *A is lower-triangular, i.e. $a_{ij} = 0$ for $i < j$.*
- *A has positive diagonal elements, i.e. $a_{ii} > 0$.*
- *$0 \leq a_{ij} < a_{ii}$ for all $1 \leq j < i \leq n$.*

Proof. (a) First of all, notice that the above conditions are invariant under multiplication of A by any positive rational number γ. Let us choose any matrix A in given right coset; replacing A by γA if necessary, we can assume

$A \in M(n;\mathbb{Z})$, $\det A \neq 0$. Then the right coset of A consists only of matrices with integer coefficients.

(b) We are free to replace A with $A' = AB$ for any $B \in GL_n(\mathbb{Z})$. This amounts to applying the same matrix B to all rows of A, or to performing some linear operations with columns of A. In particular, we are allowed to permute two or more columns, change the sign of a column, and add to a column any integer multiple of another one.

(c) Consider the matrices A from given right coset with $a_{11} > 0$, and choose one with minimal a_{11} (this is possible since $a_{11} \in \mathbb{Z}$). Then all elements a_{1i} of the first row of A must be divisible by a_{11}: otherwise we would subtract from the i-th column a multiple of the first column and make $0 < a_{1i} < a_{11}$; then we would permute the first and the i-th column and obtain a matrix A' in the same right coset with $0 < a'_{11} < a_{11}$. This contradicts the minimality of a_{11}. Therefore, any a_{1i} is divisible by a_{11}; subtracting from the i-th column a_{1i}/a_{11} times the first column, we obtain a matrix A with $a_{11} > 0$, $a_{1i} = 0$ for $1 < i \leq n$.

(d) Now consider the submatrix of A formed by its rows and columns with indices from 2 to n. Using only operations with these columns, i.e. considering the coset of A in $GL_n(\mathbb{Q})/GL_{n-1}(\mathbb{Z})$, we can make $a_{22} > 0$ and $a_{2i} = 0$ for $i > 2$ by the same reasoning as in (c). Furthermore, subtracting from the first column $[a_{21}/a_{22}]$ times the second column, we can make $0 \leq a_{21} < a_{22}$. Notice that all these manipulations do not affect the first row.

(e) Next, we can make $a_{33} > 0$ and $a_{3i} = 0$ for $i > 3$ by manipulations with columns ≥ 3, and then subtract suitable multiples of the third column from the first and the second column so as to have $0 \leq a_{3i} < a_{33}$ for $i < 3$. These operations don't affect the two first rows. We proceed further by induction, making the fourth, $\ldots$, n-th row of A of the form indicated in the proposition. This proves the existence of a representative with required properties.

(f) Now let us show the uniqueness. Let $A = (a_{ij})$ and $A' = (a'_{ij})$ be two matrices with the above properties, and suppose that they lie in the same right coset with respect to $GL_n(\mathbb{Z})$, i.e. $A' = AB$ for some $B \in GL_n(\mathbb{Z})$. First of all, B is lower-triangular with positive diagonal entries, just because A and A' are such. Next, all coefficients of B are integers, hence $b_{ii} \geq 1$, and the product of b_{ii} equals $\det B = \pm 1$, hence all $b_{ii} = 1$, i.e. B is lower-unitriangular, and $a'_{ii} = a_{ii} > 0$ for all i. Now suppose that $A \neq A'$, i.e. $B \neq E$. Then $b_{ij} \neq 0$ for some $i > j$. Choose indices (i,j), such that $b_{ij} \neq 0$, $i > j$, with minimal i. Then $a'_{ij} = \sum_k a_{ik}b_{kj} = a_{ij} + a_{ii}b_{ij}$ since $b_{kj} = 0$ for $k < j$, $b_{jj} = 1$, $b_{kj} = 0$ for $j < k < i$, and $a_{ik} = 0$ for $k > i$. On the other hand, $0 \leq a_{ij} < a_{ii}$ and $0 \leq a'_{ij} < a'_{ii} = a_{ii}$, hence $|a_{ii}b_{ij}| = |a'_{ij} - a_{ij}| < a_{ii}$. This contradicts $b_{ij} \in \mathbb{Z}$, $b_{ij} \neq 0$, q.e.d.

10.5.16. (Vector bundles of rank n.) The above result enables us to choose some canonical representatives $A \in GL_n(\mathbb{Q})$ for the isomorphism class of a vector bundle $\mathscr{E}$ of rank n. Namely, we know (cf. **10.5.6**) that the set of isomorphism classes of vector bundles of rank n over $\widehat{\operatorname{Spec}\mathbb{Z}}$ can be identified with $\mathrm{Oct}_n \backslash GL_n(\mathbb{Q})/GL_n(\mathbb{Z})$, where $\mathrm{Oct}_n = GL_n(\mathbb{Z}_{(\infty)})$ is the octahedral group, and in particular is finite. Since Oct_n is finite, any such double coset is a finite union of right cosets $GL_n(\mathbb{Q})/GL_n(\mathbb{Z})$, and these right cosets can be identified with the set of their canonical representatives in $GL_n(\mathbb{Q})$ by **10.5.15**. Therefore, any isomorphism class of vector bundles canonically corresponds to a finite set of "canonical" matrices as in **10.5.15**.

We would like to choose one canonical matrix from this finite set. In order to do this notice that when we study representatives A of such double cosets as above, we are allowed not only to multiply A from the right by matrices $B \in GL_n(\mathbb{Z})$, i.e. to permute columns of A and add multiples of one column of A to any other, but to multiply A from the left from matrices $C \in \mathrm{Oct}_n$, i.e. by permutation matrices with non-zero entries equal to ± 1. In other words, now we are allowed to permute rows and change their sign as well.

If we would find a canonical way of ordering the rows of A, preserved under multiplication by matrices from $GL_n(\mathbb{Z})$ from the right, we would reorder the rows in this way by means of a matrix from Oct_n, and then all the freedom left would be to change the sign of a row, something that must be immediately compensated by changing the sign in the corresponding column if we insist to mantain all matrices in the canonical form of **10.5.15**.

Denote by $d_i > 0$ the g.c.d. of all elements $(a_{ij})_{1 \leq j \leq n}$ of the i-th row of A. Clearly, the sequence $(d_1, \ldots, d_n)$ doesn't change when we replace A by AB with $B \in GL_n(\mathbb{Z})$. Therefore, "generically" (if all d_i are distinct), we obtain a natural ordering of the rows of A, and we can require $d_1 < d_2 < \cdots < d_n$ in order to make the choice of A more canonical. If not all d_i are distinct, we can still require $d_1 \leq d_2 \leq \cdots \leq d_n$, but now there might be some degree of freedom left: we may still permute rows with equal values of d_i.

10.5.17. (Decomposition of a vector bundle into line bundles.) Let $\mathscr{E}$ be a vector bundle of rank n. According to **10.5.6** and **10.5.15**, $\mathscr{E}$ can be described by a matrix $A \in GL_n(\mathbb{Q})$ of canonical form. In particular, we can suppose A to be lower-triangular with $a_{ii} > 0$. On the other hand, any block-triangular decomposition of A corresponds by **10.5.14** to a perfect cofibration of vector bundles. Since A is completely triangular, we obtain a finite filtration $0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_n = \mathscr{E}$ of $\mathscr{E}$ by vector bundles, such that each $\mathscr{E}_{k-1} \to \mathscr{E}_k$ is a perfect cofibration of vector bundles with cofiber equal to line bundle $\mathscr{O}(\log a_{kk})$.

In particular, $[\mathscr{E}] = \sum_{k=1}^{n}[\mathscr{O}(\log a_{kk})]$, i.e. $\hat{K}^0(\widehat{\operatorname{Spec}\mathbb{Z}})$ *is generated by line bundles.* Furthermore, $\hat{K}^0(\widehat{\operatorname{Spec}\mathbb{Z}}) \to K^0(\widehat{\operatorname{Spec}\mathbb{Z}})$ is surjective by **10.3.22**, $\widehat{\operatorname{Spec}\mathbb{Z}}$ being quasicompact, hence *the same is true for* $K^0(\widehat{\operatorname{Spec}\mathbb{Z}})$.

10.5.18. (Semistable filtration.) If all $d_i = \gcd(a_{i1}, \ldots, a_{in})$ of **10.5.16** are distinct, we can make the above filtration on $\mathscr{E}$ canonical by choosing a canonical representative A in such a way that $d_1 < d_2 < \cdots < d_n$. In general if $d_1 = \cdots = d_{i_1-1} < d_{i_1} = \cdots = d_{i_2-1} < d_{i_2} = \cdots$, then only the sub-filtration consisting of the $\mathscr{E}_{i_k}$ is canonical. Let's say that $\mathscr{E}$ *is semistable of slope* $\log d$ if all $d_i = d$. Then the sub-filtration we've just discussed has semistable cofibers, i.e. is some sort of Harder–Narasimhan filtration. Notice, however, that in general the slope $\log d$ of a semistable $\mathscr{E}$ needn't equal $\deg\mathscr{E}/\operatorname{rank}\mathscr{E}$: all we can say is that d^n is a divisor of $\det A$, i.e. $\deg\mathscr{E} - \operatorname{rank}\mathscr{E}\cdot\log d$ is "positive" (log of a natural number).

Lemma 10.5.19 (Relations between line bundles in $\hat{K}^0(\widehat{\operatorname{Spec}\mathbb{Z}})$.

(a) Let a, b, a', $b' \in \mathbb{Q}_+^*$ be such that $ab = a'b'$. Then

$$[\mathscr{O}(\log a)] + [\mathscr{O}(\log b)] = [\mathscr{O}(\log a')] + [\mathscr{O}(\log b')] \quad \textit{in } \hat{K}^0(\widehat{\operatorname{Spec}\mathbb{Z}}). \tag{10.5.19.1}$$

(b) The map $\log\lambda \mapsto [\mathscr{O}(\log\lambda)] - 1$ is an abelian group homomorphism $\log\mathbb{Q}_+^* = \operatorname{Pic}(\widehat{\operatorname{Spec}\mathbb{Z}}) \to \hat{K}^0(\widehat{\operatorname{Spec}\mathbb{Z}})$.

Proof. (a) First of all, notice that it suffices to prove (10.5.19.1) under the additional assumption $b/b' \in \mathbb{Z}$. Indeed, in the general case we can always find a common divisor $b'' > 0$ of both b and b', and put $a'' := ab/b''$; then it would suffice to show (10.5.19.1) for a, b, a'', b'' and for a', b', a'', b''.

So let us assume $b/b' \in \mathbb{N}$. Consider the line bundle $\mathscr{E}$ of rank 2 over $\widehat{\operatorname{Spec}\mathbb{Z}}$ defined by matrix $A = \left(\begin{smallmatrix} a & \\ b' & b\end{smallmatrix}\right)$. Since this matrix is triangular, we have $[\mathscr{E}] = [\mathscr{O}(\log a)] + [\mathscr{O}(\log b)]$ by **10.5.17**. On the other hand, we are free to multiply A by matrices $C \in \mathrm{Oct}_2$ from the left; taking $C = \left(\begin{smallmatrix} 0 & 1\\ 1 & 0\end{smallmatrix}\right)$, we get $A' = CA = \left(\begin{smallmatrix} b' & b\\ a & \end{smallmatrix}\right)$. Let $A'' = A'B = \left(\begin{smallmatrix} \tilde{b}' & \\ c & \tilde{a}'\end{smallmatrix}\right)$ be the canonical form of A' in the sense of **10.5.15**. Since the row g.c.d.s of A' and A'' must be equal, we get $\tilde{b}' = \gcd(b', b) = b'$; since $b'\tilde{a}' = \det A'' = \pm\det A = \pm ab = \pm a'b'$ and all numbers involved are positive, we have $\tilde{a}' = a'$ as well, i.e. $A'' = \left(\begin{smallmatrix} b' & \\ c & a'\end{smallmatrix}\right)$. Since A'' still defines the same vector bundle $\mathscr{E}$, we get $[\mathscr{E}] = [\mathscr{O}(\log b')] + [\mathscr{O}(\log a')]$ by **10.5.17**, whence (10.5.19.1).

(b) This statement is immediate from (a): we have to check $[\mathscr{O}(\log\lambda + \log\mu)] - 1 = [\mathscr{O}(\log\lambda)] - 1 + [\mathscr{O}(\log\mu)] - 1$, and this is a special case of (a) for $a = \lambda$, $b = \mu$, $a' = \lambda\mu$, $b' = 1$.

10.5.20. (λ-ring $\tilde{A}$ defined by an abelian group A.) Let A be an arbitrary abelian group. Denote by $\tilde{A}$ the ring $\mathbb{Z} \times A$, where the multiplication is determined by the requirement $A^2 = 0$ in $\tilde{A}$, i.e. $(m, x) \cdot (n, y) = (mn, nx + my)$, and in particular $(1, x) \cdot (1, y) = (1, x + y)$. The addition is of course defined componentwise: $(m, x) + (n, y) = (m + n, x + y)$. We identify $\mathbb{Z}$ with subring $\mathbb{Z} \times 0$ in $\tilde{A}$, and A with $0 \times A \subset \tilde{A}$, so we can write $n + x$ instead of (n, x) if we like. Clearly, $\tilde{A}$ is a commutative ring with unity $(1, 0)$. Furthermore, it has a unique pre-λ-structure, such that $\lambda^k(1, x) = 0$ for $k \geq 2$: indeed, uniqueness is evident since elements $(1, x)$ generate $\tilde{A}$ as an abelian group, and writing $(n, x) = (n - 1) + (1, x)$ we get $\lambda_t(n, x) = (1+t)^{n-1}\lambda_t(1, x) = (1+t)^{n-1}(1+t(1+x))$, whence $\lambda^k(n+x) = \binom{n}{k} + \binom{n-1}{k-1}x$ for any $k \geq 1$, and of course $\lambda^0(n + x) = 1$. It is immediate that these formulas do define a pre-λ-ring structure on $\tilde{A}$. Furthermore, this is actually a λ-ring structure (cf. SGA 6 V): indeed, we have to check that $\lambda_t : \tilde{A} \to 1 + \tilde{A}[[t]]^+$ is a λ-homomorphism; it is already an abelian group homomorphism, and elements $1 + x$ generate $\tilde{A}$, so by linearity it is enough to check compatibility of λ_t with λ-operations and multiplication on these generators, where it is evident since $\lambda^k(1 + x) = 0$ for $k > 1$.

Lemma 10.5.21 *Let $X = (X, \mathscr{O}_X)$ be a generalized ringed space. Put $A :=$ $\mathrm{Pic}(X)$, and construct λ-ring $\tilde{A} = \mathbb{Z} \times A$ as above. Suppose that the map $c_1 : A = \mathrm{Pic}(X) \to K^0 := \hat{K}^0(X)$ given by $\mathscr{L} \mapsto [\mathscr{L}] - 1$ is a homomorphism of abelian groups. Then:*

(a) The map $\varphi : \tilde{A} \to K^0$ given by $(n, \mathscr{L}) \mapsto n - 1 + [\mathscr{L}] = n + c_1(\mathscr{L})$ is a ring homomorphism and even a λ-homomorphism.

(b) If K^0 is generated by line bundles, then φ is surjective, and K^0 is a λ-ring.

(c) If both $\mathbb{Z} \to K^0$ and $c_1 : A \to K^0$ are injective, then φ is injective. If in addition K^0 is generated by line bundles, then $\varphi : \tilde{A} \to K^0$ is an isomorphism.

(d) All of the above applies if we put $K^0 := K^0(X)$ instead of $\hat{K}^0(X)$.

Proof. (a) Map φ is an abelian group homomorphism by construction; since the multiplication and λ_t are $\mathbb{Z}$-(bi)linear, it is enough to check compatibility of φ with these maps on any system of generators of $\tilde{A}$ as an abelian group. Elements $1 + x \in \tilde{A}$ where $x = \mathrm{cl}\,\mathscr{L}$ constitute such a system of generators, we have $\varphi(1 + x) = [\mathscr{L}]$, $(1 + x)(1 + x') = 1 + (x + x')$ and $\varphi(1 + x + x') =$ $[\mathscr{L} \otimes \mathscr{L}'] = [\mathscr{L}][\mathscr{L}'] = \varphi(1 + x)\varphi(1 + x')$ whence the compatibility with multiplication. As to the λ-operations, we notice that $s^n\mathscr{L} = \mathscr{L}^{\otimes n}$ for any $n \geq 0$, whence $s_t([\mathscr{L}]) = \sum_{n\geq 0}[\mathscr{L}]^n t^n$ and $\lambda_t([\mathscr{L}]) = s_{-t}([\mathscr{L}])^{-1} = 1+[\mathscr{L}]t$; since $\lambda_t(1+x) = 1 + (1+x)t$ in $1 + \tilde{A}[[t]]^+$, we obtain compatibility with λ-operations as well.

(b) Since $\varphi(\tilde{A})$ contains all $[\mathscr{L}]$, the first statement is obvious. Now $\tilde{A}$ is a λ-ring, K^0 is a pre-λ-ring and $\varphi : \tilde{A} \to K^0$ is a surjective λ-homomorphism, hence all necessary relations between λ-operations are fulfilled in K^0 as well, i.e. K^0 is a λ-ring.

(c) Let $\mathfrak{a} \subset \tilde{A}$ be the kernel of φ. Suppose that $\mathfrak{a} \neq 0$, and let $0 \neq (n,x) \in \mathfrak{a}$. Clearly $n \neq 0$ since $\varphi(0,x) = c_1(x) \neq 0$ for $x \neq 0$. If $n < 0$, then $-(n,x) = (-n,-x)$ also lies in $\mathfrak{a}$, so we can assume $n > 0$. Since φ is a λ-homomorphism, $\mathfrak{a}$ is a λ-ideal, hence $\lambda^n(n,x) \in \mathfrak{a}$. Clearly $\lambda^n(n,x) = (1,y)$ for some $y \in A$. Now $(1,0) = (1,-y)(1,y)$ also lies in ideal $\mathfrak{a}$, hence $\varphi(1) = \varphi(1,0) = 0$. This contradicts injectivity of $\mathbb{Z} \to K^0$.

(d) Obvious since we used only that K^0 is pre-λ-ring and the formulas for multiplication and symmetric powers of images of line bundles in K^0, valid for $K^0(X)$ as well.

Theorem 10.5.22 *Let* $A = \log\mathbb{Q}_+^*$, *denote by* $\tilde{A} = \mathbb{Z} \times \log\mathbb{Q}_+^*$ *the* λ*-ring constructed in* **10.5.20**, *and define* $\varphi : \tilde{A} \to \hat{K}^0(\widehat{\operatorname{Spec}\mathbb{Z}})$ *by* $n + \log\lambda \mapsto n - 1 + [\mathscr{O}(\log\lambda)]$. *Then* φ *is an isomorphism of* λ*-rings, and in particular* $\hat{K}^0(\widehat{\operatorname{Spec}\mathbb{Z}})$ *and* $K^0(\widehat{\operatorname{Spec}\mathbb{Z}})$ *are* λ*-rings.*

Proof. (a) Recall that $\operatorname{Pic}(\widehat{\operatorname{Spec}\mathbb{Z}}) = \log\mathbb{Q}_+^* = A$, and the isomorphism $A \to \operatorname{Pic}(\widehat{\operatorname{Spec}\mathbb{Z}})$ is given by $\log\lambda \to [\mathscr{O}(\log\lambda)]$, cf. **10.5.8**. Furthermore, the map $c_1 : A \to K^0 := \hat{K}^0(\widehat{\operatorname{Spec}\mathbb{Z}})$ given by $\log\lambda \mapsto [\mathscr{O}(\log\lambda)] - 1$ is a homomorphism by **10.5.19**, so we are in position to apply **10.5.21**. Thus $\varphi : \tilde{A} \to K^0$ is a λ-homomorphism. It is surjective by **10.5.21**,(b) and **10.5.17**. Since $\widehat{\operatorname{Spec}\mathbb{Z}}$ is quasicompact, $\hat{K}^0(\widehat{\operatorname{Spec}\mathbb{Z}}) \to K^0(\widehat{\operatorname{Spec}\mathbb{Z}})$ is also surjective; this already implies that pre-λ-rings $\hat{K}^0(\widehat{\operatorname{Spec}\mathbb{Z}})$ and $K^0(\widehat{\operatorname{Spec}\mathbb{Z}})$ are actually λ-rings.

(b) Now it remains to show injectivity of φ. According to **10.5.21**,(c), it would suffice to show injectivity of $\mathbb{Z} \to K^0$ and $c_1 : A \to K^0$. The injectivity of the first map follows from existence of map $\operatorname{Spec}\mathbb{Q} \to \widehat{\operatorname{Spec}\mathbb{Z}}$, which induces a map $K^0 \to \hat{K}^0(\mathbb{Q}) = \mathbb{Z}$. In order to show injectivity of c_1 we construct a degree map $\deg : \hat{K}^0(\widehat{\operatorname{Spec}\mathbb{Z}}) \to \log\mathbb{Q}_+^*$, such that $\deg\circ c_1 = \mathrm{id}$. Namely, for any vector bundle $\mathscr{E}$ (of some rank n) over $\widehat{\operatorname{Spec}\mathbb{Z}}$, described by a matrix A, we put $\deg\mathscr{E} := \log|\det A|$. This number is well-defined since matrices from $GL_n(\mathbb{Z})$ and $\mathrm{Oct}_n = GL_n(\mathbb{Z}_{(\infty)})$ have determinants ± 1. Next, for any cofibration sequence $\mathscr{E}' \to \mathscr{E} \to \mathscr{E}''$ we have $A = \begin{pmatrix} A' & \\ * & A'' \end{pmatrix}$ for a suitable choice of bases by **10.5.14**, hence $\log|\det A| = \log|\det A'| + \log|\det A''|$, i.e. $\deg\mathscr{E} = \deg\mathscr{E}' + \deg\mathscr{E}''$. In other words, $\mathscr{E} \mapsto \deg\mathscr{E}$ is an additive function on vector bundles, hence it induces an abelian group homomorphism $\deg : \hat{K}^0(\widehat{\operatorname{Spec}\mathbb{Z}}) \to \log\mathbb{Q}_+^*$, $\hat{K}^0(\widehat{\operatorname{Spec}\mathbb{Z}})$ being the abelian group generated by elements $[\mathscr{E}]$ and relations $[\mathscr{E}] = [\mathscr{E}'] + [\mathscr{E}'']$, and obviously $\deg\mathscr{O}(\log\lambda) = \log\lambda$, i.e. $\deg\circ c_1 = \mathrm{id}$ as claimed, q.e.d.

10.5.23. ($K^0(\widehat{\operatorname{Spec}\mathbb{Z}})$.) We have just shown that $\hat{K}^0(\widehat{\operatorname{Spec}\mathbb{Z}}) = \mathbb{Z}\oplus\log\mathbb{Q}_+^*$, and that $K^0(\widehat{\operatorname{Spec}\mathbb{Z}})$ is a quotient of this λ-ring. We would like to prove that $K^0(\widehat{\operatorname{Spec}\mathbb{Z}})$ is also isomorphic to $\mathbb{Z}\oplus\log\mathbb{Q}_+^*$. Since $\mathbb{Z}\to K^0(\widehat{\operatorname{Spec}\mathbb{Z}})$ is injective for the same reason as before, all we have to do for this is to construct a degree map $\deg : K^0(\widehat{\operatorname{Spec}\mathbb{Z}}) \to \log\mathbb{Q}_+^*$. It is quite easy to define $\deg X$ for any perfect simplicial $\mathscr{O}_{\widehat{\operatorname{Spec}\mathbb{Z}}}$-module X: we might for example consider the function $u_X : \mathbb{N}_0 \to \log\mathbb{Q}_+^*$ given by $u_X(n) = \deg X_n$, all components X_n being vector bundles by **10.2.12**. Then we can show that u_X is a polynomial function of degree $\leq \dim X$ by the same reasoning as in **10.3.28**, and put $\deg X := u_X(-1)$. The map $X\mapsto\deg X$ thus defined is easily seen to satisfy relations 0)–2) of **10.3.1**, but relation 3), which asserts that $\deg X = \deg Y$ whenever γX is isomorphic to γY in the derived category, seems much harder to prove. If we would have used Waldhausen's definition of K_0 instead, we would have to show that $\deg X = \deg Y$ whenever there is a weak equivalence $X\to Y$, a weaker but still complicated statement. Therefore, we leave this topic for the time being, even if we think equality $K^0(\widehat{\operatorname{Spec}\mathbb{Z}}) = \mathbb{Z}\oplus\log\mathbb{Q}_+^*$ to be highly plausible.

10.6. (Chow rings, Chern classes and intersection theory.) Our next step is to recover the Chow ring $CH(X)$ of a generalized ringed space or topos X (e.g. a generalized scheme) from $K^0(X)$ or $\hat{K}^0(X)$, considered as an augmented λ-ring, and obtain a theory of Chern classes $c_i : K^0(X) \to CH^i(X)$ satisfying all classical relations. We do this by an application of the classical procedure due to Grothendieck: define γ-operations, use them to define the γ-filtration on K^0, and finally define $CH(X)$ as the associated graded $\operatorname{gr}_\gamma K^0(X)$. Chern classes can be defined then by $c_i(\xi) := \operatorname{cl}\gamma^i(\xi) \in CH^i(X)$. The intersection theory thus constructed can be shown to be universal among all intersection theories admitting Chern classes (i.e. "orientable"), at least after tensoring with $\mathbb{Q}$, and, when applied to smooth algebraic varieties, coincides (again after tensoring with $\mathbb{Q}$) with any other intersection theory with rational coefficients, admitting Chern classes and such that the Riemann–Roch formula is valid. These results of Grothendieck show that $CH(X,\mathbb{Q}) = \operatorname{gr}_\gamma K^0(X)_\mathbb{Q}$ is in all respects a reasonable definition of the Chow ring of X with rational coefficients.

10.6.0. We are going to recall some of the constructions involved here; we refer to SGA 6 V for the proofs. However, there are some complications in our situation that we need to deal with:

- We have shown that $K^0(X)$ and $\hat{K}^0(X)$ are pre-λ-rings, but in order to apply Grothendieck's construction we need a λ-ring. We deal with this problem by replacing $K^0(X)$ with its largest quotient $K^0(X)_\lambda$ which is a λ-ring. In all situations where we are able to prove that $K^0(X)$ is already a λ-ring, $K^0(X)_\lambda$ coincides with $K^0(X)$, hence our result will coincide with the classical one.

- We need in fact an augmented λ-ring, i.e. we need an augmentation $\varepsilon : K^0(X) \to H^0(X, \mathbb{Z})$ given by the rank of vector bundles. For this we need to know that the rank of a vector bundle over X is a well-defined locally constant integer-valued function. Sometimes we can achieve this by considering the retract-free version of K^0 (then all vector bundles are automatically locally free), but this is not sufficient (we might have $L_{\mathscr{O}}(n) \cong L_{\mathscr{O}}(m)$ for $n \neq m$). We'll discuss what can done about it.

10.6.1. (λ-structure on $\hat{G}^t(K)$.) Let K be any commutative ring with unity. Recall that one defines a pre-λ-ring structure on the set $\hat{G}^t(K) = 1 + K[[t]]^+$ of formal series with free term equal to one, considered as an abelian group under multiplication, by imposing the following requirements:

- $\hat{G}^t(K)$ depends functorially on K.

- $\{1 + Xt\} \circ \{1 + Yt\} = \{1 + XYt\}$ in $\hat{G}^t(\mathbb{Z}[X, Y])$, where $\circ$ denotes the new multiplication on $\hat{G}^t(K)$, as opposed to the usual multiplication of formal series, which corresponds now to the addition of ring $\hat{G}^t(K)$.

- $\lambda^k\{1 + Xt\} = \{1\} = 0$ in $\hat{G}^t(\mathbb{Z}[X])$ for any $k \geq 2$.

- Operations $\circ$ and λ^k are "continuous" with respect to the natural filtration on $\hat{G}^t(K)$. In other words, the first n coefficients of $f \circ g$ and $\lambda^k(f)$ are completely determined by the first N coefficients of f and g (resp. f) for some $N = N(n) > 0$.

One can show that these conditions uniquely determine operations $\circ$ and λ^k on $\hat{G}^t$, and that $\hat{G}^t(K)$ is a pre-λ-ring with respect to these operations (cf. SGA 6 V). Furthermore, these operations can be expressed by means of certain universal polynomials with integer coefficients, which can be found by means of elementary symmetric polynomial calculus:

$$\{1+x_1t+x_2t^2+\cdots\}\circ\{1+y_1t+y_2t^2+\cdots\} = 1+P_1(x_1;y_1)t+ \\ +P_2(x_1,x_2;y_1,y_2)t^2+\cdots+P_n(x_1,\ldots,x_n;y_1,\ldots,y_n)t^n+\cdots \quad (10.6.1.1)$$

Each $P_n = P_n(x_1,\ldots,x_n;y_1,\ldots,y_n)$ is isobaric of weight n both in (x_i) and (y_i), where the weight of x_i and y_i is set equal to i. For example, $P_1(x_1;y_1) = x_1y_1$, and $P_2(x_1,x_2;y_1,y_2) = x_2y_1^2+y_2x_1^2-2x_2y_2$ (cf. *loc.cit.*). The λ-operations are also given by some universal polynomials:

$$\lambda^k\{1+x_1t+x_2t^2+\cdots\} = 1+Q_{k,1}(x_1,\ldots,x_k)t+\cdots+Q_{k,n}(x_1,\ldots,x_{kn})t^n+\cdots \quad (10.6.1.2)$$

These universal polynomials $Q_{i,j}(x_1,x_2,\ldots,x_{ij})$ are isobaric of weight ij. For example, $Q_{k,1} = x_k$, and $Q_{2,2}(x_1,x_2,x_3,x_4) = x_1x_3-x_4$ (cf. SGA 6 V 2.3).

Definition 10.6.2 (λ-rings.) *A pre-λ-ring K is a* λ-ring *if the abelian group homomorphism $\lambda_t : K \to \hat{G}^t(K)$ is a λ-homomorphism with respect to the λ-structure on $\hat{G}^t(K)$ just discussed.*

In other words, $\lambda^n(xy)$ and $\lambda^j(\lambda^i(x))$ in K must be given by the same universal polynomials:

$$\lambda^n(xy) = P_n(x,\lambda^2x,\ldots,\lambda^nx;y,\lambda^2y,\ldots,\lambda^ny) \quad (10.6.2.1)$$

$$\lambda^j(\lambda^ix) = Q_{i,j}(x,\lambda^2x,\ldots,\lambda^{ij}x) \quad (10.6.2.2)$$

Almost all pre-λ-rings considered in practice are λ-rings. For example, $\hat{G}^t(K)$ is actually a λ-ring for any commutative ring K. Another example: $K^0(X)$ is a λ-ring for any classical ringed space or topos X. Unfortunately, we cannot generalize the proof of this classical statement to our case, at least until we compute K^0 of projective bundles.

10.6.3. (λ-ring associated to a pre-λ-ring.) Given any pre-λ-ring K, we denote by K_λ the quotient of K modulo the λ-ideal generated by relations (10.6.2.1) and (10.6.2.2). Clearly, K_λ is the largest λ-ring quotient of K; if K is already a λ-ring, then $K_\lambda = K$.

10.6.4. (Binomial rings and their λ-structure.) We say that a commutative ring K is *binomial* (cf. SGA 6 V 2.7) if it is torsion-free as an abelian group, and if for any $x \in K$ and any $n \geq 1$ the element $x(x-1)\cdots(x-n+1)$ is divisible by $n!$ in K. The quotient is then denoted by $\binom{x}{n}$.

We endow a binomial ring by its canonical λ-ring structure, given by $\lambda^nx := \binom{x}{n}$. In other words, $\lambda_t(x) = (1+t)^x$, hence $s_t(x) = (1-t)^{-x}$ and $s^n(x) = (-1)^n\binom{-x}{n} = \binom{x+n-1}{n}$. For example, $\mathbb{Z}$ and $H^0(X,\mathbb{Z})$ (for any topological space or topos X) are binomial rings.

10.6.5. (Adams operations on a pre-λ-ring.) Given any pre-λ-ring A, we define the *Adams operations* $\Psi^n : A \to A$, $n \geq 1$, by means of the following identity:

$$\sum_{n=1}^{\infty}(-1)^{n-1}\Psi^n(x)t^{n-1} = \frac{d}{dt}\lambda_t(x)/\lambda_t(x) = \frac{d}{dt}\log\lambda_t(x) \tag{10.6.5.1}$$

We can rewrite the above relation coefficientwise:

$$n\lambda^n(x) = \sum_{k=1}^{n}(-1)^{k-1}\Psi^k(x)\lambda^{n-k}(x) \tag{10.6.5.2}$$

These relations allow one to express easily the Adams operations in terms of exterior power operations, or conversely. In any case, $\Psi^n(x)$ is given by a universal polynomial with integer coefficients in x, $\lambda^2(x)$, ..., $\lambda^n(x)$, isobaric of weight n.

Symmetric and exterior power operations are related by $s_t(x)\lambda_{-t}(x) = 1$, so we have

$$\sum_{n=1}^{\infty}\Psi^n(x)t^{n-1} = -\frac{d}{dt}s_t(x)/s_t(x) = -\frac{d}{dt}\log s_t(x) \tag{10.6.5.3}$$

This equality enables one to express symmetric operations in terms of Adams operations and conversely.

The usual properties of logarithmic derivatives together with $\lambda_t(x+y) = \lambda_t(x)\lambda_t(y)$ imply

$$\Psi^n(x+y) = \Psi^n(x) + \Psi^n(y), \quad \Psi^1(x) = x \tag{10.6.5.4}$$

Conversely, if $A \supset \mathbb{Q}$, then any family of abelian group homomorphisms $\{\Psi^n : A \to A\}_{n\geq 1}$, such that $\Psi^1 = \mathrm{Id}_A$, defines a pre-λ-ring structure on A by means of (10.6.5.1). A λ-homomorphism $f : B \to A$ can be described then as a ring homomorphism commuting with all Adams operations Ψ^n.

10.6.6. (Adams operations on a λ-ring.) One easily shows, starting from

$$\Psi^n\{1 + Xt\} = 1 + X^n t \quad \text{in } \hat{G}^t(\mathbb{Z}[X]) \tag{10.6.6.1}$$

that whenever A is a λ-ring, its Adams operations satisfy

$$\Psi^n(1) = 1, \quad \Psi^n(xy) = \Psi^n(x)\Psi^n(y), \quad \Psi^i(\Psi^j(x)) = \Psi^{ij}(x) \tag{10.6.6.2}$$

For example, $\Psi^n = \mathrm{id}_K$ on any binomial ring K.

Conversely, if a pre-λ-ring A is $\mathbb{Z}$-torsion-free and if its Adams operations satisfy the above relations, A can be shown to be a λ-ring. Therefore, a λ-ring structure on a commutative ring $A \supset \mathbb{Q}$ can be described as a family of ring endomorphisms $\{\Psi^n : A \to A\}_{n\geq 1}$, such that $\Psi^n \circ \Psi^m = \Psi^{nm}$ and $\Psi^1 = \mathrm{id}$. This is probably the simplest way to define a $\mathbb{Q}$-λ-algebra.

10.6.7. (γ-operations.) Given a pre-λ-ring A, we define γ-operations $\gamma^n : A \to A$, $n \geq 0$, by means of the following generating series:

$$\gamma_t(x) = \sum_{n=0}^{\infty} \gamma^n(x) t^n := \lambda_{t/(1-t)}(x) \tag{10.6.7.1}$$

Since $\lambda_t(x) = \gamma_{t/(t+1)}(x)$, the λ-operations completely determine the γ-operations, and conversely. Formal manipulations with power series, using $\lambda_t(x + n - 1) = \lambda_t(x+n)/(1+t)$, yield

$$\gamma^n(x) = \lambda^n(x + n - 1) \tag{10.6.7.2}$$

10.6.8. (Augmented K-λ-algebras.) Let K be any binomial ring. An *augmented K-λ-algebra* is by definition a K-λ-algebra A together with an *augmentation* $\varepsilon : A \to K$, supposed to be a K-λ-homomorphism. If A is $\mathbb{Z}$-torsion-free, the latter condition can be expressed with the aid of Adams operations; taking into account that all $\Psi^n_K = \mathrm{id}_K$, we get $\varepsilon \circ \Psi^n = \varepsilon$ for any $n \geq 1$. Let's identify K with a subring of A. One can put $\Psi^0 := \varepsilon$, considered as a ring homomorphism $A \to A$. Then $\Psi^n \circ \Psi^0 = \Psi^0 = \Psi^0 \circ \Psi^n$ for all $n \geq 0$, i.e. *an augmentation can be considered as an extension Ψ^0 of Adams operations Ψ^n to $n = 0$.* Conversely, K can be recovered from A and Ψ^0 as the image of Ψ^0, or as the set of all x, such that $x = \Psi^0(x)$, and relations $\Psi^n \circ \Psi^0 = \Psi^n$ imply that K is a binomial ring, at least if A is torsion-free.

10.6.9. (Augmented K-pre-λ-algebras.) If A is just an augmented K-pre-λ-algebra over a binomial ring K, λ-homomorphism $\varepsilon : A \to K$ factorizes through the largest λ-ring quotient A_λ of A, discussed in **10.6.3**, since K is a λ-ring. Therefore, A_λ becomes an augmented K-λ-algebra, and we can apply to it all the constructions that follow.

10.6.10. (Important example: augmented K^0.) Let $X = (X, \mathscr{O}_X)$ be a generalized ringed space or topos. Put $A := \hat{K}^0(X)$ or $K^0(X)$ and consider binomial ring $K := H^0(X, \mathbb{Z})$. We have natural λ-homomorphisms $K \to \hat{K}^0(X) \to K^0(X)$, given by $n \mapsto [L_{\mathscr{O}}(n)]$, i.e. A is a pre-λ-algebra. Let's assume the following:

- Any vector bundle $\mathscr{E}$ over X is locally free. Notice that this holds over any classical locally ringed topos. In general we might either prove it or use the retract-free version of the theory, where this statement is automatic.

- The rank of a vector bundle $\mathscr{E}$ over X is locally well-defined, i.e. for any non-empty open subset $U \subset X$ (object of X in the topos case) and any $n \neq m$ the free $\mathscr{O}_X|U$-modules $L_{\mathscr{O}_X|U}(n)$ and $L_{\mathscr{O}_X|U}(m)$ are not isomorphic. Notice that this condition is automatic if $\mathscr{O}_X$ is alternating (cf. **5.5.4**) and if $\mathscr{O}_X(U)$ is non-subtrivial for all $U \neq \varnothing$, since then the rank of $\mathscr{E} = L_{\mathscr{O}_X|U}(n)$ can be recovered as the largest integer k, such that $\bigwedge^k \mathscr{E} \neq 0$.

Under these conditions the rank of a vector bundle is well-defined and additive, hence it defines an additive map $\varepsilon : \hat{K}^0(X) \to H^0(X, \mathbb{Z}) = K$. In some cases it can be extended to a map $K^0(X) \to K$, for example if X is quasi-compact and has a dense open subset $U \subset X$ isomorphic to a classical scheme, such that the restriction map $H^0(X, \mathbb{Z}) \to H^0(U, \mathbb{Z})$ is an isomorphism, a condition usually fulfilled by all our "arithmetic models" $\mathscr{X}/\widehat{\operatorname{Spec}\mathbb{Z}}$ of algebraic varieties $X/\mathbb{Q}$.

Therefore, we obtain an augmented K-pre-λ-algebra A. If it is not a λ-ring, we can consider augmented K-λ-algebra A_λ instead.

10.6.11. (γ-filtration.) Let A be any augmented K-λ-algebra over a binomial ring K. We define a decreasing filtration F_γ^i on A, called the *γ-filtration*, as follows: $F_\gamma^n A$ is the ideal in A generated by elements

$$\gamma^{k_1}(x_1)\gamma^{k_2}(x_2)\cdots\gamma^{k_s}(x_s), \quad \text{where } \sum_i k_i \geq n,\ x_i \in A,\ \varepsilon(x_i) = 0 \tag{10.6.11.1}$$

In particular, $F_\gamma^0 A = A$ and $F_\gamma^1 A = \operatorname{Ker}\varepsilon$.

Since F_γ depends functorially on A, we've obtained a functorial construction $A \mapsto \operatorname{gr}_\gamma A$ of a graded commutative ring from an augmented K-λ-algebra A. We'll see in a moment that under some additional restrictions K and A can be recovered from graded commutative ring $\operatorname{gr}_\gamma A$, thus establishing an equivalence of categories.

10.6.12. (Chow ring.) In particular, if $X = (X, \mathscr{O}_X)$ satisfies the conditions of **10.6.10** we can apply the above construction to $K = H^0(\mathbb{Z}, X)$ and $A = \hat{K}^0(X)_\lambda$ or $A = K^0(X)_\lambda$, thus obtaining the *Chow ring of X:*

$$CH(X) := \operatorname{gr}_\gamma \hat{K}^0(X)_\lambda, \quad CH(X, \mathbb{Q}) := CH(X)_\mathbb{Q} = CH(X) \otimes_\mathbb{Z} \mathbb{Q} \tag{10.6.12.1}$$

Usually we cannot expect to obtain a reasonable intersection theory with integer coefficients unless X is an algebraic variety over a field, so we prefer to consider only $CH(X)_\mathbb{Q}$.

By construction $CH(X)$ and $CH(X)_\mathbb{Q}$ are contravariant in X, i.e. if $f : Y \to X$ is a morphism of generalized ringed spaces or topoi satisfying the conditions of **10.6.10**, we obtain pullback maps $f^* : CH(X) \to CH(Y)$. Under some very special conditions for f (e.g. being a "regular immersion" or "projective locally complete intersection") one can hope to construct "Gysin maps" $f_! : CH(Y) \to CH(X)$ in the opposite direction.

10.6.13. (Example: Chow ring of $\widehat{\operatorname{Spec}\mathbb{Z}}$.) Let A be any abelian group (say, $A = \log\mathbb{Q}^*_+$). Consider the λ-ring $\tilde{A} = \mathbb{Z} \times A$ of **10.5.20**, given by $(n,x)\cdot(m,y) = (nm, ny+mx)$, $\lambda^k(n,x) = \bigl(\binom{n}{k}, \binom{n-1}{k-1}x\bigr)$. Clearly $\tilde{A}$ is an augmented $\mathbb{Z}$-λ-algebra with augmentation $\varepsilon : (n,x) \mapsto n$, so we can apply the above construction to it.

First of all, we know that $F^0_\gamma\tilde{A} = \tilde{A}$ and $F^1_\gamma\tilde{A} = \operatorname{Ker}\varepsilon = 0\times A \cong A$. Since $\gamma^k(0,x) = \lambda^k(k-1,x) = (0, \binom{k-2}{k-1}x) = 0$ for $k \geq 2$ and any $x \in A$, we see that $F^2_\gamma\tilde{A}$ is the ideal generated by products $\gamma^1(x)\gamma^1(y)$ for $x, y \in \operatorname{Ker}\varepsilon = A$; since $\gamma^1 = \mathrm{id}$ and $A^2 = 0$ in $\tilde{A}$, we obtain $F^2_\gamma\tilde{A} = 0$. This implies $\operatorname{gr}_\gamma\tilde{A} = \mathbb{Z}\oplus A$, which is isomorphic as a ring to $\tilde{A}$ itself.

Now let's apply this to $\hat{K}^0(\widehat{\operatorname{Spec}\mathbb{Z}})$, equal to $\tilde{A}$ for $A = \log\mathbb{Q}^*_+$ by **10.5.22**. We obtain

$$CH(\widehat{\operatorname{Spec}\mathbb{Z}}) = CH^0(\widehat{\operatorname{Spec}\mathbb{Z}})\oplus CH^1(\widehat{\operatorname{Spec}\mathbb{Z}}) = \mathbb{Z}\oplus\log\mathbb{Q}^+_* = \mathbb{Z}\oplus\operatorname{Pic}(\widehat{\operatorname{Spec}\mathbb{Z}}) \tag{10.6.13.1}$$

We can compute $CH^0(\widehat{\operatorname{Spec}\mathbb{Z}})_{\mathbb{Q}} = \mathbb{Q}$ and $CH^1(\widehat{\operatorname{Spec}\mathbb{Z}})_{\mathbb{Q}} = \mathbb{Q}\otimes_{\mathbb{Z}}\log\mathbb{Q}^*_+$ as well, but in this special situation we obtain the "correct" result even without tensoring with $\mathbb{Q}$, just because we have $F^2_\gamma A = 0$ in this situation, i.e. "the intersection theory of $\widehat{\operatorname{Spec}\mathbb{Z}}$ is one-dimensional".

10.6.14. (Chern ring and formal computations with Chern classes.) Let $C = \oplus_{n\geq 0}C_n$ be any graded commutative K-algebra (e.g. a Chow ring $CH(X)$) over a binomial ring K (in most cases we'll have $K = C_0$). Consider the completion $\hat{C} := \prod_{n\geq 0} C_n = \{c_0 + c_1 + \cdots + c_n + \cdots \mid c_i \in C_i\}$ and multiplicative subgroup $1+\hat{C}^+ = 1\times\prod_{n\geq 1} C_n = \{1 + c_1 + c_2 + \cdots + c_n + \cdots\} \subset \hat{C}$. Define the *Chern ring* $\operatorname{Ch}(C)$ by

$$\operatorname{Ch}(C) := K \times (1 + \hat{C}^+) \tag{10.6.14.1}$$

The operations on $\operatorname{Ch}(C)$ are defined as follows:

- Addition is defined componentwise: $(a,f) + (b,g) = (a+b, fg)$.
- Action of K on $\operatorname{Ch}(C)$ (i.e. restriction of multiplication to $K\times\operatorname{Ch}(C)$) is given by

$$(a,1) * (b, 1+f) = (ab, (1+f)^a), \quad a, b \in K,\ 1+f \in 1+\hat{C}^+, \tag{10.6.14.2}$$

where the power $(1+f)^a$ is defined by the Newton binomial formula

$$(1+f)^a = \sum_{n=0}^{\infty} \binom{a}{n} f^n \tag{10.6.14.3}$$

- Multiplication satisfies

$$(1, 1+x) * (1, 1+y) = (1, 1+x+y) \quad \text{for any } x,\, y \in C_1. \tag{10.6.14.4}$$

- λ-operations satisfy

$$\lambda^n(a,1) = \Bigl(\binom{a}{n}, 1\Bigr) \quad \text{for any } a \in K,\, n \geq 0, \tag{10.6.14.5}$$

$$\lambda^n(1, 1+x) = 0 = (0,1) \quad \text{for any } x \in C_1,\, n \geq 2. \tag{10.6.14.6}$$

- $\mathrm{Ch}(C)$ is a commutative pre-λ-ring depending functorially on graded K-algebra C. Its operations $*$ and λ^n are continuous in the same sense as in **10.6.1**.

One can check that the above conditions determine a unique λ-ring structure on $\mathrm{Ch}(C)$, and that the coefficients of products and exterior powers can be expressed with the aid of some universal polynomials with integer coefficients (cf. SGA 6 V.6).

If we think of an element $\tilde{c}(\mathscr{E}) = (n, 1 + c_1 + c_2 + \cdots)$ of $\mathrm{Ch}(C)$ as the rank n and the collection c_i of Chern classes of a vector bundle $\mathscr{E}$, then these universal polynomials are nothing else than the classical rules for the Chern classes of tensor product of two vector bundles or the exterior power of a vector bundle, given in [Gr1]. In other words, if $C = CH(X)$, we can expect the completed Chern class $\tilde{c} : \hat{K}^0(X) \to \mathrm{Ch}(C)$ to be a λ-homomorphism.

10.6.15. (Formal definition of Chern classes.) Let A be again an augmented K-λ-algebra over a binomial ring K. (Our principal example is still $K = H^0(X, \mathbb{Z})$, $A = \hat{K}^0(X)_\lambda$ for a suitable generalized ringed space X.) Put $C := \mathrm{gr}_\gamma A$ and define the *Chern classes* $c_i : A \to C_i$, $i \geq 1$, by

$$c_i(x) := \mathrm{cl}_{\mathrm{gr}^i_\gamma A}\, \gamma^i(x - \varepsilon(x)) \in C_i, \quad \text{for any } x \in A. \tag{10.6.15.1}$$

We define the *completed total Chern class* $\tilde{c} : A \to \mathrm{Ch}(C)$ by

$$\tilde{c}(x) := \bigl(\varepsilon(x), 1 + c_1(x) + c_2(x) + \cdots + c_n(x) + \cdots\bigr) \tag{10.6.15.2}$$

Then $\tilde{c} : A \to \mathrm{Ch}(C)$ *is a λ-homomorphism of augmented K-λ-algebras* (cf. SGA 6 V.6.8), i.e. the formal Chern classes thus defined satisfy all classical relations of [Gr1].

In particular, applying this to $K = H^0(X, \mathbb{Z})$, $A = \hat{K}^0(X)_\lambda$ or $K^0(X)_\lambda$ as above, we obtain reasonable Chern classes $c_i : A = \hat{K}^0(X) \to CH^i(X) = \mathrm{gr}^i_\gamma \hat{K}^0(X)_\lambda$ and the completed total Chern class $\tilde{c} : \hat{K}^0(X) \to \mathrm{Ch}(CH(X))$. When $\mathscr{E}$ is a vector bundle over X, we write $c_i(\mathscr{E})$ and $\tilde{c}(\mathscr{E})$ instead of $c_i([\mathscr{E}])$ and $\tilde{c}([\mathscr{E}])$.

10.6.16. (Example: Chern classes over $\widehat{\mathrm{Spec}\,\mathbb{Z}}$.) Applying the above construction to $X = \widehat{\mathrm{Spec}\,\mathbb{Z}}$, we obtain exactly the answer we expect: $c_i(\mathscr{E}) = 0$ for $i \geq 2$, and $c_1(\mathscr{E}) = \deg \mathscr{E}$ for any vector bundle $\mathscr{E}$ over $\widehat{\mathrm{Spec}\,\mathbb{Z}}$, where the degree of a vector bundle is defined as in the proof of **10.5.22**.

10.6.17. (Additivity of c_1 on line bundles.) One has classical formula for c_1 of a product:

$$c_1(xy) = \varepsilon(x)c_1(y) + \varepsilon(y)c_1(x) \tag{10.6.17.1}$$

This implies that the restriction of c_1 onto the multiplicative subgroup $\{x : \varepsilon(x) = 1\}$ is a homomorphism of abelian groups. Applying this to our favourite example $A = \hat{K}^0(X)_\lambda$, we see that *the first Chern class induces a homomorphism* $c_1 : \mathrm{Pic}(X) \to CH^1(X)$.

10.6.18. (Chern character.) Let C be a graded commutative algebra over a binomial ring K as before. We define a functorial homomorphism $\mathrm{ch} : \mathrm{Ch}(C) \to \hat{C}_\mathbb{Q}$ (or to $K \oplus \hat{C}^+_\mathbb{Q}$), where $\hat{C}_\mathbb{Q} := \widehat{C \otimes_\mathbb{Z} \mathbb{Q}}$, by requiring ch to be "continuous" and imposing following relations:

$$\mathrm{ch}(a, 1) = a, \quad a \in K, \tag{10.6.18.1}$$

$$\mathrm{ch}(1, 1 + x) = \exp(x) = \sum_{n \geq 0} \frac{x^n}{n!}, \quad x \in C_1 \tag{10.6.18.2}$$

$$\mathrm{ch}(u + v) = \mathrm{ch}(u) + \mathrm{ch}(v), \quad u, v \in \mathrm{Ch}(C) \tag{10.6.18.3}$$

We say that ch is the *Chern character;* it satisfies $\mathrm{ch}(u * v) = \mathrm{ch}(u)\,\mathrm{ch}(v)$ for any u, $v \in \mathrm{Ch}(C)$, and it can be also defined by

$$\mathrm{ch}(a, \varphi) = a + \eta(\log \varphi) \tag{10.6.18.4}$$

where $\log : 1 + \hat{C}^+ \to \hat{C}^+_\mathbb{Q}$ is defined by the usual series, and η is the additive endomorphism of $\hat{C}^+_\mathbb{Q}$ multiplying the degree k component a_k of an element $a = a_1 + a_2 + \cdots$ of $\hat{C}^+_\mathbb{Q}$ by $(-1)^{k-1}/(k-1)!$ (cf. SGA 6 V.6.3).

One can check that ch is compatible with the multiplication on $\mathrm{Ch}(C)$, i.e. ch is a ring homomorphism (cf. *loc.cit.*). Furthermore, if $K \supset \mathbb{Q}$ (e.g. if we tensorise everything with $\mathbb{Q}$ from the very beginning), and if $C_0 = K$, then ch is a K-algebra *iso*morphism (since both η and log are isomorphisms in this case).

Applying this construction to $C = \mathrm{gr}_\gamma A$ obtained from an augmented K-λ-algebra A, we obtain a K-algebra homomorphism $\mathrm{ch} : A \xrightarrow{\tilde{c}} \mathrm{Ch}(C) \xrightarrow{\mathrm{ch}} \hat{C}_{\mathbb{Q}}$, e.g. $\mathrm{ch} : \hat{K}^0(X) \to \widehat{CH(X)}_{\mathbb{Q}}$, called *Chern character*. Thus $\mathrm{ch}(x+y) = \mathrm{ch}(x) + \mathrm{ch}(y)$ and $\mathrm{ch}(xy) = \mathrm{ch}(x)\,\mathrm{ch}(y)$ as in the classical case. Of course, if $A = \hat{K}^0(X)$, we write $\mathrm{ch}(\mathscr{E})$ or $\mathrm{ch}\,\mathscr{E}$ instead of $\mathrm{ch}([\mathscr{E}])$. In this way we obtain a reasonable Chern character on $K^0(X)$ or $\hat{K}^0(X)$ in a completely formal fashion.

10.6.19. (Adams operations on $\mathrm{Ch}(C)$.) One can compute the Adams operations Ψ^n on $\mathrm{Ch}(C)$. Since K is binomial, we have $\Psi^n(a,1) = (a,1)$ for any $n \geq 1$ and any $a \in K$. Next, let's compute over $C = \mathbb{Z}[T_1, \ldots, T_n]$, $K = \mathbb{Z}$, graded by putting $\deg T_i = 1$. Since $\lambda_t(1, 1+T_i) = 1 + (1, 1+T_i)t$, we get $\Psi^k(1, 1+T_i) = (1, 1+T_i)^{*k} = (1, 1+kT_i)$ by definition of multiplication on $\mathrm{Ch}(C)$ and by general rule $\lambda^t(x) = 1 + xt \Rightarrow \Psi^k(x) = x^k$. Since Ψ^k are additive, we can conclude $\Psi^k(0, 1+T_i) = (0, 1+kT_i)$, hence $\Psi^k(0, \prod_i(1+T_i)) = (0, \prod_i(1+kT_i))$. Taking $\mathfrak{S}_n$-invariants we obtain $\Psi^k(0, 1 + X_1 + X_2 + \cdots + X_n) = (0, 1 + kX_1 + k^2X_2 + \cdots + k^nX_n)$ over $C = \mathbb{Z}[T_1, \ldots, T_n]^{\mathfrak{S}_n} = \mathbb{Z}[X_1, \ldots, X_n]$, graded by $\deg X_i = i$. Using universality of such algebras $\mathbb{Z}[X_1, \ldots, X_n]$ together with "continuity" of Ψ^k we get

$$\Psi^k(a, 1 + c_1 + c_2 + \cdots + c_n + \cdots) = (a, 1 + kc_1 + k^2c_2 + \cdots + k^nc_n + \cdots) \tag{10.6.19.1}$$

inside any $\mathrm{Ch}(C)$. Notice that this equality extends to $k = 0$ if we define $\Psi^0 := \varepsilon$.

10.6.20. (Adams operations on $\hat{C}$ via ch.) Let's still suppose $K \supset \mathbb{Q}$, C be a graded K-algebra with $C_0 = K$, Then $\mathrm{ch} : \mathrm{Ch}(C) \to \hat{C}$ is a ring isomorphism, so we can transfer the λ-ring structure from $\mathrm{Ch}(C)$ to $\hat{C} \supset C$. Since $\hat{C} \supset K \supset \mathbb{Q}$, this λ-structure is completely determined by its Adams operations.

Proposition. *Adams operations Ψ^k on $\hat{C}$ with respect to the λ-structure just discussed are given by*

$$\Psi^k(c_0 + c_1 + c_2 + \cdots + c_n + \cdots) = c_0 + kc_1 + k^2c_2 + \cdots + k^nc_n + \cdots \tag{10.6.20.1}$$

In other words, Ψ^k are continuous on $\hat{C}$ and restrict to k^n on C_n.

Proof. Both sides are "continuous" and additive, and Ψ^k restricted to $C_0 = K$ is trivial just because K is binomial, so we can suppose $c_0 = 0$. Next, ch is an isomorphism, hence it is enough to check the statement on "universal elements" $\mathrm{ch}(0, 1 + X_1 + \cdots + X_n)$ of $C = \mathbb{Q}[X_1, \ldots, X_n]$, graded by $\deg X_i = i$.

Let's embed $\mathbb{Q}[X_1,\ldots,X_n]$ into $C' = \mathbb{Q}[T_1,\ldots,T_n]$, $\deg T_i = 1$, by means of elementary symmetric polynomials. Then $(0, 1+X_1+\cdots+X_n) = (0, 1+T_1)+\cdots+(0,1+T_n)$, so by additivity we are reduced to check (10.6.20.1) on elements $\mathrm{ch}(0,1+t)$, $t \in C_1$, or on elements $\mathrm{ch}(1,1+t) = 1+\mathrm{ch}(0,1+t)$. By definition $\mathrm{ch}(1,1+t) = \exp(t) = \sum_{n\geq 0} t^n/n!$, and $\lambda^k(1,1+t) = 0$ in $\mathrm{Ch}(C)$ for $k \geq 2$, again by definition of λ-operations in $\mathrm{Ch}(C)$, whence $\Psi^k(1,1+t) = (1,1+t)^{*k} = (1,1+kt)$ as before, and $\mathrm{ch}\circ\Psi^k(1,1+t) = \mathrm{ch}(1,1+kt) = \exp(1+kt) = \sum_{n\geq 0} k^n t^n/n! = \Psi^k \circ \mathrm{ch}(1,1+t)$, if Ψ^k on $\hat{C}$ are defined by (10.6.20.1). This proves our statement.

Notice that these Adams operations $\Psi^k : \hat{C} \to \hat{C}$ respect $C \subset \hat{C}$, i.e. $\Psi^k(C) \subset C$, hence C is a K-λ-subalgebra of $\hat{C}$ in a natural fashion.

10.6.21. (Weight decomposition with respect to Adams operations.) Notice that the homogeneous component C_n can be easily recovered from λ-ring C or $\hat{C}$ as the subgroup $\hat{C}_{(n)}$ of elements of weight n with respect to Adams operations:

$$\hat{C}_{(n)} = \{x \in \hat{C} \,:\, \Psi^k(x) = k^n x \text{ for any } k \geq 1\}. \qquad (10.6.21.1)$$

Furthermore, the above formula holds for $k \geq 0$ if we put $\Psi^0 := \varepsilon$. On the other hand, if we fix any $k \geq 2$, $\hat{C}^{(n)}$ is already completely determined by requirement $\Psi^k(x) = k^n x$ for this fixed value of k since all k^n are distinct. In particular, binomial ring $K = C_0 = \hat{C}_{(0)}$ is completely determined by the λ-structure of C or $\hat{C}$, and is the largest binomial ring contained in this λ-ring.

The above construction is valid for any $\mathbb{Q}$-λ-algebra A: we obtain weight subgroups $A_{(n)} \subset A$, and the sum $A_{(\cdot)} := \sum_{n\in\mathbb{Z}} A_{(n)} \subset A$ will be always direct (e.g. because of the non-degeneracy of Vandermond matrices), but in general it needn't coincide with A. In any case, $A_{(n)} \cdot A_{(m)} \subset A_{(n+m)}$ for any n, $m \in \mathbb{Z}$ just because all Ψ^k are ring homomorphisms, i.e. $A_{(\cdot)}$ is an augmented graded K-λ-subalgebra of A over binomial ring $K = A_{(0)}$.

Since $\mathrm{ch} : \mathrm{Ch}(C) \to \hat{C}$ is a λ-ring isomorphism (we still assume $C_0 = K \supset \mathbb{Q}$), we see that C_n might be equally easily recovered from $\mathrm{Ch}(C)$ by $C_n = \mathrm{Ch}(C)_{(n)}$.

10.6.22. (Classification of augmented K-λ-algebras with discrete γ-filtration over $K \supset \mathbb{Q}$.) Let fix a binomial ring $K \supset \mathbb{Q}$. If C is a graded K-algebra with $C_0 = K$, then the γ-filtration on $\mathrm{Ch}(C)$ is easily seen to coincide with the natural one (e.g. using the Chern character isomorphism):

$$F_\gamma^n \mathrm{Ch}(C) = \{(0, 1+c_n+c_{n+1}+\cdots),\ c_i \in C_i\} \qquad (10.6.22.1)$$

Hence $\mathrm{gr}_\gamma \mathrm{Ch}(C) \cong C$, at least as a graded K-module. In fact, this is an isomorphism of K-algebras if we define $\mathrm{gr}_\gamma \mathrm{Ch}(C) \xrightarrow{\sim} C$ with the aid of ch, i.e.

$$\mathrm{cl}_{\mathrm{gr}_\gamma^n \mathrm{Ch}(C)}(0, 1 + c_n + c_{n+1} + \cdots) \mapsto \frac{(-1)^{n-1}}{(n-1)!} \cdot c_n \qquad (10.6.22.2)$$

In other words, we have a functorial isomorphism $\mathrm{gr}_\gamma \mathrm{Ch}(C) \xrightarrow{\sim} C$.

Conversely, if A is a graded K-λ-algebra, the completed total Chern class $\tilde{c} : A \to \mathrm{Ch}(C) = \mathrm{Ch}(\mathrm{gr}_\gamma A)$ is a homomorphism of augmented K-λ-algebras, which can be shown to induce an isomorphism on associated graded with respect to the γ-filtration. Modulo some additional verifications we obtain the following statement:

Theorem. *For any integer $N \geq 0$ and any $K \supset \mathbb{Q}$ functors $A \mapsto \mathrm{gr}_\gamma A$ and $C \mapsto \mathrm{Ch}(C)$ are quasi-inverse equivalences between the category of augmented K-λ-algebras A, such that $F_\gamma^{N+1} A = 0$, and the category of graded K-algebras C, such that $C_0 = K$ and $C_n = 0$ for $n > N$.*

Proof. More details can be found in SGA 6 V 6.11.

Therefore, the category of augmented K-λ-algebras A with discrete γ-filtration is equivalent to the category of graded K-algebras C, such that $C_0 = K$, $C_n = 0$ for $n \gg 0$, i.e. any such A is isomorphic to the Chern ring $\mathrm{Ch}(C)$ of a suitable C. Furthermore, under these conditions the Chern character $\mathrm{ch} : \mathrm{Ch}(C) \to \hat{C} = C$ is an isomorphism, i.e. *the Chern character induces an isomorphism between A and a graded K-algebra C, considered as a λ-ring via Adams operations Ψ^k given by $\Psi^k|_{C_n} = k^n$.* Since the graded components C_n of C come from the weight decomposition with respect to Adams operations, and C is isomorphic to A as a λ-ring, we see that $\mathrm{gr}_\gamma^n A = C_n = C_{(n)}$ *is canonically isomorphic to abelian group $A_{(n)}$*. This is essentially the construction of Soulé (cf. [Soulé1]): $CH^i(X)_\mathbb{Q} = K^0(X)_{\mathbb{Q},(n)}$. Actually Soulé considers weight decomposition of higher algebraic K-groups as well, something we don't discuss in this work.

If A is complete with respect to its γ-filtration, i.e. $A = \varprojlim_n A/F_\gamma^n A$, then $A \cong \mathrm{Ch}(C)$ by taking projective limits, and $\mathrm{ch} : \mathrm{Ch}(C) \to \hat{C}$ is still an isomorphism, i.e. *augmented K-λ-algebras, complete with respect to γ-filtration, are isomorphic to $\mathrm{Ch}(C)$ or $\hat{C}$ for graded K-algebras C with $C_0 = K$.* In this case we still have $C_n = \mathrm{gr}_\gamma^n A \cong A_{(n)}$.

If the γ-filtration on A is just separated, we can still embed A into its completion $\hat{A}_\gamma$ and obtain $\hat{A}_\gamma \cong \mathrm{Ch}(C) \cong \hat{C}$ for $C = \mathrm{gr}_\gamma A_\gamma = \mathrm{gr}_\gamma A$. Then C_n is still equal to $\hat{A}_{\gamma,(n)}$, but $A_{(n)}$ might be smaller, i.e. Soulé's construction might give different (smaller) result.

Finally, if we don't know anything about the γ-filtration on A, we still have a K-λ-algebra homomorphism $A \to \hat{A}_\gamma$, which induces maps $A_{(n)} \to C_n = \mathrm{gr}^n_\gamma A$, as well as a homomorphism of graded K-algebras $A_{(\cdot)} \to C = \mathrm{gr}_\gamma A$. In general we cannot expect this homomorphism to be injective or surjective, so we get two different "Chow rings" $A_{(\cdot)}$ and $\mathrm{gr}_\gamma A$.

10.6.23. (Consequences for K^0.) If X is a "nice" generalized ringed space, so that $A := K^0(X)$ is an augmented K-λ-algebra for $K = H^0(X, \mathbb{Z})$ with discrete γ-filtration, then the Chow ring $C = \mathrm{gr}_\gamma A_\mathbb{Q} = CH(X, \mathbb{Q})$ is concentrated in bounded degrees ($C_n = 0$ for $n \gg 0$), and C can be recovered from A via Soulé's construction: $C_n \cong A_{\mathbb{Q},(n)} \subset A_\mathbb{Q}$, $A_\mathbb{Q} = \bigoplus_{n=0}^N A_{\mathbb{Q},(n)}$.

We know that under the above conditions the completed Chern class $\tilde{c} : K^0(X)_\mathbb{Q} \to \mathrm{Ch}(C)$ is an isomorphism, i.e. *an element* $\xi \in K^0(X)$ *is completely determined (up to $\mathbb{Z}$-torsion) by its rank* $\varepsilon(\xi)$ *and its Chern classes* $c_i(\xi) \in C_i = CH^i(X, \mathbb{Q})$, $0 < i \leq N$, *and all combinations of rank/Chern classes are possible.*

10.6.24. (Equivalent description of augmented λ-algebras with discrete γ-filtration.) We have just seen that whenever A is an augmented K-λ-algebra over a binomial ring $K \supset \mathbb{Q}$, and the γ-filtration of A is discrete (i.e. finite), then $A = \bigoplus_{n=0}^N A_{(n)}$ for some $N > 0$, and $C_n = \mathrm{gr}^n_\gamma A \cong A_{(n)}$. In particular, $K = A_{(0)}$ is also determined by the λ-structure of A.

Conversely, if A is a $\mathbb{Q}$-λ-algebra, such that $A = \bigoplus_{n=0}^N A_{(n)}$ for some $N > 0$, then A is isomorphic to graded K-algebra C, where we put $K := A_{(0)}$, $C_n := A_{(n)}$ and define the Adams operations on C by $\Psi^k|_{C_n} = k^n$ as before. This implies $A \cong C \cong \hat{C} \cong \mathrm{Ch}(C)$, i.e. A is an augmented K-λ-algebra with discrete γ-filtration.

10.6.25. (Relation to Chern character.) Let A be as above, i.e. a $\mathbb{Q}$-λ-algebra with weight decomposition $A = \bigoplus_{n=0}^N A_{(n)}$, or, equivalently, an augmented K-λ-algebra with $F_\gamma^{N+1} A = 0$ over some $K \supset \mathbb{Q}$. One might ask what the components $x_{(n)}$ in $A_{(n)} \cong C_n = \mathrm{gr}^\gamma_n A$ of an element $x \in A$ are. The answer is very simple: the isomorphism between A and C (or $A_{(n)}$ and C_n) was given by the Chern character, hence the $x_{(n)}$ are identified with the components of the Chern character $\mathrm{ch}(x)$, and x is identified with $\mathrm{ch}(x)$ itself.

10.6.26. (Duality and Adams operations.) Let A be an augmented K-λ-algebra. A *duality* on A is just an involution $x \mapsto x^*$ on A compatible with all structure. If we have such a duality, we can put $\Psi^0 := \varepsilon$, $\Psi^{-1}(x) := x^*$, $\Psi^{-n}(x) = \Psi^n(x^*)$, thus defining ring endomorphisms $\Psi^n : A \to A$ for all $n \in \mathbb{Z}$, such that $\Psi^1 = \mathrm{id}_A$ and $\Psi^{nm} = \Psi^n \circ \Psi^m$ for all $m, n \in \mathbb{Z}$.

For example, if we start from a graded K-algebra C, we can define a duality on $\mathrm{Ch}(C)$ by requiring it to be "continuous", functorial, and such that $(1, 1+x)^* = (1, 1-x)$ for any $x \in C_1$, $(a, 0)^* = (a, 0)$ for any $a \in K$. If $K \supset \mathbb{Q}$, we can transfer this duality to $\hat{C}$ via ch; we obtain $x^* = (-1)^n x$ for $x \in C_n$, i.e. the formula $\Psi^k(x) = k^n x$ for $x \in C_n$ is valid for all $k \in \mathbb{Z}$, not just for $k \geq 1$. Then $C = \bigoplus C_n$ is a weight decomposition with respect to the family of all Adams operations $\{\Psi^k\}_{k \in \mathbb{Z}}$.

In our favourite situation $A = \hat{K}^0(X)$, $K = H^0(X, \mathbb{Z})$ we have a natural duality, at least if X is additive, induced by duality of vector bundles: $[\mathscr{E}]^* := [\check{\mathscr{E}}]$. One checks immediately (e.g. reducing to the case of line bundles by Grothendieck projective bundle argument) that $c_n(x^*) = (-1)^n c_n(x)$, i.e. this duality on A is compatible via the completed Chern class $\tilde{c}$ with the duality on $\operatorname{Ch}(C)$ (and on $\hat{C}_{\mathbb{Q}}$ via Chern character) just discussed.

If X is non-additive, $\check{\mathscr{E}} = \mathbf{Hom}(\mathscr{E}, \mathscr{O})$ needn't be a vector bundle. However, if the γ-filtration on $A_{\mathbb{Q}}$ (or $A_{\mathbb{Q},\lambda}$, if A is not a λ-ring) is discrete, we get $\mathrm{ch} : A_{\mathbb{Q}} \cong C_{\mathbb{Q}}$, so we can still transfer the duality just discussed from $C_{\mathbb{Q}}$ to $A_{\mathbb{Q}}$ (or $A_{\mathbb{Q},\lambda}$) in a completely formal fashion.

10.7. (Vector bundles over $\widehat{\operatorname{Spec}\mathbb{Z}}$: further properties.) Vector bundles over $\widehat{\operatorname{Spec}\mathbb{Z}}$ and related generalized schemes appear to possess some very interesting number-theoretic properties by themselves. We cannot discuss them here in much detail, but we would like to mention at least some of their properties.

10.7.1. (Formal duality and dual parametrization.) For any $A \in GL_n(\mathbb{Q})$ put $A^* := (A^t)^{-1}$. Then $A \mapsto A^*$ is an involution on group $GL_n(\mathbb{Q})$, compatible with multiplication and preserving subgroups $GL_n(\mathbb{Z})$ and $GL_n(\mathbb{Z}_{(\infty)}) = \operatorname{Oct}_n$. Therefore, $A \mapsto A^*$ induces an involution on $\operatorname{Oct}_n \backslash GL_n(\mathbb{Q})/GL_n(\mathbb{Z})$, i.e. the moduli space of vector bundles of rank n over $\widehat{\operatorname{Spec}\mathbb{Z}}$. This involution can be used in two different ways:

- We can use it to define a "formal duality operation" $\mathscr{E} \mapsto \mathscr{E}^*$ on vector bundles: if $\mathscr{E}$ is given by some matrix A, then $\mathscr{E}^*$ will be the vector bundle of the same rank given by A^*. In general $\mathscr{E} \mapsto \mathscr{E}^*$ is not a contravariant functor, and $\mathscr{E}^* \not\cong \check{\mathscr{E}} = \mathbf{Hom}_{\mathscr{O}}(\mathscr{E}, \mathscr{O})$ as one would expect in the additive case (indeed, $\check{\mathscr{E}}_\infty$ is the dual of octahedron $\mathscr{E}_\infty$, i.e. a cube, hence $\check{\mathscr{E}}$ is not even a vector bundle if $\operatorname{rank}\mathscr{E} > 2$.) However, $\mathscr{E} \mapsto \mathscr{E}^*$ is compatible with all isomorphisms of vector bundles, and, furthermore, if $\mathscr{E}' \to \mathscr{E} \to \mathscr{E}''$ is a cofibration sequence, then $A = \left(\begin{smallmatrix} A' & \\ * & A'' \end{smallmatrix}\right)$, whence $A^* = \left(\begin{smallmatrix} A'^* & * \\ & A''^* \end{smallmatrix}\right)$, so we get a dual cofibration sequence $\mathscr{E}''^* \to \mathscr{E}^* \to \mathscr{E}'^*$. This means that $[\mathscr{E}] \mapsto [\mathscr{E}^*]$ induces an involution on $\hat{K}^0(\widehat{\operatorname{Spec}\mathbb{Z}}) = \mathbb{Z} \oplus \log \mathbb{Q}_+^*$, easily seen to coincide with the "formal duality" $n + \log\lambda \mapsto n - \log\lambda$ discussed in **10.6.26**.

- On the other hand, we can use $\mathscr{E} \mapsto A^*$ as another parametrization of vector bundles of rank n over $\widehat{\operatorname{Spec}\mathbb{Z}}$ by double coset space $\mathrm{Oct}_n \backslash GL_n(\mathbb{Q})/GL_n(\mathbb{Z})$. For example, we can reduce A^* to its canonical form of **10.5.15**, consider the set of row g.c.d.s of A^*, construct the "dual Harder–Narasimhan filtration" using the canonical form of A^*, and define the "cosemistable" vector bundles over $\widehat{\operatorname{Spec}\mathbb{Z}}$ of slope $\log\lambda$ by requiring all row g.c.d.s of A^* to be equal to λ. This dual parametrization and dual notions seem to be more convenient in some cases.

10.7.2. (Application: $\operatorname{Hom}_{\mathscr{O}}(\mathscr{E}, \mathscr{O})$ of a Mumford-antiregular $\mathscr{E}$.) Let $\mathscr{E}$ be a Mumford-antiregular vector bundle of rank n over $\widehat{\operatorname{Spec}\mathbb{Z}}$, i.e. let's suppose all elements of A^{-1} (or its transposed matrix A^*) to lie in $\mathbb{Z}$ (cf. **10.5.13**). Let's denote by (e_i) and (f_i) the bases of $E_\infty = \mathscr{E}_\infty \subset E = \mathscr{E}_\xi$ and $E_\mathbb{Z} = \Gamma(\operatorname{Spec}\mathbb{Z}, \mathscr{E}) \subset E$ used to construct A from $\mathscr{E}$, cf. **10.5.5**. Let (e_i^*) and (f_i^*) be the dual bases in E^*; since $e_i = \sum_j a_{ij} f_j$, the dual bases are related by $e_i^* = \sum_j a_{ij}^* f_j^*$, where $A^* = (a_{ij}^*)$. Clearly, (f_i^*) is a base of dual lattice $E_\mathbb{Z}^* \subset E^*$. On the other hand, since E_∞ was an octahedron with vertices $\pm e_i$, its dual E_∞^* consists of all linear forms $u = \sum u_j e_j^*$, such that $|\langle \pm e_i, u\rangle| \leq 1$ for all i. This condition is obviously equivalent to $|u_j| \leq 1$ for all j, i.e. E_∞^* is the cube with rational vertices $\pm e_1^* \pm e_2^* \pm \cdots \pm e_n^*$. Since all $a_{ij}^* \in \mathbb{Z}$, all e_i^* lie in $E_\mathbb{Z}^*$, hence the same is true for all vertices of cube E_∞^*.

Now let's compute the number $p_{\mathscr{E}}(0) = \operatorname{card}\operatorname{Hom}_{\mathscr{O}}(\mathscr{E}, \mathscr{O}) = |E_\mathbb{Z}^* \cap E_\infty^*|$, discussed in **10.5.13**. Since all e_i^* lie in lattice $E_\mathbb{Z}^*$, we can replace E_∞^* by cube $E_\infty^* + e_1^* + \cdots + e_n^* = [0, 2e_1^*] \times \cdots \times [0, 2e_n^*]$. This cube is a fundamental domain for lattice 2Λ generated by $\{2e_i^*\}_i$, and this lattice contains $E_\mathbb{Z}^*$, hence its fundamental domain contains exactly $(E_\mathbb{Z}^* : 2\Lambda) = |\det 2A^*| = 2^n |\det A^*|$ points of $E_\mathbb{Z}^*$.

However, this reasoning is slightly imprecise since we consider the closed cube $[0, 2e_1^*] \times \cdots \times [0, 2e_n^*]$, while a fundamental domain for Λ would be given e.g. by product of semi-open intervals $(0, 2e_1^*] \times \cdots \times (0, 2e_n^*]$. In order to compensate for this we introduce for any point $u = \sum_i u_i e_i^*$ its *support* $I = \operatorname{supp} u := \{i \in \mathbf{n} : u_i \neq 0\}$ and write

$$p_{\mathscr{E}}(0) = |E_\mathbb{Z}^* \cap E_\infty^*| = \sum_{I \subset \mathbf{n}} \operatorname{card}\{u \in E_\mathbb{Z}^* : 0 \leq \langle e_i, u\rangle \leq 2, \operatorname{supp} u = I\} = \sum_{I \subset \mathbf{n}} 2^{|I|} (E_I^* \cap E_\mathbb{Z}^* : \Lambda_I) = \sum_I 2^{|I|} c_I(A) \quad (10.7.2.1)$$

Here $E_I^* \subset E^*$ denotes the span of $\{e_i^*\}_{i \in I}$, and $\Lambda_I = E_I^* \cap \Lambda$ is the $\mathbb{Z}$-span of the same set. We have put

$$c_I(A) = c_I(\mathscr{E}) := (E_I^* \cap E_\mathbb{Z}^* : \Lambda_I) \quad (10.7.2.2)$$

This expression makes sense even if $\mathscr{E}$ is not Mumford-antiregular, i.e. if $E_I^* \cap E_{\mathbb{Z}}^*$ doesn't contain Λ_I: indeed, we can always take the absolute value of the determinant of any matrix relating any two bases of these two lattices in E_I^*.

10.7.3. (Computation of $c_I(A)$.) We want to explain how $c_I(A) = (E_I^* \cap E_{\mathbb{Z}}^* : \Lambda_I)$ can be computed in terms of matrix A. Consider the embedding $\kappa_I : E_I^* \to E^*$. Its dual κ_I^* is the canonical projection $E \to E_I$, $\sum x_i e_i \mapsto \sum_{i \in I^c} x_i e_i$, where $E_I \subset E$ denotes the $\mathbb{Q}$-span of $\{e_i\}_{i \in I}$. Then $E_I^* \cap E_{\mathbb{Z}}^* = \kappa_I^{-1}(E_{\mathbb{Z}}^*) = \{u \in E_I^* \,|\, \langle x, \kappa_I(u)\rangle \in \mathbb{Z} \text{ for all } x \in E_{\mathbb{Z}}\} = \{u \in E_I^* \,|\, \langle \kappa_I^*(x), u\rangle \in \mathbb{Z}$ for all $x \in E_{\mathbb{Z}}\} = (\kappa_I^*(E_{\mathbb{Z}}))^*$. In other words, for any $I \subset \mathbf{n}$, $|I| = r$, the positive rational number $c_I(A)$ can be computed as follows:

- Consider the $n \times r$-submatrix A_I^* of A^*, consisting of rows of A^* with indices in I. Since $A^* \in GL_n(\mathbb{Q})$, the r rows of A_I^* are linearly independent, i.e. $\operatorname{rank} A_I^* = r$.
- Consider the $\mathbb{Z}$-sublattice $\kappa_I^*(E_{\mathbb{Z}})$ in $\mathbb{Q}^r$ generated by the n columns of A_I^*. Since these rows span $\mathbb{Q}^r$ as a $\mathbb{Q}$-vector space, this is indeed a sublattice, so we can find a matrix $B \in GL_r(\mathbb{Q})$, the rows of which constitute a base of this lattice. In fact, the usual "integer Gauss elimination process" (similar to that used in **10.5.15**) yields a *triangular* matrix B with this property.
- Now $c_I(A) = |\det B^*|^{-1} = |\det B|$, and we are done.

Notice that the above algorithm yields a positive rational number $c_I(A)$ for any matrix $A \in GL_n(\mathbb{Q})$, and $c_I(\lambda A) = \lambda^{-|A|} c_I(A)$ for any $\lambda \in \mathbb{Q}_+^*$.

10.7.4. (Dual Hilbert polynomial of a vector bundle.) Let $\mathscr{E}$ be a vector bundle over $\widehat{\mathrm{Spec}\,\mathbb{Z}}$, given by a matrix $A \in GL_n(\mathbb{Q})$. Recall that we have defined in **10.5.13** the *dual Hilbert function* $p_{\mathscr{E}} : \log \mathbb{Q}_+^* \to \mathbb{N}$ of $\mathscr{E}$ (or A) by

$$p_{\mathscr{E}}(\log \lambda) = \operatorname{card} \operatorname{Hom}_{\mathscr{O}}(\mathscr{E}, \mathscr{O}(\log \lambda)) = \operatorname{card} |E_{\mathbb{Z}}^* \cap \lambda E_\infty^*| \tag{10.7.4.1}$$

If λ is such that $\mathscr{E}(-\log \lambda)$ is Mumford-antiregular (we know that such λs are exactly the integer multiples of λ_0, the g.c.d. of all a_{ij}^*), then

$$\begin{aligned} p_{\mathscr{E}}(\log \lambda) &= \operatorname{card} \operatorname{Hom}_{\mathscr{O}}(\mathscr{E}(-\log \lambda), \mathscr{O}) = \\ &= \sum_{I \subset \mathbf{n}} 2^{|I|} c_I(\lambda^{-1} A) = \sum_{I \subset \mathbf{n}} 2^{|I|} \lambda^{|I|} c_I(A) = \tilde{p}_{\mathscr{E}}(2\lambda) \end{aligned} \tag{10.7.4.2}$$

where $\tilde{p}_{\mathscr{E}} = \tilde{p}_A \in \mathbb{Q}[T]$ is the polynomial given by

$$\tilde{p}_{\mathscr{E}}(T) = \sum_{I \subset \mathbf{n}} c_I(A) T^{\operatorname{card} I} \tag{10.7.4.3}$$

It is natural to call this $\tilde{p}_{\mathscr{E}} = \tilde{p}_A$ the *dual Hilbert polynomial* of $\mathscr{E}$ or A.

Clearly, $\tilde{p}_{\mathscr{E}}(T)$ is a polynomial of degree $n = \operatorname{rank}\mathscr{E}$, all its coefficients are positive rational numbers, its free term $\tilde{p}_{\mathscr{E}}(0) = c_{\varnothing}(A) = 1$, and its leading coefficient is $c_{\mathbf{n}}(A) = |\det A^*| = |\det A|^{-1}$.

10.7.5. (Examples of dual Hilbert polynomials.) The dual Hilbert polynomial $\tilde{p}_{\mathscr{E}} = \tilde{p}_A$ can be easily computed for any matrix $A \in GL_n(\mathbb{Q})$, at least with the aid of a computer. If A is in canonical (or just lower-triangular) form, the computation is even more simple. For example, $\tilde{p}_{\mathscr{O}(\log\lambda)}(T) = \tilde{p}_{(\lambda)}(T) = 1 + \lambda^{-1}T$. Another example:

$$\tilde{p}_A(T) = 1 + (a + \gcd(c,d))T + adT^2 \quad \text{for } A^* = \begin{pmatrix} a & \\ c & d \end{pmatrix} \tag{10.7.5.1}$$

Consider the vector bundle $\mathscr{E}$ given by $A = \left(\begin{smallmatrix} 1 & 1/2 \\ 0 & 1 \end{smallmatrix}\right)$, $A^* = \left(\begin{smallmatrix} 1 & 0 \\ -1/2 & 1 \end{smallmatrix}\right)$. Then $\tilde{p}_{\mathscr{E}}(T) = 1 + \frac{3}{2}T + T^2 \neq (1+T)^2 = \tilde{p}_{\mathscr{O}}(T)^2$, regardless of the existence of cofibration sequence $\mathscr{O} \to \mathscr{E} \to \mathscr{O}$. Therefore, *$\mathscr{E} \mapsto \tilde{p}_{\mathscr{E}}$ is not additive.* Actually we could expect this from our knowledge of $\hat{K}^0(\widehat{\operatorname{Spec}\mathbb{Z}})$: all additive functions of $\mathscr{E}$ can be expressed with the aid of the rank $\varepsilon(\mathscr{E})$ and its degree $\deg\mathscr{E} = c_1(\mathscr{E})$, but clearly $\tilde{p}_{\mathscr{E}}(T)$ contains much more information than that.

Since $p_{\mathscr{E}}(\log\lambda)$ coincides with $p_{\mathscr{E}}(2\lambda)$ for infinitely many values of λ, we see that *$\mathscr{E} \mapsto p_{\mathscr{E}}$ is also non-additive.*

10.7.6. (Properties of dual Hilbert functions and polynomials.) By definition of coproduct $\oplus$ for any two vector bundles $\mathscr{E}$ and $\mathscr{E}'$ and any $\lambda \in \mathbb{Q}_+^*$ we have

$$\operatorname{Hom}_{\mathscr{O}}(\mathscr{E} \oplus \mathscr{E}', \mathscr{O}(\lambda)) = \operatorname{Hom}_{\mathscr{O}}(\mathscr{E}, \mathscr{O}(\lambda)) \times \operatorname{Hom}_{\mathscr{O}}(\mathscr{E}, \mathscr{O}(\lambda)) \tag{10.7.6.1}$$

Counting elements of these sets we obtain

$$p_{\mathscr{E}\oplus\mathscr{E}'}(\log\lambda) = p_{\mathscr{E}}(\log\lambda) \cdot p_{\mathscr{E}'}(\log\lambda) \tag{10.7.6.2}$$

Since we can find a rational $\lambda_0 > 0$, such that $p_{\mathscr{E}}(\log\lambda) = \tilde{p}_{\mathscr{E}}(2\lambda)$ for all integer multiples λ of λ_0, and similarly for $\mathscr{E}'$ and $\mathscr{E} \oplus \mathscr{E}'$, we obtain equality of polynomials

$$\tilde{p}_{\mathscr{E}\oplus\mathscr{E}'} = \tilde{p}_{\mathscr{E}} \cdot \tilde{p}_{\mathscr{E}'} \tag{10.7.6.3}$$

One might also prove this equality directly, by showing

$$c_I(A) = c_{I\cap\mathbf{k}}(A') \cdot c_{I\cap\mathbf{k}^c}(A'') \quad \text{for } A = A' \oplus A'' = \begin{pmatrix} A' & 0 \\ 0 & A'' \end{pmatrix}. \tag{10.7.6.4}$$

Here $A' \in GL_k(\mathbb{Q})$, $A'' \in GL_{n-k}(\mathbb{Q})$, and $\mathbf{k}^c := \mathbf{n} - \mathbf{k}$.

When $A = \left(\begin{smallmatrix} A' & 0 \\ * & A'' \end{smallmatrix}\right)$, essentially the same computation shows that $c_I(A)$ divides $c_{I\cap\mathbf{k}}(A') \cdot c_{I\cap\mathbf{k}^c}(A'')$, hence *we have a coefficientwise inequality of polynomials $\tilde{p}_{\mathscr{E}}(T) \leq \tilde{p}_{\mathscr{E}'}(T)\tilde{p}_{\mathscr{E}''}(T)$ for any cofibration sequence $\mathscr{E}' \to \mathscr{E} \to \mathscr{E}''$.*

10.7.7. ($\tilde{p}_{\mathscr{E}\otimes\mathscr{E}'}$.) One might ask whether it is possible to compute $\tilde{p}_{\mathscr{E}\otimes\mathscr{E}'}(T)$ knowing only $\tilde{p}_{\mathscr{E}}(T)$ and $\tilde{p}_{\mathscr{E}'}(T)$, and similarly for exterior and symmetric powers. If $\mathscr{E}' = \mathscr{O}(\log\mu)$, then $\mathscr{E}\otimes_{\mathscr{O}}\mathscr{E}' = \mathscr{E}(\log\mu)$ is given by matrix μA, and we get $\tilde{p}_{\mathscr{E}(\log\mu)}(T) = \tilde{p}_{\mathscr{E}}(T/\mu)$. However, in general $\tilde{p}_{\mathscr{E}}(T)$ and $\tilde{p}_{\mathscr{E}'}(T)$ are insufficient to compute $\tilde{p}_{\mathscr{E}\otimes\mathscr{E}'}(T)$, as illustrated by the following example. Let $\mathscr{E}$ and $\mathscr{E}'$ be vector bundles of rank 2 given by matrices $A^* = \left(\begin{smallmatrix}2&0\\2&6\end{smallmatrix}\right)$ and $A'^* = \left(\begin{smallmatrix}1&0\\3&12\end{smallmatrix}\right)$. Then $\tilde{p}_{\mathscr{E}}(T) = 1+4T+12T^2 = \tilde{p}_{\mathscr{E}'}(T)$; however, direct computation shows that polynomials $\tilde{p}_{\mathscr{E}\otimes\mathscr{E}}(T)$ and $\tilde{p}_{\mathscr{E}\otimes\mathscr{E}'}(T)$ differ.

10.7.8. (Difference between $p_{\mathscr{E}}$ and $\tilde{p}_{\mathscr{E}}$.) We know that the deviation $\delta_{\mathscr{E}}(\lambda) := p_{\mathscr{E}}(\log\lambda) - \tilde{p}_{\mathscr{E}}(2\lambda)$ equals zero for all λ divisible by some λ_0. However, values of $\delta_{\mathscr{E}}(\lambda)$ for $\lambda\notin\lambda_0\mathbb{Z}$ might have also some interesting number-theoretical properties, e.g. if we study $\delta_{\mathscr{E}}(\lambda)$ for $\lambda\in\mathbb{Z}$ when $\lambda_0 = N > 1$. It might be even possible to obtain $\delta_{\mathscr{E}}(a) = \left(\frac{a}{p}\right)$ for all $a>0$, $a\in\mathbb{Z}$, as illustrated by one of elementary proofs of the quadratic reciprocity law given by Gauss, based on a problem of counting points in a triangle:

$$\left(\frac{a}{p}\right) = (-1)^{d(a,p)}, \quad \text{where} \tag{10.7.8.1}$$

$$d(a,p) = \sum_{0<k<p/2}\left[\frac{2ak}{p}\right] = \operatorname{card}\left\{(x,y)\in\mathbb{N}^2 \,:\, x\equiv 1 \bmod 2,\ \frac{x}{p}+\frac{y}{a}\leq 1\right\} \tag{10.7.8.2}$$

In general it might be interesting to study the algebra of functions $\mathbb{Q}_+^*\to\mathbb{Q}$ generated by all dual Hilbert functions $p_{\mathscr{E}}$.

10.7.9. (Torsion-free finitely presented $\mathscr{O}_{\widehat{\operatorname{Spec}\mathbb{Z}}}$-modules.) Let $\mathscr{F}$ be a torsion-free finitely presented $\mathscr{O}_{\widehat{\operatorname{Spec}\mathbb{Z}}}$-module. By **10.5.2**, the category of such $\mathscr{F}$'s is equivalent to the category of triples $(F, F_{\mathbb{Z}}, F_\infty)$, where F is a finite-dimensional $\mathbb{Q}$-vector space, $F_{\mathbb{Z}}\subset F$ is a lattice in F, and $F_\infty\subset F$ is a finitely presented torsion-free $\mathbb{Z}_{(\infty)}$-submodule generating F, i.e. a symmetric convex polyhedron inside F (with rational vertices), not contained in any hyperplane of F. We see that in this respect $\widehat{\operatorname{Spec}\mathbb{Z}}$ differs from the projective line $\mathbb{P}^1$: *a torsion-free finitely presented* $\mathscr{O}_{\widehat{\operatorname{Spec}\mathbb{Z}}}$*-module* $\mathscr{F}$ *needn't be a vector bundle.* Notice that the problem of computing $\operatorname{card}\Gamma(\widehat{\operatorname{Spec}\mathbb{Z}}, \mathscr{F}(\log\lambda))$ is nothing else than the classical problem of counting points of lattice $F_{\mathbb{Z}}$ inside a convex polyhedron λF_∞.

If we cease to require finite presentation at ∞, and use $\mathbb{Z}_\infty$ instead of $\mathbb{Z}_{(\infty)}$, we obtain the problem of counting lattice points inside any symmetric convex set $F_\infty\subset F$, e.g. a ball. Even if we don't obtain an equivalence of categories, we still get a sheaf of $\mathscr{O}_{\widehat{\operatorname{Spec}\mathbb{Z}}}$-modules $\mathscr{F}$ given by

$\mathscr{F}(\operatorname{Spec}\mathbb{Z}[1/N]) = F_{\mathbb{Z}}[1/N]$, $\mathscr{F}(\operatorname{Spec}(\mathbb{Z}_{(\infty)} \cap \mathbb{Z}[1/N])) = F_{\mathbb{Z}}[1/N] \cap F_\infty$, and $\mathscr{F}|_{\operatorname{Spec}\mathbb{Z}}$ will be a vector bundle. However, in general $\mathscr{F}$ won't be even quasi-coherent in the neighbourhood of ∞.

10.7.10. (Relation to euclidean metrics.) In particular, euclidean lattices, i.e. lattices $E_{\mathbb{Z}}$ embedded into a finite-dimensional real space $E_{\mathbb{R}}$, equipped with a positive definite quadratic form Q, can be expressed in this form, by using $E_\infty := \{x \in E_{\mathbb{R}} : Q(x) \leq 1\}$. This corresponds to the classical understanding of Arakelov geometry and archimedean structure as explained in **1.5.3**.

In general, however, there are much more $\mathbb{Z}_\infty$-structures (i.e. norms) on $E_{\mathbb{R}}$ in our sense than quadratic forms, and the euclidean structures just discussed are not given by finitely presented $\mathbb{Z}_\infty$-modules. We can compensate this as follows. Given a $\mathbb{Z}_\infty$-structure E_∞, i.e. a compact convex body inside a finite-dimensional real space $E_{\mathbb{R}}$ (cf. **2.4.1**), we can construct an euclidean structure, i.e. a positive definite quadratic form Q on $E_{\mathbb{R}}$ by one of the following methods:

- If E_∞ is free, i.e. if $E_\infty = \mathbb{Z}_\infty^{(n)}$ is a octahedron centered at the origin, we can choose any its basis (e_i) and declare it an orthonormal basis for Q. Since $\operatorname{Oct}_n \subset O_n(\mathbb{R})$, Q does not depend on the choice of basis for E_∞.

- We can define Q by averaging over the dual convex set $E_\infty^* = \{u \in E^* : u(E_\infty) \subset [-1,1]\}$:

$$Q(x) = \frac{c_n}{\mu(E_\infty^*)} \int_{E_\infty^*} \langle x, u\rangle^2 \, du \tag{10.7.10.1}$$

  for some positive constant $c_n > 0$ depending on $n = \dim E_{\mathbb{R}}$.

- Dually, we can define $Q^* : E^* \to \mathbb{R}$ by averaging over E_∞, and then consider the dual quadratic form of Q^*.

In general these methods give different results. The second and the third method have the property to reproduce the original quadratic form Q' (up to a constant) if E_∞ was already quadratic (i.e. if $E_\infty = \{x : Q'(x) \leq 1\}$), so the constants may be chosen so as to have $Q = Q'$ in this case.

On the other hand, if E_∞ is a free $\mathbb{Z}_\infty$-module, and we fix any basis (e_i) for E_∞, then the resulting quadratic form will be necessarily invariant under $\operatorname{Aut}_{\mathbb{Z}_\infty}(E_\infty) \cong \operatorname{Aut}_{\mathbb{Z}_\infty}(\mathbb{Z}_\infty^{(n)}) = \operatorname{Oct}_n$, and any quadratic form Q invariant under Oct_n is easily seen to be given by a diagonal matrix. Therefore, we can always choose the constants in the second and the third method so as to make them coincide with the first method for free $\mathbb{Z}_\infty$-modules. We'll adopt this approach for simplicity.

If $E_{(\infty)}$ is a $\mathbb{Z}_{(\infty)}$-structure inside a $\mathbb{Q}$-vector space E, we can extend it to a $\mathbb{Z}_\infty$-structure E_∞ inside $E_\mathbb{R} = E \otimes_\mathbb{Q} \mathbb{R}$ (e.g. by taking the closure of $E_{(\infty)}$ in $E_\mathbb{R}$) and apply any of the above constructions. Notice that if $E_{(\infty)}$ was finitely presented, then E_∞ is a symmetric convex polyhedron with rational vertices, and any of the above methods will produce a quadratic form with rational coefficients.

10.7.11. (Relation to Shimura varieties.) The above construction induces a map on moduli spaces

$$\nu : \operatorname{Oct}_n \backslash GL_n(\mathbb{Q})/GL_n(\mathbb{Z}) \to O_n(\mathbb{R})\backslash GL_n(\mathbb{R})/GL_n(\mathbb{Z}) \tag{10.7.11.1}$$

with dense image, so, for example, a continuous function f on the RHS is completely determined by the function $g := f \circ \nu$, continuous and constant on the fibers of ν. Conversely, any continuous function g on the LHS determines a function on the RHS: we first extend it by continuity to $\operatorname{Oct}_n \backslash GL_n(\mathbb{R})/GL_n(\mathbb{Z})$, and then integrate it over the fibers. If we consider vector bundles $\mathscr{E}$ up to a Serre twist, and euclidean lattices up to similarity, we obtain another map with dense image

$$\nu' : \operatorname{Oct}_n \cdot\mathbb{Q}_+^*\backslash GL_n(\mathbb{Q})/GL_n(\mathbb{Z}) \to O_n(\mathbb{R})\cdot\mathbb{R}_+^*\backslash GL_n(\mathbb{R})/GL_n(\mathbb{Z}) \tag{10.7.11.2}$$

The target is the Shimura variety of GL_n without level structure, so we obtain some relation between e.g. automorphic forms with respect to GL_n and sections of certain bundles over the moduli space of vector bundles of rank n over $\widehat{\operatorname{Spec}\mathbb{Z}}$.

We can introduce a level N structure on this moduli space, so as to make it rigid. This means that we fix a $\mathbb{Z}/N\mathbb{Z}$-base of $E_\mathbb{Z}/NE_\mathbb{Z}$, i.e. an isomorphism $E_\mathbb{Z}/NE_\mathbb{Z} \cong (\mathbb{Z}/N\mathbb{Z})^n$. Equivalently, we consider vector bundles $\mathscr{E}$ over $\widehat{\operatorname{Spec}\mathbb{Z}}$ of rank n together with a trivialization of $\mathscr{E}|_{\operatorname{Spec}\mathbb{Z}/N\mathbb{Z}}$. Then the target of corresponding map ν'' will be the Shimura variety of GL_n with respect to the (full) level N structure. We can consider a "level ∞ structure" on vector bundles $\mathscr{E}/\widehat{\operatorname{Spec}\mathbb{Z}}$ if we like: this corresponds to choosing a $\mathbb{Z}_{(\infty)}$-base of E_∞. The corresponding moduli space is $GL_n(\mathbb{Q})/GL_n(\mathbb{Z})$; it is rigid.

10.7.12. (Operations with euclidean lattices.) Choosing a lifting $\mathscr{E}$ of an euclidean lattice $\Lambda \subset E$ with respect to the map $\nu : \operatorname{Oct}_n \backslash GL_n(\mathbb{R})/GL_n(\mathbb{Z}) \to O_n(\mathbb{R})\backslash GL_n(\mathbb{R})/GL_n(\mathbb{Z})$ roughly corresponds to fixing an orthonormal base in the euclidean space E. Then we might do some operations with $\mathscr{E}$ (which is either a vector bundle over $\widehat{\operatorname{Spec}\mathbb{Z}}$, or can be approximated by such), and in some occasions the image of the result under ν would not depend on the choice of lifting $\mathscr{E}$, thus defining an operation with euclidean lattices. For example, $\mathscr{E} \mapsto \mathscr{E}^*$ corresponds to the operation of taking dual euclidean lattice. Operations $S^k\mathscr{E}$, $\bigwedge^k \mathscr{E}$, $\mathscr{E} \oplus \mathscr{E}'$ and $\mathscr{E} \otimes \mathscr{E}'$ have this compatibility property as well, giving rise to corresponding operations with euclidean lattices. This probably explains why it was possible to establish a working Arakelov geometry using vector bundles with suitable hermitian metrics, and also establishes some relationship between classical Arakelov geometry and its version discussed in this work.

10.7.13. (Automorphisms of vector bundles.) We have already remarked that the moduli space $\mathrm{Oct}_n \backslash GL_n(\mathbb{Q})/GL_n(\mathbb{Z})$ is not rigid, i.e. a vector bundle $\mathscr{E}/\widehat{\mathrm{Spec}\,\mathbb{Z}}$ may have a non-trivial automorphism group. However, this group is always finite since $\mathrm{Aut}_{\mathscr{O}}(\mathscr{E}) \subset \mathrm{Aut}_{\mathbb{Z}_{(\infty)}}(E_\infty) = \mathrm{Oct}_n$. Fixing a "level ∞ structure" makes the moduli space $GL_n(\mathbb{Q})/GL_n(\mathbb{Z})$ rigid.

10.7.14. (Generalization to other number fields.) Let K be any number field. Then we might attempt to construct $\widehat{\mathrm{Spec}\,\mathcal{O}_K}$ in different ways. The most canonical of them is to consider the "semilocal ring" $\mathcal{O}_{K,\infty}$, the intersection of all archimedean valuation rings $\mathcal{O}_v$ inside K. Then $\mathrm{Spec}\,\mathcal{O}_{K,\infty}$ is one-dimensional, and its closed points correspond to some families of $v|\infty$ (some or all archimedean valuations can be glued together; this happens already for real quadratic fields $K/\mathbb{Q}$; but if K is a CM-field, this never happens, thus suggesting another connection to Shimura varieties). In any case, we can construct $\widehat{\mathrm{Spec}\,\mathcal{O}_K}^{(N)}$ by choosing any integer $N > 1$ and gluing $\mathrm{Spec}(\mathcal{O}_K)$ with $\mathrm{Spec}(\mathcal{O}_{K,\infty} \cap \mathcal{O}_K[1/N])$ along their common open subset $\mathrm{Spec}(\mathcal{O}_K[1/N])$. The generalized schemes thus constructed depend on the choice of $N > 1$, but they constitute a projective system over the set of integers $N > 1$ ordered by divisibility, so we can still define $\widehat{\mathrm{Spec}\,\mathcal{O}_K}$ as the projective limit of this system, either in the category of pro-generalized schemes, or in the category of generalized ringed spaces.

This $\widehat{\mathrm{Spec}\,\mathcal{O}_K}$ has one generic point, and its closed points correspond to all valuations of K, but with some archimedean valuations glued together. This suggests that the "true" $\widehat{\mathrm{Spec}\,\mathcal{O}_K}$ should be constructed in a more sophisticated way. In the fancy language of "infinite resolution of singularities" introduced in **7.1.48** one might say that this time the singularity over ∞ is even more complicated than a cusp of infinite order, so we need more complicated resolution of singularities, which will "disentagle" different points lying over ∞, thus yielding the "smooth" model of $\widehat{\mathrm{Spec}\,\mathcal{O}_K}$.

It would be interesting to classify vector bundles over $\widehat{\mathrm{Spec}\,\mathcal{O}_K}$ and to compute K^0. Unfortunately, we cannot even apply **10.4.2**, since the condition "$|x| \in \tilde{\mathbb{Q}}_+$ for all $x \in K$" is not fulfilled. This problem might become simpler after the "correct" $\widehat{\operatorname{Spec}\mathcal{O}_K}$ is constructed.

10.7.15. (Cyclotomic extensions.) Another possibility is to study the "cyclotomic extensions" $Z_n := \widehat{\operatorname{Spec}\mathbb{Z}} \otimes_{\mathbb{F}_1} \mathbb{F}_{1^n}$. Such generalized schemes are not "integral" (their generic fiber is $\operatorname{Spec}\mathbb{Q}[T]/(T^n - 1)$), but their structure (especially over ∞) seems to be simpler than that discussed in **10.7.14**. They have the following nice property: a vector bundle of rank r over Z_n corresponds to a vector bundle $\mathscr{E}$ of rank rn over $\widehat{\operatorname{Spec}\mathbb{Z}}$ together with an action $\sigma : \mathscr{E} \to \mathscr{E}$ of the cyclic group $C_n = \langle\sigma\rangle$, i.e. an element of order n inside $\operatorname{Aut}(\mathscr{E}) \subset \operatorname{Oct}_{nr}$. However, not all such couples $(\mathscr{E}, \sigma)$ correspond to a vector bundle over Z_n, i.e. we have just a fully faithful functor, not an equivalence of categories. In any case, the computation of $\hat{K}^0(Z_n)$ shouldn't be too difficult.

10.7.16. (Intersection theory of $\widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$.) Another interesting possibility is to study vector bundles and perfect cofibrations over $S_N := \widehat{\operatorname{Spec}\mathbb{Z}}^{(N)}$ for some $N > 1$, and to compute the Chow ring $CH(S_N)$ afterwards. If any vector bundle over $\operatorname{Spec} A_N$, where $A_N = \mathbb{Z}_{(\infty)} \cap \mathbb{Z}[N^{-1}]$, had been trivial, then we would obtain results similar to those obtained above for $\widehat{\operatorname{Spec}\mathbb{Z}}$: vector bundles over S_N would be parametrized by $GL_n(A_N)\backslash GL_n(B_N)/GL_n(\mathbb{Z})$, where $B_N := \mathbb{Z}[N^{-1}]$, and $CH^0(S_N) = \mathbb{Z}$, $CH^1(S_N) \cong \operatorname{Pic}(S_N) \cong \log B_{N,+}^\times \cong \mathbb{Z}^r$, where $p_1, \ldots, p_r$ are distinct prime divisors of N (cf. **7.1.35**), and $CH^i(S_N) = 0$ for $i \geq 2$. However, even proving $\operatorname{Pic}(A_N) = 0$ has been much more complicated than proving $\operatorname{Pic}(\mathbb{Z}_{(\infty)}) = 0$ (cf. **7.1.33**), and we see no reason for all finitely generated projective modules over A_N to be free, since S_N is sort of a "non-smooth version" of $\widehat{\operatorname{Spec}\mathbb{Z}}$ (cf. **7.1.48**).

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