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Non-additive geometry

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Abstract

We develop a language that makes the analogy between geometry and arithmetic more transparent. In this language there exists a base field $\mathbb{F}$, 'the field with one element'; there is a fully faithful functor from commutative rings to $\mathbb{F}$-rings; there is the notion of the $\mathbb{F}$-ring of integers of a real or complex prime of a number field K analogous to the p-adic integers, and there is a compactification of $\operatorname{Spec} O_K$; there is a notion of tensor product of $\mathbb{F}$-rings giving the product of $\mathbb{F}$-schemes; in particular there is the arithmetical surface $\operatorname{Spec} O_K \times \operatorname{Spec} O_K$, the product taken over $\mathbb{F}$.

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Introduction

The ancient idea of making arithmetic into geometry engaged the minds of great mathematicians such as Kummer, Kronecker, Dedekind, Hensel, Hasse, Minkowski, and especially Artin and Weil. It is a beautiful quest inspired by the similarity between the ring of integers $\mathbb{Z}$, and the ring of polynomials $Z = \Bbbk[x]$ over a field $\Bbbk$; for closer similarity the 'function field' case is relevant where $\Bbbk = \mathbb{F}_q$ is a finite field. There is induced similarity of the fraction fields, the field of rational numbers $\mathbb{Q}$ and the field of rational functions $Q = \Bbbk(x)$. For a prime p of $\mathbb{Z}$, we have the p-adic integers

$$\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n,$$

and its field of fractions

$$\mathbb{Q}_p = \mathbb{Z}_p\left[\frac{1}{p}\right],$$

with dense embeddings $\mathbb{Z} \subseteq \mathbb{Z}_p$ and $\mathbb{Q} \subseteq \mathbb{Q}_p$. The geometric analogues are the power series ring

$$Z_f = \varprojlim Z/f^n = \Bbbk_f[[f]],$$

and the field of Laurent series

$$Q_f = Z_f\left[\frac{1}{f}\right] = \Bbbk_f((f)),$$

for f a prime of Z, where $\Bbbk_f = \Bbbk[x]/(f)$, and the embeddings $Z \subseteq Z_f$ (respectively $Q = \Bbbk(x) \subseteq Q_f$) correspond to expanding a polynomial (respectively a rational function) into a power series (respectively a Laurent series) in f. Finite extensions of $Q = \Bbbk(x)$ correspond one-to-one with the smooth projective curves Y defined over finite extensions of $\Bbbk$, and finite extensions of $\mathbb{Q}$ are the number fields. There are two main difficulties with this analogy that we are going to describe, the problem of the real prime of $\mathbb{Q}$, and the problem of the arithmetical surface, that is defining for $\operatorname{Spec}(\mathbb{Z})$ the analogue of the geometric surface $Y \times_{\Bbbk} Y$.

From geometry we know that, in order to have theorems, we must pass from affine to projective geometry, in particular we need to add the point at infinity ∞ to the affine line, $\mathbb{P}^1_{\Bbbk} = \mathbb{A}^1_{\Bbbk} \cup \{\infty\}$. This corresponds to the ring

$$Z_\infty = \varprojlim \Bbbk\left[\frac{1}{x}\right] \Big/ \left(\frac{1}{x}\right)^n = \Bbbk\left[\left[\frac{1}{x}\right]\right],$$

and its fraction field

$$Q_\infty = Z_\infty[x] = \Bbbk\left(\left(\frac{1}{x}\right)\right);$$

the embedding $Q \subseteq Q_\infty$ is the expansion of a rational function as a Laurent series in $1/x$. The analogue of ∞ for $\mathbb{Q}$ is the real prime, which we denote by η. The associated field is $\mathbb{Q}_\eta = \mathbb{R}$, the real numbers. But there is no analogue $\mathbb{Z}_\eta$ of Z_∞. For finite primes p,

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p, |x|_p \leqslant 1\}.$$

We have to carry remainder when we add elements of $\mathbb{Z}_p$ – unlike the simple addition of power series in Z_f or Z_∞. We carry the remainder from the larger scale p^j to the smaller scale p^{j+1}, hence

$$|x + y|_p \leqslant \max\{|x|_p, |y|_p\},$$

and $\mathbb{Z}_p$ is closed under addition. In contrast, when we add real numbers, we carry the remainder from the smaller to the larger scale, we have only the weaker triangle inequality

$$|x+y|_\eta \leqslant |x|_\eta + |y|_\eta,$$

and $\{x \in \mathbb{Q}_\eta, |x|_\eta \leqslant 1\} = [-1,1]$ is not closed under addition.

The second problem is that in geometry we have products, in particular the affine plane $\mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1$, with the ring of polynomial functions $\Bbbk[x] \otimes_{\Bbbk} \Bbbk[x] = \Bbbk[x_1, x_2]$, the tensor product ($\equiv$ sum in the category of $\Bbbk$-algebras) of Z with itself. When we try to find the analogous arithmetical surface, we find $\mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}$. The integers $\mathbb{Z}$ are the initial object in the category of rings, so its tensor product ($\equiv$ sum in the category of rings) with itself is just $\mathbb{Z}$. For any geometry that is based on rings, $\operatorname{Spec}\mathbb{Z}$ will be the final object, and $\operatorname{Spec}\mathbb{Z} \times \operatorname{Spec}\mathbb{Z} = \operatorname{Spec}\mathbb{Z}$, which means the arithmetical surface reduces to the diagonal!

Motivated by the Weil conjectures, Grothendieck developed the modern language of algebraic geometry, the language of schemes [EGA], based on commutative rings. Grothendieck came from a background of functional analysis, where the paradigm of 'geometry = commutative rings' was first set. It is the famous Gelfand–Naimark theorem on the equivalence of the category of (compact, Hausdorff) topological spaces and the category of commutative (unital) $\mathbb{C}^*$-algebras. This equivalence is given by associating with the topological space X the algebra

$$\mathbb{C}(X) = \{f : X \to \mathbb{C}, f \text{ continuous}\},$$

using addition and multiplication (and conjugation, and norm) of $\mathbb{C}$ to define the similar structure on $\mathbb{C}(X)$, giving rise to the structure of ring (and $\mathbb{C}^*$-algebra structure) on $\mathbb{C}(X)$. The axioms of a commutative $\mathbb{C}^*$-algebra are generalizations of the axioms of $\mathbb{C}$: when $X = \{*\}$ reduces to a point, $\mathbb{C}(*) = \mathbb{C}$. It is clear that there is no connection between addition and multiplication of $\mathbb{C}$ and the geometry of X. The language of rings (and commutative $\mathbb{C}^*$-algebras) is just one convenient way in which to encode geometry.

With the goal of finding the arithmetical surface, the idea of abandoning addition has recently appeared in the literature. Soulé [Sou04] talks of the 'field with one element' $\mathbb{F}$, and tries to define $\mathbb{F}$-varieties as a subcollection of $\mathbb{Z}$-varieties. Kurokawa, Ochiai and Wakayama [KOW03] were the first to suggest abandoning addition, and working instead with the multiplicative monoids. This idea was further described in Deitmar [Dei05], but note that the spectra of monoids always looks like the spectra of a *local* ring: the non-invertible elements are the unique maximal ideal. For Kurokawa there is also a 'zeta world' of analytic functions that encode geometry, where the field $\mathbb{F}$ is encoded by the identity function of $\mathbb{C}$; see Manin [Man95].

Here we take our clues from the problem of the real prime to understand $\mathbb{F}$, and then develop the language of geometry based on the concept of $\mathbb{F}$-ring. Denote by $|x|_\eta$ the euclidian norm of $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, i.e.

$$|x|_\eta = \sqrt{\sum_i |x_i|_\eta^2}.$$

We have the fundamental Cauchy–Schwartz inequality

$$|x \circ y|_\eta = |x_1 y_1 + \cdots + x_n y_n|_\eta \leqslant |x|_\eta \cdot |y|_\eta.$$

Hence $[-1,1]$ will contain $x_1 y_1 + \cdots + x_n y_n = x \circ y$, whenever $|x|_\eta, |y|_\eta \leqslant 1$, although it is not closed under addition. Moreover, unlike addition, matrix multiplication behaves well in the real prime: $|a \circ b|_\eta \leqslant |a|_\eta \cdot |b|_\eta$ for real or complex matrices a, b where $|\cdot|_\eta$ is the operator norm. Within matrix multiplication there is encoded addition, but we have to take matrix multiplication as the more fundamental operation. We add also the operations of direct sum and of tensor product of matrices. Our analogue of $\mathbb{Z}_p$ (respectively of the localization $\mathbb{Z}_{(p)}$) for the real prime η is the category $\mathcal{O}_{\mathbb{R},\eta}$

(respectively $\mathcal{O}_{\mathbb{Q},\eta}$) with objects the finite sets and morphisms from X to Y given by the $Y \times X$ matrices with real (respectively rational) coefficients and with operator norm $\leqslant 1$; these matrices are closed under the operations of direct sums and tensor products (but are not closed under addition).

Remembering that the quantum area in physics started with Heisenberg's discovery of matrix multiplication as the fundamental operation describing the energy levels of microscopic systems, perhaps in the future also physics will benefit from the language of non-additive geometry.

The contribution to arithmetic is evident: the real integers $\mathbb{Z}_\eta$ become a real object, and the arithmetical surface exists and does not reduce to the diagonal. Some well-known conjectures of arithmetic (Riemann hypothesis, ABC, ...) are easy theorems in the geometric analogue of a curve C over a finite field. This is because we can form the surface $C \times C$. The knowledge of the first infinitesimal neighborhood of the diagonal C within $C \times C$, i.e. of differentials, is often sufficient to prove theorems in geometry whose arithmetic analogues are deep conjectures. Therefore, the further study of the arithmetical surface $\mathbb{F}(\mathbb{Z}) \otimes_{\mathbb{F}} \mathbb{F}(\mathbb{Z})$, its compactification using $\mathbb{F}(\mathbb{Z}) \otimes_{\mathbb{F}} \mathbb{Z}_\eta$, and the arithmetic first infinitesimal neighborhood of the diagonal are important challenges. Here we give only the foundations of the language of non-additive geometry.

In § 1 we decipher what is the 'field with one element' $\mathbb{F}$. The idea is that, while $\mathbb{F}$ degenerates into one (or two) elements, there is a whole category of '$\mathbb{F}$-valued matrices'. There are various degrees of structures one can impose on $\mathbb{F}$. In § 2 we give the basic notion of an $\mathbb{F}$-ring. As important examples of $\mathbb{F}$-rings we have: $\mathbb{F}(A)$, the $\mathbb{F}$-ring attached to a commutative ring A; $\mathcal{O}_\eta$, the $\mathbb{F}$-ring of 'integers' at a real or complex prime η of a number field; and its residue field $\mathbb{F}_\eta$, the $\mathbb{F}$-ring of partial isometries. In § 3 we give the elementary theory of modules over $\mathbb{F}$-rings, and discuss (fibred) sums and products, kernels and cokernels, free modules, tensor products, and base change. A novelty of the non-additive setting is the connection between submodules and equivalence modules of a given module.

In § 4 we give the elementary theory of ideals and primes. We associate with any $\mathbb{F}$-ring A its spectrum $\operatorname{Spec} A$, a compact sober space with respect to the 'Zariski topology'. (A topological space is sober if every closed irreducible subset has a unique generic point.) In § 5 we give the theory of localization. It gives rise to a sheaf of $\mathbb{F}$-rings over $\operatorname{Spec} A$. By gluing such spectra we get Zariski $\mathbb{F}$-schemes. In § 6 we give the theory of $\mathbb{F}$-schemes which are the pro-objects of Zariski $\mathbb{F}$-schemes. As important examples we give the compactification of $\operatorname{Spec} \mathbb{Z}$ and of $\operatorname{Spec} O_K$, K a number field. This is our solution to the problem of the real prime.

In § 7 we give the tensor product, the (fibred) sum in the category of $\mathbb{F}$-rings, and we obtain the (fibred) product in the categories of Zariski $\mathbb{F}$-schemes and of $\mathbb{F}$-schemes. As an important example we define and describe the fibred product $\mathbb{F}(\mathbb{Z}) \otimes_{\mathbb{F}} \mathbb{F}(\mathbb{Z})$, its compactification, and its generalization for number fields. This is our solution to the problem of the arithmetical surface. In § 8 we work over a fixed $\mathbb{F}$-ring F, and repeat the above constructions in the category of monoid objects in F-modules. Everything goes through, the tensor product of F-monoids is just their tensor product as modules, so we avoid the complicated product of § 7, but the functor from commutative rings to $\mathbb{F}$-monoids is not fully faithful.

1. $\mathbb{F}$, the field with one element

We define a category $\mathbb{F}$ with objects the finite sets endowed with two symmetric monoidal structures $\oplus$ and $\otimes$. The unit element $[0]$ for $\oplus$ is the initial and final object of the category, and $\otimes$ is distributive over $\oplus$.

1.1 The category $\mathbb{F}$

We consider $\mathbb{F}$-vector spaces as finite sets X with a distinguished 'zero' element $0_X \in X$, and set $X^+ = X \setminus \{0_X\}$. For a commutative ring A, we let

$$A \cdot X = \bigoplus_{x \in X^+} A \cdot x$$

denote the free A-module with basis X^+, and think about $A \cdot X$ as $A \otimes_{\mathbb{F}} X$ obtained by base extension from $\mathbb{F}$ to A. We let

$$\mathbb{F}[A]_{Y,X} = \mathrm{Hom}_A(A \cdot X, A \cdot Y),$$

the $Y^+ \times X^+$ matrices with values in A. The base extension of X from $\mathbb{F}$ to $\mathbb{Z}_\eta$ and to $\mathbb{Q}_\eta = \mathbb{R}$ gives $\mathbb{Z}_\eta \cdot X$ and $\mathbb{Q}_\eta \cdot X$: $\mathbb{Q}_\eta \cdot X$ is the real vector space with basis X^+, and $\mathbb{Z}_\eta \cdot X$ is the subset of $\mathbb{Q}_\eta \cdot X$ of vectors with norm $\leqslant 1$ in the inner product given by decreeing X^+ to be an orthonormal basis. We have

$$\mathbb{F}[\mathbb{Q}_\eta]_{Y,X} = \mathrm{Hom}_{\mathbb{Q}_\eta}(\mathbb{Q}_\eta \cdot X, \mathbb{Q}_\eta \cdot Y),$$

the $Y^+ \times X^+$ real-valued matrices, and

$$(\mathbb{Z}_\eta)_{Y,X} = \{f \in \mathbb{F}[\mathbb{Q}_\eta]_{Y,X}, f(\mathbb{Z}_\eta \cdot X) \subseteq \mathbb{Z}_\eta \cdot Y\} = \{f, |f|_\eta \leqslant 1\},$$

where $|f|_\eta$ denotes the operator norm on $\mathbb{F}[\mathbb{Q}_\eta]_{Y,X}$.

A map of finite sets $\varphi : X \to Y$, preserving the zero elements $\varphi(0_X) = 0_Y$, induces an A-linear map

$$\varphi_A : A \cdot X \to A \cdot Y, \quad \varphi_A \in \mathbb{F}[A]_{Y,X}.$$

For $\varphi_{\mathbb{Q}_\eta} : \mathbb{Q}_\eta \cdot X \to \mathbb{Q}_\eta \cdot Y$ to map $\mathbb{Z}_\eta \cdot X$ into $\mathbb{Z}_\eta \cdot Y$ it is necessary and sufficient that φ is an injection of $X \setminus \varphi^{-1}(0_Y)$ into Y. Thus we set

$$\mathbb{F}_{Y,X} = \{\varphi : X \to Y, \varphi(0_X) = 0_Y, \varphi|_{X \setminus \varphi^{-1}(0_Y)} \text{ injective}\}, \tag{1.1}$$

and we view $\mathbb{F}$ as the category with objects finite sets with a distinguished zero element, and with arrows $\mathbb{F}_{Y,X} = \mathrm{Hom}_{\mathbb{F}}(X, Y)$. In practice, we shall ignore the distinguished elements, and view $\mathbb{F}$ as the category with objects finite sets (without a distinguished zero element), and with arrows the partial bijections

$$\mathbb{F}'_{Y,X} = \{\varphi : V \xrightarrow{\sim} W \text{ bijection}, V \subseteq X, W \subseteq Y\}. \tag{1.1$'$}$$

It is clear that

$$X \mapsto X^+ := X \setminus \{0_X\}$$

and

$$\varphi \mapsto \{\varphi : X \setminus \varphi^{-1}(0_Y) \xrightarrow{\sim} \varphi(X \setminus \varphi^{-1}(0_Y))\}$$

is an isomorphism of categories

$$\mathbb{F}_{Y,X} \xrightarrow{\sim} \mathbb{F}'_{Y^+,X^+}.$$

We shall identify $\mathbb{F}$ with $\mathbb{F}'$. Thus from now on the objects of $\mathbb{F}$ are finite sets without a distinguished zero element. Alternatively, $\mathbb{F}_{Y,X}$ are the $Y \times X$ matrices with entries $0, 1$ and with at most one 1 in every row and column.

We have a functor

$$\oplus : \mathbb{F} \times \mathbb{F} \to \mathbb{F} \tag{1.2}$$

given by the *disjoint* union of sets. More formally, for sets X, Y we let

$$X \oplus Y = \{(z, i) \mid i \in \{0, 1\}; i = 0 \Rightarrow z \in X, i = 1 \Rightarrow z \in Y\} \tag{1.3}$$

and for $f_0 \in \mathbb{F}_{X',X}, f_1 \in \mathbb{F}_{Y',Y}$, we have $f_0 \oplus f_1 \in \mathbb{F}_{X'\oplus Y', X\oplus Y}$ given by

$$f_0 \oplus f_1(z,i) = (f_i(z), i). \tag{1.4}$$

(Note that in the version of $\mathbb{F}$ where the objects have a distinguished zero element, $X \oplus Y$ is obtained from the disjoint union $X \coprod Y$ by identifying 0_X with 0_Y.)

We have for $f_0' \in \mathbb{F}_{X'',X'}, f_1' \in \mathbb{F}_{Y'',Y'}$,

$$(f_0' \oplus f_1') \circ (f_0 \oplus f_1) = (f_0' \circ f_0) \oplus (f_1' \circ f_1) \tag{1.5}$$

and

$$\mathrm{id}_X \oplus \mathrm{id}_Y = \mathrm{id}_{X\oplus Y}. \tag{1.6}$$

The operation $\oplus$ makes $\mathbb{F}$ into a symmetric-monoidal category. The identity element is the empty set $[0]$ (or the set with only the distinguished zero element), which is the initial and final object of the category $\mathbb{F}$. There are canonical isomorphisms in $\mathbb{F}$:

$$X \oplus [0] \overset{l_X}{\underset{\sim}{\longleftarrow}} X \overset{r_X}{\underset{\sim}{\longrightarrow}} [0] \oplus X. \tag{1.7}$$

The commutativity isomorphism $c_{X,Y} \in \mathbb{F}_{Y\oplus X, X\oplus Y}$ is given by

$$c_{X,Y}(z,i) = (z, 1-i). \tag{1.8}$$

The associativity isomorphism $a_{X,Y,Z} \in \mathbb{F}_{X\oplus(Y\oplus Z),(X\oplus Y)\oplus Z}$ is given by

$$\begin{aligned} a((w,0),0) &= (w,0), \\ a((w,1),0) &= ((w,0),1), \\ a(w,1) &= ((w,1),1). \end{aligned} \tag{1.9}$$

We shall usually abuse notation and view $l_X, r_X, c_{X,Y}, a_{X,Y,Z}$ as identifications; thus e.g. for $f_i \in \mathbb{F}_{X_i',X_i}$ we write $f_0 \oplus f_1 = f_1 \oplus f_0$ instead of

$$c_{X_0',X_1'} \circ (f_0 \oplus f_1) = (f_1 \oplus f_0) \circ c_{X_0,X_1}. \tag{1.10}$$

We have a functor

$$\otimes : \mathbb{F} \times \mathbb{F} \to \mathbb{F} \tag{1.11}$$

given by the product of sets $X \otimes Y = \{(x,y) \mid x \in X, y \in Y\}$, and for $f_0 \in \mathbb{F}_{X',X}$, $f_1 \in \mathbb{F}_{Y',Y}$, we have $f_0 \otimes f_1 \in \mathbb{F}_{X'\otimes Y', X\otimes Y}$ given by

$$f_0 \otimes f_1(x,y) = (f_0(x), f_1(y)). \tag{1.12}$$

(Note that working with the version of $\mathbb{F}$ where the objects have a distinguished zero element, $X \otimes Y$ is obtained from the product $X \times Y$ by identifying $(x, 0_Y)$ and $(0_X, y)$ with $(0_X, 0_Y)$ for all $x \in X,\ y \in Y$.)

We have for $f_0' \in \mathbb{F}_{X'',X'}, f_1' \in \mathbb{F}_{Y'',Y'}$,

$$(f_0' \otimes f_1') \circ (f_0 \otimes f_1) = (f_0' \circ f_0) \otimes (f_1' \circ f_1) \tag{1.13}$$

and

$$\mathrm{id}_X \otimes \mathrm{id}_Y = \mathrm{id}_{X\otimes Y}. \tag{1.14}$$

The operation $\otimes$ also makes $\mathbb{F}$ into a symmetric monoidal category. The identity element is the set with one element $[1]$ (or the set with a distinguished zero element 0, and another element $[1] = \{0,1\}$). We have again isomorphisms in $\mathbb{F}$:

$$X \otimes [1] \overset{l_X^*}{\underset{\sim}{\longleftarrow}} X \overset{r_X^*}{\underset{\sim}{\longrightarrow}} X \otimes [1], \tag{1.15}$$

and

$$c^*_{X,Y} \in \mathbb{F}_{Y\otimes X, X\otimes Y}, \quad c^*_{X,Y}(x,y) = (y,x), \tag{1.16}$$

$$a^*_{X,Y,Z} \in \mathbb{F}_{X\otimes(Y\otimes Z),(X\otimes Y)\otimes Z}, \quad a^*_{X,Y,Z}((x,y),z) = (x,(y,z)). \tag{1.17}$$

We have as well the distributivity isomorphism $d_{X_0,X_1;Y} \in \mathbb{F}_{(X_0\otimes Y)\oplus(X_1\otimes Y),(X_0\oplus X_1)\otimes Y}$

$$d_{X_0,X_1;Y}((x,i),y) = ((x,y),i), i \in \{0,1\}. \tag{1.18}$$

We abuse notation and view $l^*_X, r^*_X, c^*_{X,Y}, a^*_{X,Y,Z}, d_{X_0,X_1;Y}$ as identifications; thus e.g. for $f_i \in \mathbb{F}_{X'_i,X_i}, g \in \mathbb{F}_{Y',Y}$ we write

$$(f_0 \oplus f_1) \otimes g = (f_0 \otimes g) \oplus (f_1 \otimes g) \tag{1.19}$$

which should be read as

$$d_{X'_0,X'_1;Y'} \circ [(f_0 \oplus f_1) \otimes g] = [(f_0 \otimes g) \oplus (f_1 \otimes g)] \circ d_{X_0,X_1;Y}. \tag{1.19$'$}$$

We note that there is a natural involution

$$\mathbb{F}_{Y,X} \xrightarrow{\sim} \mathbb{F}_{X,Y}, f \mapsto f^{\mathrm{t}}. \tag{1.20}$$

When viewing $\mathbb{F}_{Y,X}$ as the partial bijections $f : V \xrightarrow{\sim} W$, $V \subseteq X$ and $W \subseteq Y, f^{\mathrm{t}}$ is the inverse bijection, $f^{\mathrm{t}} = f^{-1} : W \xrightarrow{\sim} V$. When we view $\mathbb{F}_{Y,X}$ as $0,1$ matrices, f^{t} is the transpose matrix.

We have

$$(g \circ f)^{\mathrm{t}} = f^{\mathrm{t}} \circ g^{\mathrm{t}}, \tag{1.21.1}$$

$$(\mathrm{id}_X)^{\mathrm{t}} = \mathrm{id}_X, \tag{1.21.2}$$

$$(f^{\mathrm{t}})^{\mathrm{t}} = f, \tag{1.21.3}$$

$$(f_0 \oplus f_1)^{\mathrm{t}} = f_0^{\mathrm{t}} \oplus f_1^{\mathrm{t}}, \tag{1.21.4}$$

$$(f_0 \otimes f_1)^{\mathrm{t}} = f_0^{\mathrm{t}} \otimes f_1^{\mathrm{t}}. \tag{1.21.5}$$

Remark. Whenever we use the notation for composition $f \circ g$ it will always be implicitly assumed that the domain of f is the range of g; thus e.g. if we have $(f_0 \oplus f_1) \circ g$ and $f_i \in \mathbb{F}_{X'_i,X_i}$, it is implicitly assumed that g has range $X_0 \oplus X_1$.

1.2 Variants $\mathbb{F}^{\pm}$

The model $\mathbb{F}$ for the field with one element is the one we shall use here, but there is a variant $\mathbb{F}^{\pm}$ which is important, and leads to a tighter theory. The objects of the category $\mathbb{F}^{\pm}$ are finite sets X together with an action of the group $\{\pm 1\}$, without fixed points (or with a unique fixed point – the zero element). A subset $X^+ \subseteq X$ will be called a *basis* if X is the disjoint union of X^+ and $-X^+ = \{-x \mid x \in X^+\}$. The maps $f \in \mathbb{F}^{\pm}_{Y,X}$ are partial bijections

$$f : V \xrightarrow{\sim} W, \quad V \subseteq X, \quad W \subseteq Y, \quad V = -V, \quad W = -W,$$

that commute with the action $f(-x) = -f(x)$. Fixing basis $X^+ \subseteq X, Y^+ \subseteq Y$, we can identify the elements of $\mathbb{F}^{\pm}_{Y,X}$ with the $Y^+ \times X^+$ matrices of entries $0, 1, -1$, with at most one non-zero term in each row and column. The map f is identified with the matrix $\mathcal{M}(f)$, where for $x \in X^+$, $y \in Y^+$, $\mathcal{M}(f)_{y,x} = 0$ (respectively, $1, -1$) if $f(x) \neq \pm y$ (respectively, $y, -y$).

We have functors

$$\oplus, \otimes : \mathbb{F}^{\pm} \times \mathbb{F}^{\pm} \to \mathbb{F}^{\pm}, \tag{1.22}$$

$$X \oplus Y = \text{disjoint union of } X \text{ and } Y, \text{ with its natural } \{\pm 1\} \text{ action}, \tag{1.22.1}$$

$$X \otimes Y = X \times Y/_{(x,y)\sim(-x,-y)}, \text{with } \{\pm 1\} \text{ action} : -(x,y) = (-x,y) = (x,-y). \tag{1.22.2}$$

Write $x \otimes y$ for the image of (x,y) in $X \otimes Y$; we have for $f_0 \in \mathbb{F}^{\pm}_{X',X}, f_1 \in \mathbb{F}^{\pm}_{Y',Y}$,

$$f_0 \otimes f_1(x \otimes y) = f_0(x) \otimes f_1(y). \tag{1.23}$$

The unit for $\oplus$ is $[0]$, the initial and final object of $\mathbb{F}^{\pm}$. The unit for $\otimes$ is $[\pm 1]$. The analogue of formulas (1.5) to (1.19) remain true for $\mathbb{F}^{\pm}$. We have an involution $\mathbb{F}^{\pm} \to \mathbb{F}^{\pm}, f \mapsto f^{\mathrm{t}}$, where f^{t} is the inverse bijection (or transpose of an $Y^+ \times X^+$ matrix), and formulas (1.21.1–5) remain true for $\mathbb{F}^{\pm}$.

Definition. Let X be an object of $\mathbb{F}^{\pm}$ and let $X^+ \subseteq X$ be a basis. The number of elements of $X^+ : d = \# X^+$ will be called the *dimension* of X, and denoted $d = \dim X$.

For $n = 1, \dots, d$ let

$$\begin{gathered} \mathrm{P}^n(X) = \{x_1 \otimes \cdots \otimes x_n \in X \otimes \cdots \otimes X \mid x_i \neq \pm x_j \text{ for } i \neq j\} \\ \wedge^n(X) = \mathrm{P}^n(X)/\sim \end{gathered} \tag{1.24}$$

where $\sim$ is the equivalence relation

$$x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)} \sim \operatorname{sgn}(\sigma) \cdot x_1 \otimes \cdots \otimes x_n, \quad \text{for } \sigma \in S_n. \tag{1.25}$$

Write $x_1 \wedge \cdots \wedge x_n$ for the image of $x_1 \otimes \cdots \otimes x_n \in \mathrm{P}^n(X)$ in $\wedge^n(X)$. A map $f \in \mathbb{F}^{\pm}_{Y,X}$ induces a map $\mathrm{P}^n(f) \in \mathbb{F}^{\pm}_{\mathrm{P}^n(Y),\mathrm{P}^n(X)}$, which induces in turn a map $\wedge^n(f) \in \mathbb{F}^{\pm}_{\wedge^n(Y),\wedge^n(X)}$,

$$\wedge^n(f)(x_1 \wedge \cdots \wedge x_n) = f(x_1) \wedge \cdots \wedge f(x_n). \tag{1.26}$$

For $n > d$ we have $\wedge^n(X) = [0]$, and by definition we let $\wedge^0(X) = [\pm 1]$. Thus we have a sequence of functors

$$\wedge^n : \mathbb{F}^{\pm} \to \mathbb{F}^{\pm}, n = 0, 1, \dots, \tag{1.27}$$

$$\wedge^n(f \circ g) = \wedge^n(f) \circ \wedge^n(g), \tag{1.27.1}$$

$$\wedge^n(\mathrm{id}_X) = \mathrm{id}_{\wedge^n(X)}, \tag{1.27.2}$$

$$\wedge^n(f^{\mathrm{t}}) = (\wedge^n(f))^{\mathrm{t}}. \tag{1.27.3}$$

There are natural isomorphisms in $\mathbb{F}^{\pm}$ which we view as identifications

$$\wedge^n(X \oplus Y) = \bigoplus_{0 \leqslant j \leqslant n} \wedge^j(X) \otimes \wedge^{n-j}(Y). \tag{1.28}$$

Remark 1.29. When we consider the objects $|\mathbb{F}|$ of the category $\mathbb{F}$ (respectively $\mathbb{F}^{\pm}$), we assume that it contains $[n], n \geqslant 0$ (respectively $[\pm n]$), and that it contains $X \oplus Y, X \otimes Y$ (respectively and $\wedge^n(X)$) whenever it contains X, Y. Hence we may assume $|\mathbb{F}|$ and $|\mathbb{F}^{\pm}|$ are countable sets. On the other hand, we shall not use the actual realization of $\mathbb{F}$ in most of what follows. All we need is a category F with two symmetric monoidal structures $\oplus$ and $\otimes$, the unit element $[0]$ for $\oplus$ is the initial and final object of F, $\otimes$ is distributive over $\oplus$ and it respects $[0]$: $X \otimes [0] = [0]$. This opens up the possibility of introducing quantum deformations.

1.3 The 'algebraic closure' $\overline{\mathbb{F}}$ of $\mathbb{F}$

We can similarly work over the 'algebraic closure' $\overline{\mathbb{F}}$ of $\mathbb{F}$, which in arithmetic means adjoining all roots of unity $\mu \cong \mathbb{Q}/\mathbb{Z}$. The objects of $\overline{\mathbb{F}}$ are sets X with μ-action, satisfying the following two properties:

(i) set X decomposes into a finite union of μ-orbits

$$X = X_1 \sqcup \cdots \sqcup X_d, \quad X_i = \mu \cdot x_i; \tag{1.30}$$

(ii) for $x \in X$ there is a natural number N and a finite set of primes $\{p_1, \ldots, p_l\}$, $p_i \nmid N$, such that the stabilizer of x in μ is given by

$$\{\zeta \in \mu \mid \zeta \cdot x = x\} = \mu_N \times \mu_{p_1^\infty} \times \cdots \times \mu_{p_l^\infty}. \tag{1.30.1}$$

Here $\mu_N = \{\zeta \in \mu \mid \zeta^N = 1\}$ and $\mu_{p^\infty} = \bigcup_n \mu_{p^n}$.

Let $x_j \in X$, $1 \leqslant j \leqslant d$, be such that $X_j = \mu \cdot x_j$ for each j. Then the subset $X^+ = \{x_1, \ldots, x_d\} \subseteq X$ representing the μ-orbits will be called a basis for X, and $d = \#X^+ = \dim X$ the dimension of X. The maps in the category $\overline{\mathbb{F}}$ from an object X to an object Y are given by μ-covariant partial bijections

$$\overline{\mathbb{F}}_{Y,X} = \{f : V \xrightarrow{\sim} W \mid V \subseteq X, W \subseteq Y, V = \mu \cdot V, W = \mu \cdot W, f(\zeta x) = \zeta f(x), \forall x \in V, \zeta \in \mu\}. \tag{1.30.2}$$

We have functors

$$\oplus, \otimes : \overline{\mathbb{F}} \times \overline{\mathbb{F}} \to \overline{\mathbb{F}}, \tag{1.31}$$

$$X \oplus Y = \text{disjoint union of } X \text{ and } Y, \text{ with its natural } \mu\text{-action}, \tag{1.31.1}$$

$$X \otimes Y = X \times Y/_{(x,y) \sim (\zeta x, \zeta^{-1} y)}, \quad \text{with } \mu\text{-action } \zeta \cdot (x,y) = (\zeta x, y) = (x, \zeta y). \tag{1.31.2}$$

We write $x \otimes y$ for the image of (x, y) in $X \otimes Y$. For $f_i \in \overline{\mathbb{F}}_{Y_i, X_i}$ we have

$$f_0 \oplus f_1 \in \overline{\mathbb{F}}_{Y_0 \oplus Y_1, X_0 \oplus X_1}, f_0 \oplus f_1(z, i) = (f_i(z), i), \quad i = 0, 1,$$

and we have

$$f_0 \otimes f_1 \in \overline{\mathbb{F}}_{Y_0 \otimes Y_1, X_0 \otimes X_1}, f_0 \otimes f_1(x_0 \otimes x_1) = f_0(x_0) \otimes f_1(x_1).$$

Both $\oplus$ and $\otimes$ make $\overline{\mathbb{F}}$ into symmetric monoidal category; the unit for $\oplus$ is the empty set [0] which is the initial and final object of $\overline{\mathbb{F}}$; the unit for $\otimes$ is $[1] = \mu$. The analogue of formulas (1.5)–(1.19) remain true for $\overline{\mathbb{F}}$. We have an involution on $\overline{\mathbb{F}}$ satisfying (1.21). We have λ-operations: for an object X of $\overline{\mathbb{F}}$ of dimension $d = \dim X$ and for $n = 1, \ldots, d$ we let

$$\begin{gathered} \mathrm{P}^n(X) = \{x_1 \otimes \cdots \otimes x_n \in X \otimes \cdots \otimes X \mid x_i \neq \zeta x_j \text{ for } i \neq j, \zeta \in \mu\}, \\ \wedge^n(X) = \mathrm{P}^n(X)/_{x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)} \sim sgn(\sigma) \cdot x_1 \otimes \cdots \otimes x_n, \sigma \in S_n}. \end{gathered} \tag{1.32}$$

We write $x_1 \wedge \cdots \wedge x_n$ for the image of $x_1 \otimes \cdots \otimes x_n$ in $\wedge^n(X), x_i \in X$. A map $f \in \overline{\mathbb{F}}_{Y,X}$ induces a map $\wedge^n(f) \in \overline{\mathbb{F}}_{\wedge^n(Y), \wedge^n(X)}$ by

$$\begin{gathered} \wedge^n(f)(x_1 \wedge \cdots \wedge x_n) = f(x_1) \wedge \cdots \wedge f(x_n), \\ \mathrm{Domain}(\wedge^n(f)) = \{(x_1 \wedge \cdots \wedge x_n) \mid x_i \in \mathrm{Domain}(f)\}. \end{gathered} \tag{1.33}$$

We let $\wedge^0(X) = [1] = \mu, \wedge^1(X) = X$, and $\wedge^n(X) = [0]$ for $n > \dim X$. Thus we have a sequence of functors $\wedge^n : \overline{\mathbb{F}} \to \overline{\mathbb{F}}, n = 0, 1, 2, \ldots,$ and (1.28) remains valid. A novelty of $\overline{\mathbb{F}}$ is that we have a sequence of functors given by Adam's operators

$$\begin{gathered} \psi^n : \overline{\mathbb{F}} \to \overline{\mathbb{F}}, \quad n = \pm 1, \pm 2, \ldots, \\ \psi^n(X) = \text{the set } X \text{ with the new } \mu\text{-action } \zeta \cdot_{(n)} x = \zeta^n \cdot x \end{gathered} \tag{1.34}$$

(we can take n in $\{n = (n_p) \in \widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p, n_p \in \mathbb{Z}_p^*$ for all but finitely many $p\}$). These functors satisfy

$$\begin{gathered} \psi^n(X \oplus Y) = \psi^n(X) \oplus \psi^n(Y), \\ \psi^n(X \otimes Y) = \psi^n(X) \otimes \psi^n(Y), \end{gathered} \tag{1.35}$$

and are the analogue in our setting of the Frobenius endomorphisms in the theory of varieties over $\overline{\mathbb{F}_q}$. (Indeed, the action on K-theory of the Frobenius endomorphism for such varieties is given by ψ^q.)

2. $\mathbb{F}$-rings, variants, examples

We give the definition of $\mathbb{F}$-rings and of $\mathcal{R}ing$ category. We give various variants of $\mathbb{F}$-rings with involution or with λ-ring structure. We then give our main examples.

2.1 Definition of $\mathbb{F}$-rings

Definition 2.1. An $\mathbb{F}$-*ring* is a category A with objects the finite sets $|\mathbb{F}|$, and arrows $A_{Y,X} = \mathrm{Hom}_A(X,Y)$ containing $\mathbb{F}_{Y,X}$, i.e. we have a faithful functor $\mathbb{F} \to A$ which is the identity on objects. We assume $[0]$ is the initial and final object of A. We have two functors

$$\oplus, \otimes : A \times A \to A,$$

which agree with the given functors on $\mathbb{F}$, and which make A into a symmetric monoidal category with the given identity $(l_X,\, r_X;\, l^*_X,\, r^*_X)$, commutativity $(c_{X,Y}; c^*_{X,Y})$, associativity $(a_{X,Y,Z}; a^*_{X,Y,Z})$ isomorphisms of $\mathbb{F}$. We assume that $\otimes$ is distributive over $\oplus$ using the isomorphism $d_{X_0,X_1;Y}$ of $\mathbb{F}$.

Thus in explicit terms, an $\mathbb{F}$-ring is a set

$$A = \coprod_{Y,X\in|\mathbb{F}|} A_{Y,X}, \tag{2.2}$$

with operations

$$\circ : A_{Z,Y} \times A_{Y,X} \to A_{Z,X}, \tag{2.2.1}$$
$$\oplus : A_{Y_0,X_0} \times A_{Y_1,X_1} \to A_{Y_0\oplus Y_1, X_0\oplus X_1}, \tag{2.2.2}$$
$$\otimes : A_{Y_0,X_0} \times A_{Y_1,X_1} \to A_{Y_0\otimes Y_1, X_0\otimes X_1}, \tag{2.2.3}$$

satisfying

$$f \circ (g \circ h) = (f \circ g) \circ h; \tag{2.3.1}$$
$$\mathrm{id}_Y \circ f = f = f \circ \mathrm{id}_X, \quad f \in A_{Y,X}; \tag{2.3.2}$$

$$(f_0 \oplus f_1) \circ (g_0 \oplus g_1) = (f_0 \circ g_0) \oplus (f_1 \circ g_1), \quad g_i \in A_{Y_i,X_i}, f_i \in A_{Z_i,Y_i}; \tag{2.4.1}$$
$$\mathrm{id}_X \oplus \mathrm{id}_Y = \mathrm{id}_{X\oplus Y}; \tag{2.4.2}$$
$$f_0 \oplus f_1 = f_1 \oplus f_0; \tag{2.4.3}$$
$$f_0 \oplus (f_1 \oplus f_2) = (f_0 \oplus f_1) \oplus f_2; \tag{2.4.4}$$
$$f \oplus \mathrm{id}_{[0]} = f; \tag{2.4.5}$$

$$(f_0 \otimes f_1) \circ (g_0 \otimes g_1) = (f_0 \circ g_0) \otimes (f_1 \circ g_1), \quad g_i \in A_{Y_i,X_i}, f_i \in A_{Z_i,Y_i}; \tag{2.5.1}$$
$$\mathrm{id}_X \otimes \mathrm{id}_Y = \mathrm{id}_{X\otimes Y}; \tag{2.5.2}$$
$$f_0 \otimes f_1 = f_1 \otimes f_0; \tag{2.5.3}$$
$$f_0 \otimes (f_1 \otimes f_2) = (f_0 \otimes f_1) \otimes f_2; \tag{2.5.4}$$
$$f \otimes \mathrm{id}_{[1]} = f; \tag{2.5.5}$$

$$f \otimes (g_0 \oplus g_1) = (f \otimes g_0) \oplus (f \otimes g_1). \tag{2.6}$$

Remark. We remind the reader that we omit the writing of the canonical isomorphisms of $\mathbb{F}$. Thus e.g (2.4.3) should be written

$$(f_0 \oplus f_1) \circ c_{X_1,X_0} = c_{Y_1,Y_0} \circ (f_1 \oplus f_0), \quad f_i \in A_{Y_i,X_i}. \tag{2.4.3$'$}$$

We assume that $\mathbb{F}_{Y,X} \subseteq A_{Y,X}$, and that the above operations $\circ, \oplus, \otimes$ agree with the given operations on $\mathbb{F}$. In particular, we have the zero map $0_{Y,X} \in A_{Y,X}$, which is the unique map that factors through $[0]$.

We note that on $A_{[1],[1]}$, the operations of composition $\circ$ and of tensor product $\otimes$ induce the same operation, making $A_{[1],[1]}$ into a commutative monoid:

$$f \cdot d \overset{\text{def}}{=} f \circ g = (f \otimes \mathrm{id}_{[1]}) \circ (\mathrm{id}_{[1]} \otimes g) = f \otimes g = g \otimes f, \quad f, g \in A_{[1],[1]}. \tag{2.7}$$

The set $A_{[1],[1]}$ has a unit 1 coming from the map $\mathrm{id}_{[1],[1]} : [1] \to [1]$, and a zero element 0 coming from the map $0_{[1],[1]} : [1] \to [1]$, $z \mapsto 0_{[1]}$. The set $A_{[1],[1]}$ acts on the sets $A_{Y,X}$,

$$f \cdot g \overset{\text{def}}{=} f \otimes g, f \in A_{[1],[1]}, \quad g \in A_{Y,X}. \tag{2.8}$$

This action satisfies

$$(f_0 \cdot f_1) \cdot g = f_0 \cdot (f_1 \cdot g), f_i \in A_{[1],[1]}, \quad g \in A_{Y,X}; \tag{2.8.1}$$
$$1 \cdot g = g; \tag{2.8.2}$$
$$0 \cdot g = 0_{Y,X}; \tag{2.8.3}$$
$$f \cdot (g \circ h) = (f \cdot g) \circ h = g \circ (f \cdot h); \tag{2.8.4}$$
$$f \cdot (g_0 \oplus g_1) = (f \cdot g_0) \oplus (f \cdot g_1); \tag{2.8.5}$$
$$f \cdot (g_0 \otimes g_1) = (f \cdot g_0) \otimes g_1 = g_0 \otimes (f \cdot g_1). \tag{2.8.6}$$

Definition 2.9. Let A, B be $\mathbb{F}$-rings. A functor $\varphi : A \to B$ is a *homomorphism of* $\mathbb{F}$*-rings* if

$$\varphi(A_{Y,X}) \subseteq B_{Y,X}, \tag{2.9.1}$$
$$\varphi(f) = f \text{ for } f \in \mathbb{F}_{Y,X}, \tag{2.9.2}$$
$$\varphi(f \circ g) = \varphi(f) \circ \varphi(g), \tag{2.9.3}$$
$$\varphi(f_0 \oplus f_1) = \varphi(f_0) \oplus \varphi(f_1), \tag{2.9.4}$$
$$\varphi(f_0 \otimes f_1) = \varphi(f_0) \otimes \varphi(f_1). \tag{2.9.5}$$

Thus φ is a functor over $\mathbb{F}$ that respects $\oplus$ and $\otimes$. It is clear that if $\varphi : A \to B, \psi : B \to C$ are homomorphisms of $\mathbb{F}$-rings, then $\psi \circ \varphi$ is a homomorphism of $\mathbb{F}$-rings, hence we have a category $\mathbb{F}$-$\mathcal{R}ings$, with $\mathbb{F}$ as an initial object.

Remark. A (commutative) *ring category* A is a category with a symmetric monoidal structure

$$\oplus : A \times A \to A,$$

with associativity (respectively commutativity, unit) isomorphisms a (respectively, c, u), with the unit object for $\oplus$, denoted by $[0]$, being the initial and final object of A, and another symmetric monoidal structure

$$\otimes : A \times A \to A,$$

with associativity (respectively commutativity, unit) isomorphisms a^* (respectively, c^*, u^*), the unit object for $\otimes$ is denoted by [1], and distributive isomorphisms

$$d_{Y;X_0,X_1} : Y \otimes (X_0 \oplus X_1) \xrightarrow{\sim} (Y \otimes X_0) \oplus (Y \otimes X_1)$$

functorial in $Y, X_0, X_1 \in |A|$, and compatible with a, c, u, a^*, c^*, u^*. That is, we have commutative diagrams. For X_i, Y_i objects of A, we have the following.

(*a*)

$$\begin{array}{ccc} Y\otimes((X_0\oplus X_1)\oplus X_2) & \xrightarrow{\mathrm{id}\otimes a} & Y\otimes(X_0\oplus(X_1\oplus X_2)) \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d} \\ Y\otimes(X_0\oplus X_1)\oplus Y\otimes X_2 & & Y\otimes X_0\oplus Y\otimes(X_1\oplus X_2) \\ {\scriptstyle d\oplus\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\oplus d} \\ (Y\otimes X_0\oplus Y\otimes X_1)\oplus Y\otimes X_2 & \xrightarrow{a} & Y\otimes X_0\oplus(Y\otimes X_1\oplus Y\otimes X_2) \end{array}$$

(a^*)

$$\begin{array}{ccccc} (Y_1\otimes Y_0)\otimes(X_0\oplus X_1) & \xrightarrow{a^*} & Y_1\otimes(Y_0\otimes(X_0\oplus X_1)) & \xrightarrow{\mathrm{id}\otimes d} & Y_1\otimes((Y_0\otimes X_0)\oplus(Y_0\otimes X_1)) \\ {\scriptstyle d}\downarrow & & & & \downarrow{\scriptstyle d} \\ (Y_1\otimes Y_0)\otimes X_0\oplus(Y_1\otimes Y_0)\otimes X_1 & & \xrightarrow{a^*\oplus a^*} & & Y_1\otimes(Y_0\otimes X_0)\oplus Y_1\otimes(Y_0\otimes X_1) \end{array}$$

(*c*)

$$\begin{array}{ccc} Y\otimes(X_0\oplus X_1) & \xrightarrow{\mathrm{id}\otimes c} & Y\otimes(X_1\oplus X_0) \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d} \\ (Y\otimes X_0)\oplus(Y\otimes X_1) & \xrightarrow{c} & (Y\otimes X_1)\oplus(Y\otimes X_0) \end{array}$$

(u^*) With [1] denoting the unit object for $\otimes$,

$$\begin{array}{ccc} [1]\otimes(X_0\oplus X_1) & \xleftarrow{u^*} & \\ {\scriptstyle d}\downarrow & & \\ ([1]\otimes X_0)\oplus([1]\otimes X_1) & \xleftarrow{u^*\oplus u^*} & X_0\oplus X_1 \end{array}$$

(u) The canonical map gives isomorphism $Y\otimes[0]\xrightarrow{\sim}[0]$, and we have

$$\begin{array}{ccc} Y\otimes(X\oplus[0]) & \xleftarrow{\mathrm{id}\otimes u} & Y\otimes X \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle u} \\ (Y\otimes X)\oplus(Y\otimes[0]) & \underset{\sim}{\xleftarrow{\mathrm{id}\oplus[0]}} & (Y\otimes X)\oplus[0] \end{array}$$

A homomorphism of commutative ring categories $\varphi : A \to A'$ is a functor respecting $\oplus, a, c, u, \otimes, a^*, c^*, u^*, d$. Thus an $\mathbb{F}$-ring is a homomorphism of commutative ring categories $\varphi : \mathbb{F} \to A$ which is a *bijection on objects*. Most of what we do in the following works more generally for commutative ring categories, but working with $\mathbb{F}$-rings is easier and allows the suppression of the isomorphisms $a, c, u, a^*, c^*, u^*, d$. On the other hand it will be interesting to work more generally with *braided* ring categories, replacing the symmetric monoidal structure $\otimes$ by a braided monoidal structure; this might lead to the quantum geometry behind [Har01] and [Har06].

2.2 $\mathbb{F}^{\mathrm{t}}, \mathbb{F}^{\pm}, \mathbb{F}^{\lambda}, \overline{\mathbb{F}}, \overline{\mathbb{F}}^{\lambda}$-rings

Remark 2.10. We can define $\mathbb{F}^{\mathrm{t}}$-rings to be $\mathbb{F}$-rings with involution

$$A_{Y,X} \to A_{X,Y}, \quad f \mapsto f^{\mathrm{t}}, \tag{2.10.1}$$

agreeing with the given involution on $\mathbb{F}$, and satisfying

$$(f \circ g)^{\mathrm{t}} = g^{\mathrm{t}} \circ f^{\mathrm{t}}, \tag{2.10.2}$$

$$f^{\mathrm{tt}} = f, \tag{2.10.3}$$

$$(f_0 \oplus f_1)^{\mathrm{t}} = f_0^{\mathrm{t}} \oplus f_1^{\mathrm{t}}, \tag{2.10.4}$$

$$(f_0 \otimes f_1)^{\mathrm{t}} = f_0^{\mathrm{t}} \otimes f_1^{\mathrm{t}}. \tag{2.10.5}$$

A homomorphism of $\mathbb{F}^{\mathrm{t}}$-rings is a homomorphism of $\mathbb{F}$-rings φ satisfying

$$\varphi(f)^{\mathrm{t}} = \varphi(f^{\mathrm{t}}).$$

Thus we have a category of $\mathbb{F}^{\mathrm{t}}$-$\mathcal{R}ings$.

Remark 2.11. One defines $\mathbb{F}^{\pm}$-rings A as a category with objects $|\mathbb{F}^{\pm}|$, with $[0]$ as an initial and final object, and with symmetric monoidal structures

$$\oplus, \otimes : A \times A \to A, \tag{2.11.1}$$

with $[0], [\pm 1]$ as identities, with $\otimes$ distributive over $\oplus$, and with a functor $\mathbb{F}^{\pm} \to A$ which is the identity on objects and respects the symmetric monoidal structures $\oplus$ and $\otimes$. A homomorphism $\varphi : A \to B$ of $\mathbb{F}^{\pm}$-rings is a functor over $\mathbb{F}^{\pm}$ which respects the symmetric monoidal structures $\oplus$ and $\otimes$. Thus we have the category $\mathbb{F}^{\pm}$-$\mathcal{R}ings$. Replacing $\mathbb{F}^{\pm}$ by $\overline{\mathbb{F}}$, and $[\pm 1]$ by μ, one obtains the definition of the category $\overline{\mathbb{F}}$-$\mathcal{R}ings$. We can similarly define $\mathbb{F}^{\pm,\mathrm{t}}$-rings to be $\mathbb{F}^{\pm}$-rings with involution, agreeing with the given involution on $\mathbb{F}^{\pm}$, and respecting $\oplus$ and $\otimes$. Maps of $\mathbb{F}^{\pm,\mathrm{t}}$-rings are maps of $\mathbb{F}^{\pm}$-rings respecting the involution, hence we have a category $\mathbb{F}^{\pm,\mathrm{t}}$-$\mathcal{R}ings$. Similarly we have the category $\overline{\mathbb{F}}^{\mathrm{t}}$-$\mathcal{R}ings$.

Definition 2.12. An $\mathbb{F}^{\lambda}$-*ring* A is an $\mathbb{F}^{\pm}$-ring, together with functors

$$\wedge^k : A \to A, \quad k = 0, 1, \ldots \tag{2.12.1}$$

such that

$$\wedge^k : A_{Y,X} \to A_{\wedge^k(Y),\wedge^k(X)} \tag{2.12.2}$$

$$\wedge^k(\mathrm{id}_X) = \mathrm{id}_{\wedge^k(X)} \tag{2.12.3}$$

and moreover $\wedge^k$ agree with the given operation on $\mathbb{F}^{\pm}$ cf. (1.26), and

$$\wedge^0(f) = 1, \tag{2.12.4}$$

$$\wedge^1(f) = f, \tag{2.12.5}$$

$$\wedge^k(f \oplus g) = \bigoplus_{0 \leqslant j \leqslant k} \wedge^j(f) \otimes \wedge^{k-j}(g). \tag{2.12.6}$$

One similarly defines an $\mathbb{F}^{\lambda,t}$-ring to be an $\mathbb{F}^{\pm,t}$-ring and an $\mathbb{F}^{\lambda}$-ring such that

$$\wedge^k(f)^{\mathrm{t}} = \wedge^k(f^{\mathrm{t}}). \tag{2.12.7}$$

Similarly replacing $\mathbb{F}^{\pm}$ by $\overline{\mathbb{F}}$ one defines an $\overline{\mathbb{F}}^{\lambda}$-ring to be an $\overline{\mathbb{F}}$-ring A together with functors (2.12.1) satisfying (2.12.2)–(2.12.6). Similarly, $\overline{\mathbb{F}}^{\lambda,t}$-rings are $\overline{\mathbb{F}}^{\lambda}$-rings with involution satisfying (2.12.7).

Remark. It is possible to add further axioms (e.g. the ones corresponding to 'special' λ-rings). Here we shall only note the following. For X a finite set with $\{\pm\}$ action, an orientation on X is a choice of an isomorphism

$$\varepsilon : [\pm 1] \xrightarrow{\sim} \wedge^d(X), \quad d = \dim X, \tag{2.12.8}$$

i.e. it is a choice of one of the two (non-zero) elements $\varepsilon(1) \in \wedge^d(X)$. For A an $\mathbb{F}^\lambda$-ring, and for $a \in A_{X,X}$ we have $\det_X(a) \in A_{[\pm 1],[\pm 1]}$, defined by

$$\det_X(a) = \varepsilon^{-1} \circ \wedge^d(a) \circ \varepsilon. \tag{2.12.9}$$

It is independent of the choice of $\varepsilon \in \mathbb{F}_{\wedge^d(X),[\pm 1]} \subseteq A_{\wedge^d(X),[\pm 1]}$, and it satisfies

$$\det_X(a \circ a') = \det_X(a) \cdot \det_X(a'), \tag{2.12.10}$$

$$\det_X(\mathrm{id}_X) = 1 (= \mathrm{id}_{[\pm 1]}), \tag{2.12.11}$$

$$\det_{X_1 \oplus X_2}(a_1 \oplus a_2) = \det_{X_1}(a_1) \cdot \det_{X_2}(a_2). \tag{2.12.13}$$

The choice of the orientation ε on X gives also the duality isomorphism

$$\widetilde{\varepsilon} : X \xrightarrow{\sim} \wedge^{d-1}(X) \quad (\text{and } \widetilde{\varepsilon} : \wedge^j(X) \xrightarrow{\sim} \wedge^{d-j}(X)), \tag{2.12.14}$$

uniquely determined by

$$x \wedge \widetilde{\varepsilon}(x) = \varepsilon(1), \quad x \in X. \tag{2.12.15}$$

For $a \in A_{X,X}$, we have $a^{\mathrm{adj}} \in A_{X,X}$, defined by

$$a^{\mathrm{adj}} = \widetilde{\varepsilon}^{-1} \circ \wedge^{d-1}(a^{\mathrm{t}}) \circ \widetilde{\varepsilon}, \tag{2.12.16}$$

where $\widetilde{\varepsilon} \in \mathbb{F}_{\wedge^{d-1}(X),X} \subseteq A_{\wedge^{d-1}(X),X}$. It is independent of the choice of ε, and it satisfies

$$(a \circ b)^{\mathrm{adj}} = b^{\mathrm{adj}} \circ a^{\mathrm{adj}}, \tag{2.12.17}$$

$$(\mathrm{id}_X)^{\mathrm{adj}} = \mathrm{id}_X. \tag{2.12.18}$$

It is useful to have the expansion of the determinant by rows/columns,

$$a \circ a^{\mathrm{adj}} = a^{\mathrm{adj}} \circ a = \det(a) \cdot \mathrm{id}_X. \tag{2.12.19}$$

As a corollary of (2.12.19) we have that $a \in A_{X,X}$ is invertible (i.e. there exists $a^{-1} \in A_{X,X}$ with $a \circ a^{-1} = a^{-1} \circ a = \mathrm{id}_X$) if and only if $\det_X(a) \in A_{[\pm 1],[\pm 1]}$ is invertible. Indeed, if a is invertible $\det(a)$ is always invertible with inverse $\det(a^{-1})$, and conversely, if $\det(a)$ is invertible then by (2.12.19) a itself is invertible with inverse $\det(a)^{-1} \cdot a^{\mathrm{adj}}$.

Remark. For an $\mathbb{F}$-ring A (or an $\mathbb{F}^\pm$ or $\overline{\mathbb{F}}$-ring), we let $GL_X(A)$ denote the group of invertible elements in $A_{X,X}$,

$$GL_X(A) = \{a \in A_{X,X} \mid \exists a^{-1} \in A_{X,X}, a \circ a^{-1} = a^{-1} \circ a = \mathrm{id}_X\}. \tag{2.12.20}$$

We have homomorphisms,

$$GL_{X_1}(A) \times GL_{X_2}(A) \to GL_{X_1 \oplus X_2}(A), \quad (a_1, a_2) \mapsto a_1 \oplus a_2, \tag{2.12.21}$$

$$GL_{X_1}(A) \times GL_{X_2}(A) \to GL_{X_1 \otimes X_2}(A), \quad (a_1, a_2) \mapsto a_1 \otimes a_2. \tag{2.12.22}$$

In particular, we have the homomorphisms

$$GL_{[n]}(A) \to GL_{[n+1]}(A), \quad a \mapsto a \oplus \mathrm{id}_{[1]},$$

hence the direct limit

$$GL_\infty(A) = \varinjlim GL_{[n]}(A). \tag{2.12.23}$$

We can then define the higher K-groups of A following Quillen [Qui73]:

$$K_n(A) = \pi_{n+1}(BGL_\infty(A)^+). \tag{2.12.24}$$

Note that for an $\mathbb{F}$-ring A associated with a commutative ring B, $A = \mathbb{F}(B)$ (see example 1 below), we have $GL_\infty(A) = \varinjlim GL_n(B)$ and $K_n(A) = K_n(B)$.

2.3 Examples of $\mathbb{F}$-rings

Example 0. $\mathbb{F}$ is an $\mathbb{F}$-ring.

Example 1. Let A be a commutative ring (always with identity). We denote by $\mathbb{F}(A)$ the $\mathbb{F}$-ring with

$$\mathbb{F}(A)_{Y,X} = \mathrm{Hom}_A(A\cdot X, A\cdot Y) = Y\times X - \text{matrices with values in } A, \tag{2.13}$$

where $\circ$ is the usual composition of A-linear homomorphisms (or multiplication of A-valued matrices), and where $\oplus$ is the usual direct sum, and $\otimes$ the tensor product. Note that a homomorphism of commutative rings $\varphi : A \to B$ induces a map of $\mathbb{F}$-rings $\mathbb{F}(\varphi) : \mathbb{F}(A) \to \mathbb{F}(B)$, hence we have a functor

$$\mathbb{F} : \mathcal{R}ings \to \mathbb{F}\text{-}\mathcal{R}ings. \tag{2.13.1}$$

Moreover let $\varphi : \mathbb{F}(A) \to \mathbb{F}(B)$ be a map of $\mathbb{F}$-rings. For $a \in \mathbb{F}(A)_{Y,X}$ write

$$a_{y,x} = j_y^{\mathrm{t}} \circ a \circ j_x \in A = \mathbb{F}(A)_{[1],[1]} \tag{2.13.2}$$

for its matrix coefficients, where j_x, j_y^{t} are the morphisms of $\mathbb{F}$ given by

$$j_x : [1] \to X, \quad j_x(1) = x \in X,$$

and where

$$j_y^{\mathrm{t}} : Y \to [1] \text{ is the partial bijection } \{y\} \to \{1\}. \tag{2.13.3}$$

Since φ is a functor over $\mathbb{F}$, and $j_y^{\mathrm{t}}, j_x \in \mathbb{F}$, we have $\varphi(a)_{y,x} = \varphi(a_{y,x})$ and φ is determined by $\varphi : A = \mathbb{F}(A)_{[1],[1]} \to B = \mathbb{F}(B)_{[1],[1]}$. This map is multiplicative, $\varphi(a_1\cdot a_2) = \varphi(a_1)\cdot\varphi(a_2), \varphi(1) = 1$, and moreover it is additive,

$$\varphi(a_1 + a_2) = \varphi\left[(a_1, a_2)\circ\begin{pmatrix}1\\1\end{pmatrix}\right] = (\varphi(a_1), \varphi(a_2))\circ\begin{pmatrix}1\\1\end{pmatrix} = \varphi(a_1) + \varphi(a_2). \tag{2.13.4}$$

Thus the functor $\mathbb{F}$ is fully faithful.

Example 2. Let M be a commutative monoid with a unit 1 and a zero element 0. Thus we have an associative and commutative operation

$$\begin{gathered} M\times M \to M, \quad (a,b)\mapsto a\cdot b, \\ a\cdot(b\cdot c) = (a\cdot b)\cdot c, \quad a\cdot b = b\cdot a, \end{gathered} \tag{2.14}$$

and $1 \in M$ is the (unique) element such that

$$a\cdot 1 = a, \quad a\in M, \tag{2.14.1}$$

and $0 \in M$ is the (unique) element such that

$$a\cdot 0 = 0, \quad a \in M. \tag{2.14.2}$$

Let $\mathbb{F}\langle M\rangle$ denote the $\mathbb{F}$-ring with $\mathbb{F}\langle M\rangle_{Y,X}$ the $Y\times X$ matrices with values in M with at most one non-zero entry in every row and column. Note that this is indeed an $\mathbb{F}$-ring with the usual 'multiplication' of matrices $\circ$ (there is no addition involved – only multiplication in M), direct sum $\oplus$, and tensor product $\otimes$.

Denoting by $\mathcal{M}on_{0,1}$ the category of commutative monoids with unit and zero elements, and with maps respecting the operation and the elements $0, 1$, the above construction yields a functor

$$\mathcal{M}on_{0,1} \to \mathbb{F}\text{-}\mathcal{R}ings, \quad M \mapsto \mathbb{F}\langle M\rangle. \tag{2.14.3}$$

This is the functor left-adjoint to the functor

$$\mathbb{F}\text{-}\mathcal{R}ings \to \mathcal{M}on_{0,1}, \quad A\mapsto A_{[1],[1]},$$

namely

$$\mathrm{Hom}_{\mathbb{F}\text{-}\mathcal{R}ings}(\mathbb{F}\langle M\rangle, A) = \mathrm{Hom}_{\mathcal{M}on_{0,1}}(M, A_{[1],[1]}). \tag{2.14.4}$$

As a particular example, take $M = M_q$ to be the free monoid (with zero) generated by one element q,

$$M_q = q^{\mathbb{N}} \cup \{0\}.$$

Then

$$\mathrm{Hom}_{\mathbb{F}\text{-}\mathcal{R}ings}(\mathbb{F}\langle M_q\rangle, A) = A_{[1],[1]}.$$

Example 3. Let $\mathbb{S}$ denote the $\mathbb{F}$-ring of sets. The objects of $\mathbb{S}$ are the finite sets of $|\mathbb{F}|$, and we let $\mathbb{S}_{Y,X}$ be the partially defined maps of sets from X to Y,

$$\mathbb{S}_{Y,X} = \{f : V \to Y \mid V \subseteq X\}. \tag{2.15.1}$$

Notice that if A is an $\mathbb{F}$-ring, the opposite category A^{op} is again an $\mathbb{F}$-ring, since $\mathbb{F}^{\mathrm{op}} = \mathbb{F}$ and since the axioms of an $\mathbb{F}$-ring are self-dual. In particular, we have the $\mathbb{F}$-ring $\mathbb{S}^{\mathrm{op}}$ with

$$\mathbb{S}^{\mathrm{op}}_{Y,X} = \{f : V \to X \mid V \subseteq Y\}. \tag{2.15.2}$$

We have the $\mathbb{F}$-ring of relations $\mathcal{R}$ that contains both $\mathbb{S}$ and $\mathbb{S}^{\mathrm{op}}$, with

$$\mathcal{R}_{Y,X} = \{F \subseteq Y \times X \text{ a subset}\}. \tag{2.15.3}$$

The composition of $F \in \mathcal{R}_{Y,X}$ and $G \in \mathcal{R}_{Z,Y}$ is given by

$$G \circ F = \{(z,x) \in Z \times X \mid \exists y \in Y \text{ with } (z,y) \in G, (y,x) \in F\}, \tag{2.15.4}$$

and $G \circ F \in \mathcal{R}_{Z,X}$.

The sum $F_0 \oplus F_1 \in \mathcal{R}_{Y_0 \oplus Y_1, X_0 \oplus X_1}$ of $F_i \in \mathcal{R}_{Y_i,X_i}$ is given by the disjoint union of F_0 and F_1,

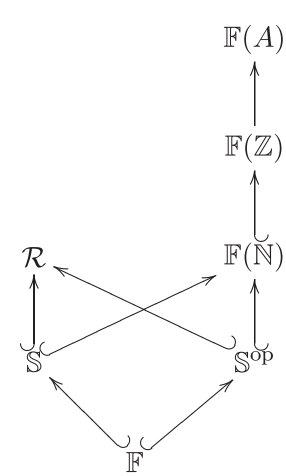
$$F_0 \oplus F_1 = \{((x,i),(y,i)) \mid (x,y) \in F_i\}, \tag{2.15.5}$$

and the product $F_0 \otimes F_1 \in \mathcal{R}_{Y_0 \otimes Y_1, X_0 \otimes X_1}$ is given by

$$F_0 \otimes F_1 = \{((x_0,x_1),(y_0,y_1)) \mid (x_0,y_0) \in F_0, (x_1,y_1) \in F_1\}. \tag{2.15.6}$$

Equivalently, $\mathcal{R}_{Y,X}$ are the $Y \times X$ matrices with values in $\{0,1\}$, and $\oplus, \otimes$ are the direct sum and the tensor product of matrices (but $\circ$ does not correspond to matrix multiplication).

We have the $\mathbb{F}$-subring of $\mathbb{F}(\mathbb{Z})$ consisting of matrices with values in $\mathbb{N}$; we denote it by $\mathbb{F}(\mathbb{N})$. This $\mathbb{F}$-ring also contains $\mathbb{S}$ and $\mathbb{S}^{\mathrm{op}}$, but composition in $\mathbb{F}(\mathbb{N})$ is matrix multiplication. We can summarize these basic $\mathbb{F}$-rings in the following diagram, where A is a commutative ring.



(2.15.7)

Example 4. Let $\Bbbk$ be a ring and $\eta : \Bbbk \to \mathbb{C}$ an embedding (e.g. η a real or complex prime of a number field). For $X \in |\mathbb{F}|$, let $\Bbbk \cdot X$ denote the free $\Bbbk$-module with inner product having X as an orthonormal basis. Thus for $a = (a_x) \in \Bbbk \cdot X$ we have its norm

$$|a|_\eta = \sqrt{\left(\sum_{x \in X} |\eta(a_x)|^2\right)}, \tag{2.16.1}$$

and for a $\Bbbk$-linear map $f \in \mathrm{Hom}_\Bbbk(\Bbbk \cdot X, \Bbbk \cdot Y)$ we have its operator norm

$$|f|_\eta = \sup_{|a|_\eta \leqslant 1} |f(a)|_\eta. \tag{2.16.2}$$

We have

$$|f \circ g|_\eta \leqslant |f|_\eta \cdot |g|_\eta, \tag{2.16.3}$$

$$|f \oplus g|_\eta = \max\{|f|_\eta, |g|_\eta\}, \tag{2.16.4}$$

$$|f \otimes g|_\eta = |f|_\eta \cdot |g|_\eta. \tag{2.16.5}$$

Let $\mathcal{O}_{\Bbbk,\eta}$ denote the $\mathbb{F}$-ring with

$$(\mathcal{O}_{\Bbbk,\eta})_{Y,X} = \{f \in \mathrm{Hom}_\Bbbk(\Bbbk \cdot X, \Bbbk \cdot Y), |f|_\eta \leqslant 1\} \tag{2.16}$$

and with the usual operations $\circ, \oplus, \otimes$.

For $X_0, X_1 \in |\mathbb{F}|$, denote by

$$j_i : X_i \to X_0 \oplus X_1, \quad j_i(x) = (x, i), \tag{2.17.1}$$

the natural inclusion, and by

$$j_i^{\mathrm{t}} : X_0 \oplus X_1 \to X_i, \quad j_i^{\mathrm{t}}(x, i') = \begin{cases} x, & i = i' \\ 0, & i \neq i' \end{cases} \tag{2.17.2}$$

its transpose. For an $\mathbb{F}$-ring A we get maps

$$A_{Y, X_0 \oplus X_1} \to A_{Y,X_0} \times A_{Y,X_1}, \quad f \mapsto (f \circ j_0, f \circ j_1), \tag{2.17.3}$$

and

$$A_{X_0 \oplus X_1, Y} \to A_{X_0,Y} \times A_{X_1,Y}, \quad f \mapsto (j_0^{\mathrm{t}} \circ f, j_1^{\mathrm{t}} \circ f). \tag{2.17.4}$$

We say that A is a *matrix ring* if these maps are always injections. Equivalently, A is a matrix ring if every element is determined by its coefficients, that is we have an injection

$$A_{Y,X} \hookrightarrow (A_{[1],[1]})^{Y \times X}, \quad f \mapsto \{j_y^{\mathrm{t}} \circ f \circ j_x\}_{y \in Y, x \in X}, \tag{2.17}$$

with j_x, j_y^{t} as in (2.13.2) and (2.13.3).

The above examples $0, 1, 2, 3$ (except for $\mathcal{R}$) and 4, all constitute matrix rings. The following gives examples of $\mathbb{F}$-rings which are not matrix rings (they are the 'residue $\mathbb{F}$-field' of the $\mathbb{F}$-rings of Example 4 (2.16)).

Example 5. Let $\Bbbk$ be a ring and let $\eta : \Bbbk \to \mathbb{C}$ be an embedding, and for $X \in |\mathbb{F}|$, let $\Bbbk \cdot X$ denote the free $\Bbbk$-module with basis X and with the inner product having X as an orthonormal basis. Let $\mathbb{F}_{\Bbbk,\eta}$ denote the $\mathbb{F}$-ring of 'partial isometries', with

$$(\mathbb{F}_{\Bbbk,\eta})_{Y,X} = \{f : V \xrightarrow{\sim} W, \text{ with } V \subseteq \Bbbk \cdot X, W \subseteq \Bbbk \cdot Y \text{ } \Bbbk\text{-submodules and } f \text{ is a } \Bbbk\text{-linear isometry}\}. \tag{2.18}$$

For $f = (f : V \xrightarrow{\sim} W) \in (\mathbb{F}_{\Bbbk,\eta})_{Y,X}, g = (g : W' \xrightarrow{\sim} U) \in (\mathbb{F}_{\Bbbk,\eta})_{Z,Y}$, we have

$$g \circ f = (g \circ f : f^{-1}(W \cap W') \xrightarrow{\sim} g(W \cap W')) \in (\mathbb{F}_{\Bbbk,\eta})_{Z,X}; \tag{2.18.1}$$

and for $f_i = (f_i : V_i \xrightarrow{\sim} W_i) \in (\mathbb{F}_{\Bbbk,\eta})_{Y_i,X_i}$, we have

$$f_0 \oplus f_1 = (f_0 \oplus f_1 : V_0 \oplus V_1 \xrightarrow{\sim} W_0 \oplus W_1), \tag{2.18.2}$$

$$f_0 \otimes f_1 = (f_0 \otimes f_1 : V_0 \otimes V_1 \xrightarrow{\sim} W_0 \otimes W_1). \tag{2.18.3}$$

We will see in Example 4.21 below that $\mathbb{F}_{\Bbbk,\eta}$ is indeed the residue field of $\mathcal{O}_{\Bbbk,\eta}$.

Remark. All of the above examples (except $\mathbb{S}$ and $\mathbb{S}^{\mathrm{op}}$) have a natural involution making them into $\mathbb{F}^{\mathrm{t}}$-rings. Moreover, all the examples have obvious analogous $\mathbb{F}^{\pm}$-rings. For example, for a commutative ring A, we have the $\mathbb{F}^{\pm}$-ring $\mathbb{F}^{\pm}(A)$ with

$$\mathbb{F}^{\pm}(A)_{Y,X} = \mathrm{Hom}_A(A \cdot X, A \cdot Y),$$

where $A \cdot X$ denotes the free A-module with basis $\{(x) \mid x \in X\}$, divided by the A-submodule generated by $\{(x) + (-x) \mid x \in X\}$:

$$A \cdot X = \bigoplus_{x \in X} A(x)/_{(-x)\sim-(x)};$$

alternatively, $A \cdot X$ is the free A-module with basis X^+, where $X^+ \subseteq X$ is a basis of the ± 1-set X. Then $A \mapsto \mathbb{F}^{\pm}(A)$ is a fully faithful functor from $\mathcal{R}ings$ to $\mathbb{F}^{\pm}$-$\mathcal{R}ings$. All the above examples of $\mathbb{F}^{\pm}$-rings are $\mathbb{F}^{\pm,\mathrm{t}}$-rings with respect to transposition. Moreover, exterior powers give them the structure of $\mathbb{F}^{\lambda}$-rings.

For a commutative ring A that contains all the roots of unity, together with a fixed map $\mu \to \mu(A)$ from our abstract group μ onto the group of roots of unity $\mu(A) \subseteq A^*$ (this map could have kernel μ_{p^∞} if A has characteristic p), we have the $\overline{\mathbb{F}}$-ring $\overline{\mathbb{F}}(A)$ with

$$\overline{\mathbb{F}}(A)_{Y,X} = \mathrm{Hom}_A(A \cdot X, A \cdot Y),$$

where $A \cdot X$ denotes the free A-module with basis X divided by the A-submodule generated by $\{\zeta \cdot (x) - (\zeta \cdot x) \mid x \in X, \zeta \in \mu\}$. Then $A \mapsto \overline{\mathbb{F}}(A)$ is a fully faithful functor from μ-$\mathcal{R}ings$ to $\overline{\mathbb{F}}$-$\mathcal{R}ings$, where μ-$\mathcal{R}ings$ is the category of such commutative rings A together with the map $\mu \to \mu(A)$, and ring homomorphisms preserving these maps. The $\overline{\mathbb{F}}(A)$ has an involution making it an $\overline{\mathbb{F}}^{\mathrm{t}}$-ring. Moreover, exterior powers give $\overline{\mathbb{F}}(A)$ the structure of $\overline{\mathbb{F}}^{\lambda}$-rings.

Remark. The categories $\mathbb{F}$-$\mathcal{R}ings$ (respectively $\mathbb{F}^{\pm}$-$\mathcal{R}ings$, $\mathbb{F}^{\mathrm{t}}$-$\mathcal{R}ings$, $\mathbb{F}^{\lambda}$-$\mathcal{R}ings$, $\overline{\mathbb{F}}$-$\mathcal{R}ings$, $\overline{\mathbb{F}}^{\mathrm{t}}$-$\mathcal{R}ings$, $\overline{\mathbb{F}}^{\lambda}$-$\mathcal{R}ings$) have fibred products. Given homomorphisms of $\mathbb{F}$-rings

$$\varphi^i : A^i \to B, \quad i = 0, 1,$$

we have the $\mathbb{F}$-ring $A^0 \prod_B A^1$, with

$$\left(A^0 \prod_B A^1\right)_{Y,X} = \{(a_0, a_1) \in A^0_{Y,X} \times A^1_{Y,X} \mid \varphi^0(a_0) = \varphi^1(a_1)\}.$$

Similarly we can construct arbitrary products $\prod_i A^i$, and arbitrary inverse limits $\varprojlim A^i$, where $i \mapsto A^i$ is a functor from a small category to $\mathbb{F}$-$\mathcal{R}ings$ (respectively $\mathbb{F}^{\mathrm{t}}$-$\mathcal{R}ings$, etc.).

Definition 2.19. Let A be an $\mathbb{F}$-ring. An *equivalence ideal* $\mathcal{E}$ is a collection of subsets

$$\mathcal{E} = \coprod_{Y,X \in |\mathbb{F}|} \mathcal{E}_{Y,X},$$

with $\mathcal{E}_{Y,X} \subseteq A_{Y,X} \times A_{Y,X}$, such that

$$\mathcal{E}_{Y,X} \text{ is an equivalence relation on } A_{Y,X}. \tag{2.19.1}$$

For $(a,a') \in \mathcal{E}_{Y,X}$, and for $b_1 \in A_{Y',Y}, b_2 \in A_{X,X''}$,

$$b_1 \circ (a,a') \circ b_2 \stackrel{\text{def}}{=} (b_1 \circ a \circ b_2, b_1 \circ a' \circ b_2) \in \mathcal{E}_{Y',X''}. \tag{2.19.2}$$

For $(a_i,a_i') \in \mathcal{E}_{Y_i,X_i}$, $i=0,1$,

$$(a_0,a_0') \oplus (a_1,a_1') \stackrel{\text{def}}{=} (a_0 \oplus a_1, a_0' \oplus a_1') \in \mathcal{E}_{Y_0 \oplus Y_1, X_0 \oplus X_1}. \tag{2.19.3}$$

For $(a,a') \in \mathcal{E}_{Y,X}$, and for $b \in A_{W,Z}$,

$$b \otimes (a,a') \stackrel{\text{def}}{=} (b \otimes a, b \otimes a') \in \mathcal{E}_{W \otimes Y, Z \otimes X}. \tag{2.19.4}$$

Given an equivalence ideal $\mathcal{E}$ of A, let

$$A/\mathcal{E} = \coprod_{Y,X \in |\mathbb{F}|} A_{Y,X}/\mathcal{E}_{Y,X},$$

and let $\pi : A \to A/\mathcal{E}$ denote the canonical map which associates with $a \in A_{Y,X}$ its equivalence class $\pi(a) \in A_{Y,X}/\mathcal{E}_{Y,X}$. It follows from (2.19.2) (respectively (2.19.3), (2.19.4)) that we have well-defined operations on $A/\mathcal{E}$,

$$\begin{gathered} \pi(f) \circ \pi(g) = \pi(f \circ g) \\ (\text{respectively } \pi(f) \oplus \pi(g) = \pi(f \oplus g), \pi(f) \otimes \pi(g) = \pi(f \otimes g)), \end{gathered} \tag{2.19.5}$$

making $A/\mathcal{E}$ into an $\mathbb{F}$-ring such that $\pi : A \to A/\mathcal{E}$ is a homomorphism of $\mathbb{F}$-rings.

Given a homomorphism of $\mathbb{F}$-rings $\varphi : A \to B$ denote by

$$\begin{gathered} \mathcal{KER}(\varphi) = \coprod_{Y,X \in |\mathbb{F}|} \mathcal{KER}_{Y,X}(\varphi), \\ \mathcal{KER}_{Y,X}(\varphi) = \{(a,a') \in A_{Y,X} \times A_{Y,X} \mid \varphi(a) = \varphi(a')\}. \end{gathered} \tag{2.20}$$

It is clear that $\mathcal{KER}(\varphi)$ is an equivalence ideal of A, and that φ induces an injection of $\mathbb{F}$-rings $\overline{\varphi} : A/\mathcal{KER}(\varphi) \hookrightarrow B$, such that $\varphi = \overline{\varphi} \circ \pi$, i.e.

$$\begin{array}{ccc} A & \xrightarrow{\quad\varphi\quad} & B \\ & {\scriptstyle\pi}\searrow \quad \nearrow{\scriptstyle\overline{\varphi}} & \\ & A/_{\mathcal{KER}(\varphi)} & \end{array} \tag{2.21}$$

is a commutative diagram. Thus every map φ of $\mathbb{F}$-rings factors as an epimorphism (π) followed by an injection ($\overline{\varphi}$).

Example 2.22. Let $A = \mathcal{O}_{\mathbb{Z}[1/N],\eta}$ be the $\mathbb{F}$-ring of Example 4, (2.16), with $\Bbbk = \mathbb{Z}[1/N]$. For a prime $\mathfrak{p}$ not dividing N there is a surjective homomorphism

$$\varphi_{\mathfrak{p}} : A \twoheadrightarrow \mathbb{F}(\mathbb{F}_{\mathfrak{p}}), \quad \varphi_{\mathfrak{p}}(a) = a(\text{mod } \mathfrak{p}).$$

We have the equivalence ideal $\mathcal{E}_{\mathfrak{p}} = \mathcal{KER}(\varphi_{\mathfrak{p}})$.

Similarly, there is a surjective homomorphism $\varphi_\eta : A \twoheadrightarrow \mathbb{F}_{\mathbb{Z}[1/N],\eta}$, cf. Example 4.21 below, with $\mathbb{F}_{\mathbb{Z}[1/N],\eta}$ the $\mathbb{F}$-ring of Example 5, (2.18), and we have the equivalence ideal $\mathcal{E}_\eta = \mathcal{KER}(\varphi_\eta)$.

3. Modules

We define the notion of an A-module for an $\mathbb{F}$-ring A. Since we gave up addition we cannot define directly the quotient M/N where N is a sub-A-module of M. We can divide A-modules only by an equivalence A-module, and we study the relationship between sub-A-modules and equivalence A-modules. We describe the standard operations on A-modules and give many examples.

3.1 Definitions and examples

DEFINITION 3.1. Let A be an $\mathbb{F}$-ring. An *A-module* M is a collection of sets $M = \{M_{Y,X}\}_{Y,X\in|\mathbb{F}|}$, together with maps

$$A_{Y',Y} \times M_{Y,X} \times A_{X,X'} \to M_{Y',X'}, \quad (a,m,a') \mapsto a \circ m \circ a', \tag{3.1.1}$$

$$A_{Y_0,X_0} \times M_{Y_1,X_1} \to M_{Y_0\otimes Y_1, X_0\otimes X_1}, \quad (a,m) \mapsto a \otimes m, \tag{3.1.2}$$

$$M_{Y_0,X_0} \times M_{Y_1,X_1} \to M_{Y_0\oplus Y_1, X_0\oplus X_1}, \quad (m_0,m_1) \mapsto m_0 \oplus m_1. \tag{3.1.3}$$

We assume $M_{[0],X} = \{0_X\}$, $M_{Y,[0]} = \{0_Y^{\mathrm{t}}\}$, and we have a distinguished zero element $0_{Y,X} \in M_{Y,X}$, such that

$$0 \circ m = 0, \quad m \circ 0 = 0, \quad a \circ 0 \circ a' = 0, \quad a \otimes 0 = 0, \quad 0 \oplus 0 = 0. \tag{3.1.4}$$

The maps $\circ, \oplus, \otimes$ satisfy: for $a, a', \overline{a}, \overline{a}', a_i, a_i' \in A,\ m, m_i \in M$,

$$\overline{a} \circ (a \circ m \circ a') \circ \overline{a}' = (\overline{a} \circ a) \circ m \circ (a' \circ \overline{a}'), \tag{3.1.5}$$

$$\mathrm{id}_Y \circ m \circ \mathrm{id}_X = m, \tag{3.1.6}$$

$$(a_0 \oplus a_1) \circ (m_0 \oplus m_1) \circ (a_0' \oplus a_1') = (a_0 \circ m_0 \circ a_0') \oplus (a_1 \circ m_1 \circ a_1'), \tag{3.1.7}$$

$$m_0 \oplus m_1 = m_1 \oplus m_0, \tag{3.1.8}$$

$$m_0 \oplus (m_1 \oplus m_2) = (m_0 \oplus m_1) \oplus m_2, \tag{3.1.9}$$

$$m \oplus 0_{[0]} = m, \tag{3.1.10}$$

$$(\overline{a} \otimes a) \circ (a_0 \otimes m) \circ (\overline{a}' \otimes a') = (\overline{a} \circ a_0 \circ \overline{a}') \otimes (a \circ m \circ a'), \tag{3.1.11}$$

$$a_0 \otimes (a_1 \otimes m) = (a_0 \otimes a_1) \otimes m, \tag{3.1.12}$$

$$\mathrm{id}_{[1]} \otimes m = m, \tag{3.1.13}$$

$$(a_0 \oplus a_1) \otimes m = (a_0 \otimes m) \oplus (a_1 \otimes m), \tag{3.1.14}$$

$$a \otimes (m_0 \oplus m_1) = (a \otimes m_0) \oplus (a \otimes m_1). \tag{3.1.15}$$

In particular, (3.1.2) induces an action of the monoid $A_{[1],[1]}$ on $M_{Y,X}$ via $(a,m) \mapsto a \otimes m$.

Example 3.2.1. Let A be a commutative ring, $\mathbb{F}(A)$ the associated $\mathbb{F}$-ring. For an A-module M let $\mathbb{F}(M)_{Y,X}$ denote the $Y \times X$ matrices with values in M. Then $\mathbb{F}(M)$ has natural operations (3.1.1), (3.1.2), (3.1.3) making it into an $\mathbb{F}(A)$-module. Note that for $M = A$ we obtain the $\mathbb{F}$-ring $\mathbb{F}(A)$ viewed as an $\mathbb{F}(A)$-module. We have, for A-modules M_1, M_2,

$$\mathbb{F}(\mathrm{Hom}_A(M_1, M_2))_{Y,X} = \mathrm{Hom}_A(M_1 \otimes_A A \cdot X, M_2 \otimes_A A \cdot Y). \tag{3.2.1}$$

Example 3.2.2. For a finite set V let $\mathbb{F}(V)_{Y,X}$ denote the $Y \times X$ matrices with values in $V \cup \{0\}$ such that every row and every column contains at most one non-zero term. Then $\mathbb{F}(V)$ has natural operations (3.1.1), (3.1.2), (3.1.3) making it into a module over the $\mathbb{F}$-ring $\mathbb{F}$. For $V = [1]$ we obtain $\mathbb{F}([1])$ which is just $\mathbb{F}$ viewed as an $\mathbb{F}$-module. We have, for finite sets V_1, V_2,

$$\mathbb{F}(\mathrm{Hom}_{\mathbb{F}}(V_1, V_2))_{Y,X} = \mathrm{Hom}_{\mathbb{F}}(V_1 \otimes X, V_2 \otimes Y). \tag{3.2.2}$$

For an $\mathbb{F}$-module W, such that $W_{Y,X}$ is a finite set for all $X, Y \in |\mathbb{F}|$, we say it has dimension $\dim_{\mathbb{F}} W$ over $\mathbb{F}$ if the following limit exists (where n, m go to infinity independently of each other):

$$\dim_{\mathbb{F}} W = \lim_{n,m\to\infty} \frac{1}{nm} \log \sharp W_{[n],[m]}.$$

Thus if V is a finite dimensional vector space over the finite field $\mathbb{F}_q$, and $\mathbb{F}(V)$ the associated

$\mathbb{F}(\mathbb{F}_q)$-module viewed as $\mathbb{F}$-module, we have

$$\dim_{\mathbb{F}} \mathbb{F}(V) = \dim_{\mathbb{F}} \mathbb{F}(\mathbb{F}_q) \cdot \dim_{\mathbb{F}_q} V$$

with $\dim_{\mathbb{F}} \mathbb{F}(\mathbb{F}_q) = \log q$.

For a finite set V, the associated $\mathbb{F}$-module $\mathbb{F}(V)$ is zero dimensional in the above sense, $\dim_{\mathbb{F}} \mathbb{F}(V) = 0$. We can use a different dimension function, $\mathrm{Dim}_{\mathbb{F}} W$ for W an $\mathbb{F}$-module (with $W_{[n],[m]}$ finite for all n, m), given by

$$\mathrm{Dim}_{\mathbb{F}} W = \lim_{x,y\to\infty} \frac{1}{xy} \log \sum_{n,m\geqslant 0} (\sharp W_{[n],[m]}) \frac{x^n}{n!} \frac{y^m}{m!}.$$

For the $\mathbb{F}$-module $W = \mathbb{F}(V)$, V a finite set, it gives

$$\begin{aligned} \mathrm{Dim}_{\mathbb{F}} \mathbb{F}(V) &= \lim_{x,y\to\infty} \frac{1}{xy} \log \sum_{n,m\geqslant k} \binom{n}{k}\binom{m}{k} k!(\sharp V)^k \frac{x^n}{n!} \frac{y^m}{m!} \\ &= \lim_{x,y\to\infty} \frac{1}{xy} \log \exp(x + y + xy(\sharp V)) = \sharp V. \end{aligned}$$

Indeed, to give an arbitrary element of $\mathbb{F}(V)_{[n],[m]}$ we have to choose k rows (respectively, k columns), and there are $\binom{n}{k}$ (respectively, $\binom{m}{k}$) choices, then we have to choose a bijection between these rows and columns (there are $k!$ possibilities for such a bijection), and finally we have to fill in the k chosen entries with elements of V (and there are $(\sharp V)^k$ such choices), hence

$$\sharp \mathbb{F}(V)_{[n],[m]} = \sum_{k\leqslant n,m} \binom{n}{k}\binom{m}{k} k!(\sharp V)^k.$$

Example 3.2.3. Let $\Bbbk$ be a field, $\eta : \Bbbk \to \mathbb{C}$ an embedding, and let V be a $\Bbbk$-vector space with an inner product $(\cdot,\cdot)_V$ and associated norm $\|\cdot\|_V$. Let $\mathbb{F}(V)_{Y,X}$ denote the $Y \times X$ matrices with values in V, $\mathrm{v} = (v_{y,x})$, such that for $\mathrm{a} = (a_x) \in \Bbbk \cdot X$, $\mathrm{b} = (b_y) \in \Bbbk \cdot Y$, we have (cf. (2.16.1))

$$\Big\| \sum_{x,y} b_y v_{y,x} a_x \Big\|_V \leqslant |\mathrm{a}|_\eta \cdot |\mathrm{b}|_\eta.$$

The set $\mathbb{F}(V)$ has natural operations (3.1.1), (3.1.2), (3.1.3) making it into an $\mathcal{O}_{\Bbbk,\eta}$-module. For $V = \Bbbk$ we obtain $\mathbb{F}(\Bbbk)$ which is $\mathcal{O}_{\Bbbk,\eta}$ viewed as an $\mathcal{O}_{\Bbbk,\eta}$-module.

Definition 3.3. Let A be an $\mathbb{F}$-ring, and M, M' be A-modules. A collection of maps

$$\varphi = \{\varphi_{Y,X} : M_{Y,X} \to M'_{Y,X} \mid Y, X \in |\mathbb{F}|\}$$

is a *homomorphism of A-modules* if it respects the operations

$$\varphi(a \circ m \circ a') = a \circ \varphi(m) \circ a', \tag{3.3.1}$$

$$\varphi(a \otimes m) = a \otimes \varphi(m), \tag{3.3.2}$$

$$\varphi(m_0 \oplus m_1) = \varphi(m_0) \oplus \varphi(m_1). \tag{3.3.3}$$

The collection of A-modules and homomorphisms form a category A-$\mathcal{M}od$. It has an initial and final object $0 = \{\{0_{Y,X}\}\}_{Y,X\in|\mathbb{F}|}$. For a commutative ring A, the construction of Example 3.2.1 gives a functor

$$\mathbb{F} : A\text{-}\mathcal{M}od \to \mathbb{F}(A)\text{-}\mathcal{M}od, \quad M \mapsto \mathbb{F}(M). \tag{3.4.1}$$

As in (2.13.4) we see that this functor is fully faithful.

Similarly, the construction of Example 3.2.2 gives us a functor

$$\mathbb{F} : \mathbb{F} \to \mathbb{F}\text{-}\mathcal{M}od, \quad V \mapsto \mathbb{F}(V). \tag{3.4.2}$$

For a field embedding $\eta : \mathbb{k} \hookrightarrow \mathbb{C}$, let $(\mathbb{k}, \eta)$-$\mathcal{V}ec$ denote the category whose objects are $\mathbb{k}$-vector spaces with an inner product and morphisms are $\mathbb{k}$-linear maps with operator norm at most 1; the construction of Example 3.2.3 gives a functor

$$\mathbb{F} : (\mathbb{k}, \eta)\text{-}\mathcal{V}ec \to \mathcal{O}_{\mathbb{k},\eta}\text{-}\mathcal{M}od, \quad V \mapsto \mathbb{F}(V). \tag{3.4.3}$$

3.2 A-submodules and equivalence A-modules

Definition 3.5. Let A be an $\mathbb{F}$-ring, M an A-module. An A-*submodule* M' of M is a collection of subsets $M' = \{M'_{Y,X} \subseteq M_{Y,X}\}$ which is closed under the operations $\circ, \oplus, \otimes$:

$$A \circ M' \circ A \subseteq M', \quad A \otimes M' \subseteq M', \quad M' \oplus M' \subseteq M'. \tag{3.5.1}$$

We denote by $\mathrm{sub}_A(M)$ the collection of A-submodules of M. The intersection of A-submodules is again an A-submodule. An A-submodule of A is called an *ideal*.

Let

$$\varphi : M \to N \tag{3.6}$$

be a homomorphism of A-modules. We have an A-submodule of M:

$$\varphi^{-1}(0) = \{m \in M \mid \varphi(m) = 0\}. \tag{3.6.1}$$

It is the kernel of φ in the category A-$\mathcal{M}od$.

We have also an A-submodule of N:

$$\varphi(M) = \{\varphi_{Y,X}(M_{Y,X})\}_{Y,X \in |\mathbb{F}|}. \tag{3.6.2}$$

The homomorphism φ induces maps

$$\varphi_* : \mathrm{sub}_A(M) \to \mathrm{sub}_A(N), \quad M' \mapsto \varphi_* M' \stackrel{\text{def}}{=} \varphi(M'), \tag{3.6.3}$$

$$\varphi^* : \mathrm{sub}_A(N) \to \mathrm{sub}_A(M), \quad N' \mapsto \varphi^* N' \stackrel{\text{def}}{=} \varphi^{-1}(N'). \tag{3.6.4}$$

The category A-$\mathcal{M}od$ has fibred products. Given A-$\mathcal{M}od$ homomorphisms

$$\varphi_0 : M_0 \to M \leftarrow M_1 : \varphi_1 \tag{3.7}$$

we have the A-module

$$M_0 \prod_M M_1$$

with

$$\left(M_0 \prod_M M_1\right)_{Y,X} = \{(m_0, m_1), \quad m_i \in (M_i)_{Y,X}, \quad \varphi_0(m_0) = \varphi_1(m_1)\}, \tag{3.7.1}$$

and the operations

$$\begin{gathered} a \circ (m_0, m_1) \circ a' = (a \circ m_0 \circ a', a \circ m_1 \circ a'), \\ a \otimes (m_0, m_1) = (a \otimes m_0, a \otimes m_1), \\ (m_0, m_1) \oplus (m'_0, m'_1) = (m_0 \oplus m'_0, m_1 \oplus m'_1). \end{gathered} \tag{3.7.2}$$

In particular we have products $M_0 \prod M_1$. We can similarly form arbitrary products $\prod_\lambda M_\lambda$, and arbitrary inverse limits

$$\varprojlim M_\lambda = \left\{(m_\lambda) \in \prod M_\lambda \;\middle|\; \varphi_{\lambda',\lambda}(m_\lambda) = m_{\lambda'}\right\}, \tag{3.7.3}$$

where $\lambda \mapsto M_\lambda$ is a functor from a small category to A-$\mathcal{M}od$.

Let $\varphi : M \to N$ be a homomorphism of A-modules. Let

$$\mathcal{KER}(\varphi)_{Y,X} = \left\{ (m, m') \in \left(M \prod M \right)_{Y,X} \middle| \varphi(m) = \varphi(m') \right\} = M \prod_N M. \tag{3.8}$$

Then $\mathcal{KER}(\varphi)$ is an A-submodule of $M \prod M$ such that, for all $Y, X \in |\mathbb{F}|, \mathcal{KER}(\varphi)_{Y,X}$ is an equivalence relation on $M_{Y,X}$.

Definition 3.9. Let M be an A-module. An *equivalence A-module* of M is an A-submodule $\mathcal{E}$ of $M \prod M$, such that $\mathcal{E}_{Y,X}$ is an equivalence relation on $M_{Y,X}$. We denote by $\mathrm{equiv}_A(M)$ the collection of equivalence A-modules of M.

For $\mathcal{E} \in \mathrm{equiv}_A(M)$ we can form the equivalence classes $(M/\mathcal{E})_{Y,X} = M_{Y,X}/\mathcal{E}_{Y,X}$. There is an induced A-module structure on $M/\mathcal{E}$ such that the canonical map $\pi : M \to M/\mathcal{E}$ is a homomorphism. We have

$$\mathrm{Hom}_{A\text{-}\mathcal{M}od}(M/\mathcal{E}, N) = \{ \varphi \in \mathrm{Hom}_{A\text{-}\mathcal{M}od}(M, N) \mid \mathcal{KER}(\varphi) \supseteq \mathcal{E} \}. \tag{3.9.1}$$

We have one-to-one order-preserving correspondence

$$\mathrm{equiv}_A(M/\mathcal{E}) \cong \{ \mathcal{E}' \in \mathrm{equiv}_A(M) \mid \mathcal{E}' \supseteq \mathcal{E} \}, \quad \mathcal{E}'/\mathcal{E} \mapsto \mathcal{E}', \tag{3.9.2}$$

and a natural isomorphism

$$(M/\mathcal{E})/(\mathcal{E}'/\mathcal{E}) \cong M/\mathcal{E}'. \tag{3.9.3}$$

Definition 3.9.4. For an equivalence A-module $\mathcal{E}$ of M, a submodule $M_0 \subseteq M$ is called *$\mathcal{E}$-stable* if for all $(m, m') \in \mathcal{E}$,

$$m \in M_0 \Leftrightarrow m' \in M_0.$$

We have a one-to-one order-preserving correspondence

$$\mathrm{sub}_A(M/\mathcal{E}) \cong \{ M_0 \in \mathrm{sub}_A(M) \mid M_0 \text{ is } \mathcal{E}\text{-stable} \}, \quad M_0/\mathcal{E} \mapsto M_0. \tag{3.9.5}$$

Every homomorphism of A-modules $\varphi : M \to N$ factors as (injection) $\circ$ (surjection), as in the diagram.

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ \downarrow & & \uparrow \\ M/_{\mathcal{KER}(\varphi)} & \xrightarrow[\sim]{} & \varphi(M) \end{array} \tag{3.9.6}$$

Definition 3.10. For an equivalence A-module $\mathcal{E} \subseteq M \prod M$ let

$$Z(\mathcal{E}) = \pi^{-1}(0) = \{ m \in M \mid (m, 0) \in \mathcal{E} \} = \mathcal{E} \cap \left(M \prod \{0\} \right). \tag{3.10.1}$$

It is an A-submodule of M. For an A-submodule $M_0 \subseteq M$ let

$$E(M_0) \subseteq M \prod M \tag{3.10.2}$$

be the equivalence A-module of M generated by $\{(m, 0) \mid m \in M_0\}$, i.e. $E(M_0)$ is the intersection $\bigcap \mathcal{E}$ of all equivalence A-modules $\mathcal{E}$ of M such that $M_0 \times \{0\} \subseteq \mathcal{E}$. We write M/M_0 for $M/E(M_0)$. For a homomorphism of A-modules $\varphi : M \to N$ we have its cokernel,

$$\mathrm{Coker}(\varphi) = N/\varphi(M) = N/E(\varphi(M)). \tag{3.10.3}$$

Lemma 3.11. *We have*

$$M_0 \subseteq M_0' \ A\text{-submodules of } M \Rightarrow E(M_0) \subseteq E(M_0'), \tag{3.11.1}$$

$$\mathcal{E} \subseteq \mathcal{E}' \subseteq M \coprod M \text{ equivalence } A\text{-modules of } M \Rightarrow Z(\mathcal{E}) \subseteq Z(\mathcal{E}'), \tag{3.11.2}$$

$$M_0 \subseteq Z(E(M_0)), \tag{3.11.3}$$

$$E(Z(\mathcal{E})) \subseteq \mathcal{E}. \tag{3.11.4}$$

Proof. The proof is straightforward. □

Corollary 3.12. *We have*

$$E(M_0) = E(Z(E(M_0))), \tag{3.12.1}$$

$$Z(\mathcal{E}) = Z(E(Z(\mathcal{E}))). \tag{3.12.2}$$

Hence we have

$$\{Z(\mathcal{E}) \mid \mathcal{E} \in \text{equiv}_A(M)\} = \{M_0 \subseteq M \mid M_0 = Z(E(M_0))\}; \tag{3.12.3}$$

we denote this set by $E\text{-sub}_A(M)$.

Similarly, we have

$$\{E(M_0) \mid M_0 \subseteq M A\text{-submodule}\} = \left\{\mathcal{E} \subseteq M \coprod M \;\middle|\; \mathcal{E} = E(Z(\mathcal{E}))\right\}; \tag{3.12.4}$$

we denote this set by $Z\text{-equiv}_A(M)$.

Moreover, there is an induced bijection

$$E\text{-sub}_A(M) \overset{\sim}{\leftrightarrow} Z\text{-equiv}_A(M),$$
$$M_0 \mapsto E(M_0),$$
$$Z(\mathcal{E}) \leftarrow\!\shortmid \mathcal{E}.$$

Lemma 3.13. *Let $M_0 \subseteq M$ be an A-submodule, and let $\mathcal{E}_{Y,X} \subseteq (M \coprod M)_{Y,X}$ denote the collection of pairs (m, m') such that there exists a 'path' $m = m_0, m_1, \dots, m_l = m'$, where for $j = 0, \dots, l-1$, $\{m_j, m_{j+1}\}$ has the form $\{a \circ (n \oplus n_0) \circ a', a \circ (n \oplus 0) \circ a'\}$ for some $a, a' \in A, n \in M, n_0 \in M_0$. Then $E(M_0) = \mathcal{E}$.*

Proof. Note that for $a, a' \in A, n \in M, n_0 \in M_0$, we have

$$(m_0, 0) \in E(M_0), (m, m) \in E(M_0)$$

and since $E(M_0) \subseteq M \coprod M$ is a submodule we get

$$(a \circ (n \oplus n_0) \circ a', a \circ (n \oplus 0) \circ a') \in E(M_0).$$

Thus if there is a path $m = m_0, \dots, m_l = m'$ as above, then $(m, m') \in E(M_0)$; so $\mathcal{E} \subseteq E(M_0)$.

For the reverse inclusion note that $\mathcal{E}_{Y,X}$ is an equivalence relation on $M_{Y,X}$. Moreover, $\mathcal{E}$ is an A-submodule of $M \coprod M$. For $(m, m') \in \mathcal{E}$ there exists a path $m = m_0, \dots, m_l = m'$ as above, hence for $a, a' \in A$,

$$a \circ m_0 \circ a', \dots, a \circ m_l \circ a'$$

is a path from $a \circ m \circ a'$ to $a \circ m' \circ a'$, hence

$$(a \circ m \circ a', a \circ m' \circ a') \in \mathcal{E}.$$

Similarly, for $\overline{a} \in A_{Y,X}$,

$$\overline{a} \otimes m_0, \dots, \overline{a} \otimes m_l$$

is a path, which shows that

$$(\overline{a}\otimes m, \overline{a}\otimes m')\in\mathcal{E}.$$

To show this use

$$\overline{a}\otimes(a\circ(n\oplus n_0)\circ a') = (\overline{a}\otimes a)\circ((\mathrm{id}_X\otimes n)\oplus(\mathrm{id}_X\otimes n_0))\circ(\mathrm{id}_X\otimes a'),$$
$$\mathrm{id}_X\otimes n_0\in M_0,\quad \mathrm{id}_X\otimes 0 = 0.$$

If also $(\overline{m},\overline{m}')\in\mathcal{E}$, we can assume the path $\overline{m}=\overline{m_0},\ldots,\overline{m_l}=\overline{m}'$ has the same length l (by adding identities $n_0=0, a=\mathrm{id}, a'=\mathrm{id}$), and then

$$m_0\oplus\overline{m_0},\ldots,m_l\oplus\overline{m_l}$$

is a path, which shows that

$$(m\oplus\overline{m}, m'\oplus\overline{m}')\in\mathcal{E}.$$

To show this use

$$(a\circ(n\oplus n_0)\circ a')\oplus(\overline{a}\circ(\overline{n}\oplus\overline{n_0})\circ\overline{a}') = (a\oplus\overline{a})\circ((n\oplus\overline{n})\oplus(n_0\oplus\overline{n_0}))\circ(a'\oplus\overline{a}').$$

Thus $\mathcal{E}$ is an equivalence A-submodule of M, and since $\{(m_0,0)\mid m_0\in M_0\}\subseteq\mathcal{E}$, we have $E(M_0)\subseteq\mathcal{E}$. ☐

Corollary 3.14. *Let $M_0\subseteq M$ be an A-submodule. We have*

$$Z(E(M_0)) = M_0$$

if and only if, for all $m_0\in M_0,\ m\in M, a, a'\in A$,

$$a\circ(m\oplus m_0)\circ a'\in M_0 \quad\Leftrightarrow\quad a\circ(m\oplus 0)\circ a'\in M_0, \tag{3.14.1}$$

i.e. $M_0\in E\text{-sub}_A(M)$ if and only if M_0 is $E(M_0)$-stable.

Proof. Assume (3.14.1) holds. By Lemma 3.13 if $(m,m')\in E(M_0)$ there exists a path $m = m_0,\ldots,m_l=m'$, and we have

$$m_j\in M_0\quad\Leftrightarrow\quad m_{j+1}\in M_0,$$

hence

$$m\in M_0\quad\Leftrightarrow\quad m'\in M_0.$$

Taking $m'=0\in M_0$, we get $(m,0)\in E(M_0)$ implies $m\in M_0$. Thus $Z(E(M_0))\subseteq M_0$, and since the reverse inclusion always holds we get $Z(E(M_0))=M_0$.

Conversely, assume $Z(E(M_0))=M_0$, then

$$a\circ(m\oplus m_0)\circ a'\in M_0\quad\Leftrightarrow\quad(a\circ(m\oplus m_0)\circ a',0)\in E(M_0), \tag{3.14.2}$$
$$a\circ(m\oplus 0)\circ a'\in M_0\quad\Leftrightarrow\quad(a\circ(m\oplus 0)\circ a',0)\in E(M_0). \tag{3.14.3}$$

Using the fact that $E(M_0)_{Y,X}$ is an equivalence relation, and that for $m_0\in M_0$

$$(a\circ(m\oplus m_0)\circ a', a\circ(m\oplus 0)\circ a')\in E(M_0),$$

we see that the statements in (3.14.2) and (3.14.3) are equivalent, hence (3.14.1) holds. ☐

For submodules $M_0\in\mathrm{sub}_A(M)$, $M'\in E\text{-sub}_A(M)$, we have $M'\supseteq M_0$ if and only if M' is $E(M_0)$-stable. We get a one-to-one order-preserving correspondence

$$E\text{-sub}_A(M/M_0) = \{M'\in E\text{-sub}_A(M)\mid M'\supseteq M_0\},\quad M'/M_0\mapsto M' \tag{3.15.1}$$

and a natural isomorphism

$$(M/M_0)/(M'/M_0)\cong M/M'. \tag{3.15.2}$$

An A-submodule of A is called an ideal, and an equivalence A-module of A is called an equivalence ideal. Thus we have the maps E, Z between ideals and equivalence ideals satisfying Lemma 3.11 and Corollary 3.12. Elements of $E\text{-sub}_A(A)$ will be called E-ideals.

Example 3.15.3. For $A = \mathcal{O}_{\mathbb{Z}[1/N],\eta}$, with the notation of Example 2.22, we have for $\mathfrak{p} \nmid N$: $EZ(\mathcal{E}_\mathfrak{p}) \underset{\neq}{\subseteq} \mathcal{E}_\mathfrak{p} \cap \mathcal{E}_\eta$.

3.3 Operations on submodules

For a family $\{M_i\}$ of A-submodules of M, we have the intersection $\bigcap_i M_i \in \text{sub}_A(M)$. Note that if $M_i \in E\text{-sub}_A(M)$ then $\bigcap_i M_i \in E\text{-sub}_A(M)$. We have also $\sum_i M_i$ the A-submodule generated by the M_i, i.e. it is the intersection $\bigcap N$ taken over all submodules N that contain all the M_i. It can be described explicitly as

$$\left(\sum_i M_i\right)_{Y,X} = \left\{ a \circ \left(\bigoplus_i m_i\right) \circ a' \;\middle|\; a \in A_{Y,\bigoplus_i Y_i},\ a' \in A_{\bigoplus_i X_i, X},\ m_i \in (M_i)_{Y_i,X_i} \right\}. \tag{3.16}$$

Indeed the right-hand side will be contained in any submodule N which contains all the M_i, it itself contains the M_i, and is closed under the module operations

$$b \circ \left(a \circ \left(\bigoplus_i m_i\right) \circ a'\right) \circ b' = (b \circ a) \circ \left(\bigoplus_i m_i\right) \circ (a' \circ b'), \tag{3.16.1}$$

$$\left(a \circ \left(\bigoplus_i m_i\right) \circ a'\right) \oplus \left(b \circ \left(\bigoplus_i m_i'\right) \circ b'\right) = (a \oplus b) \circ \left(\bigoplus_i (m_i \oplus m_i')\right) \circ (a' \oplus b'), \tag{3.16.2}$$

$$b \otimes \left(a \circ \left(\bigoplus_i m_i\right) \circ a'\right) = (\text{id}_Y \otimes a) \circ \left(\bigoplus_i (b \otimes m_i)\right) \circ (\text{id}_X \otimes a'), b \in A_{Y,X}. \tag{3.16.3}$$

More generally, given any subset $\{m_i \mid i \in I\} \subseteq M$, with $m_i \in M_{Z_i,W_i}$, the A-submodule it generates $\sum_i A \cdot m_i$ can be described explicitly as

$$\left(\sum_i A \cdot m_i\right)_{Y,X} = \left\{ a \circ \left(\bigoplus_i \text{id}_{X_i} \otimes m_i\right) \circ a' \;\middle|\; a \in A_{Y,\bigoplus_i (X_i \otimes Z_i)}, a' \in A_{\bigoplus_i (X_i \otimes W_i), X} \right\}. \tag{3.16.4}$$

Given an A-module M and an ideal $\mathfrak{a} \subseteq A$ we have their product $\mathfrak{a} \cdot M$ which is an A-submodule of M,

$$(\mathfrak{a} \cdot M)_{Y,X} = \left\{ b \circ \left(\bigoplus_i (a_i \otimes m_i)\right) \circ b' \;\middle|\; b \in A_{Y,\bigoplus_i (Y_i \otimes Z_i)}, b' \in A_{\bigoplus_i (X_i \otimes W_i), X}, a_i \in \mathfrak{a}_{Y_i,X_i}, m_i \in M_{Z_i,W_i} \right\}. \tag{3.16.5}$$

Given A-submodules M_0, M_1 of M we can form their quotient

$$(M_0 : M_1) = \{a \in A \mid a \otimes m \in M_0 \ \forall m \in M_1\}. \tag{3.16.6}$$

It is easily checked that $(M_0 : M_1)$ is an ideal of A.

3.4 Operations on modules

Sums. Given A-modules M_0, M_1, we first construct the sum (coproduct) $M_0 \coprod M_1$ in the category A-$\mathcal{M}od$. We form

$$\left(M_0 \coprod M_1\right)_{Y,X} = \{(a, m_0, m_1, a') \mid a \in A_{Y,Y_0 \oplus Y_1},\ a' \in A_{X_0 \oplus X_1, X}, m_i \in (M_i)_{Y_i,X_i}\}/\sim \tag{3.17.0}$$

where $\sim$ is the equivalence relation generated by

$$(a \circ (a_0 \oplus a_1), m_0, m_1, a') \sim (a, a_0 \circ m_0, a_1 \circ m_1, a'), \tag{3.17.1}$$
$$(a, m_0, m_1, (a'_0 \oplus a'_1) \circ a') \sim (a, m_0 \circ a'_0, m_1 \circ a'_1, a'). \tag{3.17.2}$$

Let $(a, m_0, m_1, a')/_\sim$ denote the equivalence class containing (a, m_0, m_1, a'). Define

$$b \circ (a, m_0, m_1, a')/_\sim \circ b' = (b \circ a, m_0, m_1, a' \circ b')/_\sim, \tag{3.17.3}$$
$$(a, m_0, m_1, a')/_\sim \oplus (\overline{a}, \overline{m}_0, \overline{m}_1, \overline{a}')/_\sim = (a \oplus \overline{a}, m_0 \oplus \overline{m}_0, m_1 \oplus \overline{m}_1, a' \oplus \overline{a}')/_\sim, \tag{3.17.4}$$
$$b \otimes (a, m_0, m_1, a')/_\sim = (b \otimes a, \mathrm{id}_Z \otimes m_0, \mathrm{id}_Z \otimes m_1, \mathrm{id}_Z \otimes a')/_\sim. \tag{3.17.5}$$

Proposition 3.17. *The operations (3.17.3), (3.17.4), (3.17.5) are well defined, independent of the chosen representatives, and make $M_0 \coprod M_1$ into an A-module. There are canonical homomorphisms $M_i \to M_0 \coprod M_1$, taking m in $(M_0)_{Y,X}$ (respectively, $(M_1)_{Y,X}$) into $(\mathrm{id}_Y, m, 0, \mathrm{id}_X)/_\sim$ (respectively, $(\mathrm{id}_Y, 0, m, \mathrm{id}_X)/_\sim$). These maps make $M_0 \coprod M_1$ into the sum of M_0, M_1 in the category A-$\mathcal{M}od$,*

$$\mathrm{Hom}_{A\text{-}\mathcal{M}od}(M_0, N) \times \mathrm{Hom}_{A\text{-}\mathcal{M}od}(M_1, N) = \mathrm{Hom}_{A\text{-}\mathcal{M}od}\Big(M_0 \coprod M_1, N\Big), \tag{3.17.6}$$
$$(\varphi_0, \varphi_1) \mapsto \varphi_0 \amalg \varphi_1((a, m_0, m_1, a')/_\sim) = a \circ (\varphi_0(m_0) \oplus \varphi_1(m_1)) \circ a'.$$

Proof. To show that the operations are well defined we need to show that on replacing the representatives by equivalent ones we get the same result. Since two representatives are equivalent if and only if there is a path between them whose consecutive members are related by (3.17.1) or (3.17.2), it is enough to show that we get the same result for representatives related by (3.17.1), (3.17.2). That $\circ$ in (3.17.3) is well defined follows from associativity of $\circ$. That $\oplus$ is well defined with respect to (3.17.1) follows from

$$\begin{aligned}
&((a \circ (a_0 \oplus a_1)) \oplus \overline{a}, m_0 \oplus \overline{m}_0, m_1 \oplus \overline{m}_1, a' \oplus \overline{a}')/_\sim \\
&\quad = ((a \oplus \overline{a}) \circ (a_0 \oplus a_1 \oplus \mathrm{id}_{\overline{Y}_0 \oplus \overline{Y}_1}), m_0 \oplus \overline{m}_0, m_1 \oplus \overline{m}_1, a' \oplus \overline{a}')/_\sim \\
&\quad = (a \oplus \overline{a}, (a_0 \circ m_0) \oplus \overline{m}_0, (a_1 \circ m_1) \oplus \overline{m}_1, a' \oplus \overline{a}')/_\sim.
\end{aligned}$$

Similarly $\otimes$ in (3.17.5) is well defined with respect to (3.17.1):

$$\begin{aligned}
&(b \otimes (a \circ (a_0 \oplus a_1)), \mathrm{id}_Z \otimes m_0, \mathrm{id}_Z \otimes m_1, \mathrm{id}_Z \otimes a')/_\sim \\
&\quad = ((b \otimes a) \circ (\mathrm{id}_Z \otimes a_0 \oplus \mathrm{id}_Z \otimes a_1), \mathrm{id}_Z \otimes m_0, \mathrm{id}_Z \otimes m_1, \mathrm{id}_Z \otimes a')/_\sim \\
&\quad = (b \otimes a, \mathrm{id}_Z \otimes (a_0 \circ m_0), \mathrm{id}_Z \otimes (a_1 \circ m_1), \mathrm{id}_Z \otimes a')/_\sim.
\end{aligned}$$

It is clear that $\circ$ as defined in (3.17.3) is associative (3.1.5) and unitary (3.1.6). It is clear that $\oplus$ as defined in (3.17.4) is functorial (3.1.7), commutative (3.1.8), associative (3.1.9), and unitary (3.1.10). It is clear that $\otimes$ as defined in (3.17.5) is associative (3.1.12), unitary (3.1.13), and distributive (3.1.14), (3.1.15). We check that $\otimes$ is functorial (3.1.11): on the 'left'

$$\begin{aligned}
&(d \otimes c) \circ (b \otimes (a, m_0, m_1, a')/_\sim) \\
&\quad = ((d \otimes c) \circ (b \otimes a), \mathrm{id}_Z \otimes m_0, \mathrm{id}_Z \otimes m_1, \mathrm{id}_Z \otimes a')/_\sim \\
&\quad = ((d \circ b) \otimes (c \circ a), \mathrm{id}_Z \otimes m_0, \mathrm{id}_Z \otimes m_1, \mathrm{id}_Z \otimes a')/_\sim \\
&\quad = (d \circ b) \otimes (c \circ (a, m_0, m_1, a')/_\sim);
\end{aligned}$$

and on the 'right' for $b \in A_{W,Z},\ d \in A_{Z,T}$:

$$\begin{aligned}
&(b \otimes (a, m_0, m_1, a')/_\sim) \circ (d \otimes c) \\
&\quad = (b \otimes a, \mathrm{id}_Z \otimes m_0, \mathrm{id}_Z \otimes m_1, (\mathrm{id}_Z \otimes a') \circ (d \otimes c))/_\sim \\
&\quad = (b \otimes a, \mathrm{id}_Z \otimes m_0, \mathrm{id}_Z \otimes m_1, (d \otimes \mathrm{id}_{X_0 \oplus X_1}) \circ (\mathrm{id}_T \otimes (a' \circ c)))/_\sim \\
&\quad = (b \otimes a, d \otimes m_0, d \otimes m_1, \mathrm{id}_T \otimes (a' \circ c))/_\sim \\
&\quad = ((b \otimes a) \circ (d \otimes \mathrm{id}_{Y_0 \oplus Y_1}), \mathrm{id}_T \otimes m_0, \mathrm{id}_T \otimes m_1, \mathrm{id}_T \otimes (a' \circ c))/_\sim \\
&\quad = ((b \circ d) \otimes a, \mathrm{id}_T \otimes m_0, \mathrm{id}_T \otimes m_1, \mathrm{id}_T \otimes (a' \circ c))/_\sim \\
&\quad = (b \circ d) \otimes ((a, m_0, m_1, a')/_\sim \circ c).
\end{aligned}$$

Thus $M_0 \coprod M_1$ is an A-module, and it is easy to check that it is the sum of M_0, M_1 in A-$\mathcal{M}od$. $\square$

We shall write $a \circ (m_0 \oplus m_1) \circ a'$ for $(a, m_0, m_1, a')/_\sim$. The canonical map from the sum to the product is given by

$$\begin{aligned}
M_0 \coprod M_1 &\to M_0 \prod M_1, \\
a \circ (m_0 \oplus m_1) \circ a' &\mapsto (a \circ (m_0 \oplus 0) \circ a', a \circ (0 \oplus m_1) \circ a').
\end{aligned} \tag{3.17.7}$$

Given an arbitrary family of A-modules $\{M_i\}_{i \in I}$ we can similarly form their sum

$$\coprod_i M_i.$$

Direct limits. Given homomorphisms of A-modules $\psi_i : M \to M_i,\ i = 0, 1$, let $\mathcal{E}$ be the equivalence A-module of $M_0 \coprod M_1$ generated by

$$\{(a \circ ((m_0 \oplus \psi_0(m)) \oplus m_1) \circ a',\ a \circ (m_0 \oplus (\psi_1(m) \oplus m_1)) \circ a') \mid m_i \in M_i, m \in M\}. \tag{3.17.8}$$

The quotient

$$M_0 \coprod M_1/\mathcal{E} \overset{\text{def}}{=} M_0 \coprod_M M_1,$$

is the push-out

$$\mathrm{Hom}_{A\text{-}\mathcal{M}od}\Big(M_0 \coprod_M M_1, N\Big) = \{(\varphi_0, \varphi_1) \mid \varphi_i \in \mathrm{Hom}_{A\text{-}\mathcal{M}od}(M_i, N), \varphi_0 \circ \psi_0 = \varphi_1 \circ \psi_1\}.$$

Similarly, given a functor $\lambda \mapsto M_\lambda$ from a small category to A-$\mathcal{M}od$, we can form the direct limit

$$\varinjlim M_\lambda = \coprod_\lambda M_\lambda/\mathcal{E}.$$

Here $\mathcal{E}$ is the equivalence A-module of $\coprod M_\lambda$ generated by

$$(\mathrm{id}_Y \circ (m \oplus 0 \dots) \circ \mathrm{id}_X, \mathrm{id}_Y \circ (\varphi_{\lambda',\lambda}(m) \oplus 0 \dots) \circ \mathrm{id}_X),$$

where $\varphi_{\lambda',\lambda} : M_\lambda \to M_{\lambda'}$ runs over the maps in the direct system, and $m \in M_\lambda$.

Free modules. Fix sets $Y_0,\ X_0 \in |\mathbb{F}|$. Let us form

$$M_{Y,X} = \{(a, a') \mid a \in A_{Y, Z \otimes Y_0},\ a' \in A_{Z \otimes X_0, X}\}/_\sim, \tag{3.18.0}$$

where $\sim$ is the equivalence relation generated by

$$(a \circ (c \otimes \mathrm{id}_{Y_0}), a') \sim (a, (c \otimes \mathrm{id}_{X_0}) \circ a'), \quad a \in A_{Y, Z \otimes Y_0}, c \in A_{Z, Z'}, a' \in A_{Z' \otimes X_0, X}. \tag{3.18.1}$$

Let $(a,a')/_\sim$ denote the equivalence class of (a,a'), and define

$$b\circ (a,a')/_\sim \circ b' = (b\circ a, a'\circ b')/_\sim, \tag{3.18.2}$$

$$(a,a')/_\sim \oplus (\overline{a},\overline{a}')/_\sim = (a\oplus \overline{a}, a'\oplus \overline{a}')/_\sim, \tag{3.18.3}$$

$$b\otimes (a,a')/_\sim = (b\otimes a, \mathrm{id}_Z\otimes a'), b\in A_{W,Z}. \tag{3.18.4}$$

Proposition 3.18. *The operations (3.18.2), (3.18.3), (3.18.4) are well defined, independent of the representatives, and make M into an A-module. It is the 'free' A-module in degree Y_0, X_0:*

$$\mathrm{Hom}_{A\text{-}\mathcal{M}od}(M,N) = N_{Y_0,X_0} \quad \text{for all } A\text{-modules } N. \tag{3.18.5}$$

Proof. The proof is similar to that for Proposition 3.17. It follows that $\circ$ is well defined by associativity. That $\oplus$ is well defined follows from

$$\begin{aligned}
&((a\circ(c\otimes \mathrm{id}_{Y_0}))\oplus \overline{a}, a'\oplus\overline{a}')/_\sim\\
&\quad = ((a\oplus\overline{a})\circ((c\oplus \mathrm{id}_{\overline{Z}})\otimes \mathrm{id}_{Y_0}), a'\oplus \overline{a}')/_\sim\\
&\quad = (a\oplus\overline{a}, ((c\oplus \mathrm{id}_{\overline{Z}})\otimes \mathrm{id}_{X_0})\circ(a'\oplus\overline{a}'))/_\sim\\
&\quad = (a\oplus\overline{a}, ((c\otimes \mathrm{id}_{X_0})\circ a')\oplus\overline{a}')/_\sim.
\end{aligned}$$

Similarly $\otimes$ is well defined:

$$\begin{aligned}
&(b\otimes(a\circ(c\otimes\mathrm{id}_{Y_0})), \mathrm{id}_W\otimes a')/_\sim\\
&\quad = ((b\otimes a)\circ(\mathrm{id}_W\otimes c\otimes \mathrm{id}_{Y_0}), \mathrm{id}_W\otimes a')/_\sim\\
&\quad = (b\otimes a, (\mathrm{id}_W\otimes c\otimes\mathrm{id}_{X_0})\circ(\mathrm{id}_W\otimes a'))/_\sim\\
&\quad = (b\otimes a, \mathrm{id}_W\otimes((c\otimes \mathrm{id}_{X_0})\circ a'))/_\sim.
\end{aligned}$$

It then follows that M satisfies the axioms for A-modules; for the functoriality of $\otimes$, (3.1.11), we have 'on the left':

$$\begin{aligned}
&(d\otimes c)\circ(b\otimes(a,a')/_\sim)\\
&\quad = ((d\otimes c)\circ(b\otimes a), \mathrm{id}_Z\otimes a')/_\sim\\
&\quad = ((d\circ b)\otimes(c\circ a), \mathrm{id}_Z\otimes a')/_\sim\\
&\quad = (d\circ b)\otimes(c\circ(a,a')/_\sim);
\end{aligned}$$

and 'on the right' for $b\in A_{W,Z}, d\in A_{Z,T}$:

$$\begin{aligned}
&(b\otimes(a,a')/_\sim)\circ(d\otimes c)\\
&\quad = (b\otimes a, (\mathrm{id}_Z\otimes a')\circ(d\otimes c))/_\sim\\
&\quad = (b\otimes a, (d\otimes \mathrm{id})\circ(\mathrm{id}_T\otimes(a'\circ c)))/_\sim\\
&\quad = ((b\otimes a)\circ(d\otimes\mathrm{id}), \mathrm{id}_T\otimes(a'\circ c))/_\sim\\
&\quad = ((b\circ d)\otimes a, \mathrm{id}_T\otimes(a'\circ c))/_\sim\\
&\quad = (b\circ d)\otimes((a,a')/_\sim\circ c).
\end{aligned}$$

Thus M is an A-module. Given a homomorphism $\varphi: M\to N$, we get

$$\varphi((\mathrm{id}_{Y_0},\mathrm{id}_{X_0}))\in N_{Y_0,X_0}.$$

Given $n\in N_{Y_0,X_0}$ we get homomorphism φ by

$$\varphi((a,a')/_\sim) = a\circ(\mathrm{id}_Z\otimes n)\circ a', \quad a\in A_{Y,Z\otimes Y_0},\ a'\in A_{Z\otimes X_0,X}.$$

These are inverse to one another and give the bijection (3.18.5). □

Let $f = f_{Y_0,X_0} = (\mathrm{id}_{Y_0},\mathrm{id}_{X_0})/_\sim$. We have $(a,a')/_\sim = a\circ(\mathrm{id}_Z\otimes f)\circ a'$. We write $A\cdot f_{Y_0,X_0}$ for the above module M constructed in (3.18.0). Similarly, given a collection of symbols $\{f_i = f_{Y_i,X_i} = (\mathrm{id}_{Y_i}, \mathrm{id}_{X_i})/\sim\}$, we can form the sum

$$\coprod_{i\in I} A\cdot f_i$$

of the free A-modules on the f_i, and

$$\begin{aligned}\mathrm{Hom}_{A\text{-}\mathcal{M}od}\Big(\coprod_{i\in I} A\cdot f_i, N\Big) &= \{(n_i)_{i\in I} \mid n_i \in N_{Y_i,X_i}\}\\ &= \mathrm{Hom}_{\mathcal{S}et/|\mathbb{F}|\times|\mathbb{F}|}\Big(\{f_i\}_{i\in I}, \coprod_{Y,X\in|\mathbb{F}|} M_{Y,X}\Big),\end{aligned} \tag{3.18.6}$$

i.e. the functor taking $\{f_i\}$ to $\coprod A\cdot f_i$ is the left-adjoint to the forgetful functor from A-modules to sets over $|\mathbb{F}| \times| \mathbb{F}|$, which takes an A-module M to the $|\mathbb{F}|\times|\mathbb{F}|$ set

$$\coprod_{Y,X\in|\mathbb{F}|} M_{Y,X}.$$

Tensor product. Let M_0, M_1, N be A-modules. A collection of maps

$$f = \{f_{Y_0,X_0;Y_1,X_1} : (M_0)_{Y_0,X_0} \times (M_1)_{Y_1,X_1} \to N_{Y_0\otimes Y_1, X_0\otimes X_1}\} \tag{3.19}$$

is called A-bilinear if it satisfies

$$f(m_0 \oplus m_0', m_1) = f(m_0, m_1) \oplus f(m_0', m_1), \tag{3.19.1}$$
$$f(m_0, m_1 \oplus m_1') = f(m_0, m_1) \oplus f(m_0, m_1'), \tag{3.19.2}$$
$$f(a\otimes m_0, m_1) = a\otimes f(m_0, m_1) = f(m_0, a\otimes m_1), \tag{3.19.3}$$
$$f(a_0\circ m_0\circ a_0', a_1\circ m_1\circ a_1') = (a_0\otimes a_1)\circ f(m_0, m_1)\circ(a_0'\otimes a_1'). \tag{3.19.4}$$

We denote by $\mathcal{B}ilin(M_0, M_1; N)$ the set of all such f. We can similarly define $\mathcal{B}ilin(M_0, \dots, M_l; N)$ as the set of all

$$f = \{f_{Y_0,X_0;\dots;Y_l,X_l} : (M_0)_{Y_0,X_0} \times\cdots\times (M_l)_{Y_l,X_l} \to N_{Y_0\otimes\cdots\otimes Y_l, X_0\otimes\cdots\otimes X_l}\}$$

which are 'linear' in each variable. Note that if f is bilinear with values in N, and $\varphi : N \to N'$ is a homomorphism, then $\varphi\circ f$ is bilinear with values in N'.

Lemma 3.20. *There exists a universal bilinear map*

$$M_0\times M_1 \to M_0\otimes_A M_1, (m_0, m_1)\mapsto m_0\otimes m_1,$$

such that

$$\mathcal{B}ilin(M_0, M_1; N) = \mathrm{Hom}_{A\text{-}\mathcal{M}od}(M_0\otimes M_1, N).$$

Proof. The usual construction of the tensor product goes through. We form the free A-module

$$\coprod_{m_i\in M_i} A\cdot f(m_0, m_1).$$

Here m_i runs through the elements of M_i, $i = 0, 1$, and if $m_i \in (M_i)_{Y_i,X_i}$ we view $f(m_0, m_1)$ as a formal arrow from $X_0\otimes X_1$ to $Y_0\otimes Y_1$. We divide this free module by the equivalence A-module generated by the relations (3.19.1)–(3.19.4). □

We write $m_0\otimes m_1$ for the image of $f(m_0, m_1)$ in $M_0\otimes_A M_1$.

We can similarly construct $M_0\otimes\cdots\otimes M_l$ so that

$$\mathcal{B}ilin(M_0, \dots, M_l; N) = \mathrm{Hom}_{A\text{-}\mathcal{M}od}(M_0\otimes\cdots\otimes M_l, N). \tag{3.20.1}$$

Proposition 3.21. *There are canonical isomorphisms*

$$M \otimes N \cong N \otimes M, \quad m \otimes n \mapsto n \otimes m, \tag{3.21.1}$$

$$(M \otimes N) \otimes L \cong M \otimes (N \otimes L) \cong M \otimes N \otimes L, \tag{3.21.2}$$

$$(m \otimes n) \otimes l \mapsto m \otimes (n \otimes l) \mapsto m \otimes n \otimes l,$$

$$M \otimes A = M. \tag{3.21.3}$$

If $\varphi_i : M_i \to N_i$ *are homomorphisms of* A*-modules, we get a homomorphism*

$$\varphi_0 \otimes \varphi_1 : M_0 \otimes M_1 \to N_0 \otimes N_1,$$

$$\varphi_0 \otimes \varphi_1(m_0 \otimes m_1) \mapsto \varphi_0(m_0) \otimes \varphi_1(m_1). \tag{3.21.4}$$

These are functorial,

$$\mathrm{id}_{M_0} \otimes \mathrm{id}_{M_1} = \mathrm{id}_{M_0 \otimes M_1}, \tag{3.21.5}$$

and for $\psi_i : N_i \to L_i$,

$$(\psi_0 \otimes \psi_1) \circ (\varphi_0 \otimes \varphi_1) = (\psi_0 \circ \varphi_0) \otimes (\psi_1 \circ \varphi_1). \tag{3.21.6}$$

Proof. The usual proof using the universal property goes through. □

Base change. Let $\varphi : A \to B$ be a homomorphism of $\mathbb{F}$-rings. If N is a B-module, we can consider N as an A-module via φ:

$$a \circ n \circ a' := \varphi(a) \circ n \circ \varphi(a'), \quad a, a' \in A, \ n \in N, \tag{3.22.1}$$

$$a \otimes n := \varphi(a) \otimes n, \quad a \in A, \ n \in N. \tag{3.22.2}$$

We denote this A-module by $\varphi^* N$, or by N_φ.

Given an A-module M, form

$$(M_B)_{Y,X} = \{(b, m, b') \mid b \in B_{Y,Y'}, b' \in B_{X',X}, m \in M_{Y',X'}\}/_\sim, \tag{3.23.0}$$

where $\sim$ is the equivalence relation generated by

$$(b \circ \varphi(a), m, b') \sim (b, a \circ m, b'), \quad (b, m, \varphi(a) \circ b') \sim (b, m \circ a, b'), \tag{3.23.1}$$

$$(b \circ (c \otimes \mathrm{id}_{Y'} \oplus \mathrm{id}_{Y''}), (\mathrm{id}_Z \otimes m') \oplus m'', b') \sim (b, (\mathrm{id}_W \otimes m') \oplus m'', (c \otimes \mathrm{id}_{X'} \oplus \mathrm{id}_{X''}) \circ b')$$

$$\text{for } c \in B_{W,Z}, \ m' \in M_{Y',X'}, \ m'' \in M_{Y'',X''}. \tag{3.23.2}$$

Let $(b, m, b')/_\sim$ denote the equivalence class containing (b, m, b'), and define

$$\overline{b} \circ (b, m, b')/_\sim \circ \overline{b}' = (\overline{b} \circ b, m, b' \circ \overline{b}')/_\sim, \tag{3.23.3}$$

$$(b, m, b')/_\sim \oplus (\overline{b}, \overline{m}, \overline{b}')/_\sim = (b \oplus \overline{b}, m \oplus \overline{m}, b' \oplus \overline{b}')/_\sim, \tag{3.23.4}$$

$$\overline{b} \otimes (b, m, b')/_\sim = (\overline{b} \otimes b, \mathrm{id}_{\overline{X}} \otimes m, \mathrm{id}_{\overline{X}} \otimes b')/_\sim, \ \overline{b} \in B_{\overline{Y},\overline{X}}. \tag{3.23.5}$$

Proposition 3.23. *These operations are well defined, independent of representatives. They make* M_B *into a* B*-module; and* $M \mapsto M_B$ *is a functor from* A-$\mathcal{M}od$ *to* B-$\mathcal{M}od$*, for* $f : M \to M'$ *corresponds to* $f_B : M_B \to M'_B$,

$$f_B((b, m, b')/_\sim) = (b, f(m), b')/_\sim.$$

This functor is the left-adjoint to the functor φ^*,

$$\mathrm{Hom}_{A\text{-}\mathcal{M}od}(M, \varphi^* N) = \mathrm{Hom}_{B\text{-}\mathcal{M}od}(M_B, N),$$

$$f \mapsto f((b, m, b')/_\sim) = b \circ f(m) \circ b'. \tag{3.23.6}$$

Proof. The proof is similar to those for Propositions 3.17 and 3.18. That (3.23.3) is well defined follows from associativity. The operation $\oplus$ of (3.23.4) is well defined with respect to (3.23.1):

$$\begin{aligned}
&((b\circ\varphi(a))\oplus\overline{b}, m\oplus\overline{m}, b'\oplus\overline{b}')/_\sim\\
&\quad=((b\oplus\overline{b})\circ\varphi(a\oplus\mathrm{id}), m\oplus\overline{m}, b'\oplus\overline{b}')/_\sim\\
&\quad=(b\oplus\overline{b}, (a\circ m)\oplus\overline{m}, b'\oplus\overline{b}')/_\sim.
\end{aligned}$$

It is well defined with respect to (3.23.2):

$$\begin{aligned}
&((b\circ(c\otimes\mathrm{id}_{Y'}\oplus\mathrm{id}_{Y''}))\oplus\overline{b}, \mathrm{id}_Z\otimes m'\oplus m''\oplus\overline{m}, b'\oplus\overline{b}')/_\sim\\
&\quad=((b\oplus\overline{b})\circ(c\otimes\mathrm{id}_{Y'}\oplus\mathrm{id}_{Y''}\oplus\mathrm{id}_{\overline{Y}}), \mathrm{id}_Z\otimes m'\oplus m''\oplus\overline{m}, b'\oplus\overline{b}')/_\sim\\
&\quad=(b\oplus\overline{b}, \mathrm{id}_Z\otimes m'\oplus m''\oplus\overline{m}, (c\otimes\mathrm{id}_{X'}\oplus\mathrm{id}_{X''}\oplus\mathrm{id}_{\overline{X}})\circ(b'\oplus\overline{b}'))/_\sim\\
&\quad=(b\oplus\overline{b}, \mathrm{id}_Z\otimes m'\oplus m''\oplus\overline{m}, ((c\otimes\mathrm{id}_{X'}\oplus\mathrm{id}_{X''})\circ b')\oplus\overline{b}')/_\sim.
\end{aligned}$$

The operation $\otimes$ of (3.23.5) is well defined with respect to (3.23.1):

$$\begin{aligned}
&(\overline{b}\otimes(b\circ\varphi(a)), \mathrm{id}_{\overline{X}}\otimes m, \mathrm{id}_{\overline{X}}\otimes b')/_\sim\\
&\quad=((\overline{b}\otimes b)\circ\varphi(\mathrm{id}_{\overline{X}}\otimes a), \mathrm{id}_{\overline{X}}\otimes m, \mathrm{id}_{\overline{X}}\otimes b')/_\sim\\
&\quad=(\overline{b}\otimes b, \mathrm{id}_{\overline{X}}\otimes(a\circ m), \mathrm{id}_{\overline{X}}\otimes b')/_\sim.
\end{aligned}$$

It is well defined with respect to (3.23.2):

$$\begin{aligned}
&(\overline{b}\otimes(b\circ(c\otimes\mathrm{id}_{Y'}\oplus\mathrm{id}_{Y''})), \mathrm{id}_{\overline{X}}\otimes(\mathrm{id}_Z\otimes m'\oplus m''), \mathrm{id}_{\overline{X}}\otimes b')/_\sim\\
&\quad=((\overline{b}\otimes b)\circ\mathrm{id}_{\overline{X}}\otimes(c\otimes\mathrm{id}_{Y'}\oplus\mathrm{id}_{Y''}), \mathrm{id}_{\overline{X}}\otimes(\mathrm{id}_Z\otimes m'\oplus m''), \mathrm{id}_{\overline{X}}\otimes b')/_\sim\\
&\quad=(\overline{b}\otimes b, \mathrm{id}_{\overline{X}}\otimes(\mathrm{id}_W\otimes m'\oplus m''), (\mathrm{id}_{\overline{X}}\otimes(c\otimes\mathrm{id}_{X'}\oplus\mathrm{id}_{X''}))\circ(\mathrm{id}_{\overline{X}}\otimes b'))/_\sim\\
&\quad=(\overline{b}\otimes b, \mathrm{id}_{\overline{X}}\otimes(\mathrm{id}_W\otimes m'\oplus m''), \mathrm{id}_{\overline{X}}\otimes((c\otimes\mathrm{id}_{X'}\oplus\mathrm{id}_{X''})\circ b'))/_\sim.
\end{aligned}$$

One then checks the axioms for a B-module. In particular for the functoriality of $\otimes$, (3.1.11), we have 'on the left':

$$\begin{aligned}
&(\overline{c}\otimes c)\circ(\overline{b}\otimes(b,m,b')/_\sim)\\
&\quad=((\overline{c}\otimes c)\circ(\overline{b}\otimes b), \mathrm{id}_{\overline{X}}\otimes m, \mathrm{id}_{\overline{X}}\otimes b')/_\sim\\
&\quad=((\overline{c}\circ\overline{b})\otimes(c\circ b), \mathrm{id}_{\overline{X}}\otimes m, \mathrm{id}_{\overline{X}}\otimes b')/_\sim\\
&\quad=(\overline{c}\circ\overline{b})\otimes(c\circ(b,m,b')/_\sim);
\end{aligned}$$

and 'on the right': for $\overline{b}\in B_{\overline{Y},\overline{X}}, \overline{c}\in B_{\overline{X},\overline{Z}}, m\in M_{Y',X'}$

$$\begin{aligned}
&(\overline{b}\otimes(b,m,b')/_\sim)\circ(\overline{c}\otimes c)\\
&\quad=(\overline{b}\otimes b, \mathrm{id}_{\overline{X}}\otimes m, (\mathrm{id}_{\overline{X}}\otimes b')\circ(\overline{c}\otimes c))/_\sim\\
&\quad=(\overline{b}\otimes b, \mathrm{id}_{\overline{X}}\otimes m, (\overline{c}\otimes\mathrm{id}_{X'})\circ(\mathrm{id}_{\overline{Z}}\otimes(b'\circ c)))/_\sim\\
&\quad=((\overline{b}\otimes b)\circ(\overline{c}\otimes\mathrm{id}_{Y'}), \mathrm{id}_{\overline{Z}}\otimes m, \mathrm{id}_{\overline{Z}}\otimes(b'\circ c))/_\sim\\
&\quad=((\overline{b}\circ\overline{c})\otimes b, \mathrm{id}_{\overline{Z}}\otimes m, \mathrm{id}_{\overline{Z}}\otimes(b'\circ c))/_\sim\\
&\quad=(\overline{b}\circ\overline{c})\otimes((b,m,b')/_\sim\circ c).
\end{aligned}$$

That completes the proof. ∎

The functor $M\mapsto M_B$ commutes with sums and direct limits,

$$\Big(\coprod_i M_i\Big)_B=\coprod_i (M_i)_B, (\varinjlim M_\lambda)_B=\varinjlim (M_\lambda)_B, \tag{3.23.7}$$

with tensor products,

$$(M \otimes_A N)_B = M_B \otimes_B N_B, \tag{3.23.8}$$

with the formation of free module,

$$\left(\coprod_i A \cdot f_i\right)_B = \coprod_i B \cdot f_i, \tag{3.23.9}$$

and with cokernels: for A-module homomorphism $\varphi : M' \to M$,

$$(M/\varphi(M'))_B = M_B/\varphi_B(M'_B). \tag{3.23.10}$$

Remark 3.23.11. We can consider B as an A-module and form the tensor product $B \otimes_A M$, this map to M_B; but unlike the case of commutative rings, this map is not an isomorphism.

4. Ideals and primes

We define four notions of 'ideals' (ideal, E-ideal, H-ideal, H-E-ideals), hence four notions of primes. We get four functors from $\mathbb{F}$-$\mathcal{R}ings$ to compact sober topological spaces, taking an $\mathbb{F}$-ring A to the following commutative square of spaces.

$$\begin{array}{ccc} E\text{-}\operatorname{SPEC}(A) & \hookrightarrow & \operatorname{SPEC}(A) \\ \downarrow & & \downarrow \\ E\text{-}\operatorname{Spec}(A) & \hookrightarrow & \operatorname{Spec}(A) \end{array}$$

4.1 *H*-ideals and primes

Let A be an $\mathbb{F}$-ring.

Definition 4.1. An ideal $\mathfrak{a} \subseteq A$ is called *homogeneous* if it is generated by $\mathfrak{a}_{[1],[1]}$ (i.e., $\mathfrak{a}$ is the smallest ideal containing $\mathfrak{a}_{[1],[1]}$). A subset $\mathfrak{A} \subseteq A_{[1],[1]}$ is called an *H-ideal* if for

$$a_1, \dots, a_n \in \mathfrak{A}, \quad b \in A_{[1],[n]}, \quad b' \in A_{[n],[1]} : b \circ (a_1 \oplus \cdots \oplus a_n) \circ b' \in \mathfrak{A}.$$

If $\mathfrak{a}$ is a homogeneous ideal, $\mathfrak{a}_{[1],[1]}$ is an H-ideal. If $\mathfrak{A}$ is an H-ideal it generates a homogeneous ideal $\mathfrak{a}$, and $\mathfrak{a}_{[1],[1]} = \mathfrak{A}$. Hence there is one-to-one order-preserving correspondence between homogeneous ideals and H-ideals. We denote by H-id(A) the set of H-ideals.

Proposition 4.1.1. *Given $a_1, \dots, a_n \in A_{[1],[1]}$ they generate the H-ideal*

$$(a_1, \dots, a_n) = \left\{ b \circ \left(\bigoplus_i a_i \cdot \mathrm{id}_{X_i} \right) \circ b' \;\middle|\; b \in A_{[1], \bigoplus X_i}, b' \in A_{\bigoplus X_i, [1]} \right\}. \tag{4.1.1}$$

In particular for $a \in A_{[1],[1]}, (a) = a \cdot A_{[1],[1]}$ are just the multiples of a. We have the zero ideal $(0) = \{0\}$, and the unit ideal $(1) = A_{[1],[1]}$.

Proof. Given $b \in A_{[1],[m]}, b' \in A_{[m],[1]}, b_j \circ (\bigoplus_i a_i \cdot \mathrm{id}_{X_{ij}}) \circ b'_j \in (a_1, \dots, a_n), j = 1, \dots, m$, we have

$$b \circ \left(\bigoplus_j b_j \circ \left(\bigoplus_i a_i \cdot \mathrm{id}_{X_{ij}} \right) \circ b'_j \right) \circ b' = \left(b \circ \bigoplus_j b_j \right) \circ \left(\bigoplus_i a_i \cdot \mathrm{id}_{\bigoplus_j X_{ij}} \right) \circ \left(\bigoplus_j b'_j \circ b' \right) \in (a_1, \dots, a_n).$$

□

Given $\mathfrak{a}_i \in H$-id$(A), i \in I$, the intersection $\bigcap_i \mathfrak{a}_i$ is again an H-ideal. The sum $\sum \mathfrak{a}_i$ is the H-ideal generated by $\bigcup \mathfrak{a}_i$,

$$\sum \mathfrak{a}_i = \left\{ b \circ \left(\bigoplus_j a_j \right) \circ b' \;\middle|\; a_j \in \bigcup \mathfrak{a}_i \right\}. \tag{4.1.2}$$

The product $\mathfrak{a}\cdot\mathfrak{a}'$ of two H-ideals is the H-ideal generated by the product of elements of these ideals,

$$\mathfrak{a}\cdot\mathfrak{a}' = \left\{ b\circ\left(\bigoplus_j a_j\cdot a'_j\right)\circ b' \;\middle|\; a_j\in\mathfrak{a}, a'_j\in\mathfrak{a}' \right\}. \tag{4.1.3}$$

Proposition 4.1.4. *For an A-module M, and for an element $m\in M_{Y,X}$, we have the H-ideal*

$$ann_A(m) = \{a\in A_{[1],[1]} \mid a\cdot m = 0\}. \tag{4.1.4}$$

Proof. Indeed, if $a_i\cdot m = 0$, then

$$\left(b\circ\left(\bigoplus a_i\right)\circ b'\right)\cdot m = \left(b\circ\left(\bigoplus_i a_i\right)\circ b'\right)\otimes(\mathrm{id}_Y\circ m\circ \mathrm{id}_X) = (b\otimes\mathrm{id}_Y)\circ\left(\bigoplus a_i\cdot m\right)\circ(b'\otimes \mathrm{id}_X) = 0. \quad\square$$

Similarly, for $m,m'\in M_{Y,X}$ we have the H-ideal

$$ann_A(m,m') = \{a\in A_{[1],[1]} \mid a\cdot m = a\cdot m'\}. \tag{4.1.5}$$

If $M_0, M_1\subseteq M$ are A-submodules, we have the H-ideal

$$(M_0 : M_1) = \{a\in A_{[1],[1]} \mid a\cdot M_1\subseteq M_0\}. \tag{4.1.6}$$

Let $\varphi: A\to B$ be a homomorphism of $\mathbb{F}$-rings. If $\mathfrak{b}\in H\text{-id}(B)$ then $\varphi^*(\mathfrak{b}) = \varphi^{-1}(\mathfrak{b})\in H\text{-id}(A)$, and we have the map

$$\varphi^* : H\text{-id}(B)\to H\text{-id}(A),\quad \mathfrak{b}\mapsto\varphi^{-1}(\mathfrak{b}). \tag{4.2.1}$$

If $\mathfrak{a}\in H\text{-id}(A), \varphi(\mathfrak{a})$ generates the H-ideal $\varphi_*(\mathfrak{a})$,

$$\varphi_* : H\text{-id}(A)\to H\text{-id}(B),\quad \mathfrak{a}\mapsto\varphi_*(\mathfrak{a}) = \left\{b\circ\left(\bigoplus\varphi(a_i)\right)\circ b'\right\}. \tag{4.2.2}$$

Proposition 4.2. *We have the following:*

(1) $\mathfrak{a}\subseteq\varphi^*\varphi_*\mathfrak{a}, \mathfrak{a}\in H\text{-id}(A)$;
(2) $\mathfrak{b}\supseteq\varphi_*\varphi^*\mathfrak{b}, \mathfrak{b}\in H\text{-id}(B)$;
(3) $\varphi^*\mathfrak{b} = \varphi^*\varphi_*\varphi^*\mathfrak{b}, \varphi_*\mathfrak{a} = \varphi_*\varphi^*\varphi_*\mathfrak{a}$;
(4) *there is a bijection, via $\mathfrak{a}\mapsto\varphi_*\mathfrak{a}$ (with inverse map $\mathfrak{b}\mapsto\varphi^*\mathfrak{b}$), from the set*

$$\{\mathfrak{a}\in H\text{-id}(A)\mid \varphi^*\varphi_*\mathfrak{a}=\mathfrak{a}\} = \{\varphi^*\mathfrak{b}\mid \mathfrak{b}\in H\text{-id}(B)\}$$

to the set

$$\{\mathfrak{b}\in H\text{-id}(B)\mid \varphi_*\varphi^*\mathfrak{b}=\mathfrak{b}\} = \{\varphi_*\mathfrak{a}\mid \mathfrak{a}\in H\text{-id}(A)\}.$$

Proof. The proofs of these are straightforward. □

Definition 4.3.1. For $\mathfrak{A}\in H\text{-id}(A)$, we have the homogeneous ideal $\mathfrak{a}$ generated by $\mathfrak{A}$, and the equivalence ideal $E\mathfrak{a} = E\mathfrak{A}$ it generates (cf., Definition 2.19). We write $A/\mathfrak{A}$ for $A/E\mathfrak{A}$, and let $\pi : A\to A/\mathfrak{A}$ be the canonical homomorphism.

Proposition 4.3.2. *We have a one-to-one order-preserving correspondence*

$$\pi^* : H\text{-id}(A/\mathfrak{A}) \xrightarrow{\sim} \{\mathfrak{b}\in H\text{-id}(A)\mid \mathfrak{b} \text{ is } E\mathfrak{A}\text{-stable}\}; \tag{4.3}$$

here $\mathfrak{b}$ is $E\mathfrak{A}$-stable if, for $a\in\mathfrak{A}, Z\in|\mathbb{F}|$,

$$b\circ(\mathrm{id}_Z\oplus a)\circ b'\in\mathfrak{b}\Leftrightarrow b\circ(\mathrm{id}_Z\oplus 0)\circ b'\in\mathfrak{b}.$$

Proof. The proof is clear. □

We often say H-ideal $\mathfrak{a}$ meaning *proper* H-ideal, i.e. $\mathfrak{a} \neq (1)$. Since the union of a chain of H-ideals is again an H-ideal, an application of Zorn's lemma gives the following result.

Theorem 4.4. *Every $\mathbb{F}$-ring contains a maximal (proper) H-ideal. We denote by* $\mathrm{Max}(A)$ *the set of maximal H-ideals.*

Definition 4.5. An H-ideal $\mathfrak{p} \subseteq A_{[1],[1]}$ is called *prime H-ideal* (or in short '*prime*') if $A_{[1],[1]} \setminus \mathfrak{p}$ is multiplicative closed:

$$f, g \in A_{[1],[1]} \setminus \mathfrak{p} \Rightarrow f \cdot g \notin \mathfrak{p}.$$

We denote by $\mathrm{Spec}\, A$ the set of prime H-ideals. For a homomorphism of $\mathbb{F}$-rings $\varphi : A \to B$, the pull-back $\varphi^* = \varphi^{-1}$ induces a map

$$\varphi^* = \mathrm{Spec}(\varphi) : \mathrm{Spec}\, B \to \mathrm{Spec}\, A.$$

Proposition 4.6. *If $\mathfrak{m}$ is a maximal H-ideal then $\mathfrak{m}$ is prime.*

Proof. If $f, g \in A_{[1],[1]} \setminus \mathfrak{m}$, the H-ideals (cf. (4.1.2)) $(f) + \mathfrak{m}, (g) + \mathfrak{m}$ are the unit H-ideals. So we can write (cf. (4.1.1)),

$$1 = b_1 \circ (\mathrm{id}_{X_1} \cdot f \oplus m_1) \circ b_1', \quad 1 = b_2 \circ (\mathrm{id}_{X_2} \cdot g \oplus m_2) \circ b_2',$$

with $m_i \in (\mathfrak{m}) = \{\text{ideal generated by } \mathfrak{m}\}$, $m_i \in (\mathfrak{m})_{Z_i, W_i}$, $b_i \in A_{[1], X_i \oplus Z_i}$, $b_i' \in A_{X_i \oplus W_i, [1]}$. So we have

$$1 = 1 \cdot 1 = (b_1 \otimes b_2) \circ (\mathrm{id}_{X_1 \otimes X_2} \cdot f \cdot g \oplus m) \circ (b_1' \otimes b_2'),$$

with

$$m = (\mathrm{id}_{X_1} \cdot f \otimes m_2) \oplus (m_1 \otimes \mathrm{id}_{X_2} \cdot g) \oplus (m_1 \otimes m_2) \in (\mathfrak{m}),$$

hence $f \cdot g \notin \mathfrak{m}$. ☐

More generally, given an H-ideal $\mathfrak{a} \in H\text{-id}(A)$, and given $f \in A_{[1],[1]}$ such that $f^n \notin \mathfrak{a}, n \geqslant 0$, let $\mathfrak{m}$ be a maximal element of the set

$$\{\mathfrak{b} \in H\text{-id}(A) \mid \mathfrak{b} \supseteq \mathfrak{a}, f^n \notin \mathfrak{b}\ \forall n \geqslant 0\}. \tag{4.6.1}$$

Claim 4.6.2. *We claim that $\mathfrak{m}$ is prime.*

Proof. If $g_i \in A_{[1],[1]} \setminus \mathfrak{m}, i = 1, 2$, we have for some $n_i \geqslant 0$, $f^{n_i} = b_i \circ (\mathrm{id}_{X_i} \cdot g_i \oplus m_i) \circ b_i'$, hence

$$f^{n_1 + n_2} = (b_1 \otimes b_2) \circ (\mathrm{id}_{X_1 \otimes X_2} \cdot g_1 \cdot g_2 \oplus m) \circ (b_1' \otimes b_2'),$$

with $m \in \{\text{ideal generated by } \mathfrak{m}\}$, hence $g_1 \cdot g_2 \notin \mathfrak{m}$. ☐

Definition 4.7. For $\mathfrak{a} \in H\text{-id}(A)$, the *radical* is

$$\sqrt{\mathfrak{a}} = \{f \in A_{[1],[1]} \mid f^n \in \mathfrak{a} \text{ for some } n \geqslant 1\}.$$

It is easy to see that $\sqrt{\mathfrak{a}}$ is an H-ideal. This also follows from the following proposition.

Proposition 4.7.1. *We have*

$$\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{a} \subseteq \mathfrak{p}} \mathfrak{p},$$

the intersection of prime H-ideals containing $\mathfrak{a}$.

Proof. If $f \in \sqrt{\mathfrak{a}}$, say $f^n \in \mathfrak{a}$, then for all primes $\mathfrak{p} \supseteq \mathfrak{a}$, $f \in \mathfrak{p}$. If $f \notin \sqrt{\mathfrak{a}}$, let $\mathfrak{m}$ be a maximal element of the set (4.6.1), it exists by Zorn's lemma, and it is prime by Claim 4.6.2, $\mathfrak{m} \supseteq \mathfrak{a}$ and $f \notin \mathfrak{m}$. ☐

Definition 4.8. For a set $\mathfrak{A} \subseteq A_{[1],[1]}$, we let

$$V_A(\mathfrak{A}) = \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \supseteq \mathfrak{A}\}.$$

If $\mathfrak{a}$ is the H-ideal generated by $\mathfrak{A}$, $V_A(\mathfrak{A}) = V_A(\mathfrak{a})$; we have

$$V_A(1) = \varnothing \text{ (empty set)}, \quad V_A(0) = \operatorname{Spec} A, \tag{4.8.1}$$

$$V_A\left(\sum \mathfrak{a}_i\right) = \bigcap_i V_A(\mathfrak{a}_i), \mathfrak{a}_i \in H\text{-id}(A), \tag{4.8.2}$$

$$V_A(\mathfrak{a} \cdot \mathfrak{a}') = V_A(\mathfrak{a}) \cup V_A(\mathfrak{a}'). \tag{4.8.3}$$

Hence the sets $\{V_A(\mathfrak{a}) \mid \mathfrak{a} \in H\text{-id}(A)\}$ are the closed sets for a topology on $\operatorname{Spec} A$, the *Zariski topology*.

Definition 4.9. For $f \in A_{[1],[1]}$ we let

$$D_A(f) = \operatorname{Spec}(A) \setminus V_A(f) = \{\mathfrak{p} \in \operatorname{Spec} A \mid f \notin \mathfrak{p}\}.$$

We have

$$D_A(f_1) \cup D_A(f_2) = D_A(f_1 \cdot f_2), \tag{4.9.1}$$

$$\operatorname{Spec} A \setminus V_A(\mathfrak{a}) = \bigcup_{f \in \mathfrak{a}} D_A(f). \tag{4.9.2}$$

Hence the sets $\{D_A(f) \mid f \in A_{[1],[1]}\}$ form a basis for the open sets in the Zariski topology. We have

$$D_A(f) = \varnothing \quad \Leftrightarrow \quad f \in \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} \mathfrak{p} = \sqrt{0} \quad \Leftrightarrow \quad f^n = 0 \text{ for some } n \tag{4.9.3}$$

and we say f is *nilpotent*. We have

$$D_A(f) = \operatorname{Spec} A \quad \Leftrightarrow \quad (f) = (1) \quad \Leftrightarrow \quad \exists f^{-1} \in A_{[1],[1]} : f \cdot f^{-1} = 1 \tag{4.9.4}$$

and we say f is *invertible*. We denote by $GL_{[1]}(A)$ the (commutative) group of invertible elements.

Definition. For a subset $X \subseteq \operatorname{Spec} A$, we have the associated H-ideal

$$I(X) = \bigcap_{\mathfrak{p} \in X} \mathfrak{p}.$$

Proposition 4.10. *We have*

$$IV_A(\mathfrak{a}) = \sqrt{\mathfrak{a}}, \tag{4.10.1}$$

$$V_A I(X) = \overline{X}, \text{ the closure of } X \text{ in the Zariski topology.} \tag{4.10.2}$$

Proof. Equation (4.10.1) is just a restatement of Proposition 4.7.1. For (4.10.2), $V_A I(X)$ is clearly a closed set containing X, and if $C = V_A(\mathfrak{a})$ is a closed set containing X, then $\sqrt{\mathfrak{a}} = IV_A(\mathfrak{a}) \subseteq I(X)$, hence $C = V_A(\sqrt{\mathfrak{a}}) \supseteq V_A I(X)$. ∎

Corollary 4.11. *We have a one-to-one order-reversing correspondence between closed sets $X \subseteq \operatorname{Spec} A$, and radical H-ideals $\mathfrak{a}$, via $X \mapsto I(X), V_A(\mathfrak{a}) \leftarrow \mathfrak{a}$,*

$$\{X \subseteq \operatorname{Spec} A \mid \overline{X} = X\} \overset{1:1}{\leftrightarrow} \{\mathfrak{a} \in H\text{-id}(A) \mid \sqrt{\mathfrak{a}} = \mathfrak{a}\}.$$

Under this correspondence the closed irreducible subsets correspond to the prime ideals. For $\mathfrak{p}_0, \mathfrak{p}_1 \in \operatorname{Spec} A, \mathfrak{p}_0 \in \overline{\{\mathfrak{p}_1\}} \Leftrightarrow \mathfrak{p}_0 \supseteq \mathfrak{p}_1$, we say that $\mathfrak{p}_0$ is a *Zariski specialization* of $\mathfrak{p}_1$, or that $\mathfrak{p}_1$ is a *Zariski generalization* of $\mathfrak{p}_0$. The space $\operatorname{Spec} A$ is sober : every closed irreducible subset C has the form $C = V_A(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$, and we call the (unique) prime $\mathfrak{p}$ the *generic point* of C.

Proposition 4.12. *The sets $D_A(f)$, and in particular $D_A(1) = \operatorname{Spec} A$, are compact (or 'quasi-compact': we do not include Hausdorff in compactness).*

Proof. Note that $D_A(f)$ is contained in the union $\bigcup_i D_A(g_i)$ if and only if $V_A(f) \supseteq \bigcap_i V_A(g_i) = V_A(\mathfrak{a})$, where $\mathfrak{a}$ is the H-ideal generated by $\{g_i\}$, if and only if $\sqrt{f} = IV_A(f) \subseteq IV_A(\mathfrak{a}) = \sqrt{\mathfrak{a}}$, if and only if $f^n \in \mathfrak{a}$ for some n, if and only if $f^n = b \circ (\bigoplus_i g_i \cdot \mathrm{id}_{X_i}) \circ b'$, and in any such expression only a finite number of the g_i are involved. □

Let $\varphi : A \to B$ be a homomorphism of $\mathbb{F}$-rings, $\varphi^* : \operatorname{Spec} B \to \operatorname{Spec} A$ the associated pull-back map.

Proposition 4.13. *We have*

$$\varphi^{*-1}(D_A(f)) = D_B(\varphi(f)), \quad f \in A_{[1],[1]}, \tag{4.13.1}$$

$$\varphi^{*-1}(V_A(\mathfrak{a})) = V_B(\varphi_*(\mathfrak{a})), \quad \mathfrak{a} \in H\text{-id}(A), \tag{4.13.2}$$

$$V_A(\varphi^{-1}\mathfrak{b}) = \overline{\varphi^*(V_B(\mathfrak{b}))}, \quad \mathfrak{b} \in H\text{-id}(B). \tag{4.13.3}$$

Proof. The proofs of (4.13.1) and (4.13.2) are straightforward:

$$\mathfrak{q} \in \varphi^{*-1}(D_A(f)) \Leftrightarrow \varphi^*(\mathfrak{q}) \in D_A(f) \Leftrightarrow f \notin \varphi^{-1}(\mathfrak{q}) \Leftrightarrow \varphi(f) \notin \mathfrak{q} \Leftrightarrow \mathfrak{q} \in D_B(\varphi(f)),$$
$$\mathfrak{q} \in \varphi^{*-1}(V_A(\mathfrak{a})) \Leftrightarrow \varphi^*(\mathfrak{q}) \in V_A(\mathfrak{a}) \Leftrightarrow \mathfrak{a} \subseteq \varphi^{-1}(\mathfrak{q}) \Leftrightarrow \varphi_*(\mathfrak{a}) \subseteq \mathfrak{q} \Leftrightarrow \mathfrak{q} \in V_B(\varphi_*(\mathfrak{a})).$$

For (4.13.3) we may assume $\mathfrak{b} = \sqrt{\mathfrak{b}}$ is radical since $V_B(\mathfrak{b}) = V_B(\sqrt{\mathfrak{b}}), \varphi^{-1}(\sqrt{\mathfrak{b}}) = \sqrt{\varphi^{-1}(\mathfrak{b})}$. Let $\mathfrak{a} = I(\varphi^*(V_B(\mathfrak{b})))$, so that $V_A(\mathfrak{a}) = \overline{\varphi^*(V_B(\mathfrak{b}))}$ by (4.10.2). We have

$$\begin{aligned} f \in \mathfrak{a} \quad &\Leftrightarrow \quad f \in \mathfrak{p}, \quad \forall \mathfrak{p} \in \varphi^*(V_B(\mathfrak{b})) \quad \Leftrightarrow \quad f \in \varphi^{-1}(\mathfrak{q}), \quad \forall \mathfrak{q} \supseteq \mathfrak{b} \\ &\Leftrightarrow \quad \varphi(f) \in \bigcap_{\mathfrak{q} \supseteq \mathfrak{b}} \mathfrak{q} = \sqrt{\mathfrak{b}} = \mathfrak{b} \quad \Leftrightarrow \quad f \in \varphi^{-1}(\mathfrak{b}). \end{aligned}$$ □

It follows from (4.13.1), or from (4.13.2), that $\varphi^* = \operatorname{Spec}(\varphi)$ is continuous, hence $A \mapsto \operatorname{Spec} A$ is a contravariant functor from $\mathbb{F}$-rings to compact, sober, topological spaces.

Example 4.14.1. Let A be a commutative ring, $\mathbb{F}(A)$ the associated $\mathbb{F}$-ring. An ideal $\mathfrak{a} \subseteq A = \mathbb{F}(A)_{[1],[1]}$ is an H-ideal generating the homogeneous ideal $\mathbb{F}(\mathfrak{a}) \subseteq \mathbb{F}(A)$, and conversely an H-ideal is just an ideal of A. Under this correspondence the primes of A correspond to the primes of $\mathbb{F}(A)$, and we have a homeomorphism with respect to the Zariski topologies:

$$\operatorname{Spec} A = \operatorname{Spec} \mathbb{F}(A).$$

Example 4.14.2. Let $\eta : \mathbb{k} \to \mathbb{C}$ be a real or complex prime of a number field, and let $\mathcal{O}_{\mathbb{k},\eta}$ denote the $\mathbb{F}$-ring of real or complex 'integers' of (2.16).Then

$$\mathfrak{m}_\eta = \{x \in \mathbb{k} \mid |x|_\eta < 1\}$$

is the (unique) maximal H-ideal of $\mathcal{O}_{\mathbb{k},\eta}$, the closed point of $\operatorname{Spec} \mathcal{O}_{\mathbb{k},\eta}$.

4.2 *S*-prime ideals and *H*-*E*-ideals

Definition 4.15. A (non-homogeneous) ideal $\mathfrak{p} \subseteq A$ is called a *strong prime ideal*, or *S-prime*, if $A \setminus \mathfrak{p}$ is closed with respect to $\otimes$:

$$f \in (A \setminus \mathfrak{p})_{Y_1,X_1}, g \in (A \setminus \mathfrak{p})_{Y_2,X_2} \Rightarrow f \otimes g \in (A \setminus \mathfrak{p})_{Y_1 \otimes Y_2, X_1 \otimes X_2}. \tag{4.15.1}$$

We let $\mathrm{SPEC}(A)$ denote the set of S-primes.

Every $\mathbb{F}$-ring contains a maximal ideal, and every maximal ideal is S-prime; cf. Proposition 4.6. For an ideal $\mathfrak{a}$ we have

$$\sqrt{\mathfrak{a}} = \{f \in A \mid f^{\otimes n} \in \mathfrak{a} \text{ for some } n \geqslant 1\} = \bigcap_{\mathfrak{a} \subseteq \mathfrak{p}} \mathfrak{p}, \tag{4.15.2}$$

the intersection of S-primes containing $\mathfrak{a}$; cf. Proposition 4.7.1. On SPEC(A) we have the Zariski topology, with closed sets

$$V_A(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{SPEC}(A) \mid \mathfrak{p} \supseteq \mathfrak{a}\}, \tag{4.15.3}$$

where we can take $\mathfrak{a}$ to be an ideal of A; cf. Definition 4.8. A basis for the open sets is given by the sets

$$D_A(f) = \mathrm{SPEC}(A) \setminus V_A(f) = \{\mathfrak{p} \in \mathrm{SPEC}(A) \mid f \notin \mathfrak{p}\}, \tag{4.15.4}$$

$f \in A$; cf. Definition 4.9. We have

$$D_A(f) = \varnothing \quad \Leftrightarrow \quad f \in \bigcap_{\mathfrak{p} \in \mathrm{SPEC}(A)} \mathfrak{p} = \sqrt{0} \quad \Leftrightarrow \quad f^{\otimes n} = 0 \text{ for some } n,$$

and we say f is *nilpotent*. We have

$$D_A(f) = \mathrm{SPEC}\, A \quad \Leftrightarrow \quad (f) = (1) \quad \Leftrightarrow \quad 1 = a_1 \circ (\mathrm{id}_X \otimes f) \circ a_2$$

for some $a_i \in A$, and we say f is *a unit*. For a subset $X \subseteq \mathrm{SPEC}(A)$, we have the associated ideal

$$I(X) = \bigcap_{\mathfrak{p} \in X} \mathfrak{p},$$

and Proposition 4.10 holds. Similarly Corollary 4.11 and Propositions 4.12 and 4.13 remain true. We have a continuous map

$$\mathrm{SPEC}(A) \to \mathrm{Spec}(A), \mathfrak{p} \mapsto \mathfrak{p}_{[1],[1]}. \tag{4.15.6}$$

It is always surjective. For an $\mathbb{F}$-ring $\mathbb{F}(A)$ associated to a commutative ring A, ideals of $\mathbb{F}(A)$ correspond one-to-one with ideals of A, S-primes correspond to primes of A, and the map (4.15.6) is a homeomorphism.

Definition 4.16. A non-empty subset $\mathfrak{A} \subseteq A_{[1],[1]}$ will be called an *H-E-ideal* if, for $a_i \in \mathfrak{A}$,

$$b \circ \left(\mathrm{id}_Z \oplus \bigoplus_i a_i\right) \circ b' \in \mathfrak{A} \quad \Leftrightarrow \quad b \circ (\mathrm{id}_Z \oplus 0) \circ b' \in \mathfrak{A}. \tag{4.16.1}$$

We have $0 \in \mathfrak{A}$, and hence for $a_i \in \mathfrak{A}, i = 1, \dots, n$, $b \circ (\bigoplus a_i) \circ b' \in \mathfrak{A}$, i.e. it is an H-ideal (take $Z = [0]$ in (4.16.1)). Thus an H-E-ideal $\mathfrak{A} \subseteq A_{[1],[1]}$ is just an H-ideal which is $E\mathfrak{A}$-stable. We denote by H-E-id(A) the collection of H-E-ideals of A.

Given $\mathfrak{A} \in H\text{-id}(A)$, it generates the homogeneous ideal $\mathfrak{a} = \{b \circ (\bigoplus a_i) \circ b' \mid a_i \in \mathfrak{A}\}$, which in turn generates the E-ideal $ZE\mathfrak{a} \in E\text{-sub}_A(A)$, which can be described as (cf., Lemma 3.13):

$$\begin{aligned} ZE\mathfrak{a} = \Bigg\{ & a \in A \mid \exists \text{ path } a = c_1, \dots, c_l = 0, \text{ with } \{c_j, c_{j+1}\} \text{ of the form} \\ & \left\{ b \circ \left(\mathrm{id}_Z \oplus \bigoplus_i a_i\right) \circ b', b \circ (\mathrm{id}_Z \oplus 0) \circ b' \right\}, \text{ with } a_i \in \mathfrak{A} \Bigg\}. \end{aligned} \tag{4.16.2}$$

The E-ideal $ZE\mathfrak{a} = ZE\mathfrak{A}$ is generated as E-ideal by $(ZE\mathfrak{A})_{[1],[1]} \subseteq A_{[1],[1]}$ (by the explicit description (4.16.2) and the fact that $(ZE\mathfrak{A})_{[1],[1]} \supseteq \mathfrak{A}$); we call the H-E-ideal $(ZE\mathfrak{A})_{[1],[1]}$ the E-closure of $\mathfrak{A}$. We have $\mathfrak{A} = (ZE\mathfrak{A})_{[1],[1]}$ if and only if $\mathfrak{A}$ is an H-E-ideal.

Corollary 4.16.3. *There is a one-to-one order-preserving correspondence between the set of equivalence ideal* $E\mathfrak{A}$*, generated by a subset* $\mathfrak{A} \subseteq A_{[1],[1]}$

$$\{\mathcal{E} \in \text{equiv}_A(A) \mid \mathcal{E} = E((Z\mathcal{E})_{[1],[1]})\} = \{E\mathfrak{A} \mid \mathfrak{A} \in H\text{-id}(A)\},$$

and the set of H-E-id(A)

$$H\text{-}E\text{-id}(A) = \{\mathfrak{A} \in H\text{-}E\text{-id}(A) \mid \mathfrak{A} = (ZE\mathfrak{A})_{[1],[1]}\}.$$

It is given by:

$$\mathcal{E} \mapsto (Z\mathcal{E})_{[1],[1]}, \quad E\mathfrak{A} \leftarrow\!\shortmid \mathfrak{A}.$$

For $\mathfrak{A} \in H$-E-id(A), let $\pi : A \to A/\mathfrak{A} = A/E\mathfrak{A}$ denote the canonical projection, then $\mathfrak{A} = \pi^{-1}(0_{[1],[1]})$. We have the following proposition.

Proposition 4.17. *There is a one-to-one order-preserving correspondence*

$$\{\mathfrak{B} \in H\text{-}E\text{-id}(A) \mid \mathfrak{B} \supseteq \mathfrak{A}\} \overset{1:1}{\leftrightarrow} H\text{-}E\text{-id}(A/\mathfrak{A}),$$
$$\pi^{-1}(\overline{\mathfrak{B}}) \longleftarrow\!\shortmid \overline{\mathfrak{B}}.$$

We can repeat most of our discussion of H-ideals using H-E-ideals. By Zorn's lemma, we get the next result.

Proposition 4.18. *There exist maximal (proper) H-E-ideals.*

Definition 4.19. We define E- Spec(A) = Spec$(A) \cap H$-E-id(A); its elements will be called *E-primes*. We have the following proposition.

Proposition 4.19.1. *If* $\mathfrak{m}$ *is a maximal H-E-ideal then* $\mathfrak{m}$ *is E-prime.*

More generally, given $\mathfrak{a} \in H$-E-id(A), $f \in A_{[1],[1]}$, such that $f^n \notin \mathfrak{a}$ for all n. By Zorn's lemma there exists a maximal element $\mathfrak{m}$ in the set

$$\{\mathfrak{b} \in H\text{-}E\text{-id}(A) \mid \mathfrak{b} \supseteq \mathfrak{a}, f^n \notin \mathfrak{b}\ \forall n\}.$$

Claim 4.19.2. *We claim that* $\mathfrak{m}$ *is E-prime.*

Proof. For $x \in A_{[1],[1]} \setminus \mathfrak{m}$, the H-E-ideal generated by $\mathfrak{m}$ and x contains some power f^n. Hence there is a path $f^n = c_1, \dots, c_l = 0$, with $\{c_j, c_{j+1}\}$ of the form $\{b \circ (\text{id}_Z \oplus (x \cdot \text{id}_W) \oplus m) \circ b', b \circ (\text{id}_Z \oplus 0) \circ b'\}$, with m in the ideal generated by $\mathfrak{m}$. Let $y \in A_{[1],[1]}$ with $y \cdot x \in \mathfrak{m}$. Multiplying the path $\{c_j\}$ throughout by y we see that $y \cdot f^n \in \mathfrak{m}$. If $y \in A_{[1],[1]} \setminus \mathfrak{m}$, we get similarly a path $f^m = d_1, \dots, d_l = 0$, with $\{d_j, d_{j+1}\}$ of the form $\{b \circ (\text{id}_Z \oplus y \cdot \text{id}_W \oplus m) \circ b', b \circ (\text{id}_Z \oplus 0) \circ b'\}$, and multiplying this path by f^n we see that $f^n \cdot f^m \in \mathfrak{m}$, a contradiction. □

Corollary 4.19.3. *For* $\mathfrak{A} \in H$-E-id(A),

$$\sqrt{\mathfrak{A}} = \bigcap_{\mathfrak{A} \subseteq \mathfrak{p}} \mathfrak{p},$$

the intersection of all E-primes containing $\mathfrak{A}$.

Example 4.20. For a commutative ring A, every H-ideal of $\mathbb{F}(A)$ is an H-E-ideal,

$$H\text{-}E\text{-id}(\mathbb{F}(A)) = H\text{-id}(\mathbb{F}(A)) = \text{ideals of } A. \tag{4.20.1}$$

Hence every prime is an E-prime,

$$E\text{- Spec}(\mathbb{F}(A)) = \text{Spec}(\mathbb{F}(A)) = \text{Spec}(A). \tag{4.20.2}$$

Example 4.21. The maximal H-ideal $\mathfrak{m}_\eta$ of the $\mathbb{F}$-ring of real or complex integers $\mathcal{O}_{\Bbbk,\eta}$, cf. Example 4.14.2, is an E-prime. Indeed, for $|a_i|_\eta < 1$, we have

$$\left| b \circ \left(\mathrm{id}_Z \oplus \bigoplus_i a_i \right) \circ b' \right|_\eta < 1 \quad \Leftrightarrow \quad |b \circ (\mathrm{id}_Z \oplus 0) \circ b'|_\eta < 1. \tag{4.21.1}$$

Here we may assume that $\{a_i\}$ consists of one element $a, Z = [1]$, and that $b = \overline{(b')}^{\mathrm{t}} = (b_0, b_1)$ is a vector of norm 1, so (4.21.1) reads : for $|a|_\eta < 1, |b_0|_\eta^2 + |b_1|_\eta^2 = 1$,

$$||b_0|_\eta^2 + a \cdot |b_1|_\eta^2|_\eta < 1 \quad \Leftrightarrow \quad |b_0^2|_\eta < 1.$$

Verifying this (obvious) statement shows that $\mathfrak{m}_\eta$ is indeed an E-prime.

For $a \in (\mathcal{O}_{\Bbbk,\eta})_{Y,X}$ we have the non-negative self-adjoint operators $\overline{a}^{\mathrm{t}} \circ a \in (\mathcal{O}_{\Bbbk,\eta})_{X,X}$, and $a \circ \overline{a}^{\mathrm{t}} \in (\mathcal{O}_{\Bbbk,\eta})_{Y,Y}$. We get orthogonal decompositions

$$\begin{aligned} \Bbbk \cdot X &= \bigoplus_\lambda V(\lambda), \\ \Bbbk \cdot Y &= \bigoplus_\lambda W(\lambda), \end{aligned} \tag{4.21.2}$$

where $\overline{a}^{\mathrm{t}} \circ a$ (respectively $a \circ \overline{a}^{\mathrm{t}}$) acts on $V(\lambda)$ (respectively $W(\lambda)$) as scalar multiplication by $\lambda^2, 0 \leqslant \lambda \leqslant 1$. For $\lambda > 0, a$ induces a linear isomorphism

$$a_\lambda : V(\lambda) \xrightarrow{\sim} W(\lambda), \tag{4.21.3}$$

and for $\lambda = 1$, a_1 is an isometry. This gives the singular eigenvalue decomposition of a,

$$a = b \circ \left(\bigoplus \lambda \cdot \mathrm{id}_{Z_\lambda} \right) \circ b', \tag{4.21.4}$$

where Z_λ is an orthonormal basis for $V(\lambda)$, b' is the change-of-basis matrix from X to $\{Z_\lambda\}$, and b is the change-of-basis matrix from $\{(1/\lambda) \cdot a(Z_\lambda)\}$ to Y; thus b, b' are isomorphisms in $\mathcal{O}_{\Bbbk,\eta}$ (i.e. orthogonal or unitary matrices). In particular, we see that modulo $E\mathfrak{m}_\eta$, a is equivalent to

$$a_1 = b \circ \left(\mathrm{id}_{Z_1} \oplus \bigoplus_{\lambda<1} 0_{Z_\lambda} \right) \circ b'. \tag{4.21.5}$$

It follows that the map $\pi(a) = (a_1 : V(1) \to W(1))$ is an isomorphism of $\mathcal{O}_{\Bbbk,\eta}/\mathfrak{m}_\eta$ onto $\mathbb{F}_{\Bbbk,\eta}$, the field of partial isometries of (2.18).

The last two examples give 'strong-E-primes' according to the following definition.

DEFINITION 4.22. We define E-$\mathrm{SPEC}(A) = \mathrm{SPEC}(A) \cap E\text{-sub}_A(A)$; its elements will be called *strong-E-primes*, or *S-E-primes*. Every $\mathbb{F}$-ring contains a maximal (proper) E-ideal, and such an ideal is S-E-prime. More generally, given $\mathfrak{a} \in E\text{-sub}_A(A), f \in A_{Y,X}$ such that $f^{\otimes n} \notin \mathfrak{a}$ for all n, there exists by Zorn's lemma a maximal element $\mathfrak{m}$ in the set

$$\{\mathfrak{b} \in E\text{-sub}_A(A) \mid \mathfrak{b} \supseteq \mathfrak{a}, f^{\otimes n} \notin \mathfrak{b}\ \forall n\}, \tag{4.22.1}$$

and such an $\mathfrak{m}$ is S-E-prime; cf. Claim 4.19.2. It follows that for an E-ideal $\mathfrak{a} \in E\text{-sub}_A(A)$

$$\sqrt{\mathfrak{a}} = \{f \in A \mid f^{\otimes n} \in \mathfrak{a} \text{ for some } n \geqslant 1\} = \bigcap_{\mathfrak{a} \subseteq \mathfrak{p}} \mathfrak{p}, \tag{4.22.2}$$

the intersection of all S-E-primes $\mathfrak{p}$ containing $\mathfrak{a}$.

Thus we have four functors from $\mathbb{F}$-rings to compact sober topological spaces, taking an $\mathbb{F}$-ring A to the following diagram.

$$\begin{array}{ccc} E\text{-}\operatorname{SPEC}(A) & \hookrightarrow & \operatorname{SPEC}(A) \\ \downarrow & & \downarrow \\ E\text{-}\operatorname{Spec}(A) & \hookrightarrow & \operatorname{Spec}(A) \end{array}$$

There are corresponding various notations of 'fields' for $\mathbb{F}$-rings.

Definition 4.23. An $\mathbb{F}$-ring A is called a *field*, or an *H-field*, if it satisfies the equivalent conditions:

(i) $\operatorname{Spec}(A) = \{(0)\}$;
(ii) there are no non-zero proper H-ideals $\mathfrak{a} \subseteq A_{[1],[1]}$; (4.23.1)
(iii) $GL_{[1]}(A) = A_{[1],[1]} \setminus \{0\}$: every non-zero element of $A_{[1],[1]}$ is invertible.

An $\mathbb{F}$-ring A is called an *E-field*, if it satisfies the equivalent conditions:

(i) $E\text{-}\operatorname{Spec}(A) = \{(0)\}$;
(ii) there are no non-zero proper H-E-ideals $\mathfrak{a} \subseteq A_{[1],[1]}$; (4.23.2)
(iii) every homomorphism $\varphi : A \to B$ with $B \neq 0$ and such that $\mathcal{KER}(\varphi) = E(\varphi^{-1}(0)_{[1],[1]})$ is injective, i.e., $\mathcal{KER}(\varphi)$ is trivial (= diagonal of $A \prod A$; cf., (2.20) for $\mathcal{KER}(\varphi)$).

An $\mathbb{F}$-ring A is called a *strong field* or an *S-field*, if it satisfies the equivalent conditions:

(i) $\operatorname{SPEC}(A) = \{(0)\}$;
(ii) there are no non-zero proper ideals $\mathfrak{a} \subseteq A$; (4.23.3)
(iii) for all $X, Y \in |\mathbb{F}|$, every non-zero element of $A_{Y,X}$ is a unit.

An $\mathbb{F}$-ring A is called a *strong-E-field* or an *S-E-field*, if it satisfies the equivalent conditions:

(i) $E\text{-}\operatorname{SPEC}(A) = \{(0)\}$;
(ii) there are no non-zero proper E-ideals $\mathfrak{a}$; (4.23.4)
(iii) every homomorphism $\varphi : A \to B$ with $B \neq 0$ and such that $\mathcal{KER}(\varphi) = E(\varphi^{-1}(0))$ is injective.

We have the following implications:

$$\begin{array}{ccc} H\text{-field} & \Longrightarrow & E\text{-field} \\ \Uparrow & & \Uparrow \\ S\text{-field} & \Longrightarrow & S\text{-}E\text{-field} \end{array} \tag{4.23.5}$$

In particular, if $\mathfrak{m}$ is a maximal H-ideal of an $\mathbb{F}$-ring $\mathcal{O}$, we have the residue field $\mathbb{F}_{\mathfrak{m}} = \mathcal{O}/E\mathfrak{m}$; it is an H-field by (4.3): if $\mathfrak{n}$ is an H-ideal of $\mathcal{O}$, and $\mathfrak{n}$ is $E\mathfrak{m}$-stable, (4.3), then $\mathfrak{n}$ contains $\mathfrak{m}$, and by maximality $\mathfrak{n} = \mathfrak{m}$.

5. Localization and structural sheaf $\mathcal{O}_A$

We review the theory of localization of an $\mathbb{F}$-ring A (or an A-module M) with respect to a multiplicative set $S \subseteq A_{[1],[1]}$; we obtain a sheaf of $\mathbb{F}$-rings $\mathcal{O}_A$ (respectively, an $\mathcal{O}_A$-module $\widetilde{M}$) over $\operatorname{Spec} A$.

5.1 Localization

The theory of localization of an $\mathbb{F}$-ring A, with respect to a multiplicative subset $S \subseteq A_{[1],[1]}$, goes exactly as in localization of commutative rings – since it is a multiplicative theory. We recall this theory next.

We assume that $S \subseteq A_{[1],[1]}$ satisfies

$$1 \in S, \tag{5.1.1}$$

$$s_1, s_2 \in S \quad \Rightarrow \quad s_1 \cdot s_2 \in S. \tag{5.1.2}$$

On the set

$$A \times S = \coprod_{Y,X} A_{Y,X} \times S$$

we define for $a_i \in A_{Y,X}, s_i \in S$,

$$(a_1, s_1) \sim (a_2, s_2) \quad \Leftrightarrow \quad s \cdot s_2 \cdot a_1 = s \cdot s_1 \cdot a_2 \text{ for some } s \in S. \tag{5.2}$$

It follows that $\sim$ is an equivalence relation, and we denote by a/s the equivalence class containing (a, s), and by $S^{-1}A$ the collection of equivalence classes. On $S^{-1}A$ we define the operations:

$$a_1/s_1 \circ a_2/s_2 = (a_1 \circ a_2)/s_1 s_2, \quad a_1 \in A_{Z,Y},\ a_2 \in A_{Y,X}, \tag{5.3.1}$$

$$a_1/s_1 \oplus a_2/s_2 = (s_2 \cdot a_1 \oplus s_1 \cdot a_2)/s_1 s_2, \tag{5.3.2}$$

$$a_1/s_1 \otimes a_2/s_2 = (a_1 \otimes a_2)/s_1 s_2. \tag{5.3.3}$$

Proposition 5.3. *The above operations are well defined, independent of the chosen representatives, and they satisfy the axioms of an $\mathbb{F}$-ring.*

Proof. The usual proof works. For example, replacing a_1/s_1 in (5.3.2) by $a_1'/s_1' \sim a_1/s_1$, say $s \cdot s_1' \cdot a_1 = s \cdot s_1 \cdot a_1'$, then

$$s \cdot s_1' s_2 \cdot (s_2 a_1 \oplus s_1 a_2) = s \cdot s_1 s_2 \cdot (s_2 a_1' \oplus s_1' a_2),$$

hence

$$(s_2 a_1 \oplus s_1 a_2)/s_1 s_2 = (s_2 a_1' \oplus s_1' a_2)/s_1' s_2. \qquad \square$$

The $\mathbb{F}$-ring $S^{-1}A$ comes with a canonical homomorphism

$$\phi = \phi_S : A \to S^{-1}A, \quad \phi(a) = a/1. \tag{5.4}$$

Proposition 5.5. *We have the universal property of ϕ_S:*

$$\mathrm{Hom}_{\mathbb{F}\text{-}\mathcal{R}ings}(S^{-1}A, B) = \{\varphi \in \mathrm{Hom}_{\mathbb{F}\text{-}\mathcal{R}ings}(A, B) \mid \varphi(S) \subseteq GL_{[1]}(B)\},$$
$$\widetilde{\varphi} \longmapsto \widetilde{\varphi} \circ \phi_S,$$
$$\widetilde{\varphi}(a/s) = \varphi(a) \cdot \varphi(s)^{-1} \longleftarrow \varphi.$$

Proof. The proof is clear. $\square$

Note that $S^{-1}A$ is the zero $\mathbb{F}$-ring if and only if $0 \in S$.

The main examples of localizations are as follows:

$$S_f = \{f^n\}_{n \geqslant 0}, \quad f \in A_{[1],[1]}, \tag{5.6.1}$$

and we write A_f for $S_f^{-1}A$;

$$S_{\mathfrak{p}} = A_{[1],[1]} \setminus \mathfrak{p}, \quad \mathfrak{p} \in \mathrm{Spec}(A), \tag{5.6.2}$$

and we write $A_{\mathfrak{p}}$ for $S_{\mathfrak{p}}^{-1}A$.

Similarly, for an A-module M, we have the equivalence relation $\sim$ on $M \times S$,

$$(m_1, s_1) \sim (m_2, s_2) \quad \Leftrightarrow \quad s \cdot s_2 \cdot m_1 = s \cdot s_1 \cdot m_2 \quad \text{for some } s \in S. \tag{5.7}$$

We let $m/s = (m, s)/_\sim$ denote the equivalence class containing (m, s), and $S^{-1}M = (M \times S)/_\sim$ denote the collection of equivalence classes. On $S^{-1}M$ we have the structure of an $S^{-1}A$-module, which is well defined, independent of the chosen representatives, by

$$a_1/s_1 \circ m/s \circ a_2/s_2 = (a_1 \circ m \circ a_2)/s_1 s s_2, \tag{5.7.1}$$

$$m_1/s_1 \oplus m_2/s_2 = (s_2 \cdot m_1 \oplus s_1 \cdot m_2)/s_1 s_2, \tag{5.7.2}$$

$$a_1/s_1 \otimes m/s = (a_1 \otimes m)/s_1 s. \tag{5.7.3}$$

The localization $M \mapsto S^{-1}M$ is a functor $S^{-1} : A\text{-}\mathcal{M}od \to S^{-1}A\text{-}\mathcal{M}od$; to a map $\varphi : M \to M'$ corresponds $S^{-1}\varphi : S^{-1}M \to S^{-1}M'$,

$$S^{-1}\varphi(m/s) = \varphi(m)/s. \tag{5.7.4}$$

We have an A-module homomorphism $M \to \phi_S^*(S^{-1}M), m \mapsto m/1$, which corresponds by (3.23.6) to the homomorphism of $S^{-1}A$ modules

$$M_{S^{-1}A} \to S^{-1}M, \quad (a_1/s_1, m, a_2/s_2)/_\sim \mapsto (a_1 \circ m \circ a_2)/s_1 s_2, \tag{5.7.5}$$

where $M_{S^{-1}A}$ is the $S^{-1}A$-module obtained from M via base change (3.23.0) along the homomorphism $A \to S^{-1}A$. This is clearly surjective. It is also injective. Note first that by (3.23.1), (3.23.2), $(a_1/s_1, m, a_2/s_2)/_\sim = (\mathrm{id}/1, a_1 \circ m \circ a_2, \mathrm{id}/s_1 s_2)/_\sim$, so every element of $M_{S^{-1}A}$ has the form $(\mathrm{id}/1, m, \mathrm{id}/s)/_\sim$. If $m_1/s_1 = m_2/s_2$, say $s \cdot s_2 \cdot m_1 = s \cdot s_1 \cdot m_2$, then we have the following:

$$\begin{array}{c} (\mathrm{id}/1, m_1, \mathrm{id}/s_1)/_\sim = (\mathrm{id}/1, m_1, s \cdot s_2/1 \cdot \mathrm{id}/s s_1 s_2)/_\sim = (\mathrm{id}/1, s \cdot s_2 \cdot m_1, \mathrm{id}/s s_1 s_2)/_\sim \\ \| \\ (\mathrm{id}/1, m_2, \mathrm{id}/s_2)/_\sim = (\mathrm{id}/1, m_2, s \cdot s_1/1 \cdot \mathrm{id}/s s_1 s_2)/_\sim = (\mathrm{id}/1, s \cdot s_1 \cdot m_2, \mathrm{id}/s s_1 s_2)/_\sim \end{array}$$

Thus we may identify the localization $S^{-1}M$ with the base change $M_{S^{-1}A}$, and it follows from (3.23.7), (3.23.8), (3.23.9) that we have similar properties for localization.

Corollary 5.7.6. *The functor $M \mapsto S^{-1}M$ preserves sums, direct limits, tensor products, and free modules:*

$$S^{-1}\coprod_i M_i = \coprod_i S^{-1}M_i, \quad S^{-1}\varinjlim M_\lambda = \varinjlim S^{-1}M_\lambda, \tag{5.7.7}$$

$$S^{-1}(M \otimes_A N) = S^{-1}M \otimes_{S^{-1}A} S^{-1}N; \tag{5.7.8}$$

for a formal symbol $f_{X,Y}$, any $Y, X \in |\mathbb{F}|$, and for the free A-module $A \cdot f_{Y,X}$, and the free $S^{-1}A$-module $(S^{-1}A) \cdot f_{Y,X}$, cf. (3.18.0),

$$S^{-1}(A \cdot f_{Y,X}) = (S^{-1}A) \cdot f_{Y,X}. \tag{5.7.9}$$

If $M_0 \in \mathrm{sub}_A(M)$ is an A-submodule, then $S^{-1}M_0 \in \mathrm{sub}_{S^{-1}A}(S^{-1}M)$ is an $S^{-1}A$-submodule. The map $M_0 \mapsto S^{-1}M_0$ preserves sums,

$$S^{-1}\left(\sum_i M_i\right) = \sum_i (S^{-1}M_i), \tag{5.8.1}$$

and finite intersections,

$$S^{-1}(M_1 \cap M_2) = S^{-1}M_1 \cap S^{-1}M_2. \tag{5.8.2}$$

Since we can always bring a finite sequence of elements in S-localization into 'common denominator', we have the following result.

Proposition 5.9. *The functor $M \mapsto S^{-1}M$ also preserves finite products, and finite inverse limits,*

$$S^{-1}\left(M_0 \prod_M M_1\right) = S^{-1}M_0 \prod_{S^{-1}M} S^{-1}M_1, \quad (m_0, m_1)/s \mapsto (m_0/s, m_1/s). \tag{5.9.1}$$

If $\mathcal{E} \in \mathrm{equiv}_A(M)$ is an equivalence A-module of M, then

$$S^{-1}\mathcal{E} \subseteq S^{-1}\left(M \prod M\right) = S^{-1}M \prod S^{-1}M$$

is an equivalence $S^{-1}A$-module of $S^{-1}M$, $S^{-1}\mathcal{E} \in \mathrm{equiv}_{S^{-1}A}(S^{-1}M)$. We have (with Z, E as in Definition 3.10)

$$S^{-1}(Z\mathcal{E}) = Z(S^{-1}\mathcal{E}), \quad \mathcal{E} \in \mathrm{equiv}_A(M) \tag{5.10.1}$$

and

$$S^{-1}(EM_0) = E(S^{-1}M_0), \quad M_0 \in \mathrm{sub}_A(M) \tag{5.10.2}$$

(cf., Lemma 3.13: bring a path to a common denominator).

Similarly we have the next proposition.

Proposition 5.11. *Let $\varphi : M \to N$ be a homomorphism of A-modules. The functor S^{-1} preserves kernels,*

$$S^{-1}(\varphi^{-1}(0)) = (S^{-1}\varphi)^{-1}(0), \tag{5.11.1}$$

$$S^{-1}(\mathcal{KER}(\varphi)) = \mathcal{KER}(S^{-1}\varphi), \tag{5.11.2}$$

and it preserves cokernels,

$$S^{-1}(N/\varphi(M)) = S^{-1}N/S^{-1}\varphi(S^{-1}M). \tag{5.11.3}$$

Definition 5.11.4. We write $M_{\mathfrak{p}}$ for $S_{\mathfrak{p}}^{-1}M, S_{\mathfrak{p}} = A_{[1],[1]} \setminus \mathfrak{p}, \mathfrak{p} \in \mathrm{Spec}\, A$. We write M_f for $S_f^{-1}M$, $S_f = \{f^n\}_{n \geqslant 0}, f \in A_{[1],[1]}$.

Proposition 5.12. *For an A-module M, the following are equivalent:*

$$M = 0, \tag{5.12.1}$$

$$M_{\mathfrak{p}} = 0 \ \forall \mathfrak{p} \in \mathrm{Spec}\, A, \tag{5.12.2}$$

$$M_{\mathfrak{m}} = 0 \text{ for all maximal } H\text{-ideals } \mathfrak{m}. \tag{5.12.3}$$

Proof. Clearly (5.12.1) $\Rightarrow$ (5.12.2) $\Rightarrow$ (5.12.3). If $M \neq 0$ let $m \in M_{Y,X}$ be non-zero, and let $\mathfrak{m}$ be a maximal H-ideal containing $ann_A(m)$; cf. (4.1.4). If $M_{\mathfrak{m}} = 0 \Rightarrow m/1 = 0 \in M_{\mathfrak{m}}$, which means $s \cdot m = 0$ for some $s \in A_{[1],[1]} \setminus \mathfrak{m}$, contradicting $ann_A(m) \subseteq \mathfrak{m}$. □

Proposition 5.13. *Let $\varphi : M \to N$ be a homomorphism of A-modules. The following are equivalent:*

$$\varphi \text{ is surjective}, \tag{5.13.1}$$

$$\varphi_{\mathfrak{p}} : M_{\mathfrak{p}} \to N_{\mathfrak{p}} \text{ is surjective } \forall \mathfrak{p} \in \mathrm{Spec}\, A, \tag{5.13.2}$$

$$\varphi_{\mathfrak{m}} : M_{\mathfrak{m}} \to N_{\mathfrak{m}} \text{ is surjective for all maximal } H\text{-ideals } \mathfrak{m}. \tag{5.13.3}$$

Proof. To show that (5.13.1) $\Rightarrow$ (5.13.2) $\Rightarrow$ (5.13.3) is easy. If φ is not surjective, let $n \in N \setminus \varphi(M)$, and let $\mathfrak{m}$ be a maximal H-ideal containing $(\varphi(M) : n)$; cf. (4.1.6). If $\varphi_{\mathfrak{m}}$ is surjective, $n/1 \in \varphi_{\mathfrak{m}}(M_{\mathfrak{m}})$, and we have $s \cdot n \in \varphi(M)$ for some $s \in A_{[1],[1]} \setminus \mathfrak{m}$, contradicting $(\varphi(M) : n) \subseteq \mathfrak{m}$. □

Proposition 5.14. *Let $\varphi : M \to N$ be a homomorphism of A-modules. The following are equivalent:*

$$\varphi \text{ is injective}, \tag{5.14.1}$$

$$\varphi_{\mathfrak{p}} : M_{\mathfrak{p}} \to N_{\mathfrak{p}} \text{ is injective } \forall \mathfrak{p} \in \mathrm{Spec}\, A, \tag{5.14.2}$$

$$\varphi_{\mathfrak{m}} : M_{\mathfrak{m}} \to N_{\mathfrak{m}} \text{ is injective for all maximal } H\text{-ideals } \mathfrak{m}. \tag{5.14.3}$$

Proof. To show that (5.14.1) $\Rightarrow$ (5.14.2) $\Rightarrow$ (5.14.3) is easy. If φ is not injective, let $m \neq m', \varphi(m) = \varphi(m')$, and let $\mathfrak{m}$ be a maximal H-ideal containing $ann_A(m, m')$; cf. (4.1.5). If $\varphi_{\mathfrak{m}}$ is injective, $m/1 = m'/1 \in M_{\mathfrak{m}}$, and $s \cdot m = s \cdot m'$ for some $s \in A_{[1],[1]} \setminus \mathfrak{m}$, contradicting $ann_A(m, m') \subseteq \mathfrak{m}$. ☐

Consider the canonical homomorphism $\phi = \phi_S : A \to S^{-1}A, \phi(a) = a/1$. If $\mathfrak{b} \in \mathrm{sub}_{S^{-1}A}(S^{-1}A)$ is an ideal of $S^{-1}A, \phi^{-1}(\mathfrak{b}) \in \mathrm{sub}_A(A)$ is an ideal of A; if $\mathfrak{b} = ZE(\mathfrak{b})$ is an E-ideal, so is $\phi^{-1}(\mathfrak{b})$, cf. (4.16.2) for $ZE(\mathfrak{b})$; if $\mathfrak{b}$ is homogeneous so is $\phi^{-1}(\mathfrak{b})$. If $\mathfrak{b} \in H\text{-id}(S^{-1}A)$ is an H-ideal, so is $\varphi^{-1}(\mathfrak{b})$. If $\mathfrak{a} \in \mathrm{sub}_A(A)$ is an ideal of A, then $\phi(\mathfrak{a})$ generates the ideal $S^{-1}\mathfrak{a} \in \mathrm{sub}_{S^{-1}A}(S^{-1}A)$; if $\mathfrak{a} = ZE(\mathfrak{a})$ is an E-ideal, so is $S^{-1}\mathfrak{a}$; if $\mathfrak{a}$ is homogeneous, so is $S^{-1}\mathfrak{a}$. If $\mathfrak{A} \in H\text{-id}(A)$ is an H-ideal of A then

$$S^{-1}\mathfrak{A} \overset{\text{def}}{=} \phi_*(\mathfrak{A}) = \{a/s \in (S^{-1}A)_{[1],[1]} \mid a \in \mathfrak{A}, s \in S\}$$

is an H-ideal of $S^{-1}A$.

Proposition 5.15. *For $\mathfrak{b}$ an ideal (respectively H-ideal) of $S^{-1}A$, $S^{-1}\mathfrak{b}^c = \mathfrak{b}$.*

Proof. If $a/s \in \mathfrak{b}, a \in \mathfrak{b}^c$, and $a/s \in S^{-1}(\mathfrak{b}^c)$; so $\mathfrak{b} \subseteq S^{-1}(\mathfrak{b}^c)$. The reverse inclusion is clear. ☐

Proposition 5.16. *For $\mathfrak{a}$ an ideal (respectively H-ideal) of A,*

$$(S^{-1}\mathfrak{a})^c = \{a \in A \mid \exists s \in S : s \cdot a \in \mathfrak{a}\}. \tag{5.16.1}$$

In particular,

$$S^{-1}\mathfrak{a} = (1) \quad \Leftrightarrow \quad \mathfrak{a} \cap S \neq \varnothing. \tag{5.16.2}$$

Proof. We have

$$a \in (S^{-1}\mathfrak{a})^c \quad \Leftrightarrow \quad a/1 = x/s, x \in \mathfrak{a}, s \in S \quad \Leftrightarrow \quad s \cdot a \in \mathfrak{a}, \text{ some } s \in S.$$ ☐

Proposition 5.17. *The map ϕ_S^* induces a bijection*

$$\phi_S^* : \mathrm{Spec}(S^{-1}A) \xrightarrow{\sim} \{\mathfrak{p} \in \mathrm{Spec}\, A \mid \mathfrak{p} \cap S = \varnothing\},$$

which is a homeomorphism for the Zariski topology.

Proof. If $\mathfrak{q} \in \mathrm{Spec}(S^{-1}A), \phi_S^*(\mathfrak{q})$ belongs to the right-hand side. Conversely, if $\mathfrak{p}$ belongs to the right-hand side, $S^{-1}\mathfrak{p}$ is a (proper) prime of $S^{-1}A$. By Propositions 5.15 and 5.16 these operations are inverses of each other. ☐

Corollary 5.17.1. *We have a homeomorphism for $f \in A_{[1],[1]}$,*

$$\phi_f^* : \mathrm{Spec}(A_f) \xrightarrow{\sim} D_A(f).$$

Corollary 5.17.2. *We have a homeomorphism for $\mathfrak{p} \in \mathrm{Spec}(A)$,*

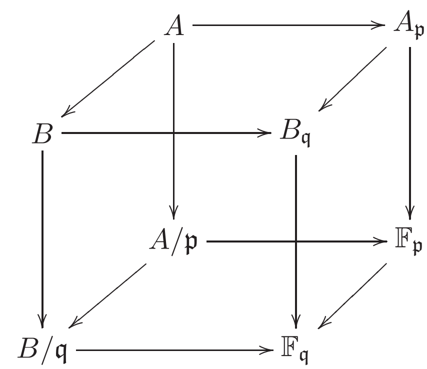
$$\phi_{\mathfrak{p}}^* : \mathrm{Spec}(A_{\mathfrak{p}}) \xrightarrow{\sim} \{\mathfrak{q} \in \mathrm{Spec}\, A \mid \mathfrak{q} \subseteq \mathfrak{p}\}.$$

In particular, $A_{\mathfrak{p}}$ contains a unique maximal H-ideal $\mathfrak{m}_{\mathfrak{p}} = S_{\mathfrak{p}}^{-1}\mathfrak{p}$; we say it is a local $\mathbb{F}$-ring.

Remark 5.17.3. For $\mathfrak{p} \in \mathrm{Spec}(A)$ we let $\mathbb{F}_{\mathfrak{p}} = A_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}$ denote the *residue field* at $\mathfrak{p}$. Let $\pi : A \to A/\mathfrak{p}$ be the canonical homomorphism, and $\overline{S}_{\mathfrak{p}} = \pi(S_{\mathfrak{p}})$; we have also $\mathbb{F}_{\mathfrak{p}} = \overline{S}_{\mathfrak{p}}^{-1}(A/\mathfrak{p})$. The commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\phi_{\mathfrak{p}}} & A_{\mathfrak{p}} \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ A/\mathfrak{p} & \longrightarrow & \mathbb{F}_{\mathfrak{p}} \end{array} \tag{5.17.4}$$

is cartesian: $\mathbb{F}_{\mathfrak{p}} = (A/\mathfrak{p}) \circledcirc_A A_{\mathfrak{p}}$ (see Theorem 7.1 for the definition of $\circledcirc$). It is also functorial: given a homomorphism of $\mathbb{F}$-rings $\varphi : A \to B, \mathfrak{q} \in \operatorname{Spec} B, \mathfrak{p} = \varphi^*(\mathfrak{q})$, we have the following commutative cube:



5.2 Structural sheaf $\mathcal{O}_A$

Next we define a sheaf $\mathcal{O}_A$ of $\mathbb{F}$-rings over $\operatorname{Spec} A$.

Definition 5.18. For an open set $U \subseteq \operatorname{Spec}(A)$, and for $Y, X \in |\mathbb{F}|$, we let $\mathcal{O}_A(U)_{Y,X}$ denote the set of functions

$$s : U \to \bigcup_{\mathfrak{p} \in U} (A_{\mathfrak{p}})_{Y,X},$$

such that $s(\mathfrak{p}) \in (A_{\mathfrak{p}})_{Y,X}$, and s is 'locally a fraction':

$$\forall \mathfrak{p} \in U, \exists \text{ a neighborhood } U_{\mathfrak{p}} \text{ of } \mathfrak{p}; \exists\ a \in A_{Y,X}; \exists\ f \in A_{[1],[1]} \setminus \bigcup_{\mathfrak{q} \in U_{\mathfrak{p}}} \mathfrak{q}$$

such that

$$s(\mathfrak{q}) = a/f \in A_{\mathfrak{q}}, \quad \forall \mathfrak{q} \in U_{\mathfrak{p}}. \tag{$\bigstar$}$$

It is clear that

$$\mathcal{O}_A(U) = \bigcup_{Y,X} \mathcal{O}_A(U)_{Y,X}$$

is an $\mathbb{F}$-ring. If $U' \subseteq U$, the natural restriction map $s \mapsto s|_{U'}$ is a homomorphism of $\mathbb{F}$-rings $\mathcal{O}_A(U) \to \mathcal{O}_A(U')$, thus $\mathcal{O}_A$ is a presheaf of $\mathbb{F}$-rings. From the local nature of ($\bigstar$) we see that $\mathcal{O}_A$ is in fact a sheaf.

Remark 5.18.1. Similarly for an A-module M we can define $\widetilde{M}(U)_{Y,X}$ as the collection of sections

$$s : U \to \bigcup_{\mathfrak{p} \in U} (M_{\mathfrak{p}})_{Y,X}$$

which are locally a fraction (replace $a \in A_{Y,X}$ in ($\bigstar$) by $\mathfrak{m} \in M_{Y,X}$). The set $\widetilde{M}$ is an $\mathcal{O}_A$-module in the following sense.

Definition 5.18.2. An $\mathcal{O}_A$-module $\mathfrak{M}$ is a sheaf of sets over $\operatorname{Spec} A$ such that $\mathfrak{M}(U)$ is an $\mathcal{O}_A(U)$-module, the structure compatible with restrictions – for open sets $U \subseteq U'$, denoting by $s \mapsto s|_U$ the restriction maps $\mathfrak{M}(U') \to \mathfrak{M}(U)$ and $\mathcal{O}_A(U') \to \mathcal{O}_A(U)$, we have

$$(a \circ m \circ a')|_U = a|_U \circ m|_U \circ a'|_U, \tag{5.18.2}$$

$$(m \oplus m')|_U = m|_U \oplus m'|_U, \tag{5.18.3}$$

$$(a \otimes m)|_U = a|_U \otimes m|_U. \tag{5.18.4}$$

For two such sheaves of $\mathcal{O}_A$-modules $\mathfrak{M}, \mathfrak{M}'$ a map of $\mathcal{O}_A$-modules $\varphi : \mathfrak{M} \to \mathfrak{M}'$ is a collection of $\mathcal{O}_A(U)$-homomorphisms

$$\varphi_U : \mathfrak{M}(U) \to \mathfrak{M}'(U)$$

for $U \subseteq \operatorname{Spec} A$ open, compatible with restrictions: $\varphi_{U'}(a)|_U = \varphi_U(a|_U)$ for $U \subseteq U'$.

Thus we have the category $\mathcal{O}_A$-$\mathcal{M}od$ of $\mathcal{O}_A$-modules.

Proposition 5.19. *For $\mathfrak{p} \in \operatorname{Spec}(A)$, the stalk*

$$\mathcal{O}_{A,\mathfrak{p}} = \varinjlim_{\mathfrak{p} \in U} \mathcal{O}_A(U)$$

of the sheaf $\mathcal{O}_A$ is isomorphic to $A_\mathfrak{p}$.

Proof. The map taking a local section s in a neighborhood of $\mathfrak{p}$ to $s(\mathfrak{p}) \in A_\mathfrak{p}$ induces a homomorphism $\mathcal{O}_{A,\mathfrak{p}} \to A_\mathfrak{p}$, which is clearly surjective. It is also injective. Let $s_1, s_2 \in \mathcal{O}_A(U)_{Y,X}$ have the same value at $\mathfrak{p}, s_1(\mathfrak{p}) = s_2(\mathfrak{p})$. Shrinking U we may assume $s_i = a_i/f_i$ on $U, a_i \in A_{Y,X}, f_i \in A_{[1],[1]}$. Then $a_1/f_1 = a_2/f_2$ in $A_\mathfrak{p}$ means $h \cdot f_2 \cdot a_1 = h \cdot f_1 \cdot a_2, h \in A_{[1],[1]} \setminus \mathfrak{p}$, but then $a_1/f_1 = a_2/f_2$ in $A_\mathfrak{q}$ $\forall \mathfrak{q} \in U \cap D_A(h)$. □

Remark 5.19.1. Similarly for an A-module M we have an isomorphism

$$(\widetilde{M})_\mathfrak{p} \stackrel{\text{def}}{=} \varinjlim_{\mathfrak{p} \in U} \widetilde{M}(U) \xrightarrow{\sim} M_\mathfrak{p}.$$

Proposition 5.20. *For $f \in A_{[1],[1]}$, the $\mathbb{F}$-ring $\mathcal{O}_A(D_A(f))$ is isomorphic to A_f. In particular, the global sections $\Gamma(\operatorname{Spec}(A), \mathcal{O}_A) \stackrel{\text{def}}{=} \mathcal{O}_A(D_A(1)) \cong A$.*

Proof. Define the homomorphism $\psi : A_f \to \mathcal{O}_A(D_A(f))$ by sending a/f^n to the section whose value at $\mathfrak{p}$ is the image of a/f^n in $A_\mathfrak{p}$.

We shall show that ψ is injective. If $\psi(a_1/f^{n_1}) = \psi(a_2/f^{n_2})$ then $\forall \mathfrak{p} \in D_A(f)$ there is $h_\mathfrak{p} \in A_{[1],[1]} \setminus \mathfrak{p}$ with

$$h_\mathfrak{p} f^{n_2} a_1 = h_\mathfrak{p} f^{n_1} a_2.$$

Let $\mathfrak{a} = ann_A(f^{n_2}a_1, f^{n_1}a_2)$. It is an H-ideal of A, and $\forall \mathfrak{p} \in D_A(f), \mathfrak{p} \notin V_A(\mathfrak{a})$, so $D_A(f) \cap V_A(\mathfrak{a}) = \varnothing$, hence $V_A(\mathfrak{a}) \subseteq V_A(f)$, hence $f \in IV_A(\mathfrak{a}) = \sqrt{\mathfrak{a}}$, hence $f^n \in \mathfrak{a}$ for some $n \geqslant 1$, showing that $a_1/f^{n_1} = a_2/f^{n_2}$ in A_f.

We show next that ψ is surjective. Let $s \in \mathcal{O}_A(D_A(f))_{Y,X}$. By Proposition 4.12, $D_A(f)$ is compact, so there exists a finite open covering

$$D_A(f) = \bigcup_{1 \leqslant i \leqslant N} D_A(h_i),$$

such that for all $\mathfrak{p} \in D_A(h_i) : s(\mathfrak{p}) = a_i/g_i \in A_\mathfrak{p}$, where $a_i \in A_{Y,X}$ and $g_i \in A_{[1],[1]}$ is such that $D_A(g_i) \supseteq D_A(h_i)$ for $1 \leqslant i \leqslant N$. We have $V_A(g_i) \subseteq V_A(h_i)$, hence

$$\sqrt{(g_i)} = IV_A(g_i) \supseteq IV_A(h_i) = \sqrt{(h_i)},$$

hence $h_i \in \sqrt{(g_i)}$ so that for some $n_i \geqslant 1$ we have $h_i^{n_i} = c_i \cdot g_i$, hence $s(\mathfrak{p}) = c_i a_i / h_i^{n_i}$. So we can replace h_i by g_i. On the set

$$D_A(g_i) \cap D_A(g_j) = D_A(g_i g_j)$$

we have $a_i/g_i = s(\mathfrak{p}) = a_j/g_j$, hence by the injectivity of ψ we find

$$a_i/g_i = a_j/g_j \text{ in } A_{g_i g_j}.$$

This means $(g_i g_j)^n \cdot g_j a_i = (g_i g_j)^n \cdot g_i a_j$, and we can choose n big enough to work for all i, j. We can replace g_i by g_i^{n+1} (since $D_A(g_i) = D_A(g_i^{n+1})$), and replace a_i by $g_i^n \cdot a_i$ (since $s(\mathfrak{p}) \equiv g_i^n a_i / g_i^{n+1}$), and then have the simpler equation $g_j \cdot a_i = g_i \cdot a_j$ for all i, j. Since the sets $D_A(g_i)$ cover $D_A(f)$ we have, cf. Proposition 4.12,

$$f^m = b \circ \left(\bigoplus_i g_i \cdot \mathrm{id}_{X_i} \right) \circ b'.$$

Set

$$a = (\mathrm{id}_Y \otimes b) \circ \left(\bigoplus_i a_i \otimes \mathrm{id}_{X_i} \right) \circ (\mathrm{id}_X \otimes b').$$

Then

$$\begin{aligned} g_j \cdot a &= (\mathrm{id}_Y \otimes b) \circ \left(\bigoplus_i g_j a_i \otimes \mathrm{id}_{X_i} \right) \circ (\mathrm{id}_X \otimes b') \\ &= (\mathrm{id}_Y \otimes b) \circ \left(\bigoplus_i a_j \otimes g_i \cdot \mathrm{id}_{X_i} \right) \circ (\mathrm{id}_X \otimes b') = f^m \cdot a_j. \end{aligned}$$

Hence $a_j / g_j = s(\mathfrak{p}) = a/f^m$ and $s = \psi(a/f^m)$. □

Remark 5.20.1. Similarly for an A-module M we have an isomorphism

$$M_f \xrightarrow{\sim} \widetilde{M}(D_A(f))$$

and in particular

$$M \xrightarrow{\sim} \widetilde{M}(\operatorname{Spec} A) \stackrel{\text{def}}{=} \Gamma(\operatorname{Spec} A, \widetilde{M}).$$

6. Schemes

We define the categories of $\mathbb{F}$-(locally)-ringed spaces, and of (Zariski) $\mathbb{F}$-schemes, and recall the theory of quasi-coherent modules. As an important example we give the ‘compactification’ $\overline{\operatorname{Spec} \mathbb{Z}}$ of $\operatorname{Spec} \mathbb{Z}$.

6.1 Locally $\mathbb{F}$-ring spaces

Definition 6.1. An $\mathbb{F}$-*ringed space* $(X, \mathcal{O}_X)$ is a topological space with a sheaf $\mathcal{O}_X$ of $\mathbb{F}$-rings. A map of $\mathbb{F}$-ringed spaces $f : X \to Y$ is a continuous map of the underlying topological spaces together with a map of sheaves of $\mathbb{F}$-rings on Y, $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$, i.e. for $U \subseteq Y$ open we have $f_U^\# : \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}U)$ a map of $\mathbb{F}$-rings, such that for

$$U' \subseteq U : f_U^\#(s)|_{f^{-1}U'} = f_{U'}^\#(s|_{U'}).$$

The $\mathbb{F}$-ringed space X is an $\mathbb{F}$-*locally-ringed space* if for all $\mathfrak{p} \in X$ the stalk $\mathcal{O}_{X,\mathfrak{p}}$ is a local $\mathbb{F}$-ring, i.e. contains a unique maximal H-ideal $\mathfrak{m}_{X,\mathfrak{p}}$. For a map of $\mathbb{F}$-ringed spaces $f : X \to Y$, and for $\mathfrak{p} \in X$, we get an induced homomorphism of $\mathbb{F}$-rings on the stalks

$$f_{\mathfrak{p}}^\# : \mathcal{O}_{Y,f(\mathfrak{p})} = \varinjlim_{f(\mathfrak{p}) \in V} \mathcal{O}_Y(V) \to \varinjlim_{\mathfrak{p} \in f^{-1}V} \mathcal{O}_X(f^{-1}V) \to \varinjlim_{\mathfrak{p} \in U} \mathcal{O}_X(U) = \mathcal{O}_{X,\mathfrak{p}}. \tag{6.1.1}$$

A map $f : X \to Y$ of $\mathbb{F}$-locally-ringed spaces is a map of $\mathbb{F}$-ringed spaces such that $f_{\mathfrak{p}}^\#$ is a *local homomorphism* for all $\mathfrak{p} \in X$, i.e.

$$f_{\mathfrak{p}}^\#(\mathfrak{m}_{Y,f(\mathfrak{p})}) \subseteq \mathfrak{m}_{X,\mathfrak{p}} \text{ or equivalently } (f_{\mathfrak{p}}^\#)^{-1} \mathfrak{m}_{X,\mathfrak{p}} = \mathfrak{m}_{Y,f(\mathfrak{p})}. \tag{6.1.2}$$

We let $\mathbb{F}$-$\mathcal{R}ings.\mathcal{S}p$ (respectively $\mathcal{L}oc$-$\mathbb{F}$-$\mathcal{R}ings$-$\mathcal{S}p$) denote the category of $\mathbb{F}$-(locally)-ringed spaces.

For a (locally) $\mathbb{F}$-ringed space $(X, \mathcal{O}_X)$ an $\mathcal{O}_X$-*module* $\mathfrak{M}$ is a sheaf of sets over X such that for $U \subseteq X$ open, $\mathfrak{M}(U)$ is an $\mathcal{O}_X(U)$-module, these structures being compatible with restrictions, cf. (5.18.2)–(5.18.4). For two $\mathcal{O}_X$-modules $\mathfrak{M}, \mathfrak{M}'$ a map of sheaves $\varphi : \mathfrak{M} \to \mathfrak{M}'$ is a *homomorphism* *of* $\mathcal{O}_X$*-modules* if for $U \subseteq X$ open the map $\varphi_U : \mathfrak{M}(U) \to \mathfrak{M}'(U)$ is a homomorphism of $\mathcal{O}_X(U)$-modules. Thus we have the category $\mathcal{O}_X$-$\mathcal{M}od$ of $\mathcal{O}_X$-modules.

For a homomorphism of $\mathbb{F}$-rings $\varphi : A \to B$, for $\mathfrak{p} \in \mathrm{Spec}(B)$, we have a unique homomorphism $\varphi_{\mathfrak{p}} : A_{\varphi^{-1}\mathfrak{p}} \to B_{\mathfrak{p}}$, such that we have a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \downarrow & & \downarrow \\ A_{\varphi^{-1}\mathfrak{p}} & \xrightarrow{\varphi_{\mathfrak{p}}} & B_{\mathfrak{p}} \end{array} \tag{6.2}$$

where $\varphi_{\mathfrak{p}}(a/s) = \varphi(a)/\varphi(s)$, and $\varphi_{\mathfrak{p}}$ is a local homomorphism. Thus $A \mapsto \mathrm{Spec}(A)$ is a contravariant functor from $\mathbb{F}$-$\mathcal{R}ings$ to $\mathcal{L}oc$-$\mathbb{F}$-$\mathcal{R}ings$-$\mathcal{S}p$. It is the adjoint of the functor Γ of taking global sections

$$\Gamma(X, \mathcal{O}_X) = \mathcal{O}_X(X), \quad \Gamma(f) = f_Y^{\#} : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X).$$

Proposition 6.3. *We have*

$$\mathrm{Hom}_{\mathcal{L}oc\text{-}\mathbb{F}\text{-}\mathcal{R}ings\text{-}\mathcal{S}p}(X, \mathrm{Spec}(A)) = \mathrm{Hom}_{\mathbb{F}\text{-}\mathcal{R}ings}(A, \mathcal{O}_X(X)).$$

Proof. For an $\mathbb{F}$-locally-ringed space X, and for a point $x \in X$, the canonical homomorphism $\phi_x : \mathcal{O}_X(X) \to \mathcal{O}_{X,x}$ gives a prime $\mathcal{P}(x) = \phi_x^{-1}(\mathfrak{m}_{X,x}) \in \mathrm{Spec}\,\mathcal{O}_X(X)$. The map $\mathcal{P} : X \to \mathrm{Spec}\,\mathcal{O}_X(X)$ is continuous:

$$\mathcal{P}^{-1}(D(f)) = \{x \in X \mid \phi_x(f) \notin \mathfrak{m}_{X,x}\}$$

is open for $f \in \mathcal{O}_X(X)$. We have an induced homomorphism

$$\mathcal{P}_{D(f)}^{\#} : \mathcal{O}_X(X)_f \to \mathcal{O}_X(\{x \in X \mid \phi_x(f) \notin \mathfrak{m}_{X,x}\}),$$

making $\mathcal{P}$ a map of $\mathbb{F}$-ringed spaces, and taking the direct limit over f with $\phi_x(f) \notin \mathfrak{m}_{X,x}$ we get

$$\mathcal{P}_x^{\#} : \mathcal{O}_X(X)_{\mathcal{P}(x)} \to \mathcal{O}_{X,x},$$

showing $\mathcal{P}$ is a map of $\mathbb{F}$-locally-ringed spaces.

To a homomorphism of $\mathbb{F}$-rings $\varphi : A \to \mathcal{O}_X(X)$ we associate the map of $\mathbb{F}$-locally-ringed spaces

$$X \xrightarrow{\mathcal{P}} \mathrm{Spec}\,\mathcal{O}_X(X) \xrightarrow{\mathrm{Spec}(\varphi)} \mathrm{Spec}\,A.$$

Conversely, to a map $f : X \to \mathrm{Spec}\,A$ of $\mathbb{F}$-locally-ringed spaces (as in Definition 6.1) we associate its action on global sections

$$\Gamma(f) = f_{\mathrm{Spec}\,A}^{\#} : A = \mathcal{O}_A(\mathrm{Spec}\,A) \to \mathcal{O}_X(X).$$

Clearly, $\Gamma(\mathrm{Spec}(\varphi) \circ \mathcal{P}) = \varphi$.

Conversely, given a map $f : X \to \mathrm{Spec}\,A$ (as in Definition 6.1), for $x \in X$ we have the following commutative diagram:

$$\begin{array}{ccc} A = \mathcal{O}_A(\mathrm{Spec}\,A) & \xrightarrow{\Gamma(f)} & \mathcal{O}_X(X) \\ \downarrow{\scriptstyle \phi_{f(x)}} & & \downarrow{\scriptstyle \phi_x} \\ A_{f(x)} & \xrightarrow{f_x^{\#}} & \mathcal{O}_{X,x} \end{array}$$

Since $f_x^{\#}$ is assumed to be local, $(f_x^{\#})^{-1}(\mathfrak{m}_{X,x}) = \mathfrak{m}_{f(x)}$, and by the commutativity of the diagram we get $\Gamma(f)^{-1}(\mathcal{P}(x)) = f(x)$, i.e. $f = (\mathrm{Spec}\,\Gamma(f)) \circ \mathcal{P}$ is the continuous map associated to the homomorphism $\Gamma(f)$. Similarly, for $g \in A$, the commutativity of the diagram

$$\begin{array}{ccc} A & \xrightarrow{\Gamma(f)} & \mathcal{O}_X(X) \\ \downarrow & & \downarrow \\ A_g & \xrightarrow{f^{\#}_{D(g)}} & \mathcal{O}_X(D_X(f^{\#}g)) \end{array}$$

gives $f^{\#}_{D(g)}(a/g^n) = \Gamma(f)(a)/(\Gamma(f)(g))^n$, hence $f = (\operatorname{Spec}\Gamma(f)) \circ \mathcal{P}$ as a map of $\mathbb{F}$-locally-ringed spaces. □

Corollary 6.3.1. *For $\mathbb{F}$-rings A, B:*

$$\operatorname{Hom}_{\mathcal{L}oc\text{-}\mathbb{F}\text{-}\mathcal{R}ings\text{-}\mathcal{S}p}(\operatorname{Spec} B, \operatorname{Spec} A) = \operatorname{Hom}_{\mathbb{F}\text{-}\mathcal{R}ings}(A, B).$$

6.2 Zariski $\mathbb{F}$-schemes

Definition 6.4. A Zariski $\mathbb{F}$-scheme is an $\mathbb{F}$-locally-ringed space $(X, \mathcal{O}_X)$, such that there is a covering by open sets $X = \bigcup_i U_i$, and the canonical maps

$$\mathcal{P} : (U_i, \mathcal{O}_X|_{U_i}) \to \operatorname{Spec} \mathcal{O}_X(U_i)$$

are isomorphisms of $\mathbb{F}$-locally-ringed spaces. A morphism of Zariski $\mathbb{F}$-schemes is a map of $\mathbb{F}$-locally-ringed spaces. We denote the category of Zariski $\mathbb{F}$-schemes by $\mathcal{Z}ar\text{-}\mathbb{F}\text{-}\mathcal{S}ch$.

Zariski $\mathbb{F}$-schemes can be glued.

Proposition 6.5. *Given a set of indices I, and for $i \in I$ given $X_i \in \mathcal{Z}ar\text{-}\mathbb{F}\text{-}\mathcal{S}ch$, and for $i \neq j, i, j \in I$, an isomorphism $\varphi_{ij} : U_{ij} \xrightarrow{\sim} U_{ji}$, with $U_{ij} \subseteq X_i$ open (and hence U_{ij} are Zariski $\mathbb{F}$-schemes), such that*

$$\varphi_{ji} = \varphi_{ij}^{-1}, \tag{6.5.1}$$

$$\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk} \quad \text{and} \quad \varphi_{jk} \circ \varphi_{ij} = \varphi_{ik} \text{ on } U_{ij} \cap U_{ik}. \tag{6.5.2}$$

There exists $X \in \mathcal{Z}ar\text{-}\mathbb{F}\text{-}\mathcal{S}ch$, and maps $\psi_i : X_i \to X$, such that

$$\psi_i \text{ is an isomorphism of } X_i \text{ onto the open set } \psi_i(X_i) \subseteq X, \tag{6.5.3}$$

$$X = \bigcup_i \psi_i(X_i), \tag{6.5.4}$$

$$\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j), \tag{6.5.5}$$

$$\psi_i = \psi_j \circ \varphi_{ij} \text{ on } U_{ij}. \tag{6.5.6}$$

Proof. The proof is clear: glue the topological spaces and glue the sheaves of $\mathbb{F}$-rings. For $V \subseteq X$ open

$$\mathcal{O}_X(V) = \ker\left\{\prod_i \mathcal{O}_{X_i}(\psi^{-1}V) \rightrightarrows \prod_{i,j} \mathcal{O}_{X_i}(\psi^{-1}V \cap U_{ij})\right\}.$$ □

Remark 6.6. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a map of $\mathbb{F}$-ringed spaces. If $\mathfrak{M}$ is an $\mathcal{O}_X$-module, then $f_*\mathfrak{M}(V) = \mathfrak{M}(f^{-1}V), V \subseteq Y$ open, gives rise to an $f_*\mathcal{O}_X$-module. Using the map $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$, we see that $f_*\mathfrak{M}$ is naturally an $\mathcal{O}_Y$-module.

If $\mathfrak{N}$ is an $\mathcal{O}_Y$-module, its inverse image $f^{-1}\mathfrak{N}$ is the sheaf on X associated with the presheaf

$$U \mapsto \varinjlim_{f(U) \subseteq V} \mathfrak{N}(V)$$

and $f^{-1}\mathfrak{N}$ is an $f^{-1}\mathcal{O}_Y$-module. To give the map $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves on Y is equivalent to giving the map $f^\natural : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ of sheaves on X. Using the map $f^\natural$ we can extend scalars, cf. Proposition 3.23, to form the presheaf

$$U \mapsto (f^{-1}\mathfrak{N}(U))_{\mathcal{O}_X(U)}.$$

The sheaf associated to this presheaf is denoted $f^*\mathfrak{N}$; it is an $\mathcal{O}_X$-module.

The functors of *direct image* f_* and of *inverse image* f^* are adjoint

$$\mathrm{Hom}_{\mathcal{O}_X\text{-}\mathcal{M}od}(f^*\mathfrak{N}, \mathfrak{M}) = \mathrm{Hom}_{\mathcal{O}_Y\text{-}\mathcal{M}od}(\mathfrak{N}, f_*\mathfrak{M}). \tag{6.6.1}$$

For a homomorphism of $\mathbb{F}$-rings $\varphi : A \to B$, and the associated map $f = \varphi^* : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$, and for any B-module M with associated $\mathcal{O}_B$-module $\widetilde{M}$, cf. Remark 5.18.1, and associated A-module $\varphi^* M$, cf. (3.22.1–2), we have

$$f_*(\widetilde{M}) = \widetilde{(\varphi^* M)}. \tag{6.6.2}$$

For an A-module N with associated B-module N_B, cf. Proposition 3.23, we have

$$f^*(\widetilde{N}) = \widetilde{(N_B)}. \tag{6.6.3}$$

The theory of quasi-coherent modules over a Zariski $\mathbb{F}$-scheme goes over as in the classical theory, incorporating the notions of § 3. We shall not give the details here, and we give only the basic definitions.

Theorem 6.7. *Let A be an $\mathbb{F}$-ring. Let $U \subseteq X = \mathrm{Spec}\, A$ be an open compact subset, and $\mathfrak{M}$ an $\mathcal{O}_X|_U$-module. The following conditions are equivalent:*

(1) *there exists an A-module M and an isomorphism $\widetilde{M}|_U \cong \mathfrak{M}$;*

(2) *there exists an open affine cover $U = \bigcup_{i \in I} U_i, U_i = \mathrm{Spec}\, A_i$, and for $i \in I$ there are A_i-modules M_i such that $\widetilde{M_i} \cong \mathfrak{M}|_{U_i}$;*

(3) *for every open affine $\mathrm{Spec}\, B \subseteq U$ there exists a B-module M such that $\widetilde{M} \cong \mathfrak{M}|_{\mathrm{Spec}\, B}$;*

(4) *for every $f \in A_{[1],[1]}$ the restriction induces an isomorphism of A_f-modules*

$$\Gamma(U, \mathfrak{M})_f \xrightarrow{\sim} \Gamma(D_A(f), \mathfrak{M}).$$

Proof. See [Gro60, 1.4]. □

If $\mathfrak{M}$ satisfies the conditions of Theorem 6.7 we say it is *quasi-coherent*. Similarly for a Zariski $\mathbb{F}$-scheme X, replacing in (2) or in (3) U by X we get the notion of *quasi-coherent $\mathcal{O}_X$-module.* We denote by $\mathcal{QC}\text{-}\mathcal{O}_X\text{-}\mathcal{M}od$ the category of quasi-coherent $\mathcal{O}_X$-modules. For an affine scheme $X = \mathrm{Spec}\, A$, the functors $M \mapsto \widetilde{M}, \mathfrak{M} \mapsto \Gamma(X, \mathfrak{M})$, give inverse isomorphisms of categories

$$A\text{-}\mathcal{M}od \xleftrightarrow{\sim} \mathcal{QC}\text{-}\mathcal{O}_A\text{-}\mathcal{M}od. \tag{6.7.1}$$

6.3 $\mathbb{F}$-schemes and the compactified $\overline{\mathrm{Spec}\, \mathbb{Z}}$

The category of $\mathbb{F}$-*locally-ringed spaces* has inverse limits. Given an inverse system $\{X_j, \mathcal{O}_{X_j}\}_{j \in J}$, where J is a partially ordered set, and for $j_1 \geqslant j_2$ in J we are given maps $\pi^{j_1}_{j_2} : X_{j_1} \to X_{j_2}$ such that $\pi^j_j = \mathrm{id}_{X_j}$, and $\pi^{j_2}_{j_3} \circ \pi^{j_1}_{j_2} = \pi^{j_1}_{j_3}$ for $j_1 \geqslant j_2 \geqslant j_3$, and where we always assume that J is directed (for $j_1, j_2 \in J$ there exists $j \in J$ with $j \geqslant j_1, j \geqslant j_2$), the inverse limit $\varprojlim_J X_j$ is constructed as follows. As a topological space it is the inverse limit topological space, i.e. $\varprojlim_J X_j$ is the inverse limit of the X_j as a set, together with the topology having as a basis for open sets the sets of the form $\pi_j^{-1}(U)$, where $\pi_j : \varprojlim_J X_j \to X_j$ are the projections, and where $U \subseteq X_j$ are arbitrary open sets in X_j (we can take the U to vary over a basis for the topology of the X_j). Then on the topological space $X = \varprojlim_J X_j$ we have the directed system of sheaves of $\mathbb{F}$-rings $\{\pi_j^{-1}\mathcal{O}_j\}$ and the direct limit, i.e. the sheaf associated with the pre-sheaf $U \mapsto \varinjlim_J \pi_j^{-1}\mathcal{O}_j(U)$ is the sheaf $\mathcal{O}_X$ on $X = \varprojlim_J X_j$

satisfying the universal property of the inverse limit in the category $\mathcal{L}oc$-$\mathbb{F}$-$\mathcal{R}ings$-$\mathcal{S}p$. For a point $x = \{x_j\} \in X$, the stalk $\mathcal{O}_{X,x}$ is the direct limit of the stalks $\mathcal{O}_{X_j,x_j}$, and hence is indeed a local $\mathbb{F}$-ring, and $(\pi_j^\sharp)_x : \mathcal{O}_{X_j,x_j} \to \mathcal{O}_{X,x}$ is a local homomorphism. For an open set $U \subseteq X$ the sections $\mathcal{O}_X(U)$ can alternatively be described as the maps

$$s : U \to \coprod_{x\in U} \mathcal{O}_{X,x}, \quad s(x) \in \mathcal{O}_{X,x},$$

such that for any $x = \{x_j\} \in U$ there exists an open set $U_j \subseteq X_j$ for some $j \in J$, such that $x_j \in U_j, \pi^{-1}(U_j) \subseteq U$, and there is a section $s_j \in \mathcal{O}_{X_j}(U_j)$, such that for all $y \in \pi^{-1}(U_j)$, we have $s(y) = (\pi_j^\sharp)_y(s_j|_{\pi_j(y)})$. In the case that the X_j are all affine Zariski $\mathbb{F}$-schemes, $X_j = \mathrm{Spec}(A_j)$, the inverse limit $X = \varprojlim_J X_j$ is again an affine Zariski $\mathbb{F}$-scheme, namely $X = \mathrm{Spec}(A)$, where $A = \varinjlim_J A_j$ is the direct limit of the $\mathbb{F}$-rings A_j. But in the case that the X_j are Zariski $\mathbb{F}$-schemes, the inverse limit $X = \varprojlim_J X_j$ need not be a Zariski $\mathbb{F}$-scheme, and the category $\mathcal{Z}ar$-$\mathbb{F}$-$\mathcal{S}ch$ does not have inverse limits ($X = \varprojlim_J X_j$ will be a Zariski $\mathbb{F}$-scheme if the maps $\pi_{j_2}^{j_1} : X_{j_1} \to X_{j_2}$ are affine).

Definition 6.8. The category of $\mathbb{F}$-schemes, $\mathbb{F}$-$\mathcal{S}ch$, is the category of pro-objects of the category of Zariski $\mathbb{F}$-schemes.

Thus the objects of $\mathbb{F}$-$\mathcal{S}ch$ are inverse systems $X = \{X_j\}_{j\in J}$, where the X_j are Zariski $\mathbb{F}$-schemes, and where J is an arbitrary directed set, and the maps in $\mathbb{F}$-$\mathcal{S}ch$ from such an object to another object $Y = \{Y_i\}_{i\in I}$ are given by

$$\mathrm{Hom}_{\mathbb{F}\text{-}\mathcal{S}ch}(X,Y) = \varprojlim_I \Big(\varinjlim_J \mathrm{Hom}_{\mathcal{Z}ar\text{-}\mathbb{F}\text{-}\mathcal{S}ch}(X_j, Y_i) \Big), \tag{6.8.1}$$

i.e. the maps $\varphi : X \to Y$ are given by a collection of maps $\varphi_i^j : X_j \to Y_i$ defined for all $i \in I$ and for $j \geqslant \sigma(i)$ sufficiently large (depending on i), and are inductive in the index j and projective in the index i: for all $i \in I$, and for $j_1 \geqslant j_2$ sufficiently large in J,

$$\varphi_i^{j_1} = \varphi_i^{j_2} \circ \pi_{j_2}^{j_1} \quad (\text{here } \pi_{j_2}^{j_1} : X_{j_1} \to X_{j_2}); \tag{6.8.2}$$

and for all $i_1 \geqslant i_2$ in I, and for j sufficiently large in J,

$$\pi_{i_2}^{i_1} \circ \varphi_{i_1}^j = \varphi_{i_2}^j \quad (\text{here } \pi_{i_2}^{i_1} : Y_{i_1} \to Y_{i_2}). \tag{6.8.3}$$

The maps $\{\varphi_i^j : X \to Y\}$ and $\{\widetilde{\varphi}_i^j : X \to Y\}$ are considered equivalent if for all $i \in I$, and for j sufficiently large in J, $\varphi_i^j = \widetilde{\varphi}_i^j$. The identity map of $\{X_j\}$ is represented by $\{\pi_{j_2}^{j_1} : X_{j_1} \to X_{j_2}\}_{j_1 \geqslant j_2}$. The composition of $\varphi = \{\varphi_i^j\}_{j\geqslant\sigma(i)} : \{X_j\}_J \to \{Y_i\}_I$ with $\widetilde{\varphi} = \{\widetilde{\varphi}_k^i\}_{i\geqslant\widetilde{\sigma}(k)} : \{Y_i\}_I \to \{Z_k\}_K$ is given by

$$\widetilde{\varphi} \circ \varphi = \{\widetilde{\varphi}_k^i \circ \varphi_i^j\}_{j\geqslant\sigma(\widetilde{\sigma}(k))}. \tag{6.8.4}$$

Note that there is always a map (with Hom in $\mathcal{L}oc$-$\mathbb{F}$-$\mathcal{R}ings$-$\mathcal{S}p$)

$$\varinjlim_J \mathrm{Hom}(X_j, Y_i) \to \mathrm{Hom}\Big(\varprojlim_J X_j, Y_i \Big), \tag{6.8.5}$$

and by definition

$$\varprojlim_I \mathrm{Hom}\Big(\varprojlim_J X_j, Y_i \Big) \xrightarrow{\sim} \mathrm{Hom}\Big(\varprojlim_J X_j, \varprojlim_I Y_i \Big). \tag{6.8.6}$$

Composing (6.8.5) and (6.8.6) we obtain

$$\mathcal{L} : \varprojlim_I \Big(\varinjlim_J \mathrm{Hom}(X_j, Y_i) \Big) \to \mathrm{Hom}\Big(\varprojlim_J X_j, \varprojlim_I Y_i \Big), \tag{6.8.7}$$

i.e. a functor

$$\mathcal{L} : \mathbb{F}\text{-}\mathcal{S}ch \to \mathcal{L}oc\text{-}\mathbb{F}\text{-}\mathcal{R}ings\text{-}\mathcal{S}p, \quad \mathcal{L}(\{X_j\}_J) = \varprojlim_J X_j. \tag{6.8.8}$$

Example 6.9. *The compactified* $\overline{\operatorname{Spec}\mathbb{Z}}$. Fix a square-free integer $N \geqslant 2$. Let $A_N = \mathbb{F}(\mathbb{Z}[1/N]) \cap \mathcal{O}_{\mathbb{Q},\eta}$; it is the $\mathbb{F}$-ring with

$$(A_N)_{Y,X} = \left\{ a \in \mathbb{F}\left(\mathbb{Z}\left[\frac{1}{N}\right]\right)_{Y,X} : |a|_\eta \leqslant 1 \right\}, \tag{6.9.1}$$

the $Y \times X$ matrices with values in $\mathbb{Z}[1/N]$ and with (real) operator norm bounded by 1. The map $j : A_N \to \mathbb{F}(\mathbb{Z}[1/N])$ defines the basic open set

$$j^* : \operatorname{Spec}\mathbb{Z}\left[\frac{1}{N}\right] \cong \operatorname{Spec}\mathbb{F}\left(\mathbb{Z}\left[\frac{1}{N}\right]\right) \cong D_{A_N}\left(\frac{1}{N}\right) \hookrightarrow \operatorname{Spec} A_N. \tag{6.9.2}$$

Indeed, it is easy to check that the map $(A_N)_{1/N} \to \mathbb{F}(\mathbb{Z}[1/N])$ is an isomorphism of $\mathbb{F}$-rings, where $(A_N)_{1/N}$ is the localization of A_N with respect to $1/N \in (A_N)_{[1],[1]}$: every matrix $a \in \mathbb{F}(\mathbb{Z}[1/N])_{Y,X}$ can be written as $a = (a/N^k)/(1/N^k)$, and for k sufficiently large $a/N^k \in (A_N)_{Y,X}$. The space $\operatorname{Spec} A_N$ contains also the closed point

$$\eta = i^*(\mathfrak{m}_{\mathbb{Q},\eta}) = \{a \in (A_N)_{[1],[1]} \mid |a|_\eta < 1\} = \mathbb{Z}\left[\frac{1}{N}\right] \cap (-1, 1), \tag{6.9.3}$$

and it is the 'real prime' given by $i : A_N \hookrightarrow \mathcal{O}_{\mathbb{Q},\eta}$. But note that η is 'very close': the only open set containing η is the hole space, since for any non-trivial basic open set $D_{A_N}(f)$, say $f = p/N^k$, p prime not dividing N and $p < N^k$, we have

$$(A_N)_f = \mathbb{F}\left(\mathbb{Z}\left[\frac{1}{N \cdot p}\right]\right)$$

(indeed, every matrix $a \in \mathbb{F}(\mathbb{Z}[1/N \cdot p])_{Y,X}$ can be written as $a = (p/N^k)^l \cdot a/f^l$, and for l sufficiently large $(p/N^k)^l \cdot a \in (A_N)_{Y,X}$), and so

$$D_{A_N}(f) = \operatorname{Spec}(A_N)_f = \operatorname{Spec}\mathbb{F}\left(\mathbb{Z}\left[\frac{1}{N \cdot p}\right]\right) \cong \operatorname{Spec}\mathbb{Z}\left[\frac{1}{N \cdot p}\right] \tag{6.9.4}$$

does not contain η. Further, η contains all the primes of A_N, it is the (unique) maximal H-ideal of A_N, and A_N is a local $\mathbb{F}$-ring (of 'Krull' dimension 2).

Let X_N be the Zariski $\mathbb{F}$-scheme obtained by gluing $\operatorname{Spec} A_N$ with $\operatorname{Spec}\mathbb{F}(\mathbb{Z})$ along the common open set $\operatorname{Spec}\mathbb{F}(\mathbb{Z}[1/N])$ cf. Proposition 6.5 or [Hart77, p. 75, Example 2.3.5]. The open sets of X_N are the open sets of $\operatorname{Spec}\mathbb{Z}$, and sets of the form $U \cup \{\eta\}$ with $\operatorname{Spec}\mathbb{Z}[1/N] \subseteq U \subseteq \operatorname{Spec}\mathbb{Z}$. For an open set $U = \operatorname{Spec}\mathbb{Z}[1/M] \subseteq \operatorname{Spec}\mathbb{Z}$, we have $\mathcal{O}_{X_N}(U) = \mathbb{F}(\mathbb{Z}[1/M])$, and for such a set $U = \operatorname{Spec}\mathbb{Z}[1/M]$ with M dividing N, we have $\mathcal{O}_{X_N}(U \cup \{\eta\}) = A_M$.

For N dividing M we have commutative diagrams

$$\begin{array}{ccc} A_N & \hookrightarrow & A_M \\ \downarrow & & \downarrow \\ \mathbb{F}\left(\mathbb{Z}\left[\frac{1}{N}\right]\right) & \hookrightarrow & \mathbb{F}\left(\mathbb{Z}\left[\frac{1}{M}\right]\right) \end{array} \qquad \begin{array}{ccc} \operatorname{Spec} A_N & \longleftarrow & \operatorname{Spec} A_M \\ \uparrow & & \uparrow \\ \operatorname{Spec}\mathbb{F}\left(\mathbb{Z}\left[\frac{1}{N}\right]\right) & \longleftarrow & \operatorname{Spec}\mathbb{F}\left(\mathbb{Z}\left[\frac{1}{M}\right]\right) \end{array} \tag{6.9.5}$$

and we obtain a map $\pi_N^M : X_M \to X_N$. Note that π_N^M is a bijection on points, and further $(\pi_N^M)_* \mathcal{O}_{X_M} = \mathcal{O}_{X_N}$, i.e. $(\pi_N^M)^\sharp$ is the identity, but there are more open sets in X_M than there are in X_N. We need all these open sets, for all N, and so we pass to the inverse limit. The 'compactified $\operatorname{Spec}\mathbb{Z}$' is the $\mathbb{F}$-scheme given by the inverse system $\{X_N\}$, we denote it by $\overline{\operatorname{Spec}\mathbb{Z}}$. The set of indices is the set of square-free integers $N \geqslant 2$, and the order relation is that of divisibility.

Note that the $\mathbb{F}$-locally-ringed space $\mathcal{L}(\overline{\operatorname{Spec}\mathbb{Z}}) = \varprojlim_N X_N$ has for points $\operatorname{Spec}\mathbb{Z} \cup \{\eta\}$, with open sets of the form U or $U \cup \{\eta\}$ with U an arbitrary open set of $\operatorname{Spec}\mathbb{Z}$ (hence $\overline{\operatorname{Spec}\mathbb{Z}}$ is of 'Krull' dimension 1). Note that each X_N is compact, cf. Proposition 4.12, and hence $\mathcal{L}(\overline{\operatorname{Spec}\mathbb{Z}})$ is compact. Furthermore, the local $\mathbb{F}$-ring$\mathcal{O}_{\overline{\operatorname{Spec}\mathbb{Z}},\eta}$ is just $\mathcal{O}_{\mathbb{Q},\eta}$ (while the local $\mathbb{F}$-ring $\mathcal{O}_{X_N,\eta}$ is only A_N). For an open set $U = \operatorname{Spec}\mathbb{Z}[1/N]$ we have

$$\mathcal{O}_{\overline{\operatorname{Spec}\mathbb{Z}}}(U) = \mathbb{F}\left(\mathbb{Z}\left[\frac{1}{N}\right]\right)$$

and

$$\mathcal{O}_{\overline{\operatorname{Spec}\mathbb{Z}}}(U \cup \{\eta\}) = A_N.$$

The global sections $\mathcal{O}_{\overline{\operatorname{Spec}\mathbb{Z}}}(\overline{\operatorname{Spec}\mathbb{Z}})$ are the $\mathbb{F}$-ring $\mathbb{F}\langle\{\pm 1\} \cup \{0\}\rangle$.

Example 6.10. Similarly for a number field K, with ring of integers $\mathcal{O}_K$, and with real primes $\{\eta_i\}, i = 1, \ldots, r(= r_{\mathbb{R}} + r_{\mathbb{C}})$, let $A_{N,i} = \mathbb{F}(\mathcal{O}_K[1/N]) \cap \mathcal{O}_{K,\eta_i}$ be the $\mathbb{F}$-ring with

$$(A_{N,i})_{Y,X} = \left\{a \in \mathbb{F}\left(\mathcal{O}_K\left[\frac{1}{N}\right]\right) : |a|_{\eta_i} \leqslant 1\right\},$$

the $Y \times X$ matrices with values in $\mathcal{O}_K[1/N]$ and with η_i-operator norm bounded by 1. Let X_N be the Zariski $\mathbb{F}$-scheme obtained by gluing $\{\operatorname{Spec} A_{N,i}\}_{i=1,\ldots,r}$ and $\{\operatorname{Spec}\mathbb{F}(\mathcal{O}_K)\}$ along the common open set $\operatorname{Spec}\mathbb{F}(\mathcal{O}_K[1/N])$. For N dividing M we obtain a map $\pi_N^M : X_M \to X_N$, with $\pi_N^M|_{\operatorname{Spec} A_{M,i}}$ induced by $A_{N,i} \subseteq A_{M,i}$. The inverse system $\{X_N\}$ is the $\mathbb{F}$-scheme $\overline{\operatorname{Spec}\mathcal{O}_K}$, the compactification of $\operatorname{Spec}\mathcal{O}_K$. The space $\mathcal{L}(\overline{\operatorname{Spec}\mathcal{O}_K}) = \varprojlim_N X_N$ has for points $\operatorname{Spec}\mathcal{O}_K \cup \{\eta_i\}_{i\leqslant r}$, and open sets are of the form $U \cup \{\eta_i\}_{i\in I}$ with U open in $\operatorname{Spec}\mathcal{O}_K$, and $I \subseteq \{1, \ldots, r\}$ a subset (and hence it is of 'Krull' dimension 1). The local $\mathbb{F}$-ring $\mathcal{O}_{\overline{\operatorname{Spec}\mathcal{O}_K},\eta_i}$ is the ring $\mathcal{O}_{K,\eta_i}$. The global sections $\mathcal{O}_{\overline{\operatorname{Spec}\mathcal{O}_K}}(\overline{\operatorname{Spec}\mathcal{O}_K})$ are the $\mathbb{F}$-ring $\mathbb{F}\langle\mu_K \cup \{0\}\rangle$, μ_K the group of roots of unity in $\mathcal{O}_K^*$.

7. Fibred products

We show that the category of $\mathbb{F}$-rings has fibred sums, and we deduce that the category of (Zariski) $\mathbb{F}$-schemes has fibred products.

7.1 Fibred sums of $\mathbb{F}$-rings

THEOREM 7.1. *The category $\mathbb{F}$-$\mathcal{R}ings$ has fibred sums: given homomorphism of $\mathbb{F}$-rings $\varphi^0 : A \to B^0$, $\varphi^1 : A \to B^1$, there exists an $\mathbb{F}$-ring $B^0 \circledcirc_A B^1$, and homomorphisms $\psi^i : B^i \to B^0 \circledcirc_A B^1$, $i = 0, 1$, such that $\psi^0 \circ \varphi^0 = \psi^1 \circ \varphi^1$ and for any $\mathbb{F}$-ring C one has*

$$\operatorname{Hom}_{\mathbb{F}\text{-}\mathcal{R}ing}(B^0 \circledcirc_A B^1, C) = \operatorname{Hom}_{\mathbb{F}\text{-}\mathcal{R}ing}(B^0, C) \prod_{\operatorname{Hom}_{\mathbb{F}\text{-}\mathcal{R}ing}(A,C)} \operatorname{Hom}_{\mathbb{F}\text{-}\mathcal{R}ing}(B^1, C),$$

$$f \mapsto (f \circ \psi^0, f \circ \psi^1).$$

So given homomorphisms $f^0 : B^0 \to C, f^1 : B^1 \to C$, such that $f^0 \circ \varphi^0 = f^1 \circ \varphi^1$, there exists a unique homomorphism $f^0 \circledcirc f^1 : B^0 \circledcirc_A B^1 \to C$, such that $(f^0 \circledcirc f^1) \circ \psi^i = f^i$.

Proof. For $X, Y \in |\mathbb{F}|$, denote by $\mathcal{B}_{Y,X}$ the set of all sequences $(b_l, b_{l-1}, \ldots, b_\delta)$, where $\delta = 0, 1$, $l \geqslant \delta$, $b_{2j} \in B^0$, $b_{2j+1} \in B^1$, the range of b_j is the domain of b_{j+1}, the range of b_l is Y, the domain of b_δ is X. On $\mathcal{B}_{Y,X}$ let $\sim$ denote the equivalence relation generated by the following relations (7.1.1), (7.1.2) and (7.1.3):

$$(\ldots, b_{j+1} \circ \varphi^{j+1}(a), b_j, \ldots) \sim (\ldots, b_{j+1}, \varphi^j(a) \circ b_j, \ldots), \quad a \in A, \tag{7.1.1}$$

where we write φ^j for $\varphi^{j(\text{mod } 2)}$;

$$(\dots, b_{j+1}, f, b_{j-1}, \dots) \sim (\dots, b_{j+1} \circ f \circ b_{j-1}, \dots), \quad f \in \mathbb{F}, \tag{7.1.2}$$

and these relations include also the boundary cases

$$(f, b_l, \dots) \sim (f \circ b_l, \dots), \quad (\dots, b_\delta, f) \sim (\dots, b_\delta \circ f), \quad f \in \mathbb{F};$$

and

$$\begin{aligned}(\dots, b_{j+1} \circ ((b \otimes \mathrm{id}_{\overline{Y}}) \oplus \mathrm{id}_{\overline{\overline{Y}}}), (\mathrm{id}_X \otimes \overline{b}_j) \oplus \overline{\overline{b}}_j, b_{j-1}, \dots) \\ \sim (\dots, b_{j+1}, (\mathrm{id}_Y \otimes \overline{b}_j) \oplus \overline{\overline{b}}_j, ((b \otimes \mathrm{id}_{\overline{X}}) \oplus \mathrm{id}_{\overline{\overline{X}}}) \circ b_{j-1}, \dots), \quad \text{for } b \in B^{j+1}_{Y,X}, \overline{b}_j \in B^j_{\overline{Y},\overline{X}}, \overline{\overline{b}}_j \in B^j_{\overline{\overline{Y}},\overline{\overline{X}}}.\end{aligned} \tag{7.1.3}$$

Write $B_{Y,X} = \mathcal{B}_{Y,X}/_\sim$ for the collection of equivalence classes,

$$\mathbf{B} = \coprod_{Y,X \in |\mathbb{F}|} B_{Y,X}.$$

We are going to prove that $\mathbf{B}$ will give the required $B^0 \circledcirc_A B^1$. We define composition $\circ$ on $\mathbf{B}$

$$B_{Z,Y} \times B_{Y,X} \to B_{Z,X}$$

by

$$(b'_{l'}, \dots, b'_{\delta'})/_\sim \circ (b_l, \dots, b_\delta)/_\sim \overset{\text{def}}{=} \begin{cases} (b'_{l'}, \dots, b'_{\delta'}, b_l, \dots, b_\delta)/_\sim, & \delta' \not\equiv l \ (\text{mod } 2), \\ (b'_{l'}, \dots, b'_{\delta'} \circ b_l, \dots, b_\delta)/_\sim, & \delta' \equiv l \ (\text{mod } 2). \end{cases} \tag{7.1.4}$$

This is well defined, independent of the chosen representatives: we have to show that changing representatives to equivalent ones on the left of (7.1.4) will give an equivalent result on the right of (7.1.4). Since elements of $\mathcal{B}_{Y,X}$ are equivalent if and only if they are connected by a 'path' made up of the 'moves' (7.1.1) or (7.1.2) or (7.1.3), it is enough to check that changing representatives by one of the three moves (7.1.1) or (7.1.2) or (7.1.3) gives equivalent results. This follows by associativity of $\circ$. It then follows that $\circ$ is associative, has identities, and $\mathbf{B}$ is a category. We have functors $\psi^i : B^i \to \mathbf{B}, \psi^i(b_i) = (b_i)/_\sim$ for $b_i \in B^i$, and $\psi^0 \circ \varphi^0 = \psi^1 \circ \varphi^1$ since by (7.1.1), (7.1.2)

$$(\varphi^0(a)) \sim (\varphi^0(a), \mathrm{id}) \sim (\mathrm{id}, \varphi^1(a)) \sim (\varphi^1(a)).$$

Since the zero map $0_{Y,X}$ is in $\mathbb{F}$, and $0_{Y,X}$ composed with anything gives again a zero map, we see that $[0]$ is the initial and final object of $\mathbf{B}$.

We next define the direct sum of two elements $(b'_{l'}, \dots, b'_{\delta'})/_\sim, (b_l, \dots, b_\delta)/_\sim$ of $\mathbf{B}$. First note that by adding identities we may assume that $l' = l,\ \delta' = \delta$. We can then define:

$$(b'_l, \dots, b'_\delta)/_\sim \oplus (b_l, \dots, b_\delta)/_\sim \overset{\text{def}}{=} (b'_l \oplus b_l, \dots, b'_\delta \oplus b_\delta)/_\sim. \tag{7.1.5}$$

We claim this is well defined, independent of the chosen representatives, and again it is enough to show that if we change the representative $(b_l, \dots, b_\delta)$ by one of the three moves (7.1.1), (7.1.2), (7.1.3), we get equivalent results.

For the move (7.1.1): since $\varphi^{j+1}(\mathrm{id}) = \mathrm{id}$ we have

$$\begin{aligned}&(\dots, b'_{j+1} \oplus (b_{j+1} \circ \varphi^{j+1}(a)), b'_j \oplus b_j, \dots) \\ &= (\dots, (b'_{j+1} \oplus b_{j+1}) \circ \varphi^{j+1}(\mathrm{id} \oplus a), b'_j \oplus b_j, \dots) \\ &\overset{(7.1.1)}{\sim} (\dots, b'_{j+1} \oplus b_{j+1}, \varphi^j(\mathrm{id} \oplus a) \circ (b'_j \oplus b_j), \dots) \\ &= (\dots, b'_{j+1} \oplus b_{j+1}, b'_j \oplus (\varphi^j(a) \circ b_j), \dots).\end{aligned}$$

For the move (7.1.2):

$$
\begin{aligned}
&(b'_l\oplus \mathrm{id},\dots,b'_j\oplus b_{j+2},b'_{j-1}\oplus(b_{j+1}\circ f\circ b_{j-1}),\dots)\\
&\quad=(\dots,b'_j\oplus b_{j+2},(\mathrm{id}\oplus b_{j+1})\circ(\mathrm{id}\oplus f)\circ(b'_{j-1}\oplus b_{j-1}),\dots)\\
&\quad\overset{(7.1.2)}{\sim}(b'_l\oplus \mathrm{id},\dots,b'_j\oplus b_{j+2},\mathrm{id}\oplus b_{j+1},\mathrm{id}\oplus f,b'_{j-1}\oplus b_{j-1},\dots)\\
&\quad=(b'_l\oplus \mathrm{id},\dots,(\mathrm{id}\oplus b_{j+3})\circ(b'_{j+1}\oplus\mathrm{id}),(\mathrm{id}\oplus b_{j+2})\circ(b'_j\oplus\mathrm{id}),\mathrm{id}\oplus b_{j+1},\mathrm{id}\oplus f,b'_{j-1}\oplus b_{j-1},\dots)\\
&\quad\overset{(7.1.3)}{\sim}(b'_l\oplus \mathrm{id},\dots,(\mathrm{id}\oplus b_{j+3})\circ(b'_{j+1}\oplus\mathrm{id}),\mathrm{id}\oplus b_{j+2},\mathrm{id}\oplus b_{j+1},b'_j\oplus f,b'_{j-1}\oplus b_{j-1},\dots)\\
&\quad\overset{(7.1.3)}{\sim}(b'_l\oplus \mathrm{id},\dots,\mathrm{id}\oplus b_{j+3},\mathrm{id}\oplus b_{j+2},b'_{j+1}\oplus b_{j+1},b'_j\oplus f,b'_{j-1}\oplus b_{j-1},\dots)\overset{(7.1.3)}{\sim}\\
&\quad\dots\overset{(7.1.3)}{\sim}(b'_l\oplus b_l,\dots,b'_{j+1}\oplus b_{j+1},b'_j\oplus f,b'_{j-1}\oplus b_{j-1},\dots).
\end{aligned}
$$

For the move (7.1.3):

$$
\begin{aligned}
&(\dots,b'_{j+1}\oplus(b_{j+1}\circ((b\otimes\mathrm{id})\oplus\mathrm{id})),b'_j\oplus((\mathrm{id}\otimes\overline{b}_j)\oplus\overline{\overline{b}}_j),\dots)\\
&\quad=(\dots,(b'_{j+1}\oplus b_{j+1})\circ(\mathrm{id}\oplus(b\otimes\mathrm{id})\oplus\mathrm{id}),b'_j\oplus(\mathrm{id}\otimes\overline{b}_j)\oplus\overline{\overline{b}}_j,b'_{j-1}\oplus b_{j-1},\dots)\\
&\quad\overset{(7.1.3)}{\sim}(\dots,b'_{j+1}\oplus b_{j+1},b'_j\oplus(\mathrm{id}\otimes\overline{b}_j)\oplus\overline{\overline{b}}_j,(\mathrm{id}\oplus(b\otimes\mathrm{id})\oplus\mathrm{id})\circ(b'_{j-1}\oplus b_{j-1}),\dots)\\
&\quad=(\dots,b'_{j+1}\oplus b_{j+1},b'_j\oplus((\mathrm{id}\otimes\overline{b}_j)\oplus\overline{\overline{b}}_j),b'_{j-1}\oplus((b\otimes\mathrm{id})\oplus\mathrm{id})\circ b_{j-1},\dots).
\end{aligned}
$$

Similarly, to define the tensor product of $(b'_{l'},\dots,b'_{\delta'})/\sim$ and $(b_l,\dots,b_\delta)/\sim$, we may assume $l=l',\delta=\delta'$, and we let

$$(b'_l,\dots,b'_\delta)/\sim\otimes(b_l,\dots,b_\delta)/\sim\overset{\text{def}}{=}(b'_l\otimes b_l,\dots,b'_\delta\otimes b_\delta)/\sim. \tag{7.1.6}$$

Again this is well defined: changing the representatives $(b_l,\dots,b_\delta)$ by one of the moves (7.1.1), (7.1.2), (7.1.3) does not change the result. The proofs for (7.1.1) and (7.1.2) are exactly as for the direct sum with $\oplus$ replaced everywhere by $\otimes$.

The proof for move (7.1.3) is similar:

$$
\begin{aligned}
&(\dots,b'_{j+1}\otimes(b_{j+1}\circ((b\otimes\mathrm{id})\oplus\mathrm{id})),b'_j\otimes((\mathrm{id}\otimes\overline{b}_j)\oplus\overline{\overline{b}}_j),b'_{j-1}\otimes b_{j-1})\\
&\quad=(\dots,(b'_{j+1}\otimes b_{j+1})\circ((\mathrm{id}\otimes b\otimes\mathrm{id})\oplus(\mathrm{id}\otimes\mathrm{id})),(b'_j\otimes\mathrm{id}\otimes\overline{b}_j)\oplus(b'_j\otimes\overline{\overline{b}}_j),b'_{j-1}\otimes b_{j-1},\dots)\\
&\quad\overset{(7.1.3)}{\sim}(\dots,b'_{j+1}\otimes b_{j+1},(b'_j\otimes\mathrm{id}\otimes\overline{b}_j)\oplus(b'_j\otimes\overline{\overline{b}}_j),((\mathrm{id}\otimes b\otimes\mathrm{id})\oplus(\mathrm{id}\otimes\mathrm{id}))\circ(b'_{j-1}\otimes b_{j-1}),\dots)\\
&\quad=(\dots,b'_{j+1}\otimes b_{j+1},b'_j\otimes((\mathrm{id}\otimes\overline{b}_j)\oplus\overline{\overline{b}}_j),b'_{j-1}\otimes(((b\otimes\mathrm{id})\oplus\mathrm{id})\circ b_{j-1}),\dots).
\end{aligned}
$$

It is now straightforward to check that $\mathbf{B}$ is an $\mathbb{F}$-ring, $\psi^i:B^i\to\mathbf{B}$ are homomorphisms of $\mathbb{F}$-rings. If $f^i:B^i\to C$ are homomorphisms of $\mathbb{F}$-rings such that $f^0\circ\varphi^0=f^1\circ\varphi^1$, we define $f^0\circledcirc f^1:\mathbf{B}\to C$ by

$$f^0\circledcirc f^1((b_l,\dots,b_\delta)/\sim)=f^l(b_l)\circ\cdots\circ f^\delta(b_\delta),\qquad f^j=f^{j(\mathrm{mod}\ 2)}. \tag{7.1.7}$$

It is well defined: changing the representatives $(b_l,\dots,b_\delta)$ by one of the moves (7.1.1), (7.1.2), (7.1.3) does not change the result of (7.1.7). For example, $f^0\circledcirc f^1$ applied to both sides of (7.1.3) gives the same result

$$\cdots\circ f^{j+1}(b_{j+1})\circ[(f^{j+1}(b)\otimes f^j(\overline{b}_j))\oplus f^j(\overline{\overline{b}}_j)]\circ f^{j-1}(b_{j-1})\circ\cdots.$$

The map $f^0\circledcirc f^1$ is a homomorphism of $\mathbb{F}$-rings, and clearly it is the unique homomorphism such that

$$(f^0\circledcirc f^1)\circ\psi^i=f^i,\quad\text{i.e. } f^0\circledcirc f^1((b_i)/\sim)=f^i(b_i)\text{ for } b_i\in B^i.$$

We write $B^0 \circledcirc_A B^1$ for $\mathbf{B}$, $b_l \circledcirc \cdots \circledcirc b_\delta$ for the equivalence class $(b_l \dots, b_\delta)/_\sim$. Thus the arithmetic in $B^0 \circledcirc_A B^1$ is governed by the three moves,

$$\begin{aligned}
&\text{(i)} \quad \cdots (b_{j+1} \circ \varphi^{j+1}(a)) \circledcirc b_j \cdots = \cdots b_{j+1} \circledcirc (\varphi^j(a) \circ b_j) \cdots \quad a \in A,\\
&\text{(ii)} \quad \cdots b_{j+1} \circledcirc f \circledcirc b_{j-1} \cdots = \cdots \circledcirc (b_{j+1} \circ f \circ b_{j-1}) \circledcirc \cdots \quad f \in \mathbb{F}, \qquad (7.1.8)\\
&\text{(iii)} \quad \cdots b_{j+1} \circ ((b \otimes \mathrm{id}) \oplus \mathrm{id}) \circledcirc ((\mathrm{id} \otimes \overline{b}_j) \oplus \overline{\overline{b}}_j) \circledcirc b_{j-1} \cdots = \cdots b_{j+1} \circledcirc ((\mathrm{id} \otimes \overline{b}_j) \oplus \overline{\overline{b}}_j) \circledcirc ((b \otimes \mathrm{id}) \oplus \mathrm{id}) \circ b_{j-1} \cdots .
\end{aligned}$$

This completes the proof of Theorem 7.1. □

Remark 7.1.9. Note that we can consider B^i as an A-module via φ^i, hence we have the tensor product $B^0 \otimes_A B^1$; cf. Lemma 3.20. We have a map

$$B^0 \otimes_A B^1 \to B^0 \circledcirc_A B^1, \quad b_0 \otimes b_1 \mapsto \psi^0(b_0) \otimes \psi^1(b_1) = (b_0 \otimes \mathrm{id}) \circledcirc (\mathrm{id} \otimes b_1),$$

but unlike the case of commutative rings this is not an isomorphism.

Remark 7.1.10. A similar construction gives the fibred sum of $\mathbb{F}^{\pm}$-rings or $\overline{\mathbb{F}}$-rings. If the $\mathbb{F}$ or $\mathbb{F}^{\pm}$ or $\overline{\mathbb{F}}$-rings A, B^0, B^1 have compatible involutions, we get a well-defined involution on $B^0 \circledcirc B^1$ by

$$((b_l, \dots, b_\delta)/_\sim)^{\mathrm{t}} = (b_\delta^{\mathrm{t}}, \dots, b_l^{\mathrm{t}})/_\sim$$

(note that the moves (7.1.1), (7.1.2), (7.1.3) are all symmetric with respect to $(\cdots)^{\mathrm{t}}$). Hence also the categories $\mathbb{F}^{\pm}$-$\mathcal{R}ings$, $\mathbb{F}^{\pm,t}$-$\mathcal{R}ings$, $\overline{\mathbb{F}}$-$\mathcal{R}ings$, $\overline{\mathbb{F}}^{\mathrm{t}}$-$\mathcal{R}ings$ have fibred sums. Similarly if A, B^0, B^1 have compatible $\mathbb{F}^{\lambda}$-structure, we get such a structure on $B^0 \circledcirc B^1$ via

$$\wedge^d((b_l, \dots, b_\delta)/_\sim) = (\wedge^d b_l, \dots \wedge^d b_\delta)/_\sim.$$

7.2 Fibred product of $\mathbb{F}$-schemes, the case of $\overline{\mathbf{Spec}\,\mathbb{Z}} \times \overline{\mathbf{Spec}\,\mathbb{Z}}$

As a corollary of Theorem 7.1 we get the following results.

Theorem 7.2. *The category $\mathcal{Z}ar$-$\mathbb{F}$-$\mathcal{S}ch$ of Zariski $\mathbb{F}$-schemes has fibred products. Given maps in $\mathcal{Z}ar$-$\mathbb{F}$-$\mathcal{S}ch$ $f^0 : X^0 \to Y, f^1 : X^1 \to Y$, there exist a Zariski $\mathbb{F}$-scheme*

$$X^0 \prod_Y X^1$$

and maps

$$\pi^j : X^0 \prod_Y X^1 \to X^j, j = 0, 1,$$

such that $f^0 \circ \pi^0 = f^1 \circ \pi^1$, and for any Zariski $\mathbb{F}$-scheme Z, and any maps $g^j : Z \to X^j$, $j = 0, 1$, such that $f^0 \circ g^0 = f^1 \circ g^1$ there exists a unique map

$$g^0 \ \Pi \ g^1 : Z \to X^0 \prod_Y X^1$$

with $\pi^j \circ (g^0 \ \Pi \ g^1) = g^j$ for $j = 0, 1$.

Proof. Write $Y = \bigcup_i \mathrm{Spec}(A_i), (f^j)^{-1}(\mathrm{Spec}(A_i)) = \bigcup_k \mathrm{Spec}(B^j_{i,k}), j = 0, 1$. Then $X^0 \prod_Y X^1$ is obtained by gluing $\{\mathrm{Spec}(B^0_{i,k_0} \circledcirc_{A_i} B^1_{i,k_1})\}_{i,k_0,k_1}$. For details, see, e.g., [Hart77, Theorem 3.3, p. 87]. □

As a corollary we obtain the following theorem.

Theorem 7.3. *The category $\mathbb{F}$-$\mathcal{S}ch$ of $\mathbb{F}$-schemes has fibred products.*

Proof. Given maps as in Definition 6.8,

$$\varphi = \{\varphi_i^j\}_{j \geqslant \sigma(i)} : X = \{X_j\}_J \to Y = \{Y_i\}_I, \quad \varphi' = \{\varphi_i'^{j'}\}_{j' \geqslant \sigma'(i)} : X' = \{X'_{j'}\}_{J'} \to Y,$$

the fibred product in $\mathbb{F}$-$\mathcal{S}ch$ is clearly given by the inverse system $\{X_j \times_{Y_i} X'_{j'}\}$, the indexing set is $\{(j, j', i) \in J \times J' \times I \mid j \geqslant \sigma(i), j' \geqslant \sigma'(i)\}$. $\square$

As an important example we have the arithmetical surface compactified

$$\overline{\operatorname{Spec}\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{F}} \overline{\operatorname{Spec}\mathbb{Z}},$$

which is represented by $\{X_N \times_{\operatorname{Spec}\mathbb{F}} X_M\}$ with indexing set $\{(N, M) \in \mathbb{N} \times \mathbb{N} | N, M \text{ square-free}\}$, and with

$$X_N = \operatorname{Spec}\mathbb{F}(\mathbb{Z}) \coprod_{\operatorname{Spec}(\mathbb{F}(\mathbb{Z}[1/N]))} \operatorname{Spec}\left(\mathbb{F}\left(\mathbb{Z}\left[\frac{1}{N}\right]\right) \cap \mathcal{O}_{\mathbb{Q},\eta}\right),$$

as in Example 6.9. The $\mathbb{F}$-scheme

$$\overline{\operatorname{Spec}\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{F}} \overline{\operatorname{Spec}\mathbb{Z}}$$

contains the open dense subset (which is a Zariski $\mathbb{F}$-scheme):

$$\operatorname{Spec}\mathbb{F}(\mathbb{Z}) \times_{\operatorname{Spec}\mathbb{F}} \operatorname{Spec}\mathbb{F}(\mathbb{Z}) = \operatorname{Spec}\mathbb{F}(\mathbb{Z}) \circledcirc_{\mathbb{F}} \mathbb{F}(\mathbb{Z}).$$

A basis for neighborhoods of (p, η) is given by

$$\mathbb{F}\left(\mathbb{Z}\left[\frac{1}{N}\right]\right) \circledcirc_{\mathbb{F}} A_M,$$

where p does not divide N, and M is arbitrary (for the definition of A_M see Example 6.9).

Similarly, for any number field K we have the compactified surface

$$\overline{\operatorname{Spec}\mathcal{O}_K} \times_{\operatorname{Spec}\mathbb{F}} \overline{\operatorname{Spec}\mathcal{O}_K}.$$

It contains the open dense subset

$$\operatorname{Spec}\mathbb{F}(\mathcal{O}_K) \circledcirc_{\mathbb{F}} \mathbb{F}(\mathcal{O}_K).$$

8. Monoids

Since we defined $\mathbb{F}$-rings to be categories, and therefore have the operation of composition $\circ$, the resulting product $\circledcirc$ (actually the sum in $\mathbb{F}$-rings) is very complicated, resembling amalgams of groups, and it is difficult to calculate for specific examples. Unlike the classical theory, it does not reduce to the tensor product $\otimes$, cf. Remark 7.1.9. Similarly, the base change functor for a map of $\mathbb{F}$-rings $\varphi : A \to B, \varphi_* : A\text{-}\mathcal{M}od \to B\text{-}\mathcal{M}od, \varphi_* M = M_B$, is complicated, and again does not reduce to the tensor product $B \otimes_A M$, as it does in the classical theory, cf. Remark 3.23.11. In this section we shall give a softer theory that overcomes these difficulties by replacing everywhere $\mathbb{F}$-rings by F-monoids, which have composition $\circ$ only with elements of a fixed base $\mathbb{F}$-ring F. Since the constructions are repetitions of the constructions of earlier sections we will be more brief. We repeat the story up to the definition of Zariski F-monoid scheme. One can define F-monoid schemes to be the category of pro-objects of Zariski F-monoid-schemes, as we did in the context of $\mathbb{F}$-rings, cf. Definition 6.8, but one does not do it for several reasons. For instance, the basic Example 6.9 of the compactification of $\operatorname{Spec}\mathbb{Z}$, and $\operatorname{Spec}\mathcal{O}_K$, does not go through in the context of F-monoids. As explained at the beginning of § 1, working with $\overline{\operatorname{Spec}\mathbb{Z}} = \operatorname{Spec}\mathbb{Z} \cup \{\eta\}$ dictates taking $F = \mathbb{F}$, and since $\mathbb{F}$ does not have the vector $(1, 1)$ addition is lost, cf. (2.13.4): the functor $A \mapsto \mathbb{F}(A)$, from $\mathcal{R}ings$ to $\mathbb{F}$-monoids, is not fully faithful (compare with Example 1 of § 2.3), and when we view $\mathbb{F}(A)$ as $\mathbb{F}$-monoid $\operatorname{Spec}\mathbb{F}(A)$ does not reduce to $\operatorname{Spec} A$. Thus for example viewing $\mathbb{F}(\mathbb{Z})$ as $\mathbb{F}$-monoid its spectrum has the cardinality of the continuum: for any set S of primes we have the prime $\mathfrak{p}_S \in \operatorname{Spec}\mathbb{F}(\mathbb{Z}), \mathfrak{p}_S = \bigcup_{p \in S} p \cdot \mathbb{Z}$.

On the other hand, using the localized version (Definition 8.20) of $\mathcal{O}_X$-monoids, one can use $\mathcal{O}_X$-monoids to define the flat (and étale) Grothendieck topologies on a given (Zariski) $\mathbb{F}$-scheme; see [TV05].

8.1 F-monoids

Definition 8.1. An F-monoid is a monoid object in the category of F-modules.

Thus an F-monoid $A = \{A_{Y,X}\}_{Y,X\in|\mathbb{F}|}$ has the operations

$$F\text{-composition:} F_{Y',Y} \times A_{Y,X} \times F_{X,X'} \to A_{Y',X'}, \quad (f, a, f') \mapsto f \circ a \circ f', \tag{8.1.1}$$

$$F\text{-tensor product:} F_{Y_0,X_0} \times A_{Y_1,X_1} \to A_{Y_0\otimes Y_1,X_0\otimes X_1}, \quad (f, a) \mapsto f \otimes a, \tag{8.1.2}$$

$$\text{direct sum:} A_{Y_0,X_0} \times A_{Y_1,X_1} \to A_{Y_0\oplus Y_1,X_0\oplus X_1}, \quad (a_0, a_1) \mapsto a_0 \oplus a_1, \tag{8.1.3}$$

satisfying the F-module axioms (3.1.4)–(3.1.15), together with the additional operation of 'tensor' product:

$$A_{Y_0,X_0} \times A_{Y_1,X_1} \to A_{Y_0\otimes Y_1,X_0\otimes X_1}, \quad (a_0, a_1) \mapsto a_0 \otimes a_1. \tag{8.1.4}$$

This operation is bilinear over F:

$$(a_0 \oplus a_0') \otimes a_1 = (a_0 \otimes a_1) \oplus (a_0' \otimes a_1), a_0 \otimes (a_1 \oplus a_1') = (a_0 \otimes a_1) \oplus (a_0 \otimes a_1'), \quad a_i, a_i' \in A, \tag{8.1.5}$$

$$(f_0 \circ a_0 \circ f_0') \otimes (f_1 \circ a_1 \circ f_1') = (f_0 \otimes f_1) \circ (a_0 \otimes a_1) \circ (f_0' \otimes f_1'), \quad f_i, f_i' \in F, a \in A, \tag{8.1.6}$$

$$(f \otimes a_0) \otimes a_1 = f \otimes (a_0 \otimes a_1) = a_0 \otimes (f \otimes a_1), \quad f \in F, a_i \in A. \tag{8.1.7}$$

It is also associative:

$$a_0 \otimes (a_1 \otimes a_2) = (a_0 \otimes a_1) \otimes a_2, \quad a_i \in A, \tag{8.1.8}$$

commutative:

$$a_0 \otimes a_1 = a_1 \otimes a_0, \quad a_i \in A, \tag{8.1.9}$$

and unital:

$$\text{there exists (a unique) } 1 = 1_A \in A_{[1],[1]} \quad \text{with } 1 \otimes a = a, \text{ for all } a \in A. \tag{8.1.10}$$

A map of F-monoids $\varphi : A \to A'$ is a map of F-modules respecting $\otimes$ and 1, i.e. we have

$$\varphi_{Y,X} : A_{Y,X} \to A'_{Y,X}, \quad Y, X \in |\mathbb{F}|, \tag{8.2}$$

satisfying

$$\varphi(f \circ a \circ f') = f \circ \varphi(a) \circ f', \quad f, f' \in F, a \in A, \tag{8.2.1}$$

$$\varphi(f \otimes a) = f \otimes \varphi(a), \quad f \in F, a \in A, \tag{8.2.2}$$

$$\varphi(a_0 \oplus a_1) = \varphi(a_0) \oplus \varphi(a_1), \quad a_i \in A, \tag{8.2.3}$$

$$\varphi(a_0 \otimes a_1) = \varphi(a_0) \otimes \varphi(a_1), \quad a_i \in A, \tag{8.2.4}$$

$$\varphi(1_A) = 1_{A'}. \tag{8.2.5}$$

Thus we have the category F-$\mathcal{M}on$ of F-monoids; it is a subcategory of F-$\mathcal{M}od$. The F-module F itself, considered as an F-monoid, is the initial object of F-$\mathcal{M}on$: for any F-monoid A we have the map $F \to A, f \mapsto f \otimes 1_A$.

All our examples of $\mathbb{F}$-rings are of course $\mathbb{F}$-monoids. Note that the functor from commutative rings to $\mathbb{F}$-monoids, $A \mapsto \mathbb{F}(A)$, is faithful, but is not fully faithful: there are more maps $\mathbb{F}(A) \to \mathbb{F}(B)$ in $\mathbb{F}$-$\mathcal{M}on$ than there are maps $A \to B$ in $\mathcal{R}ing$. On the other hand, we can consider this functor as taking values in $\mathbb{F}(\mathbb{Z})$-$\mathcal{M}on$, or $\mathbb{S}$-$\mathcal{M}on$, or $\mathbb{S}^{\text{op}}$-$\mathcal{M}on$, cf. Example 3 (2.15.1), in which case it is fully faithful.

The category of F-monoids has fibred product $A^0 \prod_B A^1$, arbitrary products $\prod_i A^i$, and arbitrary inverse limits $\varprojlim_i A^i$.

Definition 8.3. For an F-monoid A, an equivalence ideal $\mathcal{E}$ is a collection of subsets

$$\mathcal{E} = \coprod_{Y,X \in |\mathbb{F}|} \mathcal{E}_{Y,X}, \quad \mathcal{E}_{Y,X} \subset A_{Y,X} \times A_{Y,X}, \tag{8.3.1}$$

such that $\mathcal{E}_{Y,X}$ is an equivalence relation on $A_{Y,X}$, and $\mathcal{E}$ is an A-submodule of $A \prod A$, i.e. it is an F-submodule:

$$(a_0, a_1) \in \mathcal{E}_{Y,X}, \quad f \in F_{Y',Y}, \quad f' \in F_{X,X'} \Rightarrow (f \circ a_0 \circ f', f \circ a_1 \circ f') \in \mathcal{E}_{Y',X'}, \tag{8.3.2}$$

$$(a, a') \in \mathcal{E}_{Y,X}, \quad f \in F_{Y_0,X_0} \Rightarrow (f \otimes a, f \otimes a') \in \mathcal{E}_{Y_0 \otimes Y, X_0 \otimes X}, \tag{8.3.3}$$

$$(a_0, a'_0) \in \mathcal{E}_{Y_0,X_0}, \quad (a_1, a'_1) \in \mathcal{E}_{Y_1,X_1} \Rightarrow (a_0 \oplus a_1, a'_0 \oplus a'_1) \in \mathcal{E}_{Y_0 \oplus Y_1, X_0 \oplus X_1}, \tag{8.3.4}$$

and moreover it is closed under A-tensor product:

$$(a, a') \in \mathcal{E}_{Y,X}, \quad a_0 \in A_{Y_0,X_0} \Rightarrow (a_0 \otimes a, a_0 \otimes a') \in \mathcal{E}_{Y_0 \otimes Y, X_0 \otimes X}. \tag{8.3.5}$$

(Note that (8.3.3) follows from (8.3.5) by taking $a_0 = f \otimes 1_A$.)

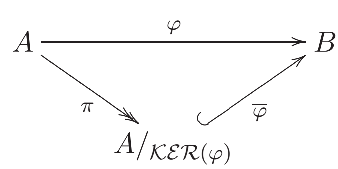
Given an equivalence ideal $\mathcal{E}$ of A we can form the quotient

$$A/\mathcal{E} = \coprod_{Y,X \in |\mathbb{F}|} A_{Y,X}/\mathcal{E}_{Y,X},$$

which has the structure of F-monoid such that the canonical projection $\pi : A \to A/\mathcal{E}$ is a homomorphism of F-monoids. For a map $\varphi : A \to B$ in F-$\mathcal{M}on$ we have the equivalence ideal of A, $\mathcal{KER}(\varphi) = \coprod_{Y,X} \mathcal{KER}(\varphi)_{Y,X}$,

$$\mathcal{KER}(\varphi)_{Y,X} = \{(a, a') \in A_{Y,X} \times A_{Y,X} \mid \varphi(a) = \varphi(a')\}, \tag{8.4.1}$$

and φ factorizes as epimorphism π followed by an injection $\overline{\varphi}$, as in the diagram.



(8.4.2)

8.2 Modules over an *F*-monoid

Definition 8.5. For an F-monoid A, an A-module M is an F-module together with maps

$$A_{Y_0,X_0} \times M_{Y_1,X_1} \to M_{Y_0 \otimes Y_1, X_0 \otimes X_1}, \quad (a, m) \mapsto a \otimes m,$$

which are bilinear over F: for $f, f_i, f'_i \in F, a, a_i \in A, m, m_i \in M$,

$$(f_1 \circ a \circ f'_1) \otimes (f_2 \circ m \circ f'_2) = (f_1 \otimes f_2) \circ (a \otimes m) \circ (f'_1 \otimes f'_2), \tag{8.5.1}$$

$$(f \otimes a) \otimes m = f \otimes (a \otimes m) = a \otimes (f \otimes m), \tag{8.5.2}$$

$$(a_0 \oplus a_1) \otimes m = (a_0 \otimes m) \oplus (a_1 \otimes m), a \otimes (m_0 \oplus m_1) = (a \otimes m_0) \oplus (a \otimes m_1), \tag{8.5.3}$$

associative:

$$a_0 \otimes (a_1 \otimes m) = (a_0 \otimes a_1) \otimes m, \tag{8.5.4}$$

and unital:

$$1_A \otimes m = m. \tag{8.5.5}$$

(Note that the first equality in (8.5.2) follows from (8.5.4) and (8.5.5) by taking $a_0 = f \otimes 1_A$.)

The above defines a 'left' A-module; we can similarly define a 'right' A-module, but these notions are equivalent: any left A-module M can be made into a right A-module by putting

$$m \otimes a \overset{\text{def}}{=} c^*_{W,Y} \circ (a \otimes m) \circ c^*_{X,Z} \text{ for } m \in M_{Y,X}, a \in A_{W,Z} (c^* \text{ as in } (1.16)). \tag{8.5.6}$$

Similarly any right A-module can be made into a left A-module; hence we shall not distinguish between left and right A-modules.

Definition 8.6. A map of A-modules $\varphi : M \to M'$ is a map of F-modules respecting $\otimes$, i.e. we have

$$\varphi_{Y,X} : M_{Y,X} \to M'_{Y,X}, \quad Y, X \in |\mathbb{F}|, \tag{8.6.1}$$

$$\varphi(f \circ m \circ f') = f \circ \varphi(m) \circ f', \quad f, f' \in F, \tag{8.6.2}$$

$$\varphi(f \otimes m) = f \otimes \varphi(m), \quad f \in F, \tag{8.6.3}$$

$$\varphi(m_0 \oplus m_1) = \varphi(m_0) \oplus \varphi(m_1), \quad m_i \in M, \tag{8.6.4}$$

$$\varphi(a \otimes m) = a \otimes \varphi(m), \quad a \in A. \tag{8.6.5}$$

(Note that (8.6.3) follows from (8.6.5).)

Thus we have a category A_F-$\mathcal{M}od$ of A-modules; it is a subcategory of F-$\mathcal{M}od$. Note that if A is an $\mathbb{F}$-ring, we can consider $A_{\mathbb{F}} := A$ as an $\mathbb{F}$-monoid, and A-$\mathcal{M}od \subseteq A_{\mathbb{F}}$-$\mathcal{M}od$. On the other hand we can consider $A_A := A$ as an A-monoid, and then A-$\mathcal{M}od = A_A$-$\mathcal{M}od$. The category of A-modules has the initial and final object $0 = \{0_{Y,X}\}_{Y,X \in |\mathbb{F}|}$. One defines the notion of an A-submodule $M_0 \subseteq M$ in the evident way; an A-submodule of A is called an ideal.

The category A_F-$\mathcal{M}od$ has fibred product

$$\left(M^0 \prod_M M^1\right)_{Y,X} = \{(m_0, m_1) \in M^0_{Y,X} \times M^1_{Y,X} \mid \varphi_0(m_0) = \varphi_1(m_1)\} \quad \text{for } \varphi_i : M^i \to M; \tag{8.7.1}$$

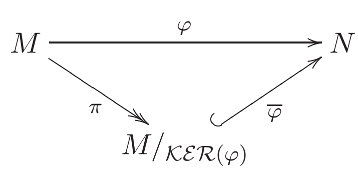
it has arbitrary products $\prod_i M^i$, and arbitrary inverse limits $\varprojlim_i M^i$,

$$\varprojlim_i M^i = \left\{(m_i) \in \prod_i M^i \mid \varphi_{i',i}(m_i) = m_{i'}\right\}. \tag{8.7.2}$$

For an F-monoid A, and A-module M, an equivalence A-module of M is an A-submodule $\mathcal{E}$ of $M \prod M$, such that $\mathcal{E}_{Y,X}$ is an equivalence relation on $M_{Y,X}$. We can form the quotient

$$M/\mathcal{E} = \coprod_{Y,X \in |\mathbb{F}|} (M/\mathcal{E})_{Y,X}, \quad (M/\mathcal{E})_{Y,X} = M_{Y,X}/\mathcal{E}_{Y,X}, \tag{8.8.1}$$

and it has the structure of an A-module such that the canonical projection $\pi : M \to M/\mathcal{E}$ is a homomorphism of A-modules. For a map $\varphi : M \to N$ in A-$\mathcal{M}od$ we have the equivalence A-module of M, $\mathcal{KER}(\varphi) = M \prod_N M$, and φ factorizes as an epimorphism π followed by an injection $\overline{\varphi}$, as in the diagram.



(8.8.2)

For a map $\varphi : M \to N$ in A-$\mathcal{M}od$ we have the A-submodule M,

$$\varphi^{-1}(0) = \{m \in M \mid \varphi(m) = 0\}, \tag{8.9.1}$$

which is the kernel of φ in A-$\mathcal{M}od$; we have also the A-submodule of N,

$$\varphi(M) = \{\varphi(m) \mid m \in M\}. \tag{8.9.2}$$

For an equivalence A-module $\mathcal{E}$ of M, we let $Z(\mathcal{E})$ denote $\pi^{-1}(0)$, $\pi : M \to M/\mathcal{E}$ the projection

$$Z(\mathcal{E}) = \pi^{-1}(0) = \{m \in M \mid (m, 0) \in \mathcal{E}\}. \tag{8.9.3}$$

For an A-submodule $M_0 \subseteq M$ we let $E(M_0)$ be the equivalence A-module of M generated by $M_0 \times \{0\}$. We write M/M_0 for $M/E(M_0)$. For a map $\varphi : M \to N$ in A-$\mathcal{M}od$ we have its cokernel

$$\text{Coker } \varphi = N/\varphi(M) = N/E(\varphi(M)). \tag{8.9.4}$$

We have for A-submodules M_i of M, and equivalence A-modules $\mathcal{E}_i$ of M,

$$M_0 \subseteq M_1 \Rightarrow E(M_0) \subseteq E(M_1), \tag{8.10.1}$$
$$\mathcal{E}_0 \subseteq \mathcal{E}_1 \Rightarrow Z(\mathcal{E}_0) \subseteq Z(\mathcal{E}_1), \tag{8.10.2}$$
$$M_0 \subseteq ZE(M_0), \tag{8.10.3}$$
$$\mathcal{E}_0 \supseteq EZ(\mathcal{E}_0), \tag{8.10.4}$$
$$E(M_0) = EZE(M_0), \tag{8.10.5}$$
$$Z(\mathcal{E}_0) = ZEZ(\mathcal{E}_0). \tag{8.10.6}$$

We have a bijection between the set

$$\left\{ Z(\mathcal{E}_0) \;\middle|\; \mathcal{E}_0 \subseteq M \prod M \right\} \equiv \{M_0 \subseteq M \mid M_0 = ZE(M_0)\}$$

and the set

$$\{E(M_0) \mid M_0 \subseteq M\} \equiv \{\mathcal{E}_0 \mid \mathcal{E}_0 = EZ(\mathcal{E}_0)\}$$

given by

$$M_0 \longmapsto E(M_0), \quad Z(\mathcal{E}_0) \longleftarrow \mathcal{E}_0. \tag{8.10.7}$$

Lemma 3.13 and its Corollary 3.14 remain valid: the equivalence A-module of M generated by the A-submodule $M_0 \subseteq M$, $\mathcal{E} = E(M_0)$, can be described explicitly:

$$\begin{aligned} \mathcal{E}_{Y,X} = \{(m, m') \in M_{Y,X} \times M_{Y,X} \mid \exists \text{ path } m = m_0, m_1, \ldots, m_l = m', \\ \text{with } \{m_j, m_{j+1}\} \text{ of the form } \{f_j \circ (\widetilde{m}_j \oplus n_j) \circ f'_j, f_j \circ (\widetilde{m}_j \oplus 0) \circ f'_j\}, \\ f_j, f'_j \in F, \widetilde{m}_j \in M, n_j \in M_0\}. \end{aligned} \tag{8.11.1}$$

We have $M_0 = ZE(M_0)$ if and only if for all $f, f' \in F, m \in M, m_0 \in M_0$,

$$f \circ (m \oplus m_0) \circ f' \in M_0 \Leftrightarrow f \circ (m \oplus 0) \circ f' \in M_0. \tag{8.11.2}$$

For A-submodules M_i of M, we have their intersection $\bigcap_i M_i \subseteq M$, and their sum

$$\sum M_i = \left\{ f \circ \left(\bigoplus_i m_i \right) \circ f' \;\middle|\; f, f' \in F, m_i \in M_i \right\}. \tag{8.12.1}$$

The A-submodule generated by a subset $\{m_i\}_{i \in I} \subseteq M$ is described explicitly as

$$\sum A m_i \stackrel{\text{def}}{=} \left\{ f \circ \left(\bigoplus_i (a_i \otimes m_i) \right) \circ f' \;\middle|\; f, f' \in F, a_i \in A \right\}. \tag{8.12.2}$$

Given an A-module M, and an ideal $\mathfrak{a} \subseteq A$, we have the A-submodule of M,

$$\mathfrak{a} \cdot M = \left\{ f \circ \left(\bigoplus_i (a_i \otimes m_i) \right) \circ f' \;\middle|\; f, f' \in F, a_i \in \mathfrak{a} \right\}. \tag{8.12.3}$$

Given A-submodules $M_0, M_1 \subseteq M$, we have the ideal of A,

$$(M_0 : M_1) = \{a \in A \mid a \otimes m \in M_0 \; \forall m \in M_1\}. \tag{8.12.4}$$

8.3 Functorial operations on modules

Sums and direct limits. Given a collection of A-modules $\{M_i\}_{i\in I}$, we have their sum in the category A-$\mathcal{M}od$ (cf. Proposition 3.17):

$$\Big(\coprod_{i\in I} M_i\Big)_{Y,X} = \{(f, \{m_i\}_{i\in I_0}, f') \mid f \in F_{Y,\bigoplus_{i\in I_0} Y_i}, f' \in F_{\bigoplus_{i\in I_0} X_i, X},$$
$$m_i \in (M_i)_{Y_i,X_i}, I_0 \subseteq I \text{ a finite subset}\}/\sim. \tag{8.13.1}$$

Here $\sim$ is the equivalence relation generated by

$$\begin{aligned}\Big(f\circ\Big(\bigoplus_{i\in I_0} f_i\Big), \{m_i\}_{i\in I_0}, \Big(\bigoplus_{i\in I_0} f_i'\Big)\circ f'\Big) &\sim (f, \{f_i\circ m_i\circ f_i'\}_{i\in I_0}, f'),\\ (f,\{m_i\}_{i\in I_0}, f') &\sim (f, \{m_i\}_{i\in I_0}\cup\{\mathrm{id}_{[0]}\}, f').\end{aligned} \tag{8.13.2}$$

The structure of A-module on $\coprod_{i\in I} M_i$ is defined by:

F-composition

$$g\circ (f,\{m_i\},f')/_\sim \circ g' = (g\circ f, \{m_i\}, f'\circ g')/_\sim \tag{8.13.3}$$

sum

$$(f_0, \{m_i\}_{i\in I_0}, f_0')/_\sim \oplus (f_1, \{m_i'\}_{i\in I_1}, f_1') = (f_0\oplus f_1, \{\widetilde{m}_i\}_{i\in I_0\cup I_1}, f_0'\oplus f_1')/_\sim \tag{8.13.4}$$

with $\widetilde{m}_i = m_i$ (respectively $m_i', m_i\oplus m_i'$) for $i\in I_0\setminus I_1$ (respectively $i\in I_1\setminus I_0, i\in I_0\cap I_1$); and A-tensor product

$$a\otimes (f,\{m_i\}_{i\in I_0}, f')/_\sim = (\mathrm{id}_W\otimes f, \{a\otimes m_i\}_{i\in I_0}, \mathrm{id}_Z\otimes f')/_\sim, \quad a\in A_{W,Z}. \tag{8.13.5}$$

Note that for $g\in F_{W,Z}$:

$$\begin{aligned} g\otimes (f,\{m_i\}_{i\in I_0}, f')/_\sim &= (g\otimes 1_A)\otimes (f,\{m_i\}_{i\in I_0}, f')/_\sim\\ &= (\mathrm{id}_W\otimes f, \{g\otimes m_i\}_{i\in I_0}, \mathrm{id}_Z\otimes f')/_\sim\\ &= (\mathrm{id}_W\otimes f, \{(g\otimes \mathrm{id}_{Y_i})\circ(\mathrm{id}_Z\otimes m_i)\}_{i\in I_0}, \mathrm{id}_Z\otimes f')/_\sim,\\ \textbf{or } &= (\mathrm{id}_W\otimes f, \{(\mathrm{id}_W\otimes m_i)\circ(g\otimes \mathrm{id}_{X_i})\}_{i\in I_0}, \mathrm{id}_Z\otimes f')/_\sim\\ &= \Big((\mathrm{id}_W\otimes f)\circ\bigoplus_{I_0}(g\otimes \mathrm{id}_{Y_i}), \{\mathrm{id}_Z\otimes m_i\}_{i\in I_0}, \mathrm{id}_Z\otimes f'\Big)\Big/_\sim,\\ \textbf{or } &= \Big(\mathrm{id}_W\otimes f, \{\mathrm{id}_W\otimes m_i\}_{i\in I_0}, \bigoplus_{I_0}(g\otimes \mathrm{id}_{X_i})\circ(\mathrm{id}_Z\otimes f')\Big)\Big/_\sim\\ &= (g\otimes f, \{\mathrm{id}_Z\otimes m_i\}_{i\in I_0}, \mathrm{id}_Z\otimes f')/_\sim,\\ \textbf{or } &= (\mathrm{id}_W\otimes f, \{\mathrm{id}_W\otimes m_i\}_{i\in I_0}, g\otimes f')/_\sim. \end{aligned} \tag{8.13.6}$$

We write $f\circ(\bigoplus_{I_0} m_i)\circ f'$ for $(f,\{m_i\}_{i\in I_0}, f')/_\sim$.

Given a functor $i\mapsto M_i$ from a small category I to A-$\mathcal{M}od$, we have the direct limit

$$\varinjlim_I M_i = \coprod_I M_i/\mathcal{E}, \tag{8.13.7}$$

where $\mathcal{E}$ is the equivalence A-module of $\coprod_I M_i$ generated by

$$\{(\mathrm{id}_Y\circ(m_i)\circ \mathrm{id}_X, \mathrm{id}_Y\circ(\varphi_{ji}(m_i))\circ \mathrm{id}_X) \mid m_i\in M_i, \varphi_{ji}: M_i\to M_j\}. \tag{8.13.8}$$

In particular we have the push-out $M_0\coprod_M M_1$ for homomorphisms $\psi_i: M\to M_i$,

$$M_0\coprod_M M_1 = \Big(M_0\coprod M_1\Big)\Big/\mathcal{E}, \tag{8.13.9}$$

where $\mathcal{E}$ is the equivalence A-module of $M_0 \coprod M_1$ generated by

$$\{(f \circ (\psi_0(m)) \circ f', f \circ (\psi_1(m)) \circ f') \mid m \in M\}. \tag{8.13.10}$$

Free modules. Let $\mathfrak{s}$ be a formal symbol representing a map $X_0 \to Y_0, X_0, Y_0 \in |\mathbb{F}|$. The free A-module on $\mathfrak{s}$ is

$$(A \cdot \mathfrak{s})_{Y,X} = \{(f, a, f') \mid f \in F_{Y, W \oplus Y_0}, f' \in F_{Z \oplus X_0, X}, a \in A_{W,Z}\}/\sim \tag{8.14.1}$$

where $\sim$ is the equivalence relation generated by

$$(f \circ (g \otimes \mathrm{id}_{Y_0}), a, (g' \otimes \mathrm{id}_{X_0}) \circ f') \sim (f, g \circ a \circ g', f'), \quad f, f', g, g' \in F. \tag{8.14.2}$$

The structure of A-module on $A \cdot \mathfrak{s}$ is defined by

$$g \circ (f, a, f')/_\sim \circ g' = (g \circ f, a, f' \circ g')/_\sim, \quad f, f', g, g' \in F, \tag{8.14.3}$$

$$(f_0, a_0, f_0')/_\sim \oplus (f_1, a_1, f_1')/_\sim = (f_0 \oplus f_0', a_0 \oplus a_1, f_0' \oplus f_1'), \quad f_i, f_i' \in F, \tag{8.14.4}$$

$$a_0 \otimes (f, a, f')/_\sim = (\mathrm{id}_W \otimes f, a_0 \otimes a, \mathrm{id}_Z \otimes f')/_\sim, \quad a_0 \in A_{W,Z}. \tag{8.14.5}$$

We write $f \circ (a \otimes \mathfrak{s}) \circ f'$ for $(f, a, f')/_\sim$. Note that for $g \in F_{W,Z}$,

$$\begin{aligned} g \otimes (f \circ (a \otimes \mathfrak{s}) \circ f') &\overset{\text{def}}{=} (g \otimes 1_A) \otimes (f \circ (a \otimes \mathfrak{s}) \circ f') \\ &= (\mathrm{id}_W \otimes f) \circ (g \otimes a \otimes \mathfrak{s}) \circ (\mathrm{id}_Z \otimes f') \\ &= (g \otimes f) \circ (\mathrm{id}_Z \otimes a \otimes \mathfrak{s}) \circ (\mathrm{id}_Z \otimes f') \\ &= (\mathrm{id}_W \otimes f) \circ (\mathrm{id}_W \otimes a \otimes \mathfrak{s}) \circ (g \otimes f'). \end{aligned} \tag{8.14.6}$$

For a set $\mathcal{S} = \{\mathfrak{s}_i\}_{i \in I}$ over $|\mathbb{F}| \times |\mathbb{F}|$, with $\mathfrak{s}_i \mapsto (Y_i, X_i)$, the free A-module on $\mathcal{S}$ is the sum

$$A \cdot \mathcal{S} = \coprod_{i \in I} A \cdot \mathfrak{s}_i, \tag{8.14.7}$$

or explicitly,

$$(A \cdot \mathcal{S})_{Y,X} = \{(f, \{a_i\}_{i \in I_0}, f') \mid f \in F_{Y, \bigoplus_{I_0}(W_i \otimes Y_i)}, f' \in F_{\bigoplus_{I_0}(Z_i \otimes X_i), X}, a_i \in A_{W_i, Z_i}, I_0 \subseteq I \text{ finite}\}/_\sim \tag{8.14.8}$$

with the equivalence relation $\sim$ generated by

$$\begin{gathered} \left(f \circ \left(\bigoplus_{I_0} (g_i \otimes \mathrm{id}_{Y_i})\right), \{a_i\}_{i \in I_0}, \left(\bigoplus_{I_0} (g_i' \otimes \mathrm{id}_{X_i})\right) \circ f'\right) \sim (f, \{g_i \circ a_i \circ g_i'\}_{i \in I_0}, f'), f, f', g_i, g_i' \in F, \\ (f, \{a_i\}_{i \in I_0}, f') \sim (f, \{a_i\}_{i \in I_0} \cup \{\mathrm{id}_{[0]}\}, f'). \end{gathered} \tag{8.14.9}$$

We write $f \circ (\bigoplus_{I_0} (a_i \otimes \mathfrak{s}_i)) \circ f'$ for $(f, \{a_i\}_{i \in I_0}, f')/_\sim$. The functor $\mathcal{S} \mapsto A \cdot \mathcal{S}$ from sets over $|\mathbb{F}| \times |\mathbb{F}|$ to A-$\mathcal{M}od$ is left-adjoint to the forgetful functor $M \mapsto \coprod_{Y,X} M_{Y,X}$,

$$\mathrm{Hom}_{A\text{-}\mathcal{M}od}(A \cdot \mathcal{S}, M) = \{(m_i)_{i \in I} \mid m_i \in M_{Y_i, X_i}\} = \mathrm{Hom}_{\mathcal{S}ets/|\mathbb{F}| \times |\mathbb{F}|}(\mathcal{S}, M). \tag{8.14.10}$$

Tensor products. For $M_0, M_1, N \in A$-$\mathcal{M}od$, we let $\mathcal{B}ilin_F^A(M_0, M_1; N)$ denote the maps $\varphi : M_0 \times M_1 \to N$ satisfying 'A_F-bilinearity':

$$\varphi((M_0)_{Y_0, X_0} \times (M_1)_{Y_1, X_1}) \subseteq N_{Y_0 \otimes Y_1, X_0 \otimes X_1}, \tag{8.15.1}$$

$$\begin{aligned} \varphi(m_0 \oplus m_0', m_1) &= \varphi(m_0, m_1) \oplus \varphi(m_0', m_1), \\ \varphi(m_0, m_1 \oplus m_1') &= \varphi(m_0, m_1) \oplus \varphi(m_0, m_1'), \quad m_i, m_i' \in M_i, \end{aligned} \tag{8.15.2}$$

$$\varphi(a \otimes m_0, m_1) = a \otimes \varphi(m_0, m_1) = \varphi(m_0, a \otimes m_1), \quad m_i \in M_i, a \in A, \tag{8.15.3}$$

$$\varphi(g_0 \circ m_0 \circ g_0', g_1 \circ m_1 \circ g_1') = (g_0 \otimes g_1) \circ \varphi(m_0, m_1) \circ (g_0' \otimes g_1') \quad g_i, g_i' \in F. \tag{8.15.4}$$

The free A-module on the set $\{\varphi(m_0, m_1)\}_{m_i\in M_i}$, divided by the equivalence A-module generated by (8.15.2), (8.15.3), (8.15.4), is denoted by $M_0 \otimes_{A/F} M_1$, and the map

$$M_0 \times M_1 \to M_0 \otimes_{A/F} M_1, (m_0, m_1) \mapsto m_0 \otimes m_1 = \text{ image of } \varphi(m_0.m_1) \tag{8.15.5}$$

is the universal bilinear map:

$$\mathcal{B}ilin_F^A(M_0, M_1; N) = \mathrm{Hom}_{A\text{-}\mathcal{M}od}(M_0 \otimes_{A/F} M_1, N). \tag{8.15.6}$$

We can describe $M_0 \otimes_A M_1$ quite explicitly as

$$\begin{aligned}(M_0 \otimes_A M_1)_{Y,X} = \{(f, \{m_0^i\}_{i\in I}, \{m_1^i\}_{i\in I}, g) \mid f \in F_{Y,\bigoplus_I (Y_0^i\otimes Y_1^i)}, g \in F_{\bigoplus_I (X_0^I\otimes X_1^i),X},\\ m_0^i \in (M_0)_{Y_0^i,X_0^i}, m_1^i \in (M_1)_{Y_1^i,X_1^i}, I \in |\mathbb{F}|\}/\sim\end{aligned} \tag{8.15.7}$$

where the equivalence relation $\sim$ is generated by

$$\begin{aligned}&\left(f \circ \left(\bigoplus_I (f_0^i \otimes f_1^i)\right), \{m_0^i\}_I, \{m_1^i\}_I, \left(\bigoplus_I (g_0^i \otimes g_1^i)\right) \circ g\right)\\ &\sim (f, \{f_0^i \circ m_0^i \circ g_0^i\}_I, \{f_1^i \circ m_1^i \circ g_1^i\}_I, g), \quad f_j^i, g_j^i \in F,\end{aligned}$$

$$(f, \{m_0^i\}_I, \{m_1^i\}_I, g) \sim (f, \{m_0^i\}_I \cup \{\mathrm{id}_{[0]}\}, \{m_1^i\}_I \cup \{\mathrm{id}_{[0]}\}, g), \tag{8.15.8}$$

$$(f, \{a^i \otimes m_0^i\}_I, \{m_1^i\}_I, g) \sim (f, \{m_0^i\}_I, \{a^i \otimes m_1^i\}_I, g), \quad a^i \in A, \tag{8.15.9}$$

$$\begin{aligned}(f, \{m_0^i \oplus m_0^{i'}\}_I, \{m_1^i\}_I, g) \sim (f, \{m_0^j\}_{I\oplus I'}, \{m_1^{\pi(j)}\}_{I\oplus I'}, g),\\ (f, \{m_0^i\}_I, \{m_1^i \oplus m_1^{i'}\}_I, g) \sim (f, \{m_0^{\pi(j)}\}_{I\oplus I'}, \{m_1^j\}_{I\oplus I'}, g),\end{aligned} \tag{8.15.10}$$

with $I \xrightarrow{\sim} I', i \mapsto i'$, and $\pi : I \oplus I' \to I, \pi(i) = i = \pi(i')$; here $m_j^i : I \to M_j, m_j^{i'} : I' \to M_j$ are two sequences of elements of M_j. One checks that the following operations are well defined, independent of the chosen representatives, and make $M_0 \otimes_A M_1$ into an A-module satisfying the universal property (8.15.6):

$$f \circ (g, \{m_0^i\}_I, \{m_1^i\}_I, g')/_\sim \circ f' \overset{\text{def}}{=} (f \circ g, \{m_0^i\}_I, \{m_1^i\}_I, g' \circ f')/_\sim, f, f', g, g' \in F, \tag{8.15.11}$$

$$a \otimes (f, \{m_0^i\}_I, \{m_1^i\}_I, g)/_\sim \overset{\text{def}}{=} (\mathrm{id}_W \otimes f, \{a \otimes m_0^i\}_I, \{m_1^i\}_I, \mathrm{id}_Z \otimes g)/_\sim, a \in A_{W,Z}, \tag{8.15.12}$$

$$(f, \{m_0^i\}_I, \{m_1^i\}_I, f')/_\sim \oplus (g, \{m_0^j\}_J, \{m_1^j\}_J, g')/_\sim \overset{\text{def}}{=} (f \oplus g, \{m_0^k\}_{k\in I\oplus J}, \{m_1^k\}_{k\in I\oplus J}, f' \oplus g'). \tag{8.15.13}$$

For $m_i \in (M_i)_{Y_i,X_i}$ we have

$$m_0 \otimes_A m_1 \overset{\text{def}}{=} (\mathrm{id}_{Y_0\otimes Y_1}, \{m_0\}, \{m_1\}, \mathrm{id}_{X_0\otimes X_1})/_\sim \in (M_0 \otimes_A M_1)_{Y_0\otimes Y_1, X_0\otimes X_1} \tag{8.15.14}$$

and hence

$$(f, \{m_0^i\}_I, \{m_1^i\}_I, g)/_\sim = f \circ \left(\bigoplus_{i\in I} (m_0^i \otimes_A m_1^i)\right) \circ g. \tag{8.15.15}$$

We can similarly define the multilinear functions $\mathcal{B}ilin_F^A(M_0, \ldots, M_l; N)$ and the A-module $M_0 \otimes_A \cdots \otimes_A M_l$ representing them.

The construction of the tensor product $M_0 \otimes_A M_1$ is functorial in the M_i and makes A-$\mathcal{M}od$ into a symmetric monoidal category with unit element A. For $\varphi_i : M_i \to N_i$ we have

$$\begin{aligned}\varphi_0 \otimes \varphi_1 : M_0 \otimes_A M_1 \to N_0 \otimes_A N_1,\\ \varphi_0 \otimes \varphi_1 \left(f \circ \left(\bigoplus_{i\in I} (m_0^i \otimes_A m_1^i)\right) \circ g\right) = f \circ \left(\bigoplus_{i\in I} (\varphi_0(m_0^i) \otimes \varphi_1(m_1^i))\right) \circ g,\end{aligned} \tag{8.16}$$

satisfying

$$\mathrm{id}_{M_0} \otimes \mathrm{id}_{M_1} = \mathrm{id}_{M_0 \otimes M_1} \tag{8.16.1}$$

and for $\psi_i : N_i \to L_i$,

$$(\psi_0 \otimes \psi_1) \circ (\varphi_0 \otimes \varphi_1) = (\psi_0 \circ \varphi_0) \otimes (\psi_1 \circ \varphi_1) \tag{8.16.2}$$

and there are canonical isomorphisms

$$M_0 \otimes M_1 \xrightarrow{\sim} M_1 \otimes M_0, \quad m_0 \otimes m_1 \mapsto m_1 \otimes m_0, \tag{8.16.3}$$

$$\begin{aligned} (M_0 \otimes M_1) \otimes M_2 \xrightarrow{\sim} M_0 \otimes (M_1 \otimes M_2) \xrightarrow{\sim} M_0 \otimes M_1 \otimes M_2, \\ (m_0 \otimes m_1) \otimes m_2 \mapsto m_0 \otimes (m_1 \otimes m_2) \mapsto m_0 \otimes m_1 \otimes m_2, \end{aligned} \tag{8.16.4}$$

$$A \otimes_A M \xrightarrow{\sim} M, a \otimes_A m \mapsto a \otimes m. \tag{8.16.5}$$

Given a homomorphism of F-monoids $\varphi : A \to B$, a B-module N can be considered as an A module N_A via $a \otimes n := \varphi(a) \otimes n$, and the functor

$$B\text{-}\mathcal{M}od \to A\text{-}\mathcal{M}od, \quad N \mapsto N_A, \tag{8.17.1}$$

has as left-adjoint the functor

$$\begin{gathered} A\text{-}\mathcal{M}od \to B\text{-}\mathcal{M}od, \quad M \mapsto B \otimes_A M, \\ \mathrm{Hom}_{A\text{-}\mathcal{M}od}(M, N_A) = \mathrm{Hom}_{B\text{-}\mathcal{M}od}(B \otimes_A M, N) \\ \varphi \mapsto \varphi\left(g \circ \left(\bigoplus_I b_i \otimes m_i\right) \circ g'\right) = g \circ \left(\bigoplus_I b_i \otimes \varphi(m_i)\right) \circ g', \\ \varphi(m) = \varphi(\mathrm{id}_Y \circ (1_B \otimes m) \circ \mathrm{id}_X) \leftarrow\!\shortmid \varphi. \end{gathered} \tag{8.17.2}$$

We have canonical isomorphisms

$$B \otimes_A \left(\coprod_{i \in I} M^i\right) = \coprod_{i \in I} (B \otimes_A M^i), \tag{8.17.3}$$

$$B \otimes_A \left(\varinjlim_I M^i\right) = \varinjlim_I (B \otimes_A M^i), \quad \text{for a functor } i \mapsto M^i, I \to A\text{-}\mathcal{M}od, \tag{8.17.4}$$

$$B \otimes_A (A \cdot \mathcal{S}) = B \cdot \mathcal{S}, \quad \text{for a set } \mathcal{S} \text{ over } |\mathbb{F}| \times |\mathbb{F}|, \tag{8.17.5}$$

$$B \otimes_A (M/\varphi(M_0)) = (B \otimes_A M)/(\mathrm{id}_B \otimes \varphi(B \otimes_A M_0)), \quad \text{for a map of } A\text{-modules } \varphi : M_0 \to M. \tag{8.17.6}$$

Given homomorphisms of F-monoids $\varphi^0 : A \to B^0$, $\varphi^1 : A \to B^1$, we can view B^0, B^1 as A-modules and form their tensor product $B^0 \otimes_A B^1$. By the universal property (8.15.6), there is a map

$$\begin{gathered} (B^0 \otimes_A B^1) \times (B^0 \otimes B^1) \to B^0 \otimes B^1, \\ (b^0 \otimes b^1, \overline{b}^0 \otimes \overline{b}^1) \mapsto (b^0 \otimes \overline{b}^0) \otimes (b^1 \otimes \overline{b}^1) \end{gathered} \tag{8.18.1}$$

making $B^0 \otimes_A B^1$ into an F-monoid, and it is the sum of B^0, B^1 over A in $F\text{-}\mathcal{M}on$: for every $C \in F\text{-}\mathcal{M}on$,

$$\begin{aligned} \mathrm{Hom}_{F\text{-}\mathcal{M}on}(B^0 \otimes_A B^1, C) &= \mathrm{Hom}_{F\text{-}\mathcal{M}on}(B^0, C) \underset{\mathrm{Hom}_{F\text{-}\mathcal{M}on}(A,C)}{\times} \mathrm{Hom}_{F\text{-}\mathcal{M}on}(B^1, C) \\ &= \{(\psi^0, \psi^1) \mid \psi^i : B^i \to C, \psi^0 \circ \varphi^0 = \psi^1 \circ \varphi^1\}. \end{aligned} \tag{8.18.2}$$

Here

$$\psi \mapsto (\psi \circ j^0, \psi \circ j^1) \tag{8.18.3}$$

with

$$j^i : B^i \to B^0 \otimes_A B^1, j^0(b) = b \otimes 1_{B^1}, j^1(b) = 1_{B^0} \otimes b;$$
$$m_C \circ (\psi^0 \otimes \psi^1) \leftarrow\!\shortmid (\psi^0, \psi^1) \tag{8.18.4}$$

with $m_C : C \otimes_A C \to C$ the multiplication map of $C, c_1 \otimes_A c_2 \mapsto c_1 \otimes c_2$.

Example 8.19. For commutative rings B^0, B^1, considering $\mathbb{F}(B^j)$ as $\mathbb{F}$-monoids, we have the tensor product $\mathbb{F}(B^0) \otimes_{\mathbb{F}} \mathbb{F}(B^1)$. It is easy to see that every element $f \circ (\bigoplus_I (b_i^0 \otimes b_i^1)) \circ g \in (\mathbb{F}(B^0) \otimes_{\mathbb{F}} \mathbb{F}(B^1))_{Y,X}$ is equivalent to an (essentially unique) such element with $b_i^j \in B^j \setminus \{0\}$, and with $f : I \hookrightarrow Y, g^{\mathrm{t}} : I \hookrightarrow X$ embeddings. Thus $\mathbb{F}(B^0) \otimes_{\mathbb{F}} \mathbb{F}(B^1)$ is the $\mathbb{F}$-monoid underlying the $\mathbb{F}$-ring $\mathbb{F}\langle B^0 \times_0 B^1\rangle$ associated with $B^0 \times_0 B^1 = B^0 \times B^1/_{(b_0,0)\sim(0,b_1)} \in \mathcal{M}on_{0,1}$. On the other hand, if the B^j are A-algebras, with A a commutative ring (e.g., $A = \mathbb{Z}$), we can consider $\mathbb{F}(A), \mathbb{F}(B^j)$ as $\mathbb{F}(A)$-monoids, and we have the tensor product $\mathbb{F}(B^0) \otimes_{\mathbb{F}(A)} \mathbb{F}(B^1)$. Every element $c \in (\mathbb{F}(B^0) \otimes_{\mathbb{F}(A)} \mathbb{F}(B^1))_{Y,X}$ is determined by its matrix coefficients

$$c_{y,x} = j_y^{\mathrm{t}} \circ c \circ j_x \in (\mathbb{F}(B^0) \otimes_{\mathbb{F}(A)} \mathbb{F}(B^1))_{[1],[1]}.$$

Every element

$$f \circ \left(\bigoplus_I (b_i^0 \otimes b_i^1) \right) \circ g \in (\mathbb{F}(B^0) \otimes_{\mathbb{F}(A)} \mathbb{F}(B^1))_{[1],[1]}$$

is equivalent to such an element with $b_i^j \in B^j$, and with $f_{1,i} = g_{i,1} = 1$, which we may denote $\sum_I b_i^0 \otimes b_i^1$. We have well-defined addition of such elements,

$$\sum_I b_i^0 \otimes b_i^1 + \sum_J b_j^0 \otimes b_j^1 = \sum_{I \oplus J} b_k^0 \otimes b_k^1,$$

and we have A-action with

$$a \cdot \sum_I b_i^0 \otimes b_i^1 = \sum_I a \cdot b_i^0 \otimes b_i^1 = \sum_I b_i^0 \otimes a \cdot b_i^1,$$

and moreover

$$b_1^0 \otimes b^1 + b_2^0 \otimes b^1 = (b_1^0 + b_2^0) \otimes b^1, b^0 \otimes b_1^1 + b^0 \otimes b_2^1 = b^0 \otimes (b_1^1 + b_2^1).$$

Thus $\mathbb{F}(B^0) \otimes_{\mathbb{F}(A)} \mathbb{F}(B^1)$ is the $\mathbb{F}(A)$-monoid $\mathbb{F}(B^0 \otimes_A B^1)$, and $B \mapsto \mathbb{F}(B)$ is a tensor functor: $\mathcal{R}ing/A \to \mathbb{F}(A)\text{-}\mathcal{M}on$.

Varying the base $\mathbb{F}$*-ring* F. We can change the base $\mathbb{F}$-ring. Given a map $\varphi : F_1 \to F_2$ of $\mathbb{F}$-rings, every F_2-monoid B is an F_1-monoid via φ, giving a functor

$$F_2\text{-}\mathcal{M}on \to F_1\text{-}\mathcal{M}on, \quad B \mapsto B_{F_1}.$$

Conversely, given an F_1-monoid A, its base change $A_{F_2} \in F_2\text{-}\mathcal{M}od$ is an F_2-monoid (using equation (3.23.8)), and we have the functor left-adjoint to the preceding one

$$F_1\text{-}\mathcal{M}on \to F_2\text{-}\mathcal{M}on, \quad A \mapsto A_{F_2}.$$

Similarly we can let F vary by working over a fixed $\mathbb{F}$-ringed space $(X, \mathcal{O}_X)$, cf. Definition 6.1.

Definition 8.20. For $X \in \mathbb{F}\text{-}\mathcal{R}ings.\mathcal{S}p$, an $\mathcal{O}_X$-monoid A is a sheaf of $\mathcal{O}_X$-modules such that for $U_2 \subseteq U_1 \subseteq X$ open, $A(U_i)$ is an $\mathcal{O}_X(U_i)$-monoid, and $A(U_1) \to A(U_2)$ is an $\mathcal{O}(U_1)$-monoid map, i.e. A is a monoid object of $\mathcal{O}_X\text{-}\mathcal{M}od$. This gives a category of $\mathcal{O}_X$-monoids, $\mathcal{O}_X\text{-}\mathcal{M}on$, and it has fibred sums.

Free monoids. Given an F-monoid A and a symbol $\mathfrak{s} : X_0 \to Y_0$, $X_0, Y_0 \in |\mathbb{F}|$, take the sequence of symbols $\mathfrak{s}^n : X_0^{\otimes n} \to Y_0^{\otimes n}, n \geqslant 0$ (where $X_0^{\otimes n} = X_0 \otimes \cdots \otimes X_0$ (n times), and $X_0^{\otimes 0} = [1]$), and form the free A-module on $\{\mathfrak{s}^n\}_{n\geqslant 0}$:

$$A[\mathfrak{s}]_{Y,X} = \left(\coprod_{n\geqslant 0} A \cdot \mathfrak{s}^n\right)_{Y,X}$$
$$= \left\{ f \circ \left(\bigoplus_{0\leqslant i\leqslant n} (a_i \otimes s^i)\right) \circ g \;\middle|\; a_i \in A_{Y_i,X_i}, f \in F_{Y,\bigoplus(Y_i\otimes Y_0^{\otimes i})}, g \in F_{\bigoplus(X_i\otimes X_0^{\otimes i}),X} \right\} \Big/ \sim \tag{8.21.1}$$

with

$$\left(f \circ \bigoplus (f_i \otimes \mathrm{id}_{Y_0^{\otimes i}})\right) \circ \left(\bigoplus a_i \otimes s^i\right) \circ \left(\bigoplus (g_i \otimes \mathrm{id}_{X_0^{\otimes i}}) \circ g\right) \sim f \circ \left(\bigoplus (f_i \circ a_i \circ g_i) \otimes s^i\right) \circ g. \tag{8.21.2}$$

Using the distributive isomorphism in $\mathbb{F}$:

$$\left[\bigoplus_i (X_i \otimes X_0^{\otimes i})\right] \otimes \left[\bigoplus_j (X'_j \otimes X_0^{\otimes j})\right] = \bigoplus_k \left[\bigoplus_{i+j=k} (X_i \otimes X'_j)\right] \otimes X_0^{\otimes k},$$

and similarly with Y, we obtain a map

$$A[\mathfrak{s}] \times A[\mathfrak{s}] \to A[\mathfrak{s}], \tag{8.21.3}$$
$$\left(f \circ \left(\bigoplus_i a_i \otimes s^i\right) \circ g, f' \circ \left(\bigoplus_j a'_j \otimes s^j\right) \circ g'\right) \mapsto (f \otimes f') \circ \left(\bigoplus_k \left(\bigoplus_{i+j=k} a_i \otimes a'_j\right) \otimes s^k\right) \circ (g \otimes g'),$$

which is well defined, A/F-bilinear, and gives $A[\mathfrak{s}]$ the structure of A-monoid. It is the free A-monoid on $\mathfrak{s}$: for any A-monoid C,

$$\mathrm{Hom}_{A\text{-}\mathcal{M}on}(A[\mathfrak{s}], C) = C_{Y_0,X_0}. \tag{8.21.4}$$

We can similarly form the free A-monoid on $\mathfrak{s}_1, \ldots, \mathfrak{s}_l$, $\mathfrak{s}_i : X_i \to Y_i$, and we have

$$A[\mathfrak{s}_1, \ldots, \mathfrak{s}_l] = A[\mathfrak{s}_1] \otimes_A \cdots \otimes_A A[\mathfrak{s}_l] = A \otimes_F F[\mathfrak{s}_1, \ldots, \mathfrak{s}_l] = \coprod_{n_1,\ldots,n_l\geqslant 0} A \cdot \mathfrak{s}^{(n_1,\ldots,n_l)} \tag{8.21.5}$$

with $\mathfrak{s}^{(n_1,\ldots,n_l)} : X_1^{\bigotimes n_1} \otimes \cdots \otimes X_l^{\bigotimes n_l} \to Y_1^{\bigotimes n_1} \otimes \cdots \otimes Y_l^{\bigotimes n_l}$. Taking the direct limit over finite subsets of a set $\mathcal{S}$ over $|\mathbb{F}| \times |\mathbb{F}|$ we have the free A-monoid on $\mathcal{S}$

$$A[\mathcal{S}] = \varinjlim_{\{\mathfrak{s}_1,\ldots,\mathfrak{s}_l\}\subseteq\mathcal{S}} A[\mathfrak{s}_1, \ldots, \mathfrak{s}_l] \tag{8.21.6}$$

and $\mathcal{S} \mapsto A[\mathcal{S}]$ is the left-adjoint to the forgetful functor $A\text{-}\mathcal{M}on \to \mathcal{S}ets/|\mathbb{F}| \times |\mathbb{F}|$.

Ideals of an F-monoid. An ideal $\mathfrak{a} \subseteq A, A \in F\text{-}\mathcal{M}on$, is an A-submodule of A. Ideals $\mathfrak{a}_i$ have intersections $\bigcap_i \mathfrak{a}_i$, sums $\sum_i \mathfrak{a}_i$, and finite products $\prod_{i=1}^n \mathfrak{a}_i$, which are again ideals. The ideal generated by $\mathfrak{s}_i \in A_{Y_i,X_i}$ is given explicitly by

$$(\mathfrak{s}_1, \ldots, \mathfrak{s}_l) = \left\{ f \circ \left(\bigoplus_i a_i \otimes \mathfrak{s}_i\right) \circ g \;\middle|\; f \in F_{Y,\bigoplus(W_i\otimes Y_i)}, g \in F_{\bigoplus(Z_i\otimes X_i),X}, a_i \in A_{W_i,Z_i} \right\}. \tag{8.22.1}$$

Homogeneous ideals $\mathfrak{a}$, i.e. $\mathfrak{a}$ is generated by $\mathfrak{a}_{[1],[1]}$, correspond one-to-one with H-ideals, i.e. $\mathfrak{a} \subseteq A_{[1],[1]}$ such that for

$$\mathfrak{s}_1, \ldots, \mathfrak{s}_l \in \mathfrak{a}, a_i \in A_{W_i,Z_i}, f \in F_{[1],\bigoplus W_i}, g \in F_{\bigoplus Z_i,[1]} \Rightarrow f \circ \left(\bigoplus a_i \otimes \mathfrak{s}_i\right) \circ g \in \mathfrak{a}. \tag{8.22.2}$$

An ideal $\mathfrak{p} \subseteq A$ (respectively H-ideal $\mathfrak{p} \subseteq A_{[1],[1]}$) is called S-prime (respectively prime) if $A \setminus \mathfrak{p}$ (respectively $A_{[1],[1]} \setminus \mathfrak{p}$) is closed under $\otimes$. We let SPEC(A) (respectively Spec(A)) denote S-primes (respectively primes).

Primes and spectra. An ideal $\mathfrak{a} \subseteq A$ (respectively H-ideal $\mathfrak{a} \subseteq A_{[1],[1]}$) is called E-ideal (respectively H-E-ideal) if $\mathfrak{a} = ZE(\mathfrak{a})$, respectively $\mathfrak{a} = (ZE(\mathfrak{a}))_{[1],[1]}$, or explicitly

$$f \circ \left(a_0 \oplus \bigoplus_i a_i \otimes \mathfrak{s}_i \right) \circ g \in \mathfrak{a} \quad \Leftrightarrow \quad f \circ (a_0 \oplus 0) \circ g \in \mathfrak{a}, \quad \text{for } \mathfrak{s}_i \in \mathfrak{a}, a_i \in A. \tag{8.22.3}$$

We let E-SPEC(A) (respectively E-Spec(A)) denote the S-E-primes (respectively E-primes). We have the sets

$$\begin{gathered} V_A(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{SPEC}(A) \mid \mathfrak{p} \supseteq \mathfrak{a}\}, \quad \mathfrak{a} \subseteq A \text{ ideal}; \\ D_A(f) = \{\mathfrak{p} \in \mathrm{SPEC}(A) \mid \mathfrak{p} \not\ni f\}, \quad f \in A; \\ V_A(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid \mathfrak{p} \supseteq \mathfrak{a}\}, \mathfrak{a} \subseteq A_{[1],[1]} \ H\text{-ideal}; \\ D_A(f) = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid \mathfrak{p} \not\ni f\}, \quad f \in A_{[1],[1]}. \end{gathered} \tag{8.22.4}$$

The $V_A(\mathfrak{a})$ define the closed sets, the $D_A(f)$ define a basis for the open sets of the Zariski topology on SPEC(A) and on Spec(A).

For an ideal $\mathfrak{a}$ (respectively E-ideal, H-ideal, H-E-ideal), and for $f \in A$ (respectively $f \in A_{[1],[1]}$ for H-ideals) with $f^{\otimes n} \notin \mathfrak{a}$ for all n, a maximal element of the set of ideals (respectively E-ideals, H-ideals, H-E-ideals) containing $\mathfrak{a}$ and not containing any $f^{\bigotimes n}$ is S-prime (respectively S-E-prime, prime, E-prime), and it follows that

$$\sqrt{\mathfrak{a}} = \{f \in A (\text{respectively } A_{[1],[1]}) \mid f^{\bigotimes n} \in \mathfrak{a} \text{ for some } n \geqslant 1\} = \bigcap_{\mathfrak{a} \subseteq \mathfrak{p}} \mathfrak{p}, \tag{8.22.5}$$

where the intersection is taken over all S-primes (respectively S-E-primes, primes, E-primes) containing $\mathfrak{a}$. For a subset $X \subseteq$ SPEC(A) (respectively E-SPEC(A), Spec(A), E-Spec(A)) we have the ideal (respectively E-ideal, H-ideal, H-E-ideal),

$$I(X) = \bigcap_{\mathfrak{p} \in X} \mathfrak{p}, \tag{8.22.6}$$

and

$$\sqrt{\mathfrak{a}} = I(V_A(\mathfrak{a})), \quad V_A(I(X)) = \overline{X}. \tag{8.22.7}$$

It follows that we have bijections between the closed sets in SPEC(A) (respectively E-SPEC(A), Spec(A), E-Spec(A)), and the radical $\mathfrak{a} = \sqrt{\mathfrak{a}}$ ideals (respectively E-ideals, H-ideals, H-E-ideals); the irreducible closed sets correspond to S-primes (respectively S-E-primes, primes, E-primes). There is a commutative diagram of spaces

$$\begin{array}{ccc} E\text{-}\mathrm{SPEC}(A) & \hookrightarrow & \mathrm{SPEC}(A) \\ \big\downarrow & & \big\downarrow \\ E\text{-}\mathrm{Spec}(A) & \hookrightarrow & \mathrm{Spec}(A) \end{array} \tag{8.22.8}$$

with the horizontal arrows embedding of subspaces (with the subspace topology), and the vertical arrows are surjective continuous maps (to show surjectivity one needs localization, cf. below). For a map of F-monoids $\varphi : A \to B$, the pull-back along φ, $\varphi^*(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$, gives a map from the diagram (8.22.8) associated with B to the one associated with A. Since

$$\varphi^{*-1} V_A(\mathfrak{a}) = V_B(\varphi(\mathfrak{a})), \quad \varphi^{*-1} D_A(f) = D_B(\varphi(f)), \tag{8.22.9}$$

φ^* is continuous, and we have four functors from F-$\mathcal{M}on$ to the category of compact sober spaces and continuous maps.

Localization and structure sheaf $\mathcal{O}_A$. The theory of localization of an F-monoid A (respectively A-module M) with respect to a multiplicative subset $S \subseteq A_{[1],[1]}$ goes exactly as in § 5, and produces an F-monoid $S^{-1}A$ (respectively $S^{-1}A$-module $S^{-1}M$). The functor $A \mapsto S^{-1}A$ (respectively $M \mapsto S^{-1}M$) commutes with direct limits, tensor products, finite inverse limits, free objects; in particular it preserves kernels $\varphi^{-1}(0)$, $\mathcal{KER}(\varphi)$, cokernels, and commutes with the operations E, Z. Propositions 5.12, 5.13, and 5.14 remain valid: surjectivity or injectivity of a map in A-$\mathcal{M}od$ can be checked locally at every prime (or maximal H-ideal). For an F-monoid A (respectively A-module M), and for $\mathfrak{p} \in \operatorname{Spec}(A)$, or for $f \in A_{[1],[1]}$, we put

$$\begin{aligned} A_{\mathfrak{p}} = S_{\mathfrak{p}}^{-1}A \text{ (respectively } M_{\mathfrak{p}} = S_{\mathfrak{p}}^{-1}M), \quad S_{\mathfrak{p}} = A_{[1],[1]} \setminus \mathfrak{p}; \\ A_f = S_f^{-1}A \text{ (respectively } M_f = S_f^{-1}M), \quad S_f = \{f^n | n \geqslant 0\}. \end{aligned} \tag{8.22.10}$$

Over $\operatorname{Spec}(A)$ we have a sheaf $\mathcal{O}_A$ of F-monoids (respectively a sheaf of $\mathcal{O}_A$-modules $\widetilde{M}$) such that

$$\mathcal{O}_A(D_A(f)) = A_f \text{ (respectively } \widetilde{M}(D_A(f)) = M_f), \quad f \in A_{[1],[1]}, \tag{8.23.1}$$

and with stalks at $\mathfrak{p} \in \operatorname{Spec}(A)$ given by

$$\mathcal{O}_{A,\mathfrak{p}} = A_{\mathfrak{p}} \text{ (respectively } \widetilde{M}|_{\mathfrak{p}} = M_{\mathfrak{p}}). \tag{8.23.2}$$

The proof of (8.23.2) goes exactly as for $\mathbb{F}$-rings, cf. Proposition 5.19. The proof of (8.23.1) goes as for $\mathbb{F}$-rings, cf. Proposition 5.20, with only a minor change at the end: since the sets $D_A(g_i)$ cover $D_A(f)$ we have

$$f^m = f_1 \circ \left(\bigoplus_i g_i \otimes b_i \right) \circ f_2, \quad \text{with} f_i \in F, b_i \in A,$$

and we let $a = (\mathrm{id}_Y \otimes f_1) \circ (\bigoplus_i a_i \otimes b_i) \circ (\mathrm{id}_X \otimes f_2)$, giving again $g_j \otimes a = f^m \otimes a_j$.

Stalk $\mathcal{O}_{A,\mathfrak{p}}$ is a *local* F-monoid, having a unique maximal H-ideal $\mathfrak{m}_{\mathfrak{p}}$, and $F_{\mathfrak{p}} = \mathcal{O}_{A,\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}$ is a field in F-$\mathcal{M}on$. We remark that we have again four notions of fields (and four notions of local F-monoids) given by the conditions in Definition 4.23.

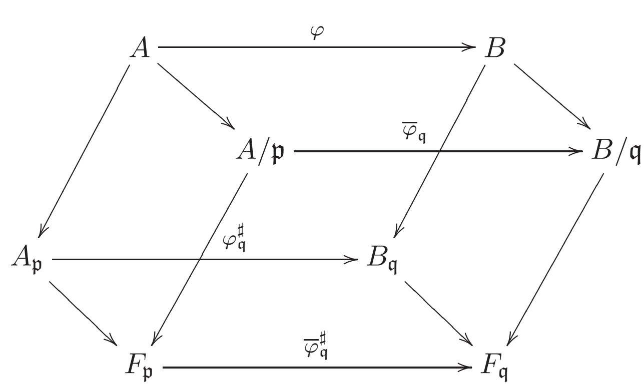
A map of F-monoid $\varphi : A \to B$ induces the localized map

$$\varphi^\sharp : \mathcal{O}_A(U) \to \mathcal{O}_B(\varphi^{*-1}U), \quad U \subseteq \operatorname{Spec}(A),$$

giving rise to a local homomorphism

$$\varphi_{\mathfrak{q}}^\sharp : A_{\varphi^*(\mathfrak{q})} \to B_{\mathfrak{q}}, \quad \varphi_{\mathfrak{q}}^\sharp(\mathfrak{m}_{\varphi*(\mathfrak{q})}) \subseteq \mathfrak{m}_{\mathfrak{q}}, \quad \mathfrak{q} \in \operatorname{Spec}(B), \tag{8.23.3}$$

and a commutative diagram with $\mathfrak{p} = \varphi^*(\mathfrak{q}), \mathfrak{q} \in \operatorname{Spec}(B)$ as follows:



(8.23.4)

One defines the categories of F-monoid spaces, $F\text{-}\mathcal{M}on.\mathcal{S}p$ (respectively of local F-monoid spaces, $\mathcal{L}oc.F\text{-}\mathcal{M}on.\mathcal{S}p$) as the category with objects $(X, \mathcal{O}_X)$, X a topological space, $\mathcal{O}_X$ a sheaf of F-monoids on X (respectively with local F-monoids for stalks $\mathcal{O}_{X,x}, x \in X$), and with maps $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, continuous maps $f : X \to Y$, and maps of sheaves of F-monoids over Y, $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ (respectively such that the induced map on stalks $f^\sharp_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local homomorphism). Then $A \mapsto (\mathrm{Spec}(A), \mathcal{O}_A)$ is a contravariant functor

$$\mathrm{Spec} : F\text{-}\mathcal{M}on \to \mathcal{L}oc.F\text{-}\mathcal{M}on.\mathcal{S}p \tag{8.23.5}$$

which is the adjoint to the global section functor

$$\Gamma : \mathcal{L}oc.F\text{-}\mathcal{M}on.\mathcal{S}p \to F\text{-}\mathcal{M}on, \quad \Gamma(X, \mathcal{O}_X) = \mathcal{O}_X(X), \tag{8.23.6}$$

so that we have

$$\mathrm{Hom}_{\mathcal{L}oc.F\text{-}\mathcal{M}on.\mathcal{S}p}(X, \mathrm{Spec}\, A) = \mathrm{Hom}_{F\text{-}\mathcal{M}on}(A, \Gamma(X, \mathcal{O}_X)). \tag{8.23.7}$$

Definition 8.24. A Zariski F-monoid scheme is a local F-monoid space $(X, \mathcal{O}_X)$ such that there exists an open covering $X = \bigcup_i U_i$, with $(U_i, \mathcal{O}_X|_{U_i}) \simeq \mathrm{Spec}\, \mathcal{O}_X(U_i)$. Maps of Zariski F-monoid schemes are maps of $\mathcal{L}oc.F\text{-}\mathcal{M}on.\mathcal{S}p$, thus we have a full subcategory of $\mathcal{L}oc.F\text{-}\mathcal{M}on.\mathcal{S}p$ consisting of Zariski F-monoid schemes.

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