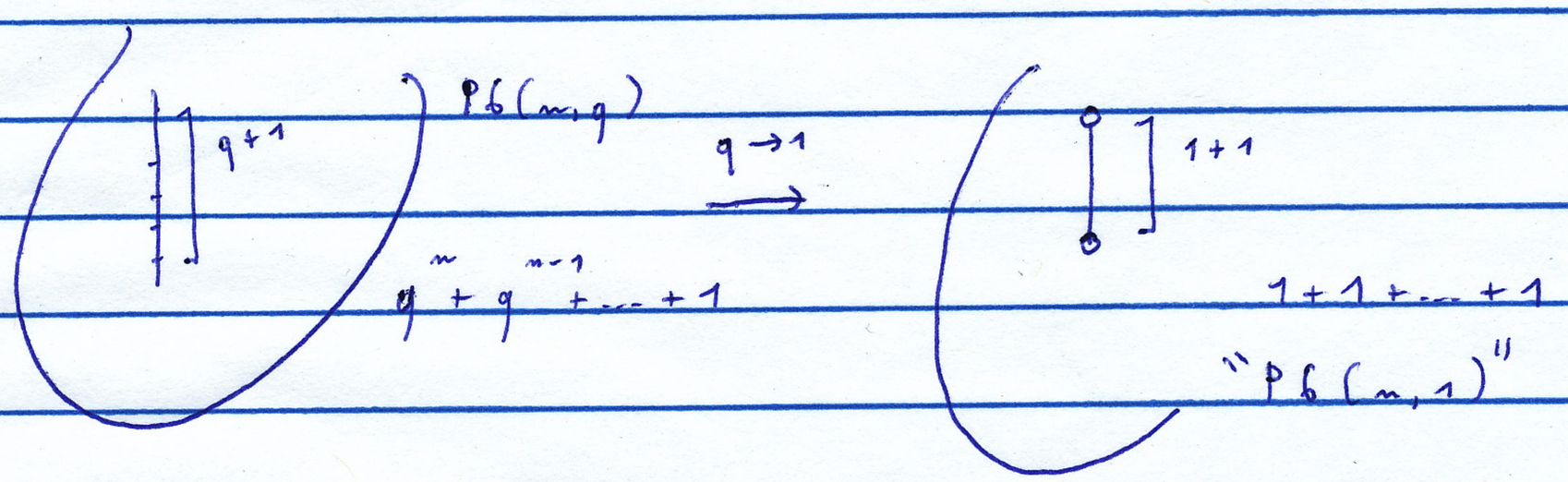
Introduction to absolut arithmetic

— Consider a projective geometry $PG(n,q)$ over the finite field $\mathbb{F}_q$.
In a 1957 paper, Tit remarked that some considerations still make sense if we let $q \to 1$, that is, if we suppose there is a field "of characteristic one".



$PG(n,1)$ becomes a set of $n+1$ points, together with all it subset (an m-subset is an $(m-1)$-space). The automorphism grp is the set of all permutations on $n+1$ pts $\Rightarrow S_{n+1}$. (Tit considers S_{n+1} as a "Chevalley group over the field of characteristic one", $\mathbb{F}_1$.)

Other examples = (see Tit's paper)

classical generalized n-gons (defined over a field, say finite) $\xrightarrow{q \to 1}$ ordinary n-gons

(Note that over $\mathbb{F}_1$, they exist for all n!)

$\downarrow$ aut

automorphism groups = dihedral groups D_n (also Chevalley over $\mathbb{F}_1$)

(Note that for $n=3$, we obtain triangles $\to PG(2,1)$)

More generally:

(spherical) buildings $\xrightarrow{q \to 1}$ apartment

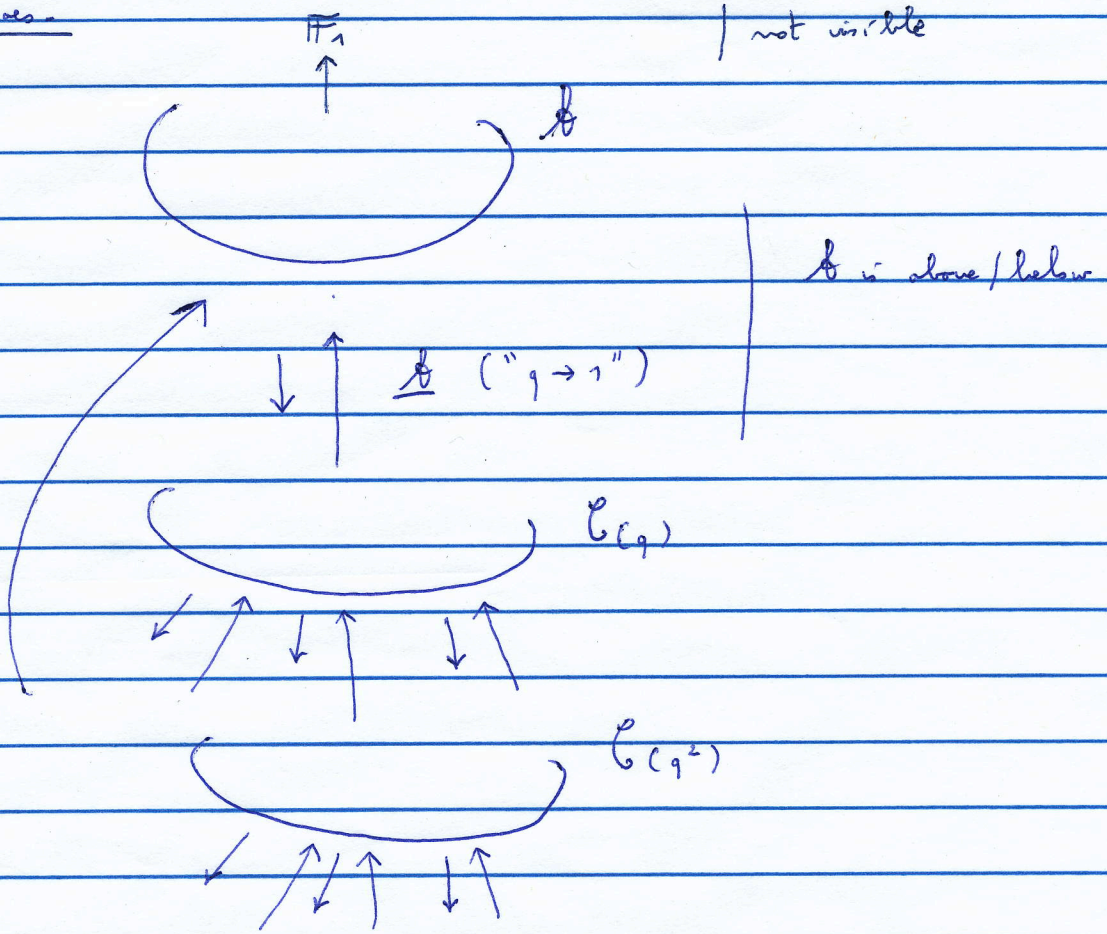
$\downarrow$ aut

W = Weyl groups (= automorphism groups of apartment)

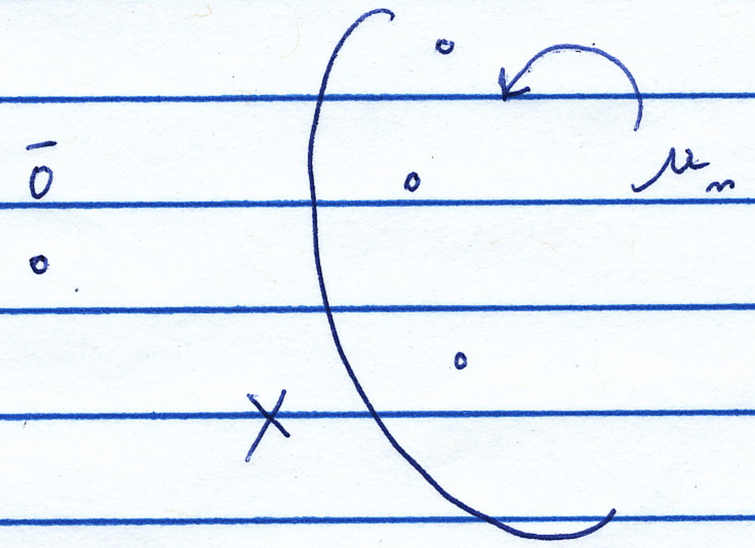
($W = N/N \cap B$, (B,N) "standard BN-pair")

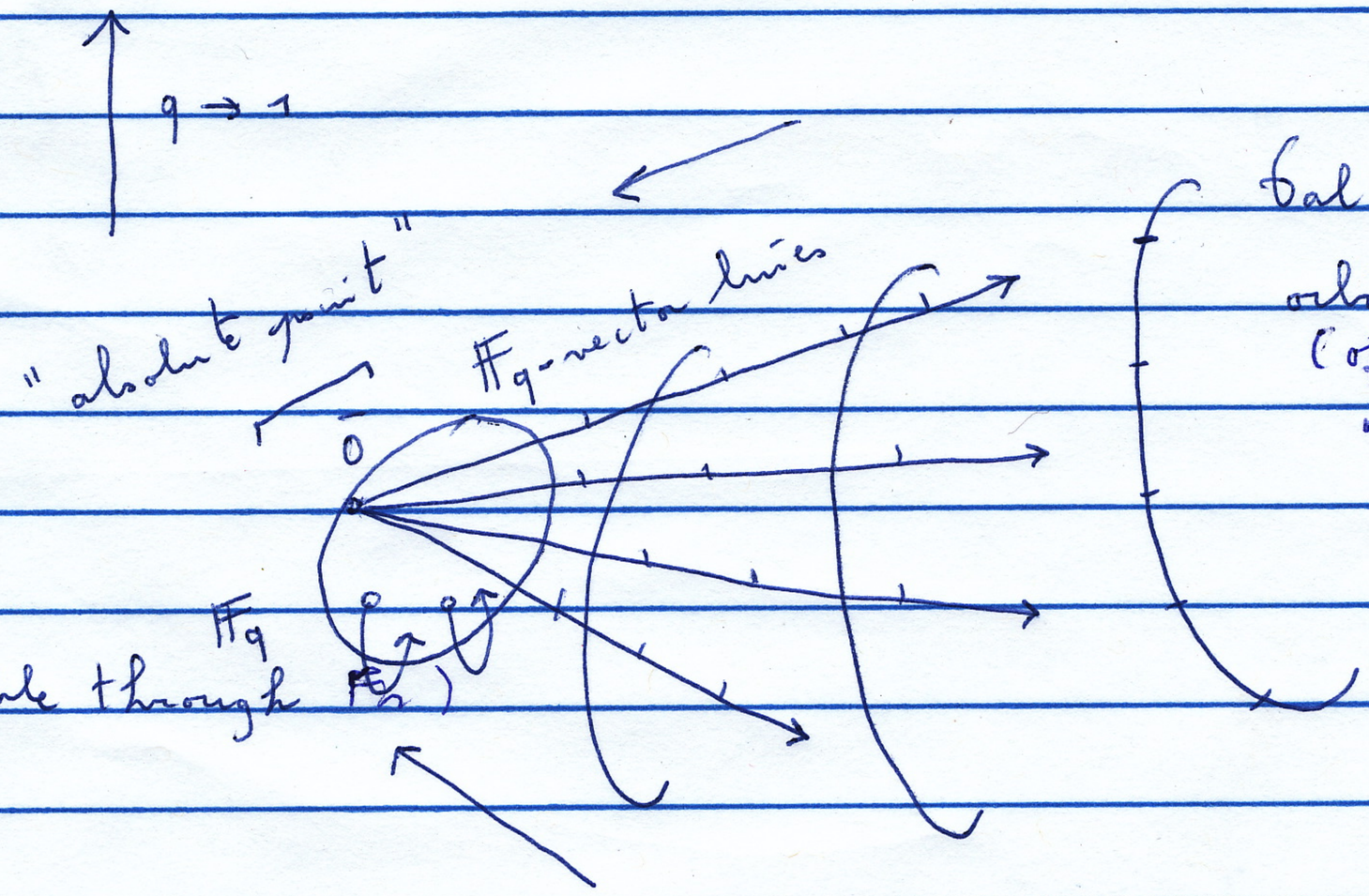
Note that "$\xrightarrow{q\to 1}$" also makes sense for non-classical polygons (since they also are constructed by apartment), although a priori there isn't necessarily a field around.

Philosophy (here): "$q\to 1$" is a functor from a category of geometries to a subcategory $\mathcal{B}$; elements of $\mathcal{B}$ reflect the "essence" of the geometries, and still satisfy the governing axioms. (It singles out the orig. building bricks.) $\mathbb{F}_1$ does not exist, but geometry "over $\mathbb{F}_1$" does.



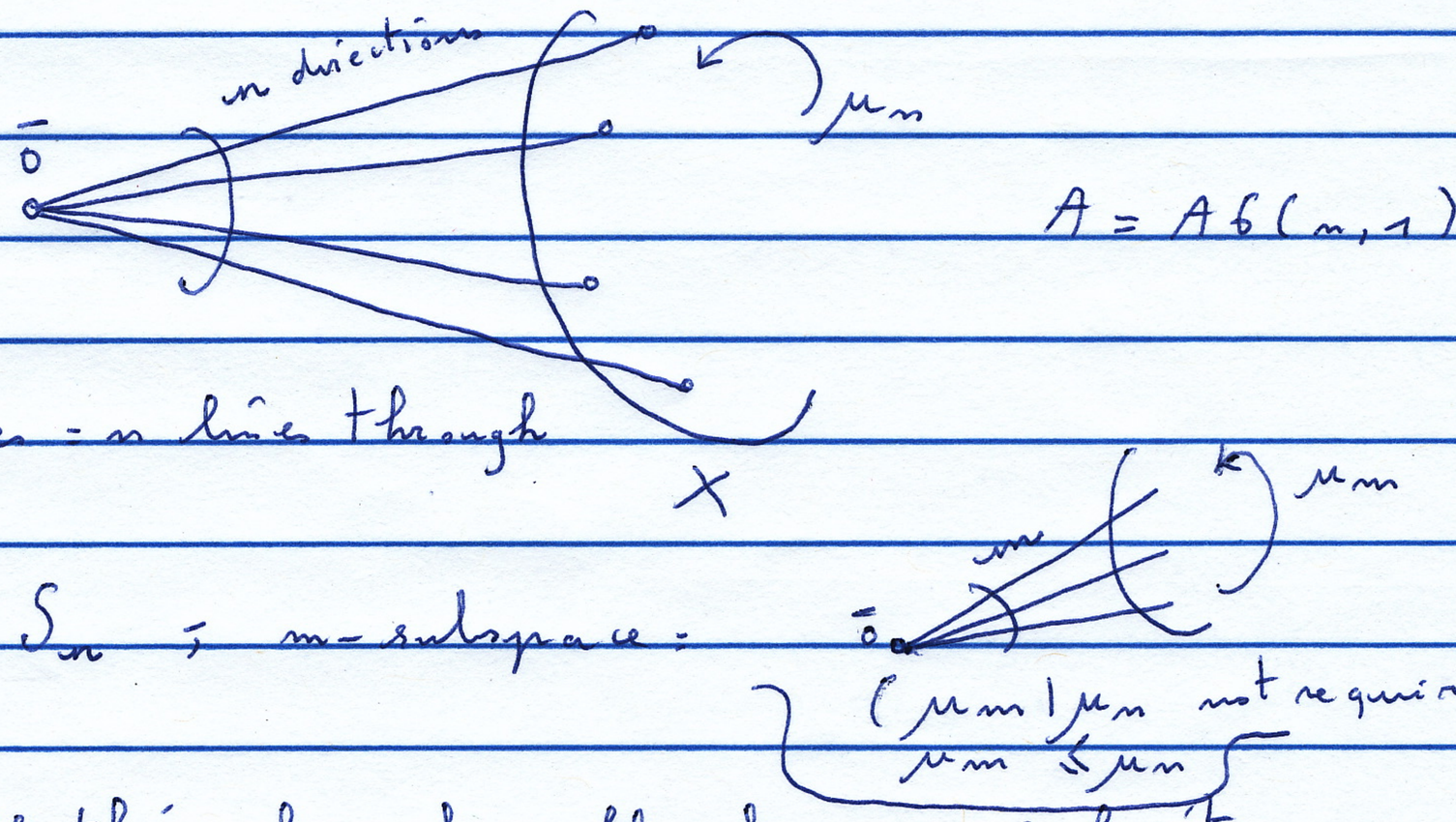
— Consider a prime power q; then $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$, $n \in \mathbb{N}_0$, is cyclic of order n. Now as we can see $\mathbb{F}_{q^n}$ as an n-dim. vector space over $\mathbb{F}_q$ (or an affine space), a vector space of dimension n over $\mathbb{F}_q$ is identified as a 3-tuple $(\bar{0}, X, \mu_n)$, with $\bar{0}$ a distinguished point, X a set (of size n), and μ_n a cyclic group of size n acting freely and transitively on $X \setminus \mu_n$.





Although we do not have a 0 in $\mathbb{F}_1$, often it makes sense to add it anyhow. In fact, we will see later that the $\bar{0}$ of above is actually "the absolute point".

Seeing affine space (n-dim.) over $\mathbb{F}_1$ as a torsor, and noting that $\bar{0}$ cannot move, an <u>affine space</u> will be something like:



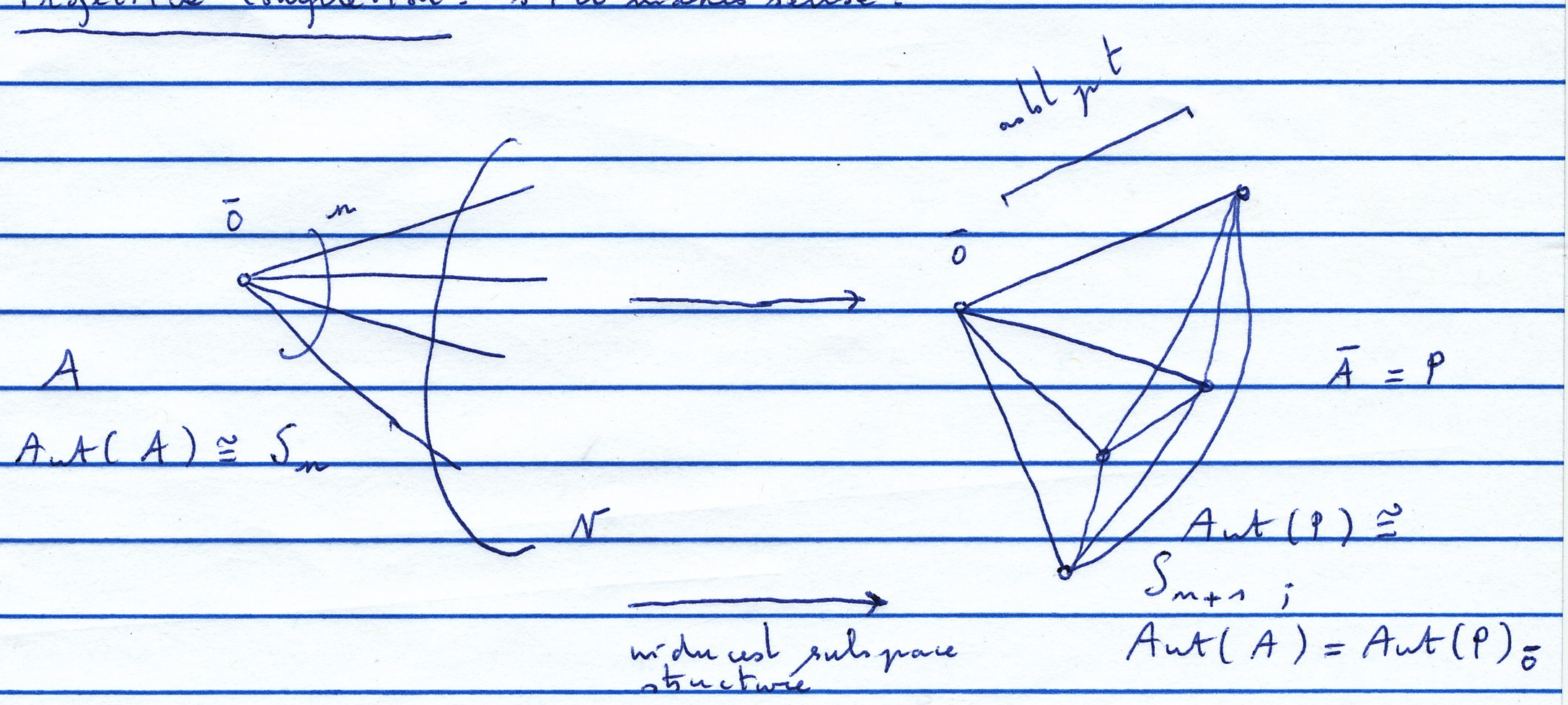
$A = Af(n,1)$

Points = $\bar{0}$ Lines = n lines through $\bar{0}$

$Aut(A) \cong S_n$; m-subspace:

<u>CAUTION</u>: All this only works well when n is finite or countable, since uncountable cyclic groups do not exist. An ω-dim. affine space over $\mathbb{F}_1$ is a pt $\bar{0}$ with ω one-point-lines through it, with induced subspaces as above; $Aut(A) \cong S(\omega)$, S_ω.

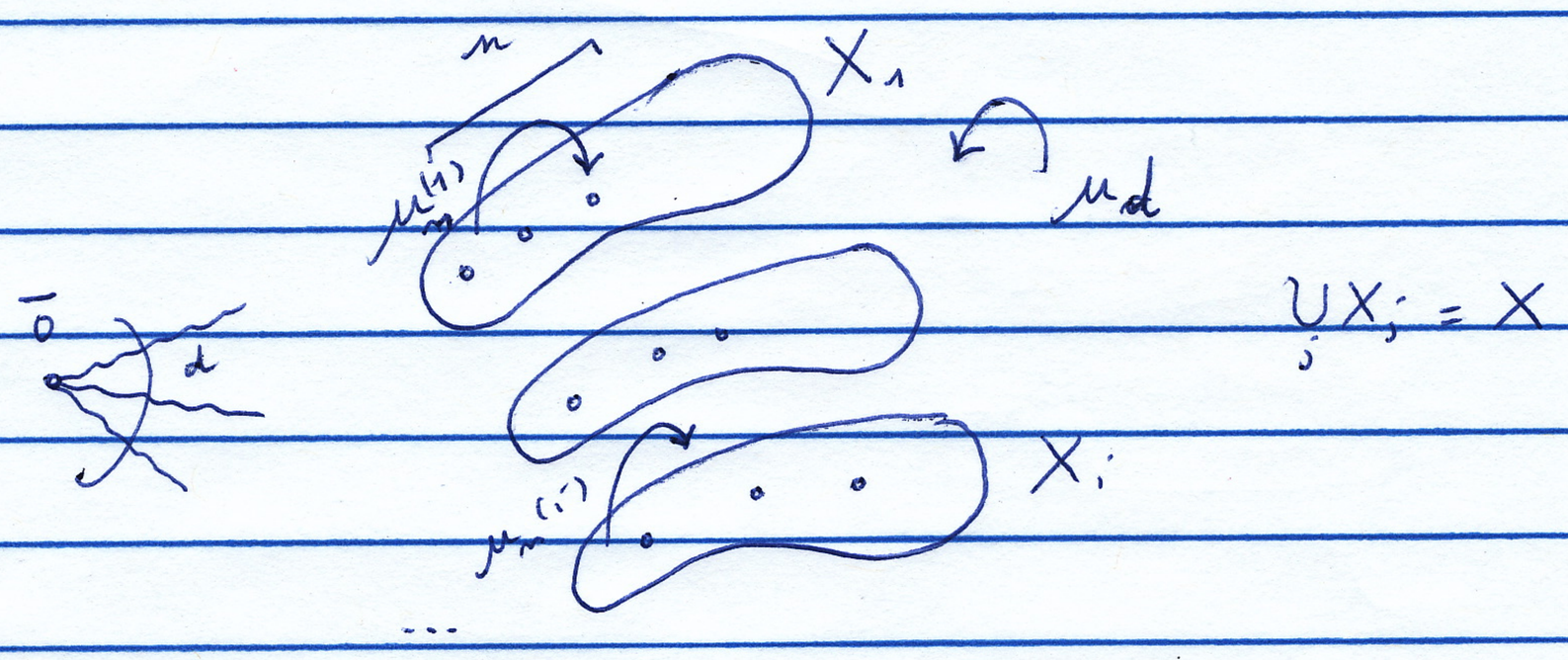
<u>Projective completion</u>: still makes sense:



FIELD EXTENSION / VECTOR SPACES.

⇓

Vector space over $\mathbb{F}_m$, of dimension d ($\in \mathbb{N}_0$) = a set X of size m^d, together with point $\bar{0}$; an action of μ_d ↷ subsets of X of size m ((free, transitive)), minimal subspaces



A basis is a set of d elements, each for each μ_d-orbit ; for $n=1$, there is only one basis – the absolute basis. (Any element $x \in X$ can be written uniquely as $b_j^{\alpha_j}$ for some $b_j \in X_j$ a basis element, and $\alpha_j \in \mu_m^{(-)}$.) ⇒ a vector $x \in X$ can be represented as

$$d \left[\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \alpha_j b_j^{\alpha_j} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right. \longleftarrow j \text{ (only one } \neq 0 \text{ element)}$$

Linear automorphisms are of the form

$$\begin{cases} (b_1, \dots, b_d) \to (b_{\sigma(1)}^{\alpha_1}, \dots, b_{\sigma(d)}^{\alpha_d}) \\ \quad \sigma \in Sym_d \\ \quad \alpha_i \in \mu_n \text{ (or rather, } \alpha_i \in \mu_m^{(\sigma(i))}) \end{cases}$$

⇒ $d \times d$-matrices with precisely one "$\neq 0$ entry" in each row/column, over μ_n ⇒ $GL(d, \mathbb{F}_1^n) \cong S_d \wr (\mu_m)^d$

Setting $n=1$, we obtain that $Aut(V) \cong S_d$ (group of all $d \times d$ permutation matrices). (V ↓ the vector space)

- Philosophy: Over $\mathbb{F}_1$, we cannot add vectors, we only have scalar multiplication; in particular, $\mathbb{F}_1$ "has" a monoidal structure. (So we can only work with permutation like matrices.)

KT
28-4-11