Scheme theory over the field of characteristic one

– Let $\left[\begin{array}{l} X \text{ be a scheme of finite type over } \mathbb{Z}, \text{ etc.} \\ \mathcal{C} \text{ be a (smooth abs. irreducible) curve over } \mathbb{F}_q, q \text{ a prime power} \end{array}\right.$

(= non-singular)

Its zeta function is

note suggestive form

$$Z(\mathcal{C}, s) = \prod_{\substack{x \\ (x \text{ closed pt})}} \frac{1}{1 - N(x)^{-s}} . \qquad (N(x) : \text{nr. of pts in residue fields } k$$

Let $\mathcal{C}(\mathbb{F}_{q^f})$ be the set of $\mathbb{F}_{q^f}$-rational points, and define

$$\mathcal{C}(\mathbb{F}_{q^f})^\circ = \{ x \in \mathcal{C}(\mathbb{F}_{q^f}) \mid \mathbb{F}_q(x) = \mathbb{F}_{q^f} \}; \text{ then}$$

(everything is w.r.t. some fixed closure $\overline{\mathbb{F}_q}$)

$$Z(\mathcal{C}, s) = \prod_{f=1}^{\infty} \left(\frac{1}{1 - q^{-fs}} \right)^{\frac{|\mathcal{C}(\mathbb{F}_{q^f})^\circ|}{f}} , \text{ and}$$

Lefschetz

$$|\mathcal{C}(\mathbb{F}_{q^f})| = \sum_{\omega=0}^{2} (-1)^\omega \, \mathrm{Tr}\left(Fr^d \mid H^\omega(\mathcal{C}) \right) = 1 - \sum_{j=0}^{2g} \phi_i^d + q$$

$\hookrightarrow \mathcal{C}(\mathbb{F}_{q^f})^\circ$ can be calculated by this formula, $d \mid n$

(Fr is the Frobenius endomorphism acting on the étale l-adic cohomology of $\mathcal{C}$, ϕ_i are its eigenvalues, g is the genus, $H^\omega(\mathcal{C}$ of aforementioned cohomology)

Weil cohomology

$$\Rightarrow \quad Z(\mathcal{C}, s) = \prod_{\omega=0}^{2} \det\left((\mathrm{id} - Fr . q^{-s}) \mid H^\omega(\mathcal{C}) \right)^{(-1)^{\omega-1}}$$

Deninger (90's): A similar decomposition of the classical Riemann zeta function $\zeta(-)$ should be something like

$\prod_\rho \frac{s-\rho}{2\pi}$

$\frac{s}{2\pi}\left(\frac{s-1}{2\pi}\right)$

1

$$= Z(\mathrm{Spec}(\overline{\mathbb{Z}}), s) := 2^{-\frac{1}{2}} \pi^{-s/2} \, \Gamma(s/2) \, \zeta(s) \overset{?}{=}$$

$$\prod_{\omega=0}^{2} \mathrm{DET}\left(\frac{s . \mathrm{id} - \phi}{2\pi} \mid H^\omega_?(\overline{\mathrm{Spec}(\mathbb{Z})}) \right)^{(-1)^{\omega-1}} \qquad (*$$

⇒ postulates the existence of a new cohomology theory H, endowed with a canonical "absolute Frobenius endomorphism", ϕ.

Combined work of Kurokawa, and Deninger, motivates Manin to state that there might be an interpretation of these formulae over $\mathbb{F}_1$: there are certain objects h^0, h^1, h^2 s.t. they have suitably defined zeta functions

"absolute point" ↳ "affine line over $\mathbb{F}_1$"

$$\omega = 0,1,2: \quad \zeta_{h^\omega}(s) = \mathrm{DET}\left(\frac{1}{2\pi}(s - \Theta) \,\Big|\, H^\omega(\overline{\mathrm{Spec}(\mathbb{Z})})\right).$$

↳ certain functor

In other words: there is an algebraic geometric theory over a hypothetical field, $\mathbb{F}_1$, such that Deninger's prediction finds a natural environment to be true ⇒ there is, in this theory, an analogous expression for Riemann zeta as there is for zeta over finite fields ⇒ $\overline{\mathrm{Spec}(\mathbb{Z})}$ is a curve over $\mathbb{F}_1$
⇒ can one adapt Weil to solve Riemann zeta?

— Recall that Spec (as a functor) is a contravariant equivalence between the category of comm. rings (with 1) and the category of affine schemes; $\mathbb{Z}$ is an initial object in the category of rings with 1 (unique morphisms $\mathbb{Z} \to R$ $\forall R$) ⇒ $\mathrm{Spec}(\mathbb{Z})$ is a terminal object for affine schemes.

GOAL:

$$\begin{array}{c} \mathbb{F}_1 \to \mathbb{Z} \to R \quad \forall R \\ \big\downarrow \mathrm{Spec} \\ \mathrm{Spec}(R) \to \mathrm{Spec}(\mathbb{Z}) \to \mathrm{Spec}(\mathbb{F}_1) \end{array}$$

($\mathbb{Z}$ must be some "a ring over $\mathbb{F}_1$")

"under $\mathrm{Spec}(\mathbb{Z})$"

→ cf. also Kazuya Kato

- $\mathbb{D}$-schemes. Let M be a monoid (1, binary operation, ass., abelian); $M^\times$ is the group of invertible elements. We suppose there is an absorbing element 0. $\Rightarrow$ M is an $\mathbb{F}_1$-ring

base extension to $\mathbb{Z}$: $A \otimes_{\mathbb{D}} \mathbb{Z} = \mathbb{Z}[A]$ (defined similarly as group ring); monoidal integral ring over A. $\underbrace{}_{\text{(commutative)}}$

Let A be a monoid as above, $S \subseteq A$ submonoid.

$$\begin{cases} S^{-1}A = A \times S / \sim, & \text{monoid} \\ (a,s) \sim (a',s') \text{ iff } s''s'a = s''s a' \text{ for some} \\ s'' \in S. \quad (\text{mult. is componentwise}; [(a,s)] = \text{"}\tfrac{a}{s}\text{"}) \end{cases}$$

An ideal $\mathfrak{a}$ of a monoid M is a subset s.t. $M\mathfrak{a} \subseteq \mathfrak{a}$.
An ideal is prime if $S_{\mathfrak{p}} = M \setminus \mathfrak{p}$ is a monoid.
Denote ${}^{\text{?pos}}$ $M_{\mathfrak{p}} = (S_{\mathfrak{p}}^{-1})M$; localization at $\mathfrak{p}$.
$\mathrm{Spec}(M)$ is the set of all prime ideals endowed with Zariski topology; closed subsets are of form

$$\begin{cases} V(\underset{\text{ideal}}{\mathfrak{a}}) = \{\mathfrak{p} \in \overbrace{\mathrm{Spec}(M)}^{\text{prime}} \mid \mathfrak{a} \subseteq \mathfrak{p}\}, \\ \emptyset. \end{cases}$$

$\emptyset$ ($=$) $\mathfrak{I}$ is in every $\neq \emptyset$ open set, $\underbrace{M \setminus M^\times}_{\text{(closed) pt (prime ideal)}}$ is closed and in every $\neq \emptyset$ closed set.

(→ see notes) generic unique

Consider an $\mathbb{F}_1$-ring A, and $\mathrm{Spec}(A)$. For U open in $\mathrm{Spec}(A)$, define $\mathcal{O}_X(U) := \{ s : U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}} \mid s(\mathfrak{p}) \in A_{\mathfrak{p}},\ s$ is locally a quotient of elements in $A \}$

$\Rightarrow$ $\mathcal{O}_X$ is structure sheaf of A; it is a sheaf of monoids (contravariant functor from $\tau(X)_{,\subseteq} \longrightarrow \mathcal{M}_0$ (category of monoids)) $\hookrightarrow$ presheaf + compatibility properties

A monoidal space $(X, \mathcal{O}_X)$ is a $\mathcal{D}$-scheme if for all x there is a $U \ni x$, U open, s.t. $(U, \mathcal{O}_{X|U})$ is an affine $\mathcal{D}$-scheme. An affine $\mathcal{D}$-scheme is of the form $(\mathrm{Spec}(A), \mathcal{O}_X)$ as above.

— Example. $\mathrm{Spec}(\mathbb{F}_1)$; $(\mathbb{F}_1 = \{0,1\}, \cdot\,;)$ $\{0\}$ is unique prime ideal $\Rightarrow$ unique pt. $\Rightarrow$ terminal object for $\mathcal{D}$-schemes ($\mathbb{F}_1$ is initial object)

not necessary formally

Base extension to $\mathbb{Z}$ $\Rightarrow$ $\mathrm{Spec}(\mathbb{Z})$: terminal object for $\mathbb{Z}$-schemes (Grothendieck schemes); $\mathbb{Z}$ is initial object for comm. rings (with 1)

— Example. Consider $(\{0\} \cup) \overbrace{\mathbb{F}_1[X_1, \dots, X_m]}^{A}$; monoidal polynomial ring over $\mathbb{F}_1$ (abelian). Prime ideals are of form

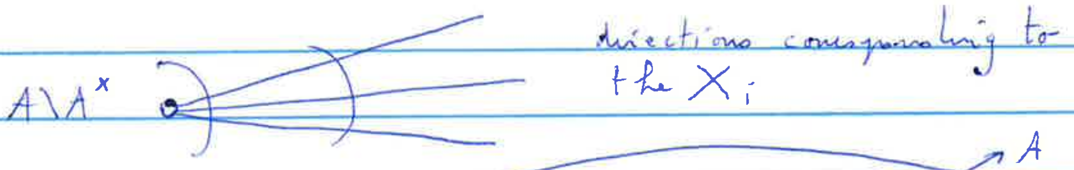
$$\mathfrak{p}_I = \bigcup_{i \in I} X_i A \,, (\{0\})$$

Only unit is $\{1\} = A^{\times} \Rightarrow A \setminus A^{\times} = \bigcup_m X_i A$ is closed pt. (it obviously prime ideal) $\leftarrow$ maximal ideal so prime

Any ideal is in $A \setminus A^{\times}$ $\Rightarrow$ only maximal ideal; $\bigcup_m X_i A$ only is contained in itself as prime ideal $\Rightarrow$ $\{A \setminus A^{\times}\}$ is closed. Each maximal ideal defines a closed pt, more generally

$\Rightarrow$ $\mathrm{Spec}(A)$ has a unique closed point.
(also more generally for affine $\mathbb{F}_1$-schemes)

closed pts $\rightsquigarrow$ "real points"



- <u>Proj. construction.</u> Consider $\overbrace{\mathbb{F}_1[X_0, X_1, \dots, X_m]}^{A}$;

$$\| \qquad \bigoplus_{i \geq 0} \underbrace{R_i}_{\text{elements of total degree } i} \quad (\text{grading})$$

<u>Irr</u> (irrelevant ideal) $= \bigoplus_{i \geq 1} R_i \quad (\cup \{0\})$

$$\| \qquad \bigcup_{i=1,m} X_i A$$

$\mathrm{Proj}(\mathbb{F}_1[X_0, \dots, X_m])$ = set of prime ideals not containing Irr (only Irr is left out); closed sets, etc. one as usual.

Each ideal $(X_i) = X_i A$ defines an open set $D((X_i))$ s.t. restriction of the scheme to $\mathrm{Spec}(\mathbb{F}_1[X_0, \dots, X_m])$ (i left out)

Closed pts = next to maximal ideals (maximal w.r.t. not being in Irr) $\underset{\text{form}}{\Longrightarrow} \bigcup_I AX_i \quad (|I| = m)$.

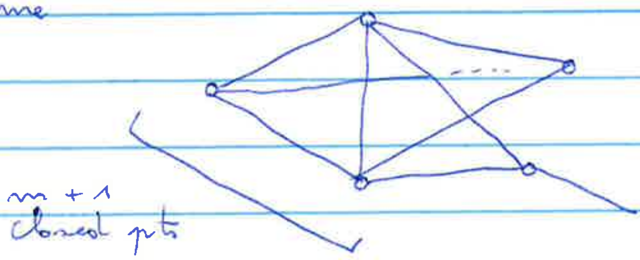
$\Rightarrow$ $m+1$ closed points

<u>Linear subspaces</u> correspond to ideals $D = \bigcup_{j \in J \subsetneq I} AX_j$

$\Rightarrow$ closed set $V(D)$ with coordinate ring

$$\mathbb{F}_1[X_0, \dots, X_m] / D$$

$\underset{\text{induced scheme}}{\Longrightarrow}$ $\mathrm{Proj}(\mathbb{F}_1[X_j]_{j \in J})$.



→ after Deitmar, Soulé

– CC-schemes. It appears that D-schemes do not suffice for a good theory: because there is no addition allowed, one usually needs too much equations to describe, e.g., a variety over $\mathbb{F}_1$ ⇒ after base extension to $\mathbb{Z}$, the corresponding $\mathbb{Z}$-schemes are very restricted (cf. toric varieties).

D-schemes $\hookrightarrow$ CC-schemes

Connes – Consani: an $\mathbb{F}_1$-scheme, or CC-scheme, is a triple $(X, \tilde{X}, e_X)$, with X a $\mathbb{Z}$-scheme, $\tilde{X}$ a D-scheme, and e_X a certain morphism between X and $\tilde{X} \times_D \mathbb{Z}$ (inducing bijections over fields); e_X shows / describes how in some larger category, how certain pieces of $\tilde{X} \times_D \mathbb{Z}$ are glued "over X" (essentially this replaces addition at least partially) ⇒ richer. (base change: $\tilde{X} \otimes_{\mathbb{F}_1} \mathbb{Z} = X$)

All other approaches are variations on D-, CC-schemes.

– KT: every (poss. infinite) graph defines a D-scheme for which the closed pts are the pts of the graph, and the edges determine the monoidal space completely.

(Ex. Complete graph → Proj-scheme

→ affine scheme)

The automorphism grp of the graph coincides with the scheme theoretic automorphism group of the D-scheme.

⇒ interpretation of Chevalley groups as $\mathbb{F}_1$-group schemes (cf. Soulé), and much more general grps.

KT
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