Degree functions and integrally closed ideals in dimension two

Degree functions were introduced by Rees. With an m-primary ideal I of a d-dimensional Noetherian local domain (R, m), Rees associated a function d_I, the so-called degree function associated with I. Rees proved the following formula for d_I: for every non-zero element x of m, one has

$$d_I(x) = \sum_v d(I, v) \ v(x)$$

where the sum is over all prime divisors v of R and $d(I, v) = 0$ for almost every v. The theory was further developed by Rees and Sharp who used it among others to study a conjecture of Teissier.

We examine the effect of a quadratic transformation of a 2-dimensional regular local ring (R, m) on the integers $d(I, v)$. This enables us to use the degree functions in the theory of complete ideals in 2-dimensional regular local rings. Besides the theory of Rees and Sharp, we also use the length formula of Hoskin-Deligne, of which we give a new, short and elementary proof. Our analysis of the birational behaviour of the integers $d(I, v)$, allows us to yield particularly short proofs of some well-known results concerning complete ideals in 2-dimensional regular local rings, a.o. Zariski's Unique Factorization Theorem, Zariski's one-to-one Correspondence, Lipman's Multiplicity and Reciprocity Formula. We also prove a criterion for regularity.

Moreover, the theory of degree functions has the advantage of not being restricted to the class of 2-dimensional regular local rings, it suffices that the ring is Cohen-Macaulay. This enables us to extend some results of Noh to a larger class of local rings than the class of 2-dimensional regular local rings. Finally, we provide some examples which give a concrete illustration of the theory.