Smallest poles of Igusa's and topological zeta functions and solutions of polynomial congruences

Igusa's p-adic zeta function belongs to the domain of number theory. It was introduced by Weil in 1965 and the basic properties of this function were studied by Igusa. I consider here the relevant situation in which Igusa's p-adic zeta function is associated to a polynomial $f \in \mathbb{Z}[x_1, \ldots, x_n]$ and in which it is defined as the meromorphic continuation of the function that associates to a complex number s with positive real part the integral of $|f|_p^s$ over $\mathbb{Z}_p^n$. If we denote the number of solutions of the congruence $f \equiv 0 \bmod p^i$ by M_i, then we have that all the M_i's together form exactly the information which is contained in this zeta function. The intensive study of Igusa's p-adic zeta function by using an embedded resolution of f led to the introduction of the topological zeta function. This geometric invariant of the zero locus of a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ was introduced in the early nineties by Denef and Loeser. It is a rational function which they obtained as a limit of Igusa's p-adic zeta functions and which is defined by using an embedded resolution.

The smallest real parts of the poles of these zeta functions are studied in the thesis. Note that the poles of the topological zeta function are rational numbers. For this zeta function, it is thus unnecessary to take the real part.

For $n = 2$ and $n = 3$ these zeta functions are studied by using embedded resolutions. We determine all values less than $-1/2$ if $n = 2$ and less than -1 if $n = 3$ which are the real part of a pole. If $n = 2$ there are no poles with real part less than $-1/2$ and different from -1 if there is no singular point of multiplicity two or three. If $n = 3$ we prove that there are no poles with real part less than -1 if there is no singular point of multiplicity two. For this we have to prove that some candidate poles are not a pole. We use a formula of Veys to treat the topological zeta function, for Igusa's p-adic zeta function we have to deduce a formula for the residue at a candidate pole of expected order one ourselves.

We also give a description of the smallest real part of a pole of Igusa's p-adic zeta function in terms of the M_i. Using this approach, we prove for arbitrary n that there are no poles with real part less than $-n/2$. We get this lower bound also for the topological zeta function by taking the limit. An example shows us that this bound is optimal.