The cohomology of Monsky and Washnitzer
(Wouter Castryck)

Determining the number of solutions to a polynomial equation $f(x_1, \ldots, x_n) = 0$ over a finite field $\mathbb{F}_q$ is an old problem that dates back to the work of Gauss and Jacobi. However, for the big technical tool to attack the problem, one had to wait until 1949, when Weil defined his Zeta function (following previous work of Artin):

$$Z_f(X) = \exp\left(\sum_{k=1}^{\infty} N_k \frac{X^k}{k}\right) \quad \in \mathbb{Q}[[X]],$$

where N_k denotes the number of solutions to $f(x_1, \ldots, x_n) = 0$ over the extension field $\mathbb{F}_{q^k}$. Weil conjectured some properties of Z_f, the most famous among them being its rationality and an analogon of the classical Riemann hypothesis. He believed that these could be proven by developing an appropriate cohomology theory for varieties over finite fields. And indeed, about a decade later, Grothendieck e.a. developed the celebrated étale cohomology, by means of which Deligne settled the Weil conjectures in 1973.

However, already in 1959, Dwork was able to prove the rationality of the Zeta function. He implicitly made use of another type of cohomology: p-adic cohomology (whereas étale cohomology is often referred to as ℓ-adic cohomology). Attempts to make this explicit lead to various cohomology theories, such as Dwork cohomology, Monsky-Washnitzer cohomology, rigid cohomology, crystalline cohomology, ... Last year, Kedlaya was able to give a completely p-adic proof of the Weil conjectures.

The main advantage of p-adic cohomology is that it is often well suited for computation. In 2001, Kedlaya used Monsky-Washnitzer cohomology to develop an efficient algorithm for counting the number of points on a hyperelliptic curve defined over a finite field of odd characteristic.

In the first lecture, I will define the Monsky-Washnitzer cohomology spaces $H^i(\overline{X}, \mathbb{Q}_q)$ of a smooth affine hypersurface $\overline{X}/\mathbb{F}_q$, with emphasis on why they are defined the way they are. Next, I will go through the proof of the trace formula, expressing $\#\overline{X}$ by means of the traces of Frobenius acting on the cohomology spaces. The third lecture will be devoted to Kedlaya's algorithm.