Polynomial Equivalence of Generalized Ehrhart Series and Enumerator Functions

Sven Verdoolaege, Kevin Woods

sven@cs.kuleuven.ac.be, kwoods@math.berkeley.edu

K.U.Leuven—Dept. of Computer Science

UC-Berkeley—Mathematics

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Motivational Example 1

How many times is `S1` executed ?

```
for(i=1; i<=N; i++)
    for(j=1; j<=i; j++)
        S1;
```

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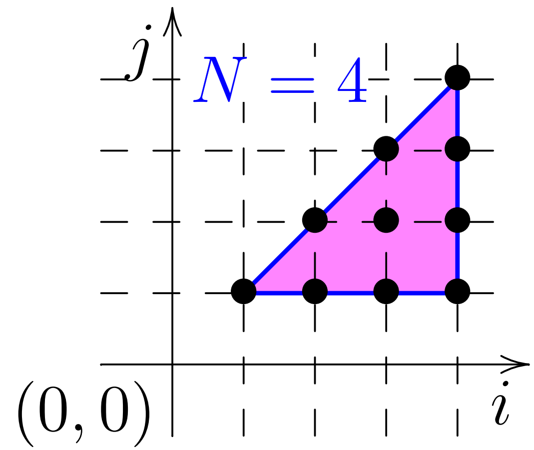

$$\#\{\,(i,j) \in \mathbb{Z}^2 \mid 1 \leq i \leq N \wedge 1 \leq j \leq i\,\} = \frac{N(N+1)}{2}$$

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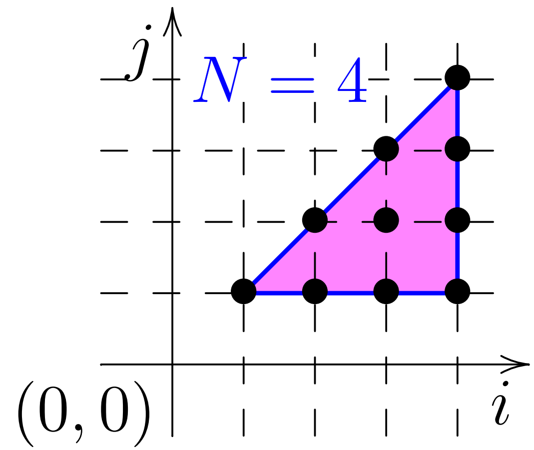

$$\#\{\, (i,j) \in \mathbb{Z}^2 \mid 1 \le i \le N \wedge 1 \le j \le i \,\} = \frac{N(N+1)}{2}$$

$$= \# \left\{ \begin{pmatrix} i \\ j \end{pmatrix} \in \mathbb{Z}^2 \mid \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix} \ge \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} N \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

2.4%

Motivational Example 2

How many times is S1 executed ?

```
for(i=max(0,N-M); i<=N-M+3; i++)
    for(j=0; j<=N-2*i; j++)
        S1;
```

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Motivational Example 2

How many times is `S1` executed ?

```
for(i=max(0,N-M); i<=N-M+3; i++)
    for(j=0; j<=N-2*i; j++)
        S1;
```

Equal to the number of elements in $P_{\left(\begin{smallmatrix} N \\ M \end{smallmatrix}\right)} =$

$$\left\{ \begin{pmatrix} i \\ j \end{pmatrix} \in \mathbb{Z}^2 \mid \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix} \geq \begin{pmatrix} 0 & 0 \\ 1 & -1 \\ -1 & 1 \\ 0 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} N \\ M \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -3 \\ 0 \\ 0 \end{pmatrix} \right\}$$

4.8%

Motivational Example 2

$$\#P_{\binom{N}{M}} =$$

$$\begin{cases} -4N + 8M - 8 & \text{if } M \le N \le 2M - 6 \\ MN - 2N - M^2 + 6M - 8 & \text{if } N \le M \le N + 3 \wedge N \le 2M - 6 \\ \frac{N^2}{4} + \frac{3}{4}N + \frac{1}{2}\left\lfloor \frac{N}{2} \right\rfloor + 1 & \text{if } 0 \le N \le M \wedge 2M \le N + 6 \\ \frac{N^2}{4} - MN - \frac{5}{4}N + M^2 + 2M + \frac{1}{2}\left\lfloor \frac{N}{2} \right\rfloor + 1 & \\ & \text{if } M \le N \le 2M \le N + 6 \end{cases}$$

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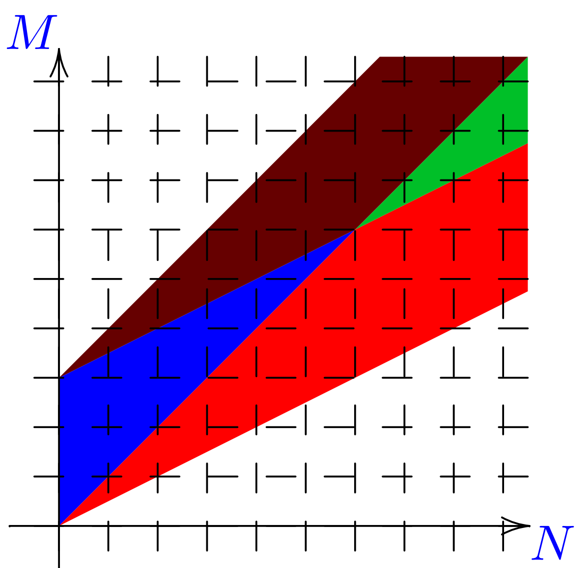
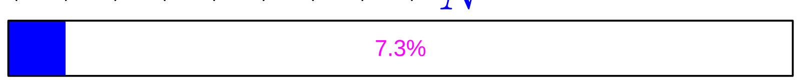
Chambers:





Motivational Example 3

How many elements of array a are accessed ?

```
for (j = 1; j <= p; ++j)
    for (i = 1; i <= 8; ++i)
        a[6i+9j-7] = a[6i+9j-7] + 5;
```

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Equal to the number of elements in

$$S_p = \{ l \in \mathbb{Z} \mid \exists i, j \in \mathbb{Z} : l = 6i + 9j - 7 \wedge 1 \leq j \leq p \wedge 1 \leq i \leq 8 \}$$

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$$\#S_p = \begin{cases} 8 & \text{if } p = 1 \\ 3p + 10 & \text{if } p \ge 2 \end{cases}$$

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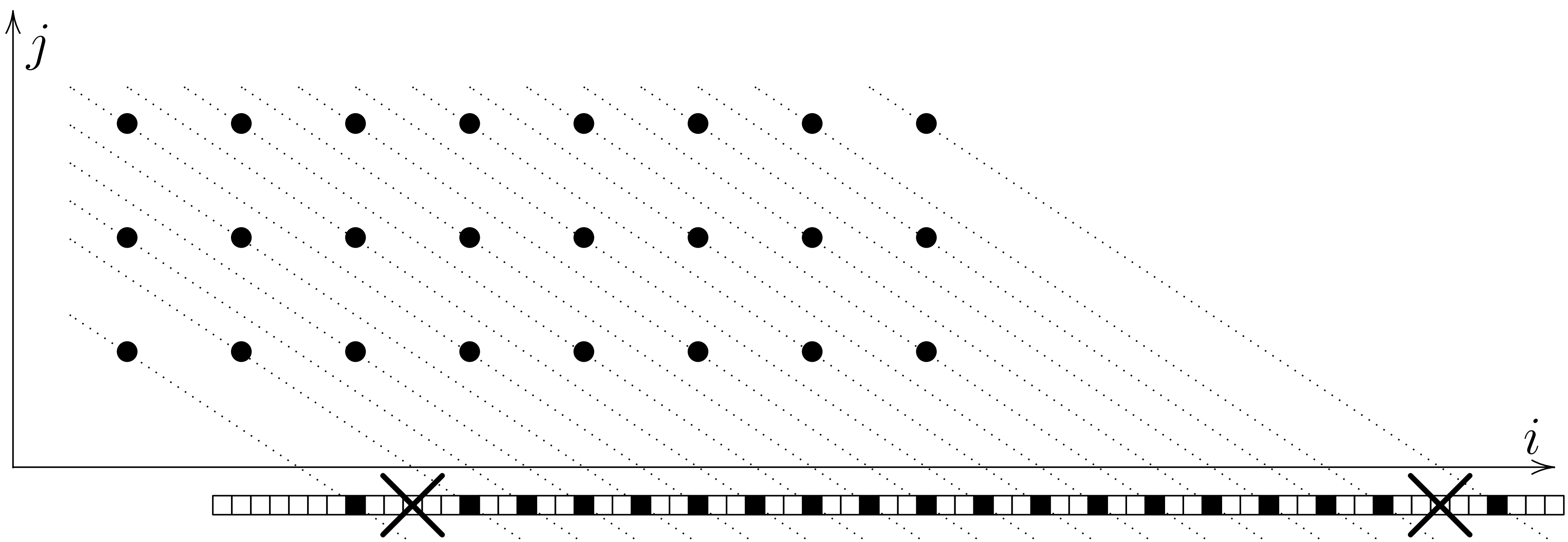
Motivational Example 3

Array reference: `a[6i+9j-7]`
Array elements accessed for $p = 3$:



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Overview

- Motivational Examples
- Parametric Counting Problems
- Two Representations + Main Theorem
- Barvinok's Algorithm
  - Computing Generating Functions
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Ehrhart Quasi-Polynomials

$P \subset \mathbb{Q}$ rational polytope
sP: P dilated by a factor of s

$$c_P(s) = \#(sP \cap \mathbb{Z}^d)$$

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sP: P dilated by a factor of s

$$c_P(s) = \#(sP \cap \mathbb{Z}^d)$$

Ehrhart (1962): $c_P(s)$ is a quasi-polynomial

- $\exists \mathcal{D}$: period of quasi-polynomial
- $\exists g_i \in \mathbb{Q}[T]$ for $0 \le i < \mathcal{D}$

$$c_P(s) = g_j(s) \qquad s \equiv j \mod \mathcal{D}$$

- $\mathcal{D}$ divides the lcm of the denominators of the vertices
- degree of g_i is at most d

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Ehrhart Quasi-Polynomials

$$P = \left[0, \frac{1}{2}\right] \in \mathbb{Q}^1$$

$$c(s) = \left\lfloor \frac{s}{2} \right\rfloor + 1$$

Ehrhart series:

$$C(x) = \frac{1}{(1-x)(1-x^2)}$$

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Vector Partition Functions

$A \in \mathbb{N}^{d \times n}$ of rank d

$$c(\mathbf{s}) = \#\left\{ \boldsymbol{\lambda} \in \mathbb{N}^d \mid \mathbf{s} = \lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 + \cdots + \lambda_d \mathbf{a}_d \right\}$$

$$C(\mathbf{x}) = \frac{1}{(1 - \mathbf{x}^{\mathbf{a}_1})(1 - \mathbf{x}^{\mathbf{a}_2}) \cdots (1 - \mathbf{x}^{\mathbf{a}_d})} = \sum_{\mathbf{s} \in \mathbb{Z}^d} c(\mathbf{s}) \mathbf{x}^{\mathbf{s}}$$

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Sturmfels (1995):

- Chamber complex $\{ C_i \}$: polyhedral subdivision of $\operatorname{pos} A$; common refinement of $\operatorname{pos} A_\sigma$ with $\sigma \subset \{ 1 \ldots, n \}$
- On each chamber C_i: (P, Q_i polynomials)

$$c(\mathbf{s}) = P(\mathbf{s}) + \sum_{\sigma \in \Delta(C_i)} \Omega_\sigma(\bar{\mathbf{s}}) Q_\sigma(\mathbf{s})$$

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Vector Partition Functions

$$c(\mathbf{s}) = \#\left\{\boldsymbol{\lambda} \in \mathbb{N}^4 \mid \begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \boldsymbol{\lambda} = \mathbf{s}\right\}$$

$$C(\mathbf{x}) = \frac{1}{(1-\mathbf{x}^{(1,1)})(1-\mathbf{x}^{(2,1)})(1-\mathbf{x}^{(1,0)})(1-\mathbf{x}^{(0,1)})}$$

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Vector Partition Functions

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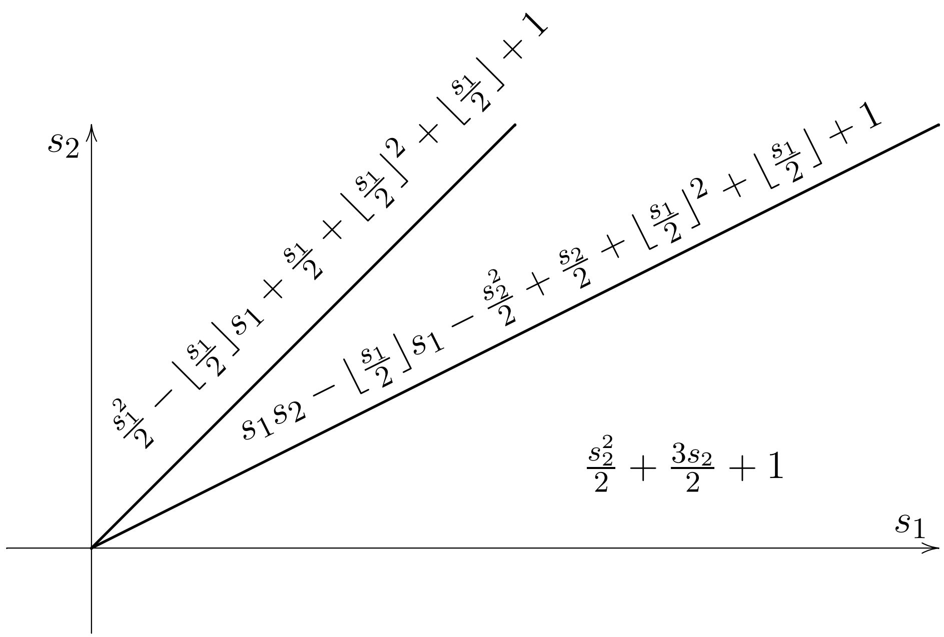

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Parametric Polytopes

$$c(\mathbf{p}) = \#(\mathbb{Z}^d \cap \{\mathbf{x} \in \mathbb{Q}^d \mid A\mathbf{x} + C\mathbf{p} + \mathbf{b} \geq 0\})$$

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Parametric Polytopes

$$c(\mathbf{p}) = \#(\mathbb{Z}^d \cap \{\mathbf{x} \in \mathbb{Q}^d \mid A\mathbf{x} + C\mathbf{p} + \mathbf{b} \geq 0\})$$

- Vertices $V_j(\mathbf{p}) = \sum_i \lambda_{ji} p_i + f_j$
- Chamber decomposition $\{ C_i \}$
- On each chamber C_i
  - Fixed set of active vertices $V(\mathbf{p})$

$$P_{\mathbf{p}} = \left\{ \mathbf{x} \in \mathbb{Q}^d \mid V(\mathbf{p})\boldsymbol{\nu}, \boldsymbol{\nu} \geq \mathbf{0}, \sum \nu = 1 \right\}$$

  - Multivariate quasi-polynomial of degree d in $\mathbf{p}$
  - Period in dimension i is lcm of denominators of λ_{ji}

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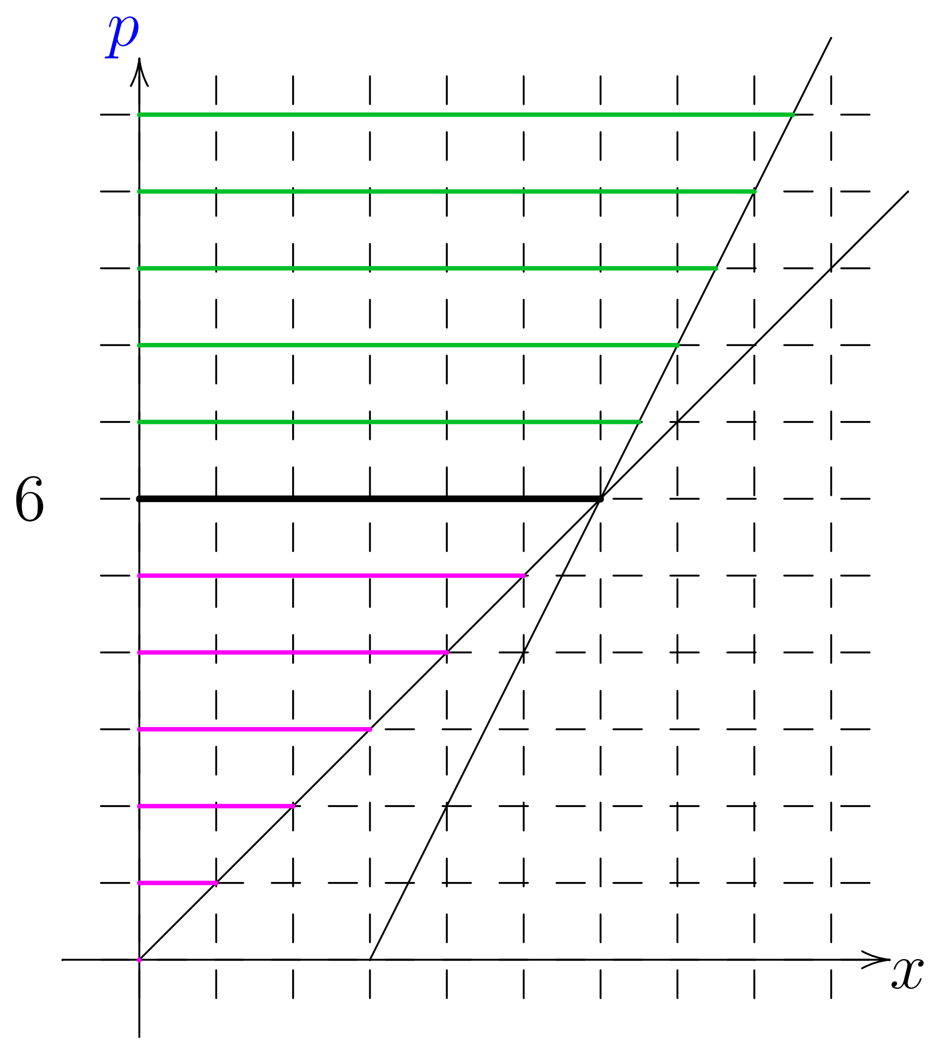
Vertices and Chambers

$$P = \{\, x \mid x \geq 0, 2x \leq p + 6, x \leq p \,\}$$

Vertices   Active if



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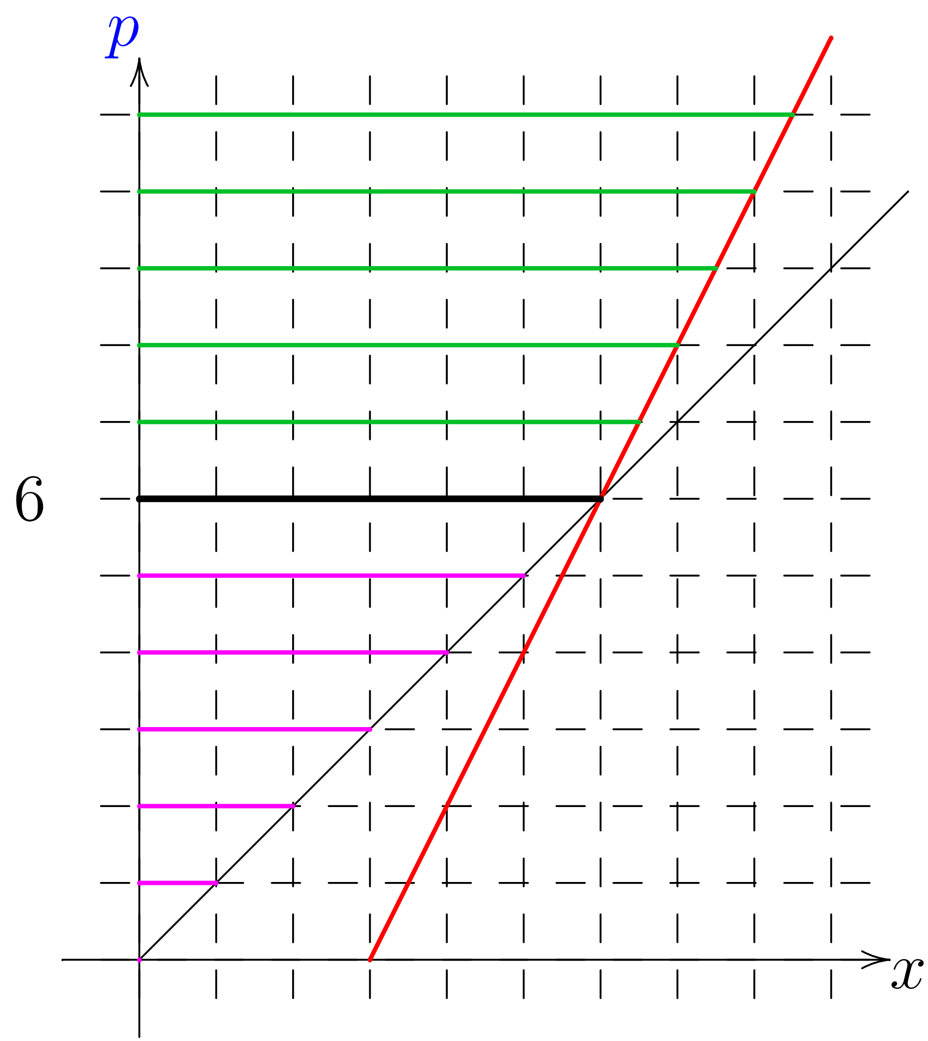
Vertices	Active if
$x = \frac{p}{2} + 3$	$p \geq 6$



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Vertices and Chambers

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Vertices	Active if
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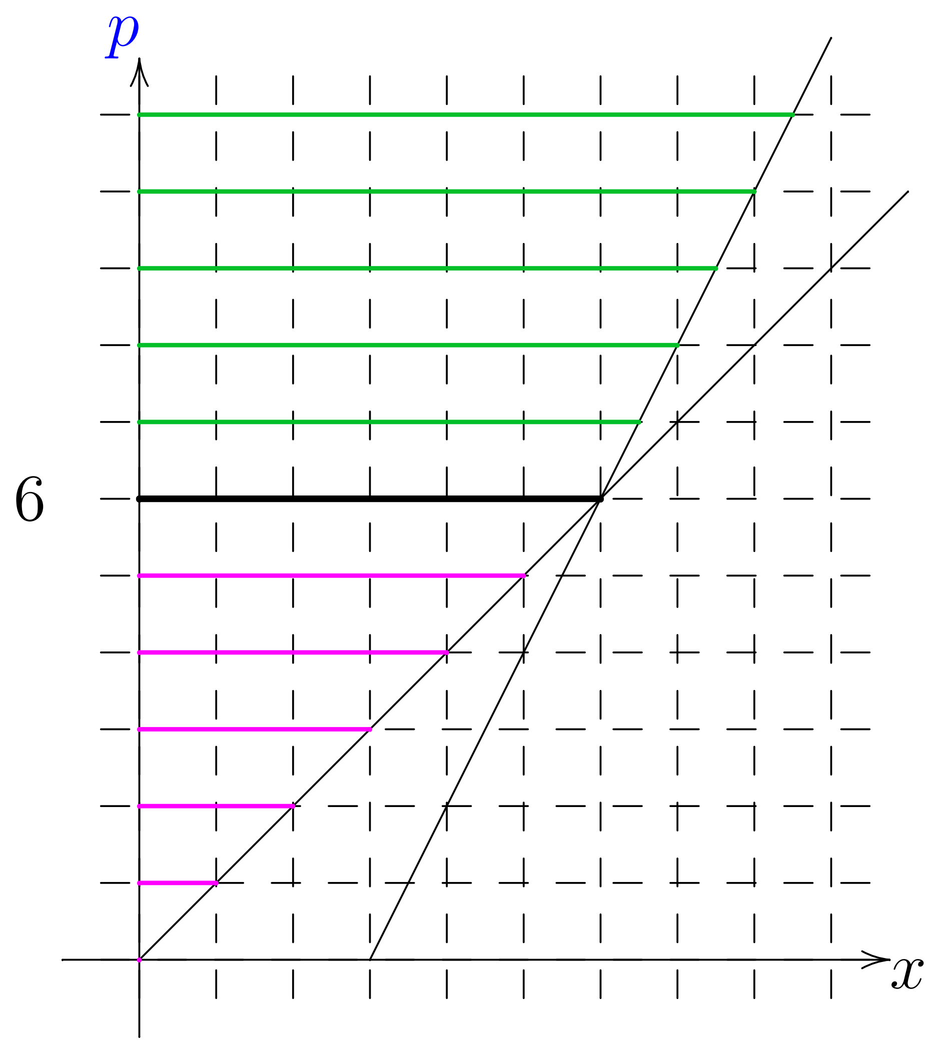

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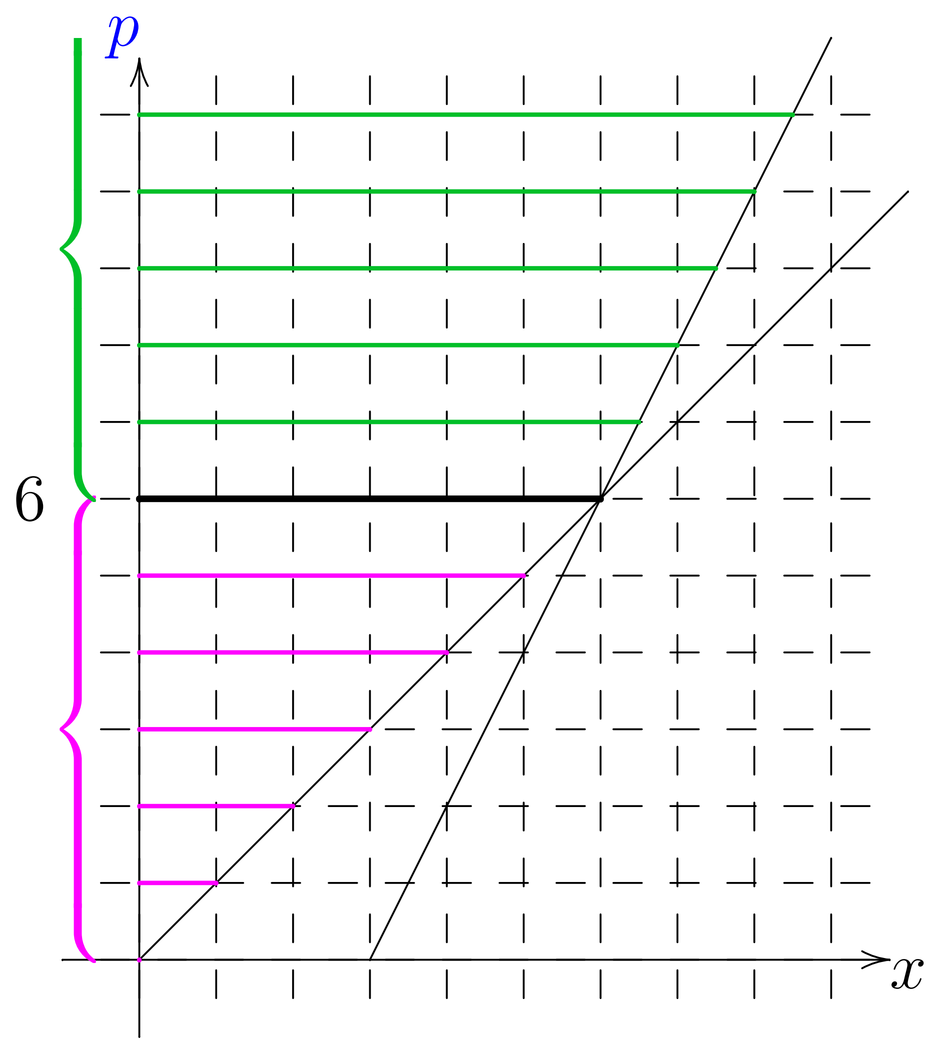
Chambers:
fixed set of vertices

$0 \leq p \leq 6$ $\quad$ $p \geq 6$



General Problem

Enumeration of projection of integer points in a polytope

$$
\begin{aligned}
c(\mathbf{p}) &= \#(\{\mathbf{x} \in \mathbb{Z}^d \mid \exists \mathbf{y} \in \mathbb{Z}^{d'} : A\mathbf{x} + D\mathbf{y} + C\mathbf{p} + \mathbf{b} \geq 0\}) \\
&= \#(\mathbb{Z}^d \cap \pi_d\{(\mathbf{x}, \mathbf{y}) \in \mathbb{Q}^{d+d'} \mid A\mathbf{x} + D\mathbf{y} + C\mathbf{p} + \mathbf{b} \geq 0\})
\end{aligned}
$$

31.7%

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Piecewise Step-Polynomials

$f : \mathbb{Z}^d \to \mathbb{Q}$ is a *step-polynomial* iff

$$f(\mathbf{s}) = \sum_{j=1}^{n} \alpha_j \prod_{k=1}^{d_j} \left\lfloor \langle \mathbf{a}_{jk}, \mathbf{s} \rangle + \mathbf{b}_{jk} \right\rfloor \qquad \alpha_j \in \mathbb{Q}, \mathbf{a}_{jk} \in \mathbb{Q}^d, \mathbf{b}_{jk} \in \mathbb{Q}$$

degree of f: $\deg f = \max_j \{ d_j \}$

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degree of f: $\deg f = \max_j \{ d_j \}$

$c : \mathbb{Z}^d \to \mathbb{Q}$ is a *piecewise step-polynomial* iff there exists a polyhedral subdivision $\{ C_i \}$ and step-polynomials f_i such that

$$c(\mathbf{s}) = \begin{cases} f_i(\mathbf{s}) & \text{if } \mathbf{s} \in C_i \cap \mathbb{Z}^d \\ 0 & \text{if } \mathbf{s} \notin (\bigcup_i C_i) \cap \mathbb{Z}^d \end{cases}$$

$\deg c = \max_i \deg f_i$

34.1%

Rational Generating Functions

$C(\mathbf{x})$ is a *rational generating function* iff

$$C(\mathbf{x}) = \sum_{i \in I} \alpha_i \frac{\mathbf{x}^{\mathbf{p}_i}}{(1 - \mathbf{x}^{\mathbf{b}_{i1}})(1 - \mathbf{x}^{\mathbf{b}_{i2}}) \cdots (1 - \mathbf{x}^{\mathbf{b}_{ik_i}})}$$

$\mathbf{x} \in \mathbb{C}^d$, $\alpha_i \in \mathbb{Q}$, $\mathbf{p}_i \in \mathbb{Z}^d$, $\mathbf{b}_{ij} \in \mathbb{Z}^d \setminus \{0\}$

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Comparison

$$C(\mathbf{x}) = \sum_{\mathbf{s}\in\mathbb{Z}^d} c(\mathbf{s})\mathbf{x}^{\mathbf{s}}$$

Evaluation:	Trivial	Not so trivial
Summation:	$d(\mathbf{t}) = \sum_{\mathbf{s}} c(\mathbf{s},\mathbf{t})$ New counting problem(s)	$D(\mathbf{x}) = C(\mathbf{1},\mathbf{x})$ Plug in $\mathbf{1}$
Projection:	Hard	Polynomial

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Main Theorem

For fixed dimensions:

- The generating function of a piecewise step-polynomial can be computed in polynomial time in the form of a rational generating function.
- The explicit function corresponding to a rational generating function can be computed in polynomial time in the form of a piecewise step-polynomial.

Strategy: reduce problem to a set of counting problems and apply Barvinok's algorithm

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Definitions

- Indicator function of a set A:

$$[A] : \mathbb{Q}^d \to \mathbb{Q} : [A](\mathbf{s}) = \begin{cases} 1 & \text{if } \mathbf{s} \in A \\ 0 & \text{if } \mathbf{s} \notin A \end{cases}$$

- Generating function of A (or $A \cap \mathbb{Z}^d$):
  "generating function of $[A]|_{\mathbb{Z}^d}$"

$$f(A; \mathbf{x}) = \sum_{\mathbf{s} \in A \cap \mathbb{Z}^d} \mathbf{x}^{\mathbf{s}}$$

  If A contains a line then $f(A; \mathbf{x}) \equiv 0$
  Note: $f(A; \mathbf{1}) = \#(A \cap \mathbb{Z}^d)$

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Barvinok's Algorithm

Given a polytope $P \subset \mathbb{Q}^d$:

- Compute generating function $f(P; \mathbf{x})$
- Evaluate $f(P; \mathbf{1})$
  $\Rightarrow$ constant term in Laurent expansion of each term

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Parametric polytope $P_{\mathbf{p}}$:

- Piecewise step-polynomial
  - Compute parametric vertices and chambers
  - Apply Barvinok's algorithm on each chamber

45.5%

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Parametric polytope $P_{\mathbf{p}}$:

- Piecewise step-polynomial
  - Compute parametric vertices and chambers
  - Apply Barvinok's algorithm on each chamber
- Generating function
  - Consider polyhedron $P = \{ (\mathbf{p}, \mathbf{t}) \mid \mathbf{t} \in P_{\mathbf{p}} \}$
  - Apply Barvinok's algorithm with *partial* evaluation
    $f(P; (\mathbf{x}, \mathbf{y})) \rightsquigarrow f(P; (\mathbf{x}, \mathbf{1}))$

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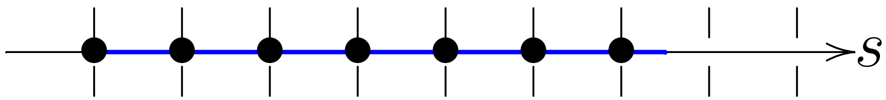
Barvinok Example



$$S = \{ s \mid s \geq 0 \wedge 2s \leq 13 \}$$

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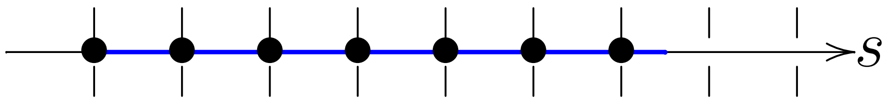
Barvinok Example



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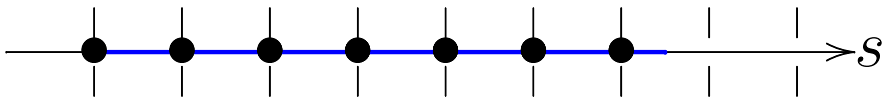
Barvinok Example



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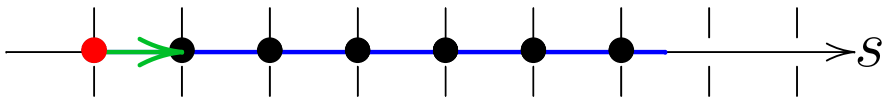
Barvinok Example



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Generating function of a polytope $=$ sum of generating functions of its “supporting cones”

$$f(S;\mathbf{x}) = \frac{x^0}{1 - x}$$

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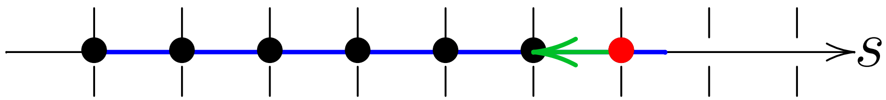
Barvinok Example



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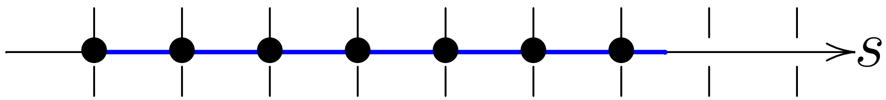
Generating function of a polytope $=$ sum of generating functions of its "supporting cones"

$$f(S;\mathbf{x}) = \frac{x^0}{1-x} + \frac{x^6}{1-x^{-1}}$$

47.8%

Barvinok Example



$S = \{ s \mid s \geq 0 \wedge 2s \leq 13 \}$

Generating function: $f(P; \mathbf{x}) = \sum_{\boldsymbol{\alpha} \in P \cap \mathbb{Z}^d} \mathbf{x}^{\boldsymbol{\alpha}}$

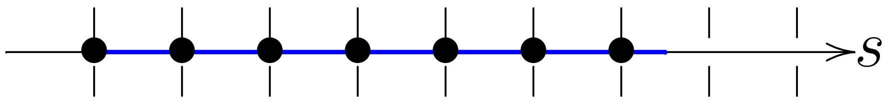
$$f(S; x) = x^0 + x^1 + x^2 + x^3 + x^4 + x^5 + x^6$$
$$f(S; 1) = 1^0 + 1^1 + 1^2 + 1^3 + 1^4 + 1^5 + 1^6 = 7 = \#S$$

Generating function of a polytope $=$ sum of generating functions of its “supporting cones”

$$f(S; \mathbf{x}) = \frac{x^0}{1-x} + \frac{x^6}{1-x^{-1}} \left(= \frac{x^0}{1-x} - \frac{x^6 x}{1-x} \right)$$

48.1%

Barvinok Example



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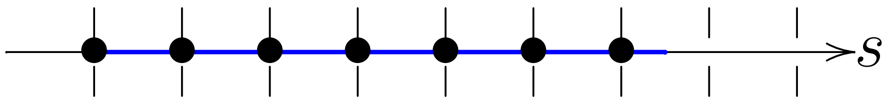
$$-1(x-1)^{-1} + 0(x-1)^0 + \cdots \qquad +1(x-1)^{-1} + 7(x-1)^0 + \cdots$$

48.4%

Barvinok Example



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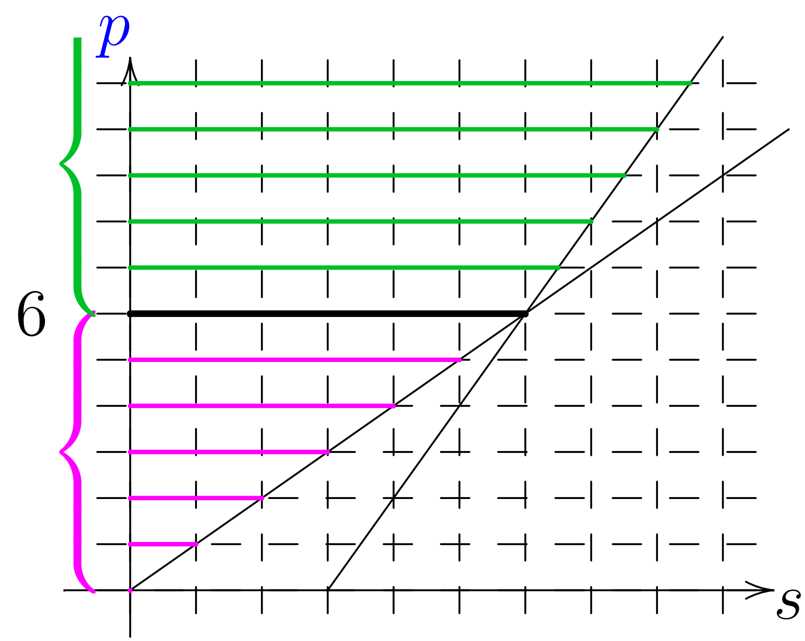
$$-1(x-1)^{-1} + 0(x-1)^0 + \cdots \qquad +1(x-1)^{-1} + 7(x-1)^0 + \cdots$$
$$f(S;\mathbf{1}) = 0 + 7 = 7$$

Parametric Barvinok Example

$$P = \{ s \mid s \geq 0, 2s \leq p + 6, s \leq p \}$$



Vertex	0	p	$\frac{p}{2} + 3$
Generator supporting cone	$\frac{x^0}{1-x}$	$\frac{x^p}{1-x^{-1}}$	$\frac{x^{\lfloor \frac{p}{2} \rfloor + 3}}{1-x^{-1}}$
Laurent coefficient	0	$p+1$	$\lfloor \frac{p}{2} \rfloor + 4$

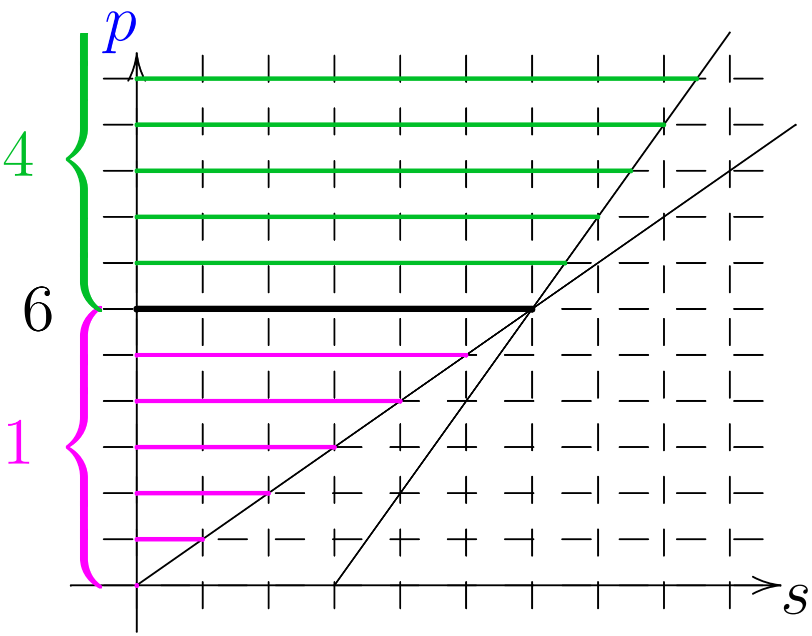
50.0%

Parametric Barvinok Example

$$P = \{\, s \mid s \geq 0, 2s \leq p + 6, s \leq p \,\}$$

$$p \geq 6 \Rightarrow \mathcal{E}(p) = 0 + \left\lfloor \frac{p}{2} \right\rfloor + 4$$

$$0 \leq p \leq 6 \Rightarrow \mathcal{E}(p) = 0 + p + 1$$



Vertex	0	p	$\frac{p}{2} + 3$
Generator supporting cone	$\frac{x^0}{1 - x}$	$\frac{x^p}{1 - x^{-1}}$	$\frac{x^{\left\lfloor \frac{p}{2} \right\rfloor + 3}}{1 - x^{-1}}$
Laurent coefficient	0	$p + 1$	$\left\lfloor \frac{p}{2} \right\rfloor + 4$

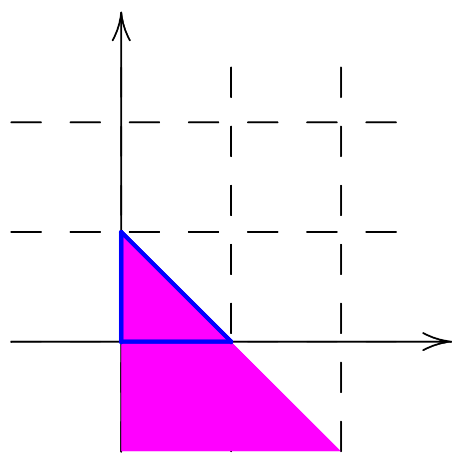
Brion's Theorem

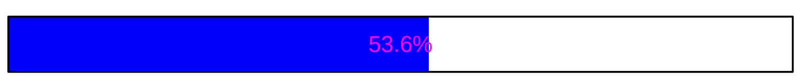
$$f(P;\mathbf{x}) = \sum_{\mathbf{v} \text{ a vertex of } P} f(\mathrm{cone}(P,\mathbf{v});\mathbf{x})$$

$P \subset \mathbb{Q}^d$ a polyhedron; $\mathrm{cone}(P,\mathbf{v})$ the supporting cone at $\mathbf{v}$

$$\mathrm{cone}(P,F) = \left\{ \mathbf{x} \in \mathbb{Q}^d \mid \langle \mathbf{a}_i, \mathbf{x} \rangle \geq \mathbf{b}_i \text{ for } i \in I_F \right\}$$

where I_F is the set of inequalities active on face F.

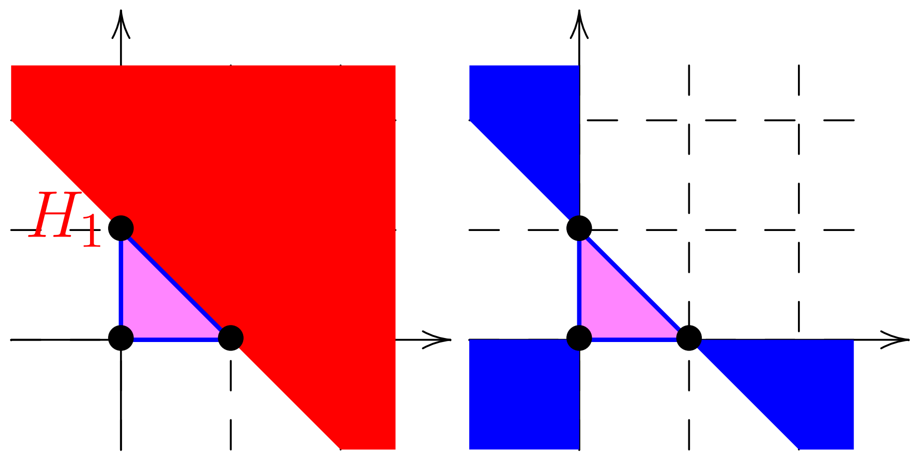




Brion's Theorem

Intuition:

$$[\mathbb{Q}^2] - [T] = [H_1] + [H_2] + [H_3] - [H_1 \cap H_2] - [H_2 \cap H_3] - [H_3 \cap H_1]$$



$$[T] = [H_1 \cap H_2] + [H_2 \cap H_3] + [H_3 \cap H_1] \qquad \text{mod } \mathcal{L}$$

$$f(H_1 \cap H_2; \mathbf{x}) = f(\text{cone}(T, \mathbf{v}_1); \mathbf{x})$$

56.0%

Unimodular Cone

$K = \text{pos}\{\, \mathbf{u}_i \,\}$ is unimodular iff

$$\left|\det \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_d \end{bmatrix}\right| = 1$$

$$K \cap \mathbb{Z}^d = \left\{\, \mathbf{s} = \lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{u}_2 + \cdots + \lambda_d \mathbf{u}_d \mid \boldsymbol{\lambda} \in \mathbb{N}^d \,\right\}$$

$$\begin{aligned} c(\mathbf{s}) &= \#\left\{\, \boldsymbol{\lambda} \in \mathbb{N}^d \mid \mathbf{s} = \lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{u}_2 + \cdots + \lambda_d \mathbf{u}_d \,\right\} \\ &= 1 \qquad \text{if } \mathbf{s} \in K \cap \mathbb{Z}^d \end{aligned}$$

$$f(K \cap \mathbb{Z}^d; \mathbf{x}) = \frac{1}{(1 - \mathbf{x}^{\mathbf{u}_1})(1 - \mathbf{x}^{\mathbf{u}_2}) \cdots (1 - \mathbf{x}^{\mathbf{u}_d})}$$

58.5%

Arbitrary Cone

- Non-simplicial cone
  $\Rightarrow$ Triangulate
- Simplicial cone
  $\Rightarrow$ Barvinok's decomposition

$$[K] = \sum_{i \in I} \epsilon_i \, [K_i]$$

with K_i unimodular, $\epsilon_i \in \{ -1, 1 \}$ and $|I|$ bounded by a polynomial in the input size of K

60.9%

Barvinok's Decomposition

$$K = \operatorname{pos}\{\mathbf{u}_i\}_{1\le i\le d}$$

Consider

$$B = \left\{ \sum_{i=1}^{d} \alpha_i \mathbf{u}_i \mid \forall i : |\alpha_i| \le |\det(K)|^{-1/d} \right\}$$

B is centrally symmetric and has volume 2^d
$\Rightarrow B$ contains an integer point $\mathbf{w}$
Let

$$K_j = \operatorname{pos}(\{\mathbf{u}_i\} \setminus \{\mathbf{u}_j\} \cup \{\mathbf{w}\})$$

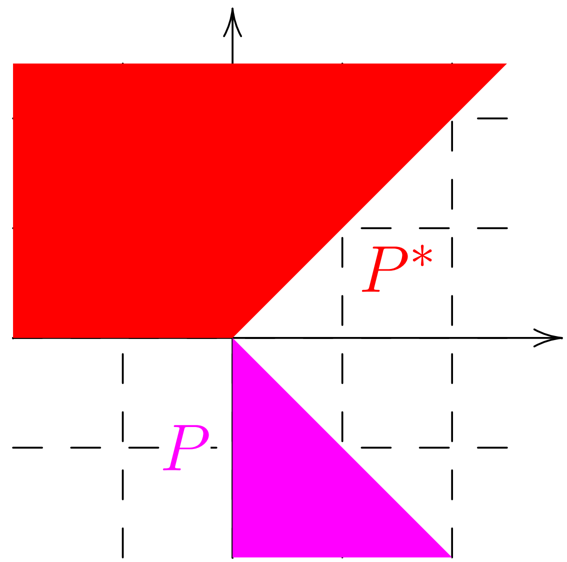
$$|\det K_j| \le |\alpha_j| |\det K| \le |\det K|^{(d-1)/d}$$

63.4%

Polar Cone

- Polar cone

$$P^* = \{\, \mathbf{y} \mid \forall \mathbf{x} \in P : \langle \mathbf{x}, \mathbf{y} \rangle \leq 0 \,\}$$



Or

$$P^* = \{\, \mathbf{y} \mid \forall \mathbf{x} \in P : \langle \mathbf{x}, \mathbf{y} \rangle \geq 0 \,\}$$

65.8%

Le Truc de Brion

Barvinok's decomposition yields:

$$[K] = \sum_j \epsilon_j \, [K_j] + \sum_F \epsilon_F \, [F]$$

Brion's trick: apply decomposition on polar of K

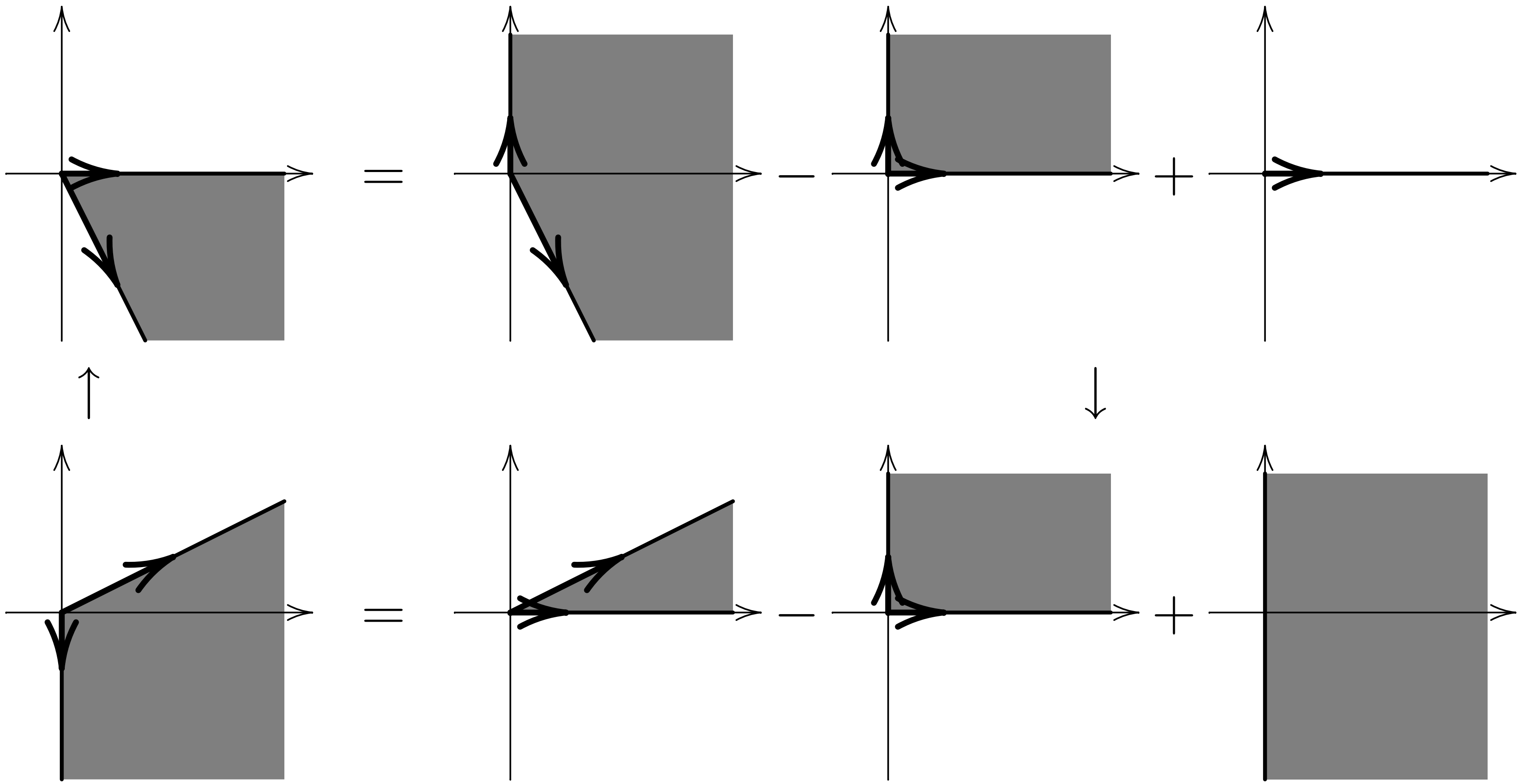
$$[K^*] = \sum_j \epsilon_j \, [K_j] + \sum_F \epsilon_F \, [F]$$

$$[K] = \sum_j \epsilon_j \, [K_j^*] + \sum_F \epsilon_F \, [F^*] = \sum_j \epsilon_j \, [K_j^*]$$

(F is lowerdimensional $\Rightarrow$ F^* contains a line)

68.2%

Polar Decomposition



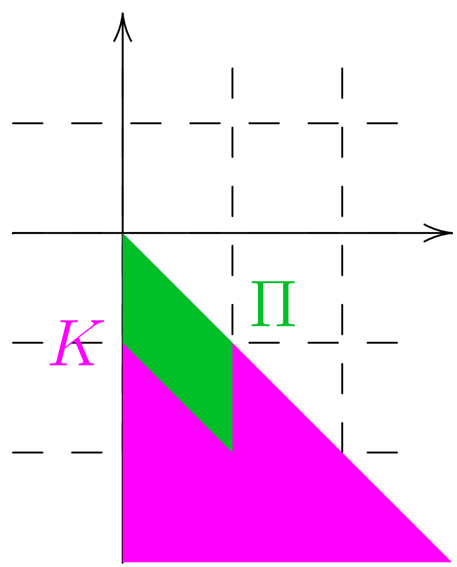
70.7%

Fundamental Parallelepiped

- Fundamental parallelepiped Π of cone K

$$K = \text{pos}\,\{\mathbf{u}_i\}_{1\le i\le k}$$

$$\Pi = \left\{ \sum_i \lambda_i \mathbf{u}_i \mid 0 \le \lambda_i < 1 \right\}$$



73.1%

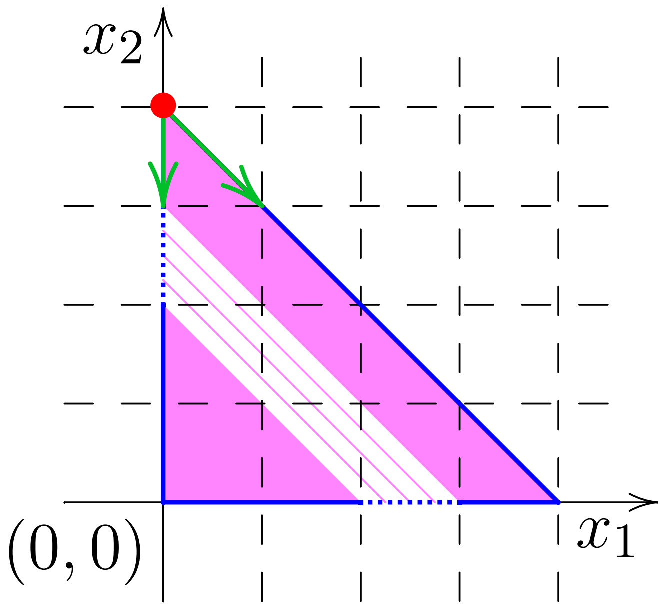
Translation

In general, supporting cone does not have apex $\mathbf{0}$, but $\mathbf{v}$

- Compute $K = \mathrm{cone}(P, \mathbf{v}(\mathbf{p})) - \mathbf{v}(\mathbf{p})$
- Compute Barvinok's decomposition $\{\, (\epsilon_i, K_i) \,\}$
- $f((K_i + \mathbf{v}(\mathbf{p})); \mathbf{x}) = \mathbf{x}^{\mathbf{v}'(\mathbf{p})} f(K_i; \mathbf{x})$
  - $\mathbf{v}'(\mathbf{p})$ is (single) point in fundamental parallelepiped of $K_i + \mathbf{v}(\mathbf{p})$
  - $\mathbf{v}'(\mathbf{p}) = \lfloor \langle \boldsymbol{\alpha}, \mathbf{p} \rangle + \beta \rfloor$ for some $\boldsymbol{\alpha} \in \mathbb{Q}^d, \beta \in \mathbb{Q}$
- $f(\mathrm{cone}(P, \mathbf{v}(\mathbf{p})); \mathbf{x}) = \sum_i f((K_i + \mathbf{v}(\mathbf{p})); \mathbf{x})$

75.6%

2D Example



$$\{\, \mathbf{x} \mid x_1 \geq 0 \wedge x_2 \geq 0 \wedge 2x_1 + 2x_2 \leq p \,\}$$

$$\frac{x_2^{\lfloor \frac{p}{2} \rfloor}}{(1 - x_2^{-1})(1 - x_1 x_2^{-1})} + \frac{x_1^{\lfloor \frac{p}{2} \rfloor}}{(1 - x_1^{-1})(1 - x_1^{-1} x_2)} + \frac{1}{(1 - x_1)(1 - x_2)}$$

$$f(T; \mathbf{1}) = \frac{1}{2} \left\lfloor \frac{p}{2} \right\rfloor^2 + \frac{3}{2} \left\lfloor \frac{p}{2} \right\rfloor + 1$$

78.0%

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78.0%

Specialization

- Given $f(\mathbf{x})$ as a rational generating function
- Compute $g(\mathbf{z}) = f(z_1, \ldots, z_m, 1, \ldots, 1)$ as a rational generating function

$C(\mathbf{x})$ is a rational generating function iff

$$C(\mathbf{x}) = \sum_{i \in I} \alpha_i \frac{\mathbf{x}^{\mathbf{p}_i}}{(1 - \mathbf{x}^{\mathbf{b}_{i1}})(1 - \mathbf{x}^{\mathbf{b}_{i2}}) \cdots (1 - \mathbf{x}^{\mathbf{b}_{ik_i}})},$$

$\mathbf{x} \in \mathbb{C}^d$, $\alpha_i \in \mathbb{Q}$, $\mathbf{p}_i \in \mathbb{Z}^d$, $\mathbf{b}_{ij} \in \mathbb{Z}^d \setminus \{0\}$

$(z_1, \ldots, z_m, 1, \ldots, 1)$ may be a pole of some term !

80.4%

Specialization

- Compute $\boldsymbol{\lambda} \in \mathbb{Z}^{d-m}$ such that for each $i \in I$
  - $(b_{i1}, \ldots, b_{im}) \neq \mathbf{0}$ or
  - $\langle (b_{i(m+1)}, \ldots, b_{id}), \boldsymbol{\lambda} \rangle \neq 0$.
- Substitute $(1+t)^{\lambda_i}$ for x_{m+i}

$$h(t) = f(z_1, \ldots, z_m, (1+t)^{\lambda_1}, \ldots (1+t)^{\lambda_{d-m}})$$

- Compute constant term in Laurent expansion of each term

$$h_i(t) = \alpha_i \frac{\mathbf{z}^{\mathbf{p}_i'}(t+1)^{q_i}}{\prod_{j=1}^{k_i} \left(1 - \mathbf{z}^{\mathbf{b}_{ij}'}(t+1)^{v_{ij}}\right)}$$

82.9%

Specialization

$$
\begin{aligned}
h_i(t) &= \alpha_i \frac{\mathbf{z}^{\mathbf{p}'_i}(t+1)^{q_i}}{\prod_{j=1}^{k_i}\left(1-\mathbf{z}^{\mathbf{b}'_{ij}}(t+1)^{v_{ij}}\right)} \\
&= C(\mathbf{z})\frac{(t+1)^{q_i}}{\prod_r((t+1)^{\beta_j}-1)\prod_s((t+1)^{\gamma_j}-\mathbf{z}^{\mathbf{c}_j})} \\
&= C(\mathbf{z})\frac{P(t)}{t^r\prod Q_j(t;\mathbf{z})} \\
&= C(\mathbf{z})\frac{P(t)}{t^r Q(t)}
\end{aligned}
$$

Constant term of $h_i(t)$ is coefficient of t^r in $\frac{P(t)}{t^r Q(t)}$

85.3%

Specialization

$$\frac{P(t)}{Q(t)} =: c_0 + c_1 t + c_2 t^2 + \cdots$$

$$\begin{aligned} P(t) &=: a_0 + a_1 t + a_2 t^2 + \cdots \\ Q(t) &=: b_0 + b_1 t + b_2 t^2 + \cdots \end{aligned}$$

$$c_j = \frac{1}{b_0}\left(a_j - \sum_{i=1}^{j} b_i c_{j-i}\right)$$

$$b_0 = 1 - \mathbf{z}^{\mathbf{c}_j} \qquad P(t) = (1+t)^r = \sum_k \binom{r}{k} t^k$$

87.8%

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87.8%

Conversion

- Rational generating function to piecewise step-polynomial

$$
\begin{aligned}
C(\mathbf{x}) &= \sum_{i \in I} \alpha_i \frac{\mathbf{x}^{\mathbf{p}_i}}{(1-\mathbf{x}^{\mathbf{b}_{i1}})(1-\mathbf{x}^{\mathbf{b}_{i2}})\cdots(1-\mathbf{x}^{\mathbf{b}_{ik_i}})} \\
&= \sum_{i \in I} \alpha_i \mathbf{x}^{\mathbf{p}_i} \left(\frac{1}{(1-\mathbf{x}^{\mathbf{b}_{i1}})(1-\mathbf{x}^{\mathbf{b}_{i2}})\cdots(1-\mathbf{x}^{\mathbf{b}_{ik_i}})} \right)
\end{aligned}
$$

$\Rightarrow |I|$ vector partition functions

$$
D(\mathbf{x}) = \alpha \mathbf{x}^{\mathbf{p}} C(\mathbf{x}) \Rightarrow d(\mathbf{s}) = \alpha \cdot c(\mathbf{s} - \mathbf{p})
$$

90.2%

Conversion

- Piecewise step-polynomial to rational generating function
  Base case:

$$c(\mathbf{s}) = \begin{cases} \prod_{j=1}^{d} \lfloor \langle \mathbf{a}_j, \mathbf{s} \rangle + \mathbf{b}_j \rfloor & \text{if } \mathbf{s} \in C \cap \mathbb{Z}^d \\ 0 & \text{if } \mathbf{s} \notin C \cap \mathbb{Z}^d \end{cases}$$

$$c(\mathbf{s}) = \#\{\mathbf{t} \in \mathbb{Z}^d \mid (\mathbf{s}, \mathbf{t}) \in Q\}$$

$$Q = \{(\mathbf{s}, \mathbf{t}) \in C \times \mathbb{Q}^d \mid 1 \le t_j \le \langle \mathbf{a}_j, \mathbf{s} \rangle + b_j, \text{ for } 1 \le j \le d\}$$

$\Rightarrow Q_{\mathbf{s}}$ is a parametric polytope

92.6%

Projection

$$P_{\mathbf{p}} = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{Q}^{d+d'} \mid A\mathbf{x} + D\mathbf{y} + C\mathbf{p} + \mathbf{b} \geq 0\}$$

$$Q_{\mathbf{p}} = \pi_d P_{\mathbf{p}}$$

How to compute $c(\mathbf{p}) = \#(\mathbb{Z}^d \cap Q_{\mathbf{p}})$?

93.9%

Projection

$$P_{\mathbf{p}} = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{Q}^{d+d'} \mid A\mathbf{x} + D\mathbf{y} + C\mathbf{p} + \mathbf{b} \geq 0\}$$

$$Q_{\mathbf{p}} = \pi_d P_{\mathbf{p}}$$

How to compute $c(\mathbf{p}) = \#(\mathbb{Z}^d \cap Q_{\mathbf{p}})$?

- Compute generating function $f(P; \mathbf{x})$
- Apply projection (Barvinok and Woods 2003)
  $\Rightarrow f(Q; \mathbf{x}) = C(\mathbf{x})$
- Apply conversion to piecewise step-polynomial
  $\Rightarrow c(\mathbf{p})$

95.1%

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95.1%

Conclusions and Future Work

- “Equivalence” of explicit functions and generating functions
- Polynomial algorithm for enumerating projections of the integer points in parametric polytopes
- Future Work
  - Simplification of step-polynomials
  - Implementation of projection algorithm

Available from

`http://freshmeat.net/projects/barvinok/`

97.5%

References

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100.0%