

CYCLOTOMY AND ANALYTIC GEOMETRY OVER F_1

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Abstract. Geometry over non-existent “field with one element” F_1 conceived by Jacques Tits [Ti] half a century ago recently found an incarnation, in at least two related but different guises. In this paper I analyze the crucial role of roots of unity in this geometry and propose a version of the notion of “analytic functions” over F_1 . The paper combines a focused survey with some new constructions.

To Alain Connes, for his sixtieth anniversary

0. Introduction: many faces of cyclotomy

0.1. Roots of unity and field with one element. The basics of algebraic geometry over an elusive “field with one element F_1 ” were laid down recently in [So], [De1], [De2], [TV], fifty years after a seminal remark by J. Tits [Ti]. There are many motivations to look for F_1 ; a hope to imitate Weil’s proof for Riemann’s zeta is one of them, cf [CCMa3], [Ku], [Ma1].

An important role in the formalization of F_1 -geometry was played by the suggestion made in [KS] that one should simultaneously consider all the “finite extensions” F_{1^n} . This resulted in the approach of [So], where a geometric object, say a scheme, V over F_1 , acquired flesh after a base extension to \mathbf{Z} , and the F_1 -geometry of V was reflected in (and in fact, formally defined in terms of) the geometry of “cyclotomic” points of an appropriate ordinary scheme $V_{\mathbf{Z}}$. In [De1] and [TV], schemes over F_1 are defined in categorical terms independently of cyclotomy, but the latter reappears soon: see the Definition 1.7.1 below and the following discussion.

All these approaches are interrelated but lead to different versions of basic definitions, and develop the initial intuition in different directions, so that their divergence can be fruitfully exploited. With this goal in mind, I have chosen topics to be discussed in sec. 1, where three known definitions of F_1 -schemes are sketched and compared.

Of course, roots of unity appear naturally in many different geometric contexts, not motivated by geometry over F_1 : some of these contexts are reviewed below in the subsections 0.2–0.6 of this Introduction.

One motivation of this note was my desire to understand, how the insights gained within these contexts could help develop F_1 -geometry.

Seemingly, a similar desire moved the authors of [CCMa2] to put the theory of Bost–Connes in the framework of F_1 -geometry.

I show in sec. 2 and sec. 3 that the results of [Ha1], the preparatory part to K. Habiro’s work on Witten–Reshetikhin–Turaev invariants of homology spheres [Ha2], and discoveries about these invariants made in [Law] and [LawZ], can be naturally viewed as a contribution to the rudiments of analytic geometry over F_1 .

Finally, in sec. 4 I discuss Witt vectors and a series of F_1 -models of moduli spaces.

Acknowledgements. V. Golyshev’s note [Go] prompted me to think about cyclotomy in the F_1 context. K. Habiro read a preliminary version of this paper and suggested several complements and simplifications. H. Lenstra kindly referred me to [van D] and other useful sources on profinite numbers. A. Connes and C. Consani sent me a copy of their new paper [CC] which was being written during the same weeks as this article. I deeply appreciate their interest and help.

0.2. Roots of unity and Morse–Smale diffeomorphisms. This aspect of cyclotomy is described by D. Grayson in [Gr].

Let M be a compact smooth manifold, f a diffeomorphism of M . It is called *Morse–Smale*, if it is structurally stable, and only a finite number of points x are non-wandering, i. e., for any neighborhood U of x , we have $U \cap f^n(U) \neq \emptyset$ infinitely often.

Assume that all eigenvalues of the action of f on integral cohomology of M are roots of unity and put the question: *when f is isotopic to a Morse–Smale map?*

There is an obstruction to this, lying in the group $SK_1(\mathcal{R})$, where \mathcal{R} is the ring obtained by localizing $\mathbf{Z}[q]$ with respect to $\Phi_0(q) := q$ and all cyclotomic polynomials

$$\Phi_n(q) := \prod_{\eta} (q - \eta)$$

where η runs over all primitive roots of unity of degree $n \geq 1$.

This ring turns out to be a principal ideal domain. The reason for this is that each closed point (a prime ideal of depth two) of the “arithmetical plane” $\text{Spec } \mathbf{Z}[q]$ is situated on an arithmetic curve $\Phi_n(q) = 0, n \geq 0$, because all finite fields consist of roots of unity and zero.

Localization cuts all these curves off, and all closed points go with them. Remaining prime ideals are of height one, and they are principal.

The same effect can be achieved by localizing wrt all primes $p \in \mathbf{Z}$, thus getting the principal ideal domain $\mathbf{Q}[q]$. This localization cuts away the closed fibers of the projection $\text{Spec } \mathbf{Z}[q] \rightarrow \text{Spec } \mathbf{Z}$, and all the closed points with them.

This suggests that the union of cyclotomic arithmetic curves $\Phi_n(q) = 0$ can be imagined as the union of closed fibers of the projection $\text{Spec } \mathbf{Z}[q] \rightarrow \text{Spec } F_1[q]$, and the arithmetic plane itself as the product of two coordinate axes, arithmetic one $\text{Spec } \mathbf{Z}$ and geometric one, $\text{Spec } F_1[q]$, over the “absolute point” $\text{Spec } F_1$.

In sec. 1 below, I review several versions of algebraic F_1 -geometry where this intuition can be made precise.

0.2.1. Question. Is there a context in which diffeomorphisms f , acting on integral cohomology of M with eigenvalues roots of unity, could be interpreted as “Frobenius maps in characteristic 1”, and their fixed (or non-wandering) points in a Morse–Smale situation as F_1^n -points of an appropriate variety?

0.3. Roots of unity and the Witten–Reshetikhin–Turaev invariants. Apparently totally different line of thought led to the consideration of completions of $\mathbf{Z}[q]$ with respect to various linear topologies generated by the cyclotomic polynomials $\Phi_n(q)$. Namely, it turned out that the invariants of 3-dimensional homology spheres, introduced first by E. Witten by means of path integrals, and mathematically constructed by Reshetikhin and Turaev, can be unified into objects lying in completions of the kind described above.

0.3.1. Question. Can these completions be interpreted in a framework of F_1 -geometry?

We try to answer this question affirmatively in sec. 2 and 3 below.

(Similar completions along the arithmetical axis produce for example direct products of p -adic integers $\prod \mathbf{Z}_{p_i}$ and the ring $\varprojlim_N \mathbf{Z}/(p_1 \dots p_N)$, in which \mathbf{Z} can be embedded.)

We suggest two interpretations, one in the framework of Soulé’s axiomatics, and another more in the spirit of Toën–Vaquié and Deitmar’s definitions. Here are brief explanations.

Soulé’s definition of an F_1 -scheme X involves, besides $X_{\mathbf{Z}}$, a \mathbf{C} -algebra \mathcal{A}_X , and each cyclotomic point of $X_{\mathbf{Z}}$ coming from X must assign “values” to the elements of \mathcal{A}_X . His choice of \mathcal{A}_X for the multiplicative group G_{m, F_1} is that of continuous

functions on the unit circle in \mathbf{C} (cf. [So], 5.2.2). For the affine line he uses holomorphic functions in the open unit circle continuous on the boundary.

We suggest to consider respectively *the ring of Habiro's analytic functions* and *the ring of Habiro's functions admitting an analytic continuation in the open unit disc*. The first one consists of formal series

$$f(q) = a_0 + \sum_{n=1}^{\infty} a_n(1-q)\dots(1-q^n). \quad (0.1)$$

where a_n are polynomials in q of degree $\leq n-1$. At any root of unity, only a finite number of terms do not vanish, so f is a well defined function on cyclotomic points.

The second option consists in considering holomorphic functions $\varphi(q)$ in the unit circle, such that for any root of unity ζ , a radial limit $\lim_{r \rightarrow 1_-} r\zeta$ exists, and the family of such limits can be given by the series (0.1).

Versions of this choice might involve functions holomorphic inside variable rings with outer boundary $|q| = 1$ admitting radial limits at roots of unity, or even pairs of functions φ_- , resp. φ_+ , holomorphic inside narrow rings with outer, resp. inner, boundary unit circle, which restrict to a C^∞ -function on all small radial intervals $(1 - \varepsilon_\zeta, 1 + \varepsilon_\zeta) \cdot \zeta$ containing roots of unity ζ . In particular, they must satisfy

$$\lim_{r \rightarrow 1_-} \varphi_-(r\zeta) = \lim_{r \rightarrow 1_+} \varphi_+(r\zeta)$$

The limit values should admit the representation (0.1).

The fact that there exist highly nontrivial and interesting examples of such functions, was discovered in the theory of Witten–Reshetikhin–Turaev invariants: cf. [Law], [LawZ]. Don Zagier says that φ_\pm “leak through” roots of unity.

On the other hand, if \mathcal{A}_X is not a part of the definition of a F_1 -scheme, as in the versions of [TV] and [De1], one can still imagine that a ring of the type discussed above would form a part of the structure of analytic F_1 -varieties when F_1 -geometry becomes mature enough to include analytic geometry.

[CCMa2] also suggests that time is ripe for such generalizations.

0.4. Roots of unity and the Bost–Connes system. In the paper [BoCo] roots of unity appears in the following setting. Consider the Hecke algebra \mathcal{H} with involution over \mathbf{Q} given by the following presentation. The generators are denoted μ_n , $n \in \mathbf{Z}_+$, and $e(\gamma)$, $\gamma \in \mathbf{Q}/\mathbf{Z}$. The relations are

$$\mu_n^* \mu_n = 1, \quad \mu_{mn} = \mu_m \mu_n, \quad \mu_m^* \mu_n = \mu_n \mu_m^* \text{ for } (m, n) = 1;$$

$$e(\gamma)^* = e(-\gamma), \quad e(\gamma_1 + \gamma_2) = e(\gamma_1) e(\gamma_2);$$

$$e(\gamma)\mu_n = \mu_n e(n\gamma), \quad \mu_n e(\gamma)\mu_n^* = \frac{1}{n} \sum_{n\delta=\gamma} e(\delta).$$

The idèle class group $\widehat{\mathbf{Z}}^*$ of \mathbf{Q} acts upon \mathcal{H} in a very explicit and simple way: on $e(\gamma)$'s the action is induced by the multiplication $\widehat{\mathbf{Z}}^* \times \mathbf{Q}/\mathbf{Z} \rightarrow \mathbf{Q}/\mathbf{Z}$, whereas on μ_n 's it is identical.

The algebra \mathcal{H} admits an involutive representation ρ in $l^2(\mathbf{Z}_+)$: denoting by $\{\epsilon_k\}$ the standard basis of this space, we have

$$\rho(\mu_n) \epsilon_k = \epsilon_{nk}, \quad \rho(e(\gamma)) \epsilon_k = e^{2\pi i k \gamma} \epsilon_k.$$

From this, one can produce the whole $\text{Gal}(\mathbf{Q}^{ab}/\mathbf{Q})$ -orbit $\{\rho_g\}$ of such representations, applying $g \in \text{Gal}(\mathbf{Q}^{ab}/\mathbf{Q})$ to all roots of unity occuring at the right hand sides of the expressions for $\rho(e(\gamma))\epsilon_k$. All these representations can be canonically extended to the C^* -algebra completion C of \mathcal{H} constructed from the regular representation of \mathcal{H} . Let us denote them by the same symbol ρ_g .

To formulate the main theorem of [BoCo], we need some more explanations. The algebra C admits a canonical action of \mathbf{R} , which can be interpreted as *time evolution* represented on the algebra of observables. This is a general (and deep) fact in the theory of C^* -algebras, but for C the action of \mathbf{R} can be quite explicitly described on the generators. Let us denote by σ_t the action of $t \in \mathbf{R}$. A *KMS $_\beta$ state at inverse temperature β* on (C, σ_t) is defined as a state φ on C such that for any $x, y \in C$ there exists a bounded holomorphic function $F_{x,y}(z)$ defined in the strip $0 \leq \text{Im } z \leq \beta$ and continuous on the boundary, satisfying

$$\varphi(x\sigma_t(y)) = F_{x,y}(t), \quad \varphi(\sigma_t(y)x) = F_{x,y}(t + i\beta).$$

Now denote by H the positive operator on $l^2(\mathbf{Z}_+)$: $H\epsilon_k = (\log k) \epsilon_k$. Then for any $\beta > 1$, $g \in \text{Gal}(\mathbf{Q}^{ab}/\mathbf{Q})$ one can define a *KMS $_\beta$ state $\varphi_{\beta,g}$* on (C, σ_t) by the following formula:

$$\varphi_{\beta,g}(x) := \zeta(\beta)^{-1} \text{Trace}(\rho_g(x) e^{-\beta H}), \quad x \in C$$

where ζ is the Riemann zeta-function. The map $g \mapsto \varphi_{\beta,g}(x)$ is a homeomorphism of $\text{Gal}(\mathbf{Q}^{ab}/\mathbf{Q})$ with the space of extreme points of the Choquet simplex of all *KMS $_\beta$ states*.

To the contrary, for $\beta < 1$ there is a unique KMS_β state. This is a remarkable “arithmetical symmetry breaking” phenomenon.

The description of the Hecke algebra above involves denominators in the last relation. In [CCMa2], the authors construct \mathbf{Z} -models of finite layers of this object and natural morphisms between them, and show that the resulting system is a lift to \mathbf{Z} of an F_1 -tower.

0.5. Witt vectors. It is desirable to consider the arithmetical axis $\text{Spec } \mathbf{Z}$ as an F_1 -space as well, but in the current framework it is certainly not a scheme of finite type. In fact, its base extension to \mathbf{Z} is elusive, being precisely what we would like to see as the spectrum of $\mathbf{Z} \otimes \mathbf{Z}$.

Nevertheless, in a certain sense primes can be considered as cyclotomic points of $\text{Spec } \mathbf{Z}$, at which the “cyclotomic coordinates”, all integers, take values that are roots of unity or zero.

In fact, roots of unity of degree $q = p^n - 1$ (and zero), considered together with their embedding into a fixed unramified extension \mathbf{Z}_p^{nr} of \mathbf{Z}_p rather than \mathbf{C} , appear as natural coefficients of p -adic expansions discovered by Teichmüller and Witt. Namely, each residue class in $\mathbf{Z}_p^{nr}/(p)$ has a unique (Teichmüller) representative ζ which is either a root of unity or 0 in \mathbf{Z}_p^{nr} , so that an element of such an extension can be written as a well defined series $\sum_{i=0}^{\infty} \zeta_i p^i$. Moreover, coefficients of a sum or a product of two such series are given by Witt’s universal polynomials in the coefficients of the summands/factors in the following sense: one must reduce Witt’s coefficients modulo p , apply these polynomials (which are defined over \mathbf{Z}), and lift results back to roots of unity.

This can be generalized to the so called “big Witt ring” and interpreted in the following way. On affine spaces $A_{\mathbf{Z}}^k = \text{Spec } \mathbf{Z}[u_1, u_2, \dots, u_k]$ there exists a natural system of “cyclotomic coordinates” (in the p -adic context sometimes called “ghost coordinates”). In terms of these coordinates, one can define an F_1 -gadget à la Soulé, requiring that in the subfunctor of points, these coordinates took cyclotomic values (including zero). However, Witt’s addition/multiplication becomes well defined only after extension to \mathbf{Z} , unless the notion of morphism over F_1 is drastically extended.

To me, this looks like a strong argument for considering options for such an extension.

We supply some more details in sec. 4.

0.6. Roots of unity and the analogy between Hilbert polynomials and

zeta functions. There were several suggestions that Hilbert polynomials $H(n)$, say, of graded commutative rings behave like toy zeta functions.

Rather precise recent observations by V. Golyshev in [Go] can be summarized as follows.

a) The comparison to zetas becomes most striking if one restricts oneself to the following Hilbert polynomials of projective smooth manifolds X :

(i) If X is of general type or Fano: consider $H_{-K_X}(n) := \chi(-nK_X)$.

(ii) If X is a Calabi–Yau manifold embedded as an anticanonical section in a Fano manifold: consider the Euler characteristic of the powers of the induced anticanonical sheaf.

b) With this normalization, the Serre duality leads to a functional equation for the Hilbert polynomial of the $s \mapsto -1 - s$ type.

c) In many cases, the well known inequalities for Chern numbers of X imply that all roots of $H(s)$ lie in the critical strip $-1 < \operatorname{Re} s < 0$, and sometimes even more precise statements. For example, Yau’s inequality $c_1^3 \geq 8/3c_1c_2$ for Fano’s threefolds shows this fact for them.

0.6.1 Question. Is there a systematic relationship between Hilbert polynomials and zeta functions of schemes (or more general spaces) over F_1 ?

The existence of such zeta functions and their structure in certain cases was heuristically suggested in [Ma1] (cf. also [Ku]). They make precise sense for some specimens in Soulé’s category, and are indeed polynomials; see also [CC] for essential complements. An obvious attack on question 0.6.1 might start with comparing the counting of F_{1^n} -points with counting of monomials in cyclotomic coordinates.

Roots of unity appear in this context via the following beautiful observation due to F. Rodriguez–Villegas, [RV].

Consider first an arbitrary polynomial $H(q) \in \mathbf{Z}[q]$. Define another polynomial $P(t)$ such that

$$\sum_{n=0}^{\infty} H(n)t^n = \frac{P(t)}{(1-t)^d}$$

where $P(1) \neq 0$. Let $d := \deg H + 1$, $e := \deg P$.

Rodriguez–Villegas proves that *if all roots of P lie on the unit circle, then $H(q)$ has simple roots at $q = -1, \dots, e + 1 - d$ and possible additional roots at the middle of this critical strip $\operatorname{Re} q = \frac{e-d}{2}$.*

This result can be applied to the case when $H(n) = \dim A_n$ is the Hilbert polynomial of a graded algebra $\bigoplus_{n=0}^{\infty} A_n$ generated by A_1 , of Krull dimension d , complete intersection of polydegree (n_1, \dots, n_s) .

It turns out that $e = n_1 + \dots + n_s$ and $P(t) = \prod_{j=1}^s (1 + t + \dots + t^{n_j})$ so that all roots of P are in fact roots of unity.

The critical strip in [Go] has width 1, because Golyshev differently normalizes the grading via $-K_X$. However, the Rodriguez–Villegas grading agrees with the motivic philosophy involving weights and Tate’s motives over F_1 , see [Ma1].

0.7. Summary. We are guided by the following heuristics. Each time that roots of unity appear in a certain context, we try to interpret the functions whose values are these roots of unity as cyclotomic coordinates on a relevant F_1 –scheme, in the sense of the Definition 1.7.1 below, or a version thereof.

An appropriate version of (big) Witt vectors must furnish the basic F_1 –analytic (or formal) approximation to the arithmetic line $\text{Spec } \mathbf{Z}$.

1. Geometry over F_1 : generalities

1.1. Affine schemes over F_1 according to Toën and Vaquié. Affine schemes over F_1 arise in the most straightforward (and allowing vast generalizations) manner in the framework of [TV], according to which algebraic geometry over F_1 is a special case of algebraic geometry relative to a monoidal symmetric category $(C, \otimes, \mathbf{1})$, which is assumed to be complete, cocomplete, and to admit internal Hom ’s.

Such a category C gives rise to the category of commutative, associative and unitary monoids $\text{Comm}(C)$ which serves as a substitute for the category of ordinary commutative rings. Each object A of $\text{Comm}(C)$ determines the category of A –modules $A\text{-Mod}$ consisting of pairs (M, μ) where M is an object of C together with action $\mu : A \otimes M \rightarrow M$ and satisfying the usual formalism.

The opposite category $\text{Aff}_C := \text{Comm}(C)^{\text{opp}}$ is called the category of C –affine schemes, and the tautological functor $\text{Comm}(C) \rightarrow \text{Aff}_C$ is called Spec .

According to [TV], we obtain F_1 –geometry as the geometry relative to the monoidal category of sets and direct products $(\text{Ens}, \times, *)$.

Commutative rings relative to $(\text{Ens}, \times, *)$ are just the ordinary commutative (associative, unital) monoids written multiplicatively: this explains the popular motto that to do F_1 –algebra one must forget the additive structure: cf. [Har]. This structure is restored when one applies the functor “base change” $\otimes_{F_1} \mathbf{Z}$: a

monoid M turns into the commutative associative unital ring $\mathbf{Z}[M]$. The opposite to monoids category will be denoted Aff_{F_1} .

More generally, any commutative ring R determines the base extension functor

$$\otimes_{F_1} R : Aff_{F_1} \rightarrow Aff_R, M \mapsto R[M],$$

from affine schemes over F_1 to affine schemes over $Spec R$.

Elements of monoids M will be called *cyclotomic coordinates* on the respective affine scheme. The same term will refer to their images in $R[M]$. On more general schemes, we may speak about local cyclotomic coordinates.

1.2. Deitmar's affine schemes. A. Deitmar in [De1] adopts the same definition of the category Aff_{F_1} . Moreover, he associates to a monoid M a topological space which we will denote $spec M$ (to distinguish it from the spectrum of prime ideals of a ring $Spec(*)$), and which is endowed with a structure sheaf. Points of this space are prime ideals $P \subset M$: such submonoids that $xy \in P$ implies $x \in P$ or $y \in P$. Basic open sets and the structure (pre)sheaf of monoids are determined via localization, just as in the classical case of commutative rings. Moreover, Deitmar characterizes morphisms in Aff_{F_1} in terms of appropriate morphisms of topological spaces $spec$ with structure sheaves.

1.3. Examples. (i) *Affine F_1 -schemes associated to abelian groups.* Let M be an abelian group considered as a monoid in Ens . We have

$$spec M \otimes_{F_1} \mathbf{Z} = Spec \mathbf{Z}[M].$$

In particular, [TV] define F_{1^n} as the monoid (group) $\mathbf{Z}/n\mathbf{Z}$, and its spectrum after lifting to \mathbf{Z} becomes

$$spec F_{1^n} \otimes_{F_1} \mathbf{Z} := Spec \mathbf{Z}[q]/(q^n - 1) = Spec \mathbf{Z}[q]/\{n\}_q. \quad (1.1)$$

In [So], the study of $\mathbf{Z}[q]/\{n\}_q$ -points of an a priori given ordinary scheme X gives clues to finding its F_1 -forms identified with certain subfunctors of F_{1^n} -points.

In our paper, formula (1.1) motivates the introduction of analytic functions on (certain) F_1 -schemes via Habiro's formalism: morally, they are functions that are defined *at all F_{1^n} -points, but nowhere else*. (In fact, the latter stricture should not be taken too literally: some functions have very interesting p -adic, and sometimes complex, arguments and values as well).

(ii). *Affine scheme* G_{m,F_1} . Over F_1 , it is represented by the spectrum of the infinite cyclic group \mathbf{Z} . Lifted to \mathbf{Z} , it becomes the ordinary $G_m = \text{Spec } \mathbf{Z}[q, q^{-1}]$.

1.4. Affine spaces. Affine line $A_{F_1}^1$ is the spectrum of the infinite cyclic monoid \mathbf{N} . Its lift to \mathbf{Z} is $A_{\mathbf{Z}}^1 := \text{Spec } \mathbf{Z}[q]$. Similarly, $A_{F_1}^k$ is the spectrum of \mathbf{N}^k , $k \geq 1$.

The space $\text{spec } \mathbf{N}$ consists of one closed point (q) and one generic point.

One can also consider \mathbf{N}^\times that is, the free monoid freely generated by all primes. Its lift to \mathbf{Z} is the ring of polynomials in infinitely many variables indexed by primes.

1.5. Affine scheme $GL(n)_{F_1}$. According to [TV], Proposition 4.1, the natural sheaf (in the Grothendieck flat topology, see [TV] and 1.6 below) of automorphisms of a free module of rank n is represented, after lifting to \mathbf{Z} , by the semidirect product of G_m^n and \mathbf{S}_n (the symmetric group).

In the more down-to-earth language of [De1], sec. 5, this is expressed as follows. Let A be a commutative monoid. Define the (set-theoretic) group of “ A -valued points of $GL(n)_{F_1}$ ” as

$$GL(n)_{F_1}(A) := \text{Aut}_A(A^n).$$

This can be identified with the group of (n, n) matrices with entries in $A \subset \mathbf{Z}[A]$, having exactly one non-zero element in each row and each column. This is precisely the description of [TV] quoted above.

The reader should be warned that, unlike to what happened with A^k and G_m , after lifting $GL(n)_{F_1}$ to \mathbf{Z} we *do not* get the usual $GL(n)_{\mathbf{Z}}$. This caused a difficulty in the framework of [So], where it was not obvious how to choose “cyclotomic points of $GL(n)_{\mathbf{Z}}$.” In fact, according to [TV], Proposition 4.1, $GL(n)_{\mathbf{Z}}$ for $n > 1$ is not a lift of an F_1 -scheme in their sense.

1.6. General schemes over F_1 . Glueing general schemes from affine ones is defined differently in [TV] and [De1] respectively.

For Deitmar, an F_1 -scheme is a topological space with a sheaf of monoids that is everywhere locally affine, that is, locally isomorphic to some $\text{spec } M$.

Toën and Vaquié endow the category Aff_C with a natural Grothendieck’s topology, which is called *the flat topology*. Using it, one can define general *schemes* relative to C , as functors that can be obtained from disjoint unions of affine schemes X by taking the quotient with respect to an equivalence relation $R \subset X \times X$ such that projections $R \rightarrow X$ are local Zariski isomorphisms. Such schemes form a category denoted $\text{Sch}(C)$.

1.7. Schemes over F_1 à la Soulé. The idea of Soulé’s definitions in [So] can be succinctly formulated as the project of direct reconstruction of F_1 –schemes X of *finite type* from certain schemes $X_{\mathbf{Z}}$ over \mathbf{Z} endowed with an additional data \mathcal{A}_X which is a \mathbf{C} –algebra, morally an algebra of functions on the “ ∞ –adic completion” of X .

This latter structure embeds F_1 –geometry into a wider context, potentially containing also rich structures of Arakelov, or ∞ –adic geometry. Some hints that this should be necessary and possible can be glimpsed in the remark made in [Ma1], 1.7. Namely, in [Ma1] it was suggested that the zeta function of $\mathbf{P}_{F_1}^k$ must be $(2\pi)^{-(k+1)}s(s-1)\dots(s-k)$.

Combining this with Deninger’s representation of the basic Euler Γ –factor at arithmetical infinity as a regularized product

$$\Gamma_{\mathbf{C}}(s)^{-1} := \frac{(2\pi)^s}{\Gamma(s)} = \prod_{n \geq 0} \frac{s+n}{2\pi}$$

we see that this gamma–factor should be understood as the zeta–function of the (motivic dual of) an infinite dimensional projective space over F_1 .

However, the existing framework is too narrow to make sense of this statement: although the zeta of $\mathbf{P}_{F_1}^k$ is now defined in [So] and agrees with expectations of [Ma1] (up to a power of 2π), the infinite–dimensional case and its connections with ∞ –adic geometry still elude us. A promising approach extensively elaborated in the thesis by N. Durov (cf. [Du]) might pave the road to this unification. The treatment of the Bost–Connes dynamical system in [CCMa2] provides another bridge between F_1 –geometry and the archimedean world.

Returning to [So], we will now sketch his version of F_1 –schemes of finite type.

The data defining such a scheme X consist of:

- (i) A \mathbf{Z} –scheme of finite type $X_{\mathbf{Z}}$.
- (ii) A subfunctor $X(R)$ of the functor of points of $X_{\mathbf{Z}}$ from a category of rings to the category of sets:

$$X_{\mathbf{Z}}(R) := \text{Hom}(\text{Spec } R, X_{\mathbf{Z}}).$$

Here R runs over rings that are direct summands of $\otimes_i \mathbf{Z}[q]/(\{n_i\}_q)$, and each $X(R)$ is required to be a finite set. We will call elements of $X(R)$ “cyclotomic points”.

- (iii) A \mathbf{C} –algebra \mathcal{A}_X , and an assignment of complex values to each element $f \in \mathcal{A}_X$ at each pair consisting of point of $X(R)$ and a ring homomorphism $R \rightarrow \mathbf{C}$.

We will not spell out here the compatibility requirements between these data, which are pretty straightforward.

Morphisms of schemes over F_1 are pairs, consisting of functor morphisms of cyclotomic points and contravariant homomorphisms of function algebras, compatible with the rest of the data.

It is natural to call an F_1 -scheme X affine, if $X_{\mathbf{Z}}$ is affine. But without further restrictions, one would get many schemes over \mathbf{Z} into which the cyclotomic points could be embedded as a subfunctor. The restriction that restores the uniqueness of $X_{\mathbf{Z}}$ once X is known declares that $X_{\mathbf{Z}}$ must be the initial object in the category of such embeddings (see [So], sec. 4, Definition 3, for a precise statement). A similar universality requirement defines general F_1 -schemes (loc. cit., Definition 5).

We will now formally define the notion of *cyclotomic coordinates* on Soulé's F_1 -schemes. Let X be an affine F_1 -scheme, $X_{\mathbf{Z}} = \text{Spec } A$.

1.7.1. Definition. *A cyclotomic coordinate on the affine F_1 -scheme X in the sense of Soulé is any element $f \in A$ whose values at all cyclotomic points $X(R)$ are either 0, or roots of unity.*

Clearly, cyclotomic coordinates in this sense form a commutative monoid with unit. If the scheme X is not affine, local cyclotomic coordinates can be defined, forming a (pre)sheaf of commutative monoids.

Recall that in the framework of [TV] and [De1], where \mathbf{Z} -lifts of F_1 -spaces are patched from spectra of monoid ring $\mathbf{Z}[S]$, the elements of S themselves were called cyclotomic coordinates. However, since these versions of F_1 -schemes are not equivalent to Soulé's one, we should use this term being aware of its context.

Notice also that

a) To reconstruct cyclotomic coordinates in the sense 1.7.1, it is sufficient to know $X_{\mathbf{Z}}$ and the functor $R \rightarrow X(R) \subset X_{\mathbf{Z}}(R)$. This is a part of the structure of *a gadget*, as Soulé's *truc* was translated in [CCMa2]).

b) The rings R used by Soulé to probe schemes over F_1 are essentially group rings of finite abelian groups.

Conceivably, one could replace finite abelian groups by finite commutative unital monoids, thus narrowing the gap between [So] and [TV], [De].

c) Moreover, one could sketch rudiments of *supergeometry* over F_1 , by requiring Z_2 -grading of our monoids, a structure subgroup $\{\pm 1\}$, and the anticommutation rule for odd elements.

The following example from [So] serves as a good illustration of similarities and differences between affine schemes in the sense of [So] and [TV] respectively, and of relationships of F_1 to Arakelov geometry.

1.7.2. Arakelov vector bundles over $\text{Spec } \mathbf{Z}$ as affine F_1 -schemes. An Arakelov vector bundle $\bar{\Lambda}$ over $\text{Spec } \mathbf{Z}$ is defined as a pair consisting of a free abelian group Λ of finite rank and an hermitean norm $\|\cdot\|$ on $\Lambda_{\mathbf{C}} := \Lambda \otimes \mathbf{C}$, “integral structure at arithmetical infinity”. The global sections of $\bar{\Lambda}$ over the “compactification” $\text{Spec } \mathbf{Z} \cup \infty$ are defined as $B \cap \Lambda$, where $B := \{x \in \Lambda_{\mathbf{C}} \mid \|x\| \leq 1\}$.

In order to produce a Soulé’s affine scheme $X(\bar{\Lambda})$ out of $\bar{\Lambda}$, make an additional choice (of which the final product will not depend): choose a finite subset $\Phi \subset B \cap \Lambda \setminus \{0\}$ such that if $v \in B \cap \Lambda \setminus \{0\}$, then exactly one element of the pair $\{v, -v\}$ belongs to Φ . Let Λ_0 be the sublattice of Λ generated by Φ , Λ_0^t the dual lattice.

Now we can define the structure data.

- (i) $X(\bar{\Lambda})_{\mathbf{Z}} := \mathbf{Z}[\Lambda_0^t]$.
- (ii) The points of $X(\bar{\Lambda})(R)$ are given by the following prescription:

$$X(\bar{\Lambda})(R) := \left\{ x = \sum_{v \in \Phi} v \otimes \zeta_v \mid x \in \Lambda \otimes_{\mathbf{Z}} R, \zeta_v \in \mu(R) \cup \{0\} \right\}.$$

Equivalently, coefficients at $v \in \Phi$ are cyclotomic coordinates.

(iii) $\mathcal{A}_{X(\bar{\Lambda})}$ is defined as the algebra of functions holomorphic and continuous on the boundary of the following domain:

$$C := \{x \in \Lambda_0 \otimes \mathbf{C} \mid \|x\| \leq \text{card } \Phi\}.$$

Given a homomorphism $\sigma : R \rightarrow \mathbf{C}$, a cyclotomic R -valued point x and a function $f \in \mathbf{Z}[\Lambda_0^t]$, we get its value at (x, σ) in an obvious way.

Comparing this example to the definitions of [TV] and [De1], we see that the algebra $X(\bar{\Lambda})_{\mathbf{Z}} := \mathbf{Z}[\Lambda_0^t]$ fits in their framework, but that other elements of the structure significantly change morphisms and points.

1.7.3. A non-affine case: toric varieties. The treatment of this case in [TV], [De1] and [So] leads to essentially the same object (although Soulé produces his \mathbf{C} -algebra only in the smooth case i. e. for regular fans).

Let Δ be a fan. Each element $\sigma \in \Delta$ determines the dual cone σ^* . Let M_{σ} be the commutative monoid of integer points of σ^* , U_{σ} is its spectrum as F_1 -scheme. If τ

is a face of σ , we get a morphism of monoids $M_\sigma \rightarrow M_\tau$. The respective morphism of schemes $U_\tau \rightarrow U_\sigma$ is Zariski open. Put $X := \coprod_{\sigma \in \Delta} U_\sigma$. According to [TV], 4.2, the quotient of $X \times X$ modulo equivalence relation $R := \coprod_{\sigma, \tau \in \Delta} U_{\sigma \cap \tau}$ defines an F_1 -scheme $X(\Delta)_{F_1}$. Lifting it to \mathbf{Z} , we get the classical toric scheme $X(\Delta)$.

In [De1], the same quotient is straightforwardly interpreted as a glueing of monoid spectra. In [So] the picture is enhanced by an appropriate \mathbf{C} -algebra.

1.8. A summary. Deitmar’s definition of the category of schemes over F_1 is, as he himself stresses in the opening paragraph of [De2], a minimalistic one. It is quite transparent, but obviously does not allow one to treat some more sophisticated situations, such as Soulé’s scheme 1.7.2. In fact, the Theorem 4.1 of [De2] shows that if X is a connected integral F_1 -scheme of finite type, then its lift to \mathbf{C} , $X_{\mathbf{C}}$ consists of a finite union of mutually isomorphic toric varieties.

The richness of Toën and Vaquié’s definition becomes apparent, when it is applied to other basic symmetric monoidal categories. Especially remarkable is the extension of F_1 -geometry $\mathbf{S}_1\text{-Sch}$ which is the category of schemes relative to the category $(SEns, \times, *)$ of simplicial sets with direct product. There is a canonical functor “base extension” $\mathbf{S}_1\text{-Sch} \rightarrow F_1\text{-Sch}$, so that this geometry lies “below” F_1 -geometry, in the same sense as F_1 -geometry lies below \mathbf{Z} -geometry. Another extension with great future is the algebraic geometry over “brave new rings”.

One outstanding problem is to extend cyclotomy to the homotopical framework.

This is an appropriate place to stress that in a wider context of [TV], or eventually in noncommutative F_1 -geometry, the spectrum of F_1 loses its privileged position of a final object of a geometric category. For example, in noncommutative geometry, or in an appropriate category of stacks, the quotient of this spectrum modulo the trivial action of a group must lie below this spectrum.

Soulé’s algebras \mathcal{A}_X are a very important element of the structure, in particular, because they form a bridge to Arakelov geometry. Soulé uses concrete choices of them in order to produce “just right” supply of morphisms, without *a priori* constraining these choices formally. In this work, we use these algebras and their version also to pave a way to the analytic (and possibly non-commutative) geometry over F_1 .

In [CC], Soulé’s approach is clarified and generalized; it is shown that Chevalley groups and Tits geometries can be accomodated in this extended version as well.

2. Habiro's analytic functions of many variables: statements of results

2.1. Notations. *Rings* in this and the next sections are associative, commutative and unital, unless a context suggests otherwise. *Ring homomorphisms* are unital. Letters $R, R_0, R_1 \dots$ denote rings, $q, q_0, q_1 \dots$ are independent commuting variables.

Let R be a ring, $\mathcal{I} = \{I_\alpha\}$ a family of ideals filtered by inclusion. The ring projective limit $\varprojlim_\alpha R/I_\alpha$ is called the completion of R with respect to \mathcal{I} and denoted $\widehat{R}_\mathcal{I}$ or some version of this notation. When \mathcal{I} is (cofinal to) the family of powers of one ideal I , the respective limit is called the I -adic completion.

We say that R is \mathcal{I} - (resp. I -adically) *separated*, if $\bigcap_\alpha I_\alpha = \emptyset$. Equivalently, the canonical homomorphism $R \rightarrow \widehat{R}_\mathcal{I}$ is injective. Example: $R = \mathbf{Z}$, \mathcal{I} any infinite filtering system.

When q is considered as a “quantization parameter”, our quantized (Gaussian) versions of integers and factorials are, as in [Ha2],

$$\{N\}_q := q^N - 1, \quad \{N\}_q! := \{N\}_q \{N-1\}_q \dots \{1\}_q. \quad (2.1)$$

Fix an integral domain R_0 of characteristic zero and put $R_n := R_0[q, \dots, q_n]$, with natural embeddings $R_0 \subset R_1 \subset R_2 \subset \dots$

Denote by $I_{n,N} \subset R_n$ the ideal $(\{N\}_{q_1}!, \dots, \{N\}_{q_n}!)$, $N \geq 1$. Clearly, $I_{n,N} \subset I_{n,N+1}$ so that the rings $R_n^{(N)} := R_n/I_{n,N}$, n being fixed, form an inverse system.

2.2. Definition. *The ring of Habiro's analytic functions of n variables over R_0 is defined as*

$$\widehat{R}_n := \varprojlim_N R_n^{(N)}.$$

2.3. Taylor series of analytic functions. Choose a vector of roots of unity $\zeta = (\zeta_1, \dots, \zeta_n)$ such that all ζ_i are in R_0 . For any integer $M > 0$, there exists $N_0 = N_0(\zeta, M)$ such that $I_{n,N} \subset (q_1 - \zeta_1, \dots, q_n - \zeta_n)^M$ for all $N \geq N_0$. In fact, $\{N\}_{q_i}!$ is divisible by any fixed monomial $(q_i - \zeta)^M$, $\zeta \in \mu$, if N is large enough.

The completion $\varprojlim_M R_n/(q_1 - \zeta_1, \dots, q_n - \zeta_n)^M$ is $R[[q_1 - \zeta_1, \dots, q_n - \zeta_n]]$.

Therefore we obtain a ring homomorphism “Taylor expansion at the point ζ ”:

$$T_n(\zeta) : \widehat{R}_n \rightarrow R_0[[q_1 - \zeta_1, \dots, q_n - \zeta_n]].$$

2.3.1. Theorem. *If R_0 is an integral domain, p -adically separated for all primes p , then the same is true for \widehat{R}_n , and the Habiro–Taylor homomorphism $T_n(\zeta)$ is injective.*

More generally, let $F = \{F_1, \dots, F_n\} \in \mathbf{Z}[q]$ be a family of monic polynomials in $R_0[q]$ whose all roots are roots of unity. Denote by (F) the ideal generated $F_1(q_1), \dots, F_n(q_n)$ in R_n . In place of the formal series ring above, we can consider the completion

$$\widehat{R}_F := \varprojlim_M R_n / (F)^M$$

and the respective Taylor expansion homomorphism:

$$T_n(F) : \widehat{R}_n \rightarrow \widehat{R}_F.$$

2.3.2. Theorem. *If R_0 is an integral domain, p -adically separated for all p , $R[[F]]$ is as well p -adically separated, and the homomorphism $T_n(F)$ is injective.*

K. Habiro proved these results, as well as their generalizations, for $n = 1$, and we build upon his proof.

2.3.3. Differential calculus. Divided powers of partial derivatives with respect to q_k are continuous in the linear topologies generated by $I_{n,N}$, resp. by all $(q_1 - \zeta_1, \dots, q_n - \zeta_n)^M$. Hence these derivatives make sense in \widehat{R}_n , and their values at $(\zeta_1, \dots, \zeta_n)$ are the Taylor coefficients of the respective series.

(In order to check the continuity with respect to $I_{n,N}$ it suffices to notice that as N tends to infinity, $\{N\}_q!$ as a polynomial of q vanishes at a growing set of roots of unity with infinitely growing multiplicity at each root of unity. Taking a derivative of such a sequence of polynomials does not destroy this property).

Thus we can develop for \widehat{R}_n the conventional formalism of tangent and cotangent modules, differential forms etc.

2.4. Elements of \widehat{R}_n as functions on roots of unity. Let $R'_0 \supset R_0$ be an integral domain flat over R_0 and containing all roots of unity (that is, all cyclotomic polynomials $q^n - 1$ completely split in R'_0). Denote by μ the set of all roots of unity in R'_0 . Choose $\zeta := (\zeta_1, \dots, \zeta_n) \in \mu^n$. Any element of R_n , being a polynomial in (q_1, \dots, q_n) , takes a certain value at ζ belonging to R'_0 . If $N \geq N_0(\zeta)$, all elements of $I_{n,N}$ vanish at ζ . Hence any element $f \in \widehat{R}_n$ defines a map $\bar{f} : \mu^n \rightarrow R'_0$. This map is R_0 -linear and compatible with pointwise addition and multiplication of functions.

Besides assuming that R_0 is p -adically separated for all primes p , impose the following separatedness condition: *for any infinite sequence of pairwise distinct primes p_1, \dots, p_k, \dots , we have*

$$\bigcap_{m=1}^{\infty} R_{p_1 \dots p_m} = \{0\}. \quad (2.2)$$

2.4.1. Theorem. *Under these assumptions, the map $f \mapsto \bar{f}$ is injective.*

One can also formulate this statement without adjoining to R_0 roots of unity.

2.4.2. Theorem. *The natural map $\widehat{R}_n \rightarrow \prod_{m=1}^{\infty} \widehat{R}_n \bmod (\Phi_m(q_1), \dots, \Phi_m(q_n))$ is injective.*

For $n = 1$, these results were established by K. Habiro. He has also shown that vanishing of \bar{f} on certain sufficiently large subsets of μ suffices to establish the vanishing of f .

More precisely, *Habiro's topology* on the set μ of all roots of unity is defined as follows (cf. [Ha2], 1.2).

Two roots of unity ξ, η are called *adjacent*, if $\xi\eta^{-1}$ is of order p^m , $m \in \mathbf{Z}$, p a prime; or equivalently, if $\xi - \eta$ is not a unit (as an algebraic number). Clearly, the action of $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ preserves adjacency.

2.4.3. Definition. *A subset $U \subset \mu$ is called open, if for any point $\xi \in U$, all except of finitely many $\eta \in \mu$, adjacent to ξ , belong to U .*

The Galois action is continuous in this topology, in marked contrast to the topology induced from \mathbf{C} .

Let now μ' be an infinite set of roots of unity. A point $\xi \in \mu'$ is a limit point of μ' , if for any open neighborhood U of ξ we have $\mu' \cap (U \setminus \xi) \neq \emptyset$. In Habiro's topology, this means that μ' contains infinitely many points, adjacent to ξ .

2.4.4. Theorem. *Under the notations and assumptions of Theorem 2.3.1, let $\nu = \nu_1 \times \dots \times \nu_n \subset \mu^n$ be a set, such that each $\nu_i \subset \mu$ has a limit point. Let $f \in \widehat{R}_n$. If the restriction $\bar{f}|_{\nu}$ is identical zero, then $f = 0$.*

In the next section, we will prove this last result; Theorems 2.4.1 and 2.4.2 follow from it.

2.5. Analogs of Habiro's functions on the arithmetic axis and analytic continuation. The Habiro ring of one variable $\varprojlim_N \mathbf{Z}[q]/(\{N\}_q!)$ "is" the lift to \mathbf{Z} of an imaginary ring $\varprojlim_N F_1[q]/(\{N\}_q!)$.

Along the arithmetical axis, the straightforward analog of the latter exists: this is the topological ring of profinite integers $\widehat{\mathbf{Z}} := \varprojlim_N \mathbf{Z}/(N!)$. Its elements can be uniquely represented by infinite series $\sum_{n=1}^{\infty} c_n n!$ where c_n are integers with $0 \leq c_n \leq n$, cf. [van D].

H. Lenstra in [Le] discusses profinite Fibonacci numbers: continuous extrapolation to $n \in \widehat{\mathbf{Z}}$ of the Fibonacci function $n \mapsto u_n$.

An analog of the profinite number $1 + \sum_{n=1}^{\infty} (-1)^n n!$ is the remarkable example of Habiro function of one variable

$$1 + \sum_{n=1}^{\infty} (-1)^n \{n\}_q! = 1 + \sum_{n=1}^{\infty} (1-q) \dots (1-q^n).$$

As a function on roots of unity, it emerged in a work of M. Kontsevich on Feynman integrals (talk at MPIM, 1997). Don Zagier in [Za] proved that its values, as well as values of its derivatives, are radial limits of the function (resp. its derivatives) holomorphic in the unit circle

$$\frac{1}{2} \sum_{n=1}^{\infty} n \chi(n) q^{(n^2-1)/24},$$

where χ is the quadratic character of conductor 12.

2.6. Habiro's functions on F_1 -schemes. Let X be an F_1 -scheme in the sense of one of the definitions from sec. 1. Let (x_1, \dots, x_n) be a finite family of local cyclotomic coordinates on X . For any ring R as in 1.7. (ii), denote by $U(R) \subset X(R)$ the set of cyclotomic points, at which all x_i are defined and take non-zero values.

Consider an analytic function $f \in \widehat{R}_n$ in the sense of Habiro. This function then defines a map

$$f_R : U(R) \rightarrow R, f_R(r) := \bar{f}(x_1(r), \dots, x_n(r)),$$

with evident functorial properties.

In an appropriate setting such functions must be local sections of a global sheaf. I hope to return to this problem in another paper. Here I will restrict myself to the following observations.

(i) We have to exclude zero values, because q_1 is invertible in \widehat{R}_1 , and hence each monomial $q_1^{m_1} \dots q_n^{m_n}$ is invertible in \widehat{R}_n . In fact,

$$q^{-1} = 1 + \sum_{n=1}^{\infty} (-1)^n q^n \{n\}_q!,$$

see [Ha1], Proposition 7.1.

(ii) From the perspective of this paper, it seems quite natural to consider localizations with respect to functions such as $q_1^{m_1} \dots q_n^{m_n} - 1$, deleting sets of roots of unity closed in Habiro's topology. However, such functions are generally not cyclotomic coordinates. This runs counter the spirit of Toën–Vaqué's definitions, and requires rethinking of their framework.

3. Habiro's analytic functions of many variables: proofs and generalizations

3.1. The case $n = 1$. Assuming that a ring R is \mathcal{I} -separated for each member \mathcal{I} of some set of filters \mathcal{S}_R , we can deduce that the ring $R[q]$, and certain its completions, are separated with respect to the members of another set of filters, say $\mathcal{S}_{R[T]}$. Results of this type are collected and proved in [Ha1]. They will allow us to perform inductive steps, passing from n to $n + 1$,

3.2. Proof of the Theorem 2.3.1. We will perform induction on n , using Habiro's theorem for $n = 1$ ([Ha1], Theorem 5.2) as the basis of induction.

Assuming the theorem proved for \widehat{R}_n , we will proceed by decomposing the Taylor series map $\widehat{R}_{n+1} \rightarrow R_0[[q_1 - \zeta_1, \dots, q_n - \zeta_n, q_{n+1} - \zeta_{n+1}]]$ into the product of two ring homomorphisms and checking injectivity of each one:

$$\widehat{R}_{n+1} \xrightarrow{\alpha} \widehat{R}_n[[q_{n+1} - \zeta_{n+1}]] \xrightarrow{\beta} R_0[[q_1 - \zeta_1, \dots, q_n - \zeta_n]] [[q_{n+1} - z_{n+1}]]$$

Now we will define the arrows α, β and check their properties.

The arrow β is continuous in $(q_{n+1} - \zeta_{n+1})$ -adic topology, acts identically on $q_{n+1} - \zeta_{n+1}$, and sends each element of \widehat{R}_n to its Taylor series at $(\zeta_1, \dots, \zeta_n)$. In view of the inductive assumption, β is injective.

To define α , consider an element $g \in \widehat{R}_{n+1}$. It can be represented as the limit of a sequence of polynomials $g_1, g_2, \dots, g_N, \dots$, where $g_i \in R_0[q_1, \dots, q_{n+1}]$ such that $g_{N+1} \equiv g_N \pmod{I_{n+1, N}}$.

From the definition it follows that

$$I_{n+1, N} = I_{n, N}[q_{n+1}] + R_{n+1} \cdot \{N\}_{q_{n+1}}!$$

Therefore,

$$g_{N+1} = g_N + i_N + r_N \cdot \{N\}_{q_{n+1}}!, \tag{3.1}$$

where

$$i_N \in I_{n,N}[q_{n+1}], \quad r_N \in R_{n+1}.$$

Now consider a point $(\zeta_1, \dots, \zeta_{n+1})$ as above. Clearly,

$$I_{n,N}[q_{n+1}] = I_{n,N}[q_{n+1} - \zeta_{n+1}].$$

Write $g_N, i_N, \{N\}_{q_{n+1}}!$ as polynomials in $q_{n+1} - \zeta_{n+1}$ with coefficients in R_n . When N becomes large enough, $\{N\}_{q_{n+1}}!$ starts with arbitrary large power of $q_{n+1} - \zeta_{n+1}$. Therefore for any given M , the coefficient at $(q_{n+1} - \zeta_{n+1})^M$ in g_{N+1} is the same as in $g_N + i_N$ if $N \geq N_1(M, \zeta)$. Hence the sequence of these coefficients (M being fixed and N growing) converges to a certain element $a_M \in \widehat{R}_n$.

Put $\alpha(g) := \sum_{M=0}^{\infty} a_M (q_{n+1} - \zeta_{n+1})^M$. One can routinely check that $\alpha(g)$ depends only on $g \in \widehat{R}_{n+1}$ and not on the system (g_N) chosen to represent g . Moreover, we get a ring homomorphism

$$\alpha : \widehat{R}_{n+1} \rightarrow \widehat{R}_n[[q_{n+1} - \zeta_{n+1}]]. \quad (3.2)$$

Let us check that α is injective. In fact, take a nonzero element $g = \lim g_N$. Then there exist arbitrarily large N such that $g_N \notin I_{n+1,N}$. Representing g_N as a polynomial in $q_{n+1} - \zeta_{n+1}$ with coefficients in R_n , we can find in this polynomial a coefficient, not belonging to $I_{n,N}$. In the limit, it will produce a nonvanishing a_M .

Finally, $\beta \circ \alpha = T_{n+1}(\zeta)$ by construction.

3.3. Proof of the Theorem 2.4.4. We first remark, that the case $n = 1$ is essentially covered by the Theorem 6.1 of [Ha1], if one weakens the assumption $R_0 \subset \overline{\mathbf{Q}}$ in the statement of this Theorem. In fact, this assumption is used only at the end of the proof, in order to ensure the validity of the separatedness condition (2.2). Instead, we will simply postulate (2.2) for R_0 , and then deduce it for each \widehat{R}_n using the Taylor embedding of R_n into $R_0[[q_1 - \zeta_1, \dots, q_{n+1} - \zeta_{n+1}]]$.

To pass from n to $n + 1$, I will start with the following remarks.

Let R be a ring endowed with a filtering family of ideals $\mathcal{I} = \{I_\alpha\}$. Consider the following two families of ideals in the polynomial ring $R[q]$:

- (i) $\mathcal{I}_1 := \{I_\alpha[q] + (\{N\}_q!) \mid \alpha, N \text{ arbitrary}\}$.
- (ii) $\mathcal{I}_2 := \{(\{N\}_q!) \mid N \text{ arbitrary}\}$.

Denote by $R[q]^\sim$ (resp. $R[q]^\widehat{}$) the completion of $R[q]$ with respect to \mathcal{I}_1 (resp. \mathcal{I}_2).

For any N and α , we have natural surjections

$$R[q]/(\{N\}_q!) \rightarrow R[q]/(I_\alpha[q] + (\{N\}_q!)).$$

Passing to the limit, we get a canonical surjection

$$\varphi : R[q]^\wedge \rightarrow R[q]^\sim.$$

3.3.1. Lemma. *Consider the case $R = \widehat{R}_n$, $\mathcal{I} = \{\widehat{I}_{n,N}\}$ where*

$$\widehat{I}_{n,N} := (\{N\}_{q_1!}, \dots, \{N\}_{q_n!}) \subset \widehat{R}_n.$$

Then the homomorphism

$$\varphi : \widehat{R}_n[q_{n+1}]^\wedge \rightarrow \widehat{R}_n[q_{n+1}]^\sim = \widehat{R}_{n+1}$$

is an isomorphism.

Proof. It suffices to check that $\text{Ker } \varphi = \{0\}$. In fact, as in 3.2, we have an injection

$$\alpha : \widehat{R}_n[q_{n+1}]^\sim \rightarrow \widehat{R}_n[[q_{n+1} - 1]]$$

and an one-variable Taylor series injection

$$T : \widehat{R}_n[q_{n+1}]^\wedge \rightarrow \widehat{R}_n[[q_{n+1} - 1]].$$

By construction, $\alpha \circ \varphi = T$, hence φ is an injection as well.

3.3.2. End of proof of the Theorem 2.4.4. Suppose now that $g \in \widehat{R}_{n+1}$ vanishes at all points $(\zeta_1, \dots, \zeta_{n+1})$, $\zeta_i \in \nu_i \subset \mu$, each of ν_i having a limit point. To simplify notation, assume that all roots of unity are in R_0 .

The evaluation of g at $(\zeta_1, \dots, \zeta_{n+1})$ can be decomposed into the composition of two arrows:

$$ev_{(\zeta_1, \dots, \zeta_n)} \circ ev_{\zeta_{n+1}} : \widehat{R}_{n+1} \rightarrow \widehat{R}_n \rightarrow R_0,$$

where the first arrow $ev_{\zeta_{n+1}}$ is obtained by taking the constant term in $\alpha(g)$, (3.2), and the second one is the evaluation at $(\zeta_1, \dots, \zeta_n)$.

First, fix $(\zeta_1, \dots, \zeta_n)$ and vary $\zeta_{n+1} \in \nu_{n+1}$. We have already identified \widehat{R}_{n+1} with $\widehat{R}_n[q_{n+1}]^\wedge$ in a way which is clearly compatible with evaluation maps.

From the Habiro Theorem 6.1, [Ha1], we obtain that

$$ev_{(\zeta_1, \dots, \zeta_n)}(g) = 0$$

for all

$$(\zeta_1, \dots, \zeta_n) \in \nu_1 \times \dots \times \nu_n.$$

By the inductive assumption, $g = 0$. This finishes the proof.

3.4. General monoids, coordinate independence, and functoriality. Let M be a commutative monoid with unit.

We can consider the completion $R'_0[M]$ of $R_0[M]$ with respect to the system of ideals I_N , where I_N is generated by all elements $\{N\}_m!$ for $m \in M$.

Obviously, any morphism $\psi : M \rightarrow N$ induces the respective morphism of the completed rings. In particular, the diagonal morphism $M \rightarrow M \times M$ produces a structure of Hopf algebra on $R_0[M]$ and its completed version on $R'_0[M]$.

As K. Habiro noticed in a message to the author (Aug. 23, 2008), applying this construction to $M = \mathbf{Z}^n$, we get precisely \widehat{R}_n (if q_i corresponds to the basic vector $(0, \dots, 1, 0, \dots, 0)$, with 1 at i -th place.) Since q_i are invertible in \widehat{R}_n , we could as well start with $R_0[q_1, q_1^{-1}, \dots, q_n, q_n^{-1}]$, but it seemed more natural to me to deduce the invertibility at the end of the construction.

4. Schemes with natural cyclotomic coordinates: Witt vectors and moduli spaces

In this section we treat two disjoint constructions.

4.1. Witt functors. The (big) Witt ring scheme W can be defined as an infinite dimensional affine space $\text{Spec } \mathbf{Z}[u_1, u_2, u_3, \dots]$, whose polynomial algebra of functions A is endowed with two homomorphisms $A \rightarrow A \otimes A$, “coaddition” α and “comultiplication” μ .

The functor of its R -points, for a variable commutative ring R , set theoretically is $W(R) = \prod_{k=1}^{\infty} R$ where the k -th coordinate of the product is the value of u_k at the respective R -point. The maps α and μ induce on $W(R)$ the structure of commutative ring, functorial in R . This structure can be described quite explicitly, if we use in place of $\{u_k\}$ the “ghost coordinates”

$$q_n := \sum_{d|n} du_d^{n/d}.$$

In this coordinates, α and μ induce respectively componentwise addition and multiplication (cf. [Haz], sec. 9 and 14, in particular (14.3)).

The N -truncated Witt scheme $W^{(N)}$ is obtained if we apply this to the subring $\mathbf{Z}[u_1, \dots, u_N]$ with induced α_N and μ_N . For a prime p , the scheme W_p is obtained by taking the subring generated by all u_{p^k} , $k \geq 0$. The truncated version $W_p^{(N)}$ jumps only at powers of p as well. In this way we get quotient functors of the Witt functor, valued in commutative algebras.

In place of subrings, one can consider quotients by the ideals generated by the complementary coordinates.

4.1.1. Definition. *The (truncated) Witt gadget $\mathcal{W}^{(N)}$ is defined by the following data:*

(i) $\mathcal{W}_{\mathbf{Z}}^{(N)} := W^{(N)}$.

(ii) *For a ring R as in 1.7, (ii), the subfunctor of cyclotomic points $\mathcal{W}_{\mathbf{Z}}^{(N)}(R)$ of $W^{(N)}(R)$ is defined as consisting of points, whose ghost coordinates are 0 or roots of unity.*

Thus, ghost coordinates are cyclotomic coordinates in the sense of 1.7.1.

4.2. Moduli spaces $\overline{L}_{0;2,B}$. In an ideal world, not only schemes allowing “finite combinatorial” description ([So], p. 217) must be extensions of objects over F_1 , but perhaps “all” rigid structures as well. An obvious challenge is presented by $\overline{M}_{0,n}$, moduli spaces of stable curves of genus zero with n marked points forming the basic operad of quantum cohomology.

As the first approximation, we look in this subsection to some moduli spaces introduced in [LoMa1] and studied further in [LoMa2] and [Ma2]. Generally, they parametrize curves of genus g , with marked points, a part of which (carrying “black” labels) being allowed to merge between them, although not with singular or “white labeled” points. There is an appropriate notion of stability and a representability theorem. (Both were vastly generalized in the study [BaMa].)

Here we will focus on the case of genus zero, two white points and arbitrary (≥ 1) number of black points. The resulting moduli spaces turn out to be toric, based upon permutohedral fans. Therefore they are certainly lifts to \mathbf{Z} of toric F_1 -schemes. We discuss which of the canonical morphisms between them descend to F_1 .

It is convenient to label the black points by elements of a finite set B , carrying no additional structure (rather than, say, by $\{1, \dots, n\}$, which suggests a complete order on labels).

Below we will give a toric description of the respective moduli space that we will now denote \overline{L}_B . For proofs, see [LoMa1].

4.3. Partitions. A partition $\{\sigma\}$ of a finite set B is a *totally ordered set of non-empty subsets of B whose union is B and whose pairwise intersections are empty*. If a partition consists of N subsets, it is called N -partition. If its components are denoted $\sigma_1, \dots, \sigma_N$, or otherwise listed, this means that they are listed in their structure order.

Let τ be an $N + 1$ -partition of B . If $N \geq 1$, it determines a well ordered family of N 2-partitions $\sigma^{(a)}$:

$$\sigma_1^{(a)} := \tau_1 \cup \dots \cup \tau_a, \quad \sigma_2^{(a)} := \tau_{a+1} \cup \dots \cup \tau_{N+1}, \quad a = 1, \dots, N. \quad (4.1)$$

In reverse direction, call a family of 2-partitions $(\sigma^{(i)})$ *good* if for any $i \neq j$ we have $\sigma^{(i)} \neq \sigma^{(j)}$ and either $\sigma_1^{(i)} \subset \sigma_1^{(j)}$, or $\sigma_1^{(j)} \subset \sigma_1^{(i)}$. Any good family is naturally well-ordered by the relation $\sigma_1^{(i)} \subset \sigma_1^{(j)}$, and we will consider this ordering as a part of the structure. If a good family of 2-partitions consists of N members, we will usually choose superscripts $1, \dots, N$ to number these partitions in such a way that $\sigma_1^{(i)} \subset \sigma_1^{(j)}$ for $i < j$.

Such a good family produces one $(N + 1)$ -partition τ :

$$\tau_1 := \sigma_1^{(1)}, \quad \tau_2 := \sigma_1^{(2)} \setminus \sigma_1^{(1)}, \quad \dots, \quad \tau_N := \sigma_1^{(N)} \setminus \sigma_1^{(N-1)}, \quad \tau_{N+1} = \sigma_2^{(N)}. \quad (4.2)$$

This correspondence between good N -element families of 2-partitions and $(N + 1)$ -partitions is one-to-one, because clearly $\sigma_1^{(i)} = \tau_1 \cup \dots \cup \tau_i$ for $1 \leq i \leq N$.

4.4. The fan F_B . Now we will describe a fan F_B in the space $N_B \otimes \mathbf{R}$, where $N_B := \text{Hom}(\mathbf{G}_m, T_B)$, $T_B := \mathbf{G}_m^B / \mathbf{G}_m$. Clearly, N_B can be canonically identified with $\mathbf{Z}^B / \mathbf{Z}$, the latter subgroup being embedded diagonally. Similarly, $N_B \otimes \mathbf{R} = \mathbf{R}^B / \mathbf{R}$. We will write the vectors of this space (resp. lattice) as functions $B \rightarrow \mathbf{R}$ (resp. $B \rightarrow \mathbf{Z}$) considered modulo constant functions. For a subset $\beta \subset B$, let χ_β be the function equal 1 on β and 0 elsewhere.

4.4.1. Definition. *The fan F_B consists of the following l -dimensional cones $C(\tau)$ labeled by $(l + 1)$ -partitions τ of B .*

If τ is the trivial 1-partition, $C(\tau) = \{0\}$.

If σ is a 2-partition, $C(\sigma)$ is generated by χ_{σ_1} , or, equivalently, $-\chi_{\sigma_2}$, modulo constants.

Generally, let τ be an $(l+1)$ -partition, and $\sigma^{(i)}$, $i = 1, \dots, l$, the respective good family of 2-partitions (4.1). Then $C(\tau)$ as a cone is generated by all $C(\sigma^{(i)})$.

4.5. Toric varieties $\overline{\mathcal{L}}_B$ and forgetful morphisms. We denote by $\overline{\mathcal{L}}_B$ the variety associated with the fan F_B . It is smooth and proper, in fact projective.

Assume that $B \subset B'$. Then we have the projection morphism $\mathbf{Z}^{B'} \rightarrow \mathbf{Z}^B$ which induces the morphism $f^{B',B} : N_{B'} \rightarrow N_B$. It satisfies the following property: for each cone $C(\tau') \in F_{B'}$, there exists a cone $C(\tau) \in F_B$ such that $f^{B',B}(C(\tau')) \subset C(\tau)$. In fact, τ is obtained from τ' by deleting elements of $B' \setminus B$ and then deleting the empty subsets of the resulting partition of B .

Therefore, we have a morphism $f_*^{B',B} : \overline{\mathcal{L}}_{B'} \rightarrow \overline{\mathcal{L}}_B$ which we will call *forgetful one* (it forgets elements of $B' \setminus B$). The forgetful morphism is flat, because locally in toric coordinates it is described as adjoining variables and localization.

4.6. $\overline{\mathcal{L}}_B$ as families of curves with two white and B black points. This structure can be defined in terms of forgetful morphisms forgetting just one point B . Let $B \subset B'$, $\text{card } B' \setminus B = 1$.

We start with describing structure sections.

In order to define the two white sections of the forgetful morphism, consider two partitions $(B' \setminus B, B)$ and $(B, B' \setminus B)$ of B' and the respective closed strata. The forgetful morphism restricted to these strata identifies them with $\overline{\mathcal{L}}_B$. We will call them x_0 and x_∞ respectively.

Finally, to define the j -th black section, $j \in B$, consider the morphism of lattices $s_j : N_B \rightarrow N_{B'}$ which extends a function χ on B to the function $s_j(\chi)$ on B' taking the value $\chi(j)$ at the forgotten point. This morphism satisfies the following condition: each cone $C(\tau)$ from F_B lands in an appropriate cone $C(\tau')$ from $F_{B'}$. Hence we have the induced morphisms $s_{j*} : \overline{\mathcal{L}}_B \rightarrow \overline{\mathcal{L}}_{B'}$ which obviously are sections. Moreover, they do not intersect x_0 and x_∞ .

4.6.1. Proposition. *With the notations and assumptions above, the forgetful morphism is a universal family of (painted stable) marked curves of genus zero with two white points and B black points.*

In order to see the structure of fibers of the forgetful morphism, one should notice that the inverse image of any point $x \in \overline{\mathcal{L}}_\tau$ is acted upon by the multiplicative group $\mathbf{G}_m = \text{Ker}(T_{B'} \rightarrow T_B)$. This action breaks the fiber into a finite number of orbits which coincide with the intersections of this fiber with various $\overline{\mathcal{L}}_{\tau'}$ described above. When τ' is obtained by adding the forgotten point to one of the parts, this intersection is a torsor over the kernel, otherwise it is a point. As a result, we get

that the fiber is a chain of \mathbf{P}^1 's, whose components are labeled by the components of τ and singular points by the neighboring pairs of components.

4.7. Clutching morphisms. They are morphisms of the type $\overline{L}_{B_1} \times \overline{L}_{B_2} \rightarrow \overline{L}_{B_1} \amalg B_2$ whose fiberwise description is this: glue ∞ of the first curve to 0 of the second curve. They admit an obvious toric description.

About their operadic role, see [Ma2].

4.8. Proposition. *Forgetful and clutching morphisms descend to the F_1 -models of the toric varieties \overline{L}_B .*

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