The packing problem in statistics, coding theory and finite projective spaces: update 2001

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Abstract

This article updates the authors' 1998 survey [133] on the same theme that was written for the Bose Memorial Conference (Colorado, June 7-11, 1995). That article contained the principal results on the packing problem, up to 1995. Since then, considerable progress has been made on different kinds of subconfigurations.

1 Introduction

1.1 The packing problem

The packing problem in statistics, coding theory and finite projective spaces regards the determination of the maximal or minimal sizes of given subconfigurations of finite projective spaces. This problem is not only interesting from a geometrical point of view; it also arises when coding-theoretical problems and problems from the design of experiments are translated into equivalent geometrical problems.

The geometrical interest in the packing problem and the links with problems investigated in other research fields have given this problem a central place in *Galois geometries*, that is, the study of finite projective spaces.

In 1983, a historical survey on the packing problem was written by the first author [126] for the 9th British Combinatorial Conference. A new survey article stating the principal results up to 1995 was written by the authors for the Bose Memorial Conference [133].

Since 1995, many interesting results have appeared: for instance, the result of Szőnyi [213] confirming the $1 \pmod p$ conjecture of Blokhuis on small minimal blocking k-sets in $PG(2,q)$, $q = p^h$, p prime; the classification of small Rédei-type blocking k-sets in $PG(2,q)$ by Blokhuis, Ball, Brouwer, Storme and Szőnyi [30]; the unified construction of Cherowitzo, O'Keefe and Penttila for hyperovals in $PG(2,q)$, q even [58]; greatly improved upper bounds on the sizes of caps in $PG(N,q)$ by Meshulam [158], and Bierbrauer and Edel [26]; and the determination of the maximal size of a cap in $PG(4,4)$ by Edel and Bierbrauer [70].

These recent results are the motivation for updating [133]. We collect in this article the known results up to 2001. For connections with Coding Theory and Statistics, see [133].

With the aim of making this survey both easily accessible and self-contained, in the next subsection the exact definition of the packing problem and the notation used in this article are repeated.

When comparing this survey to the one of 1995, it is a pleasure to see both the progress made on different subconfigurations and the new techniques. It is hoped that these results may motivate others to study the packing problem.

The authors also wish to thank all the colleagues for their suggestions and remarks in writing this update.

1.2 Subsets in projective spaces

Let $PG(N,q)$ be the projective space of N dimensions over the finite field $\mathbf{F}_q$ of q elements, $q = p^h$ with p prime, and let $|PG(N,q)| = \theta_N = (q^{N+1}-1)/(q-1)$.

In $PG(N,q)$, subspaces will be denoted by Π_l, where l is the dimension of the subspace. A Π_0 is a *point*, a Π_1 is a *line*, a Π_2 is a *plane*, a Π_3 is a *solid*, and a Π_{N-1} is a *hyperplane* or *prime*.

Let $e_i = (0,\ldots,0,1,0,\ldots,0)$, for $i = 0,\ldots,N$, where 1 is in the $(i+1)$-th position and let $e = (1,\ldots,1)$.

For a unified notion of subsets of projective spaces, an $(n;r,s;N,q)$-set K is defined to be a set satisfying the following properties:

(a) the set K consists of n points of $PG(N,q)$ and is not contained in a proper subspace;

(b) some subspace Π_s contains r points of K, but no Π_s contains $r+1$ points of K;

(c) there is a subspace Π_{s+1} containing $r+2$ points of K.

An $(n;r,s;N,q)$-set K is called *complete* if it is not contained in an $(n+1;r,s;N,q)$-set. Several natural problems on $(n;r,s;N,q)$-sets K arise immediately.

(I) Find the maximum value $m(r, s; N, q)$ of n for which a $(n; r, s; N, q)$-set exists.

(II) Characterize the sets with this size $m(r, s; N, q)$.

(III) Find the size $m'(r, s; N, q)$ of the second largest complete $(n; r, s; N, q)$-set.

(IV) Find the size $t(r, s; N, q)$ of the smallest complete $(n; r, s; N, q)$-set.

The following examples of these sets will be considered in subsequent sections.

(1) An $(n; 2, 1; N, q)$-set is an n-set with at most two points on any line of $PG(N, q)$ and is called an *n-cap* when $N \geq 3$ and a (plane) n-arc when $N = 2$. The numbers $m(2, 1; N, q)$, $m'(2, 1; N, q)$ and $t(2, 1; N, q)$ are respectively denoted by $m_2(N, q)$, $m'_2(N, q)$ and $t_2(N, q)$.

(2) An $(n; N, N-1; N, q)$-set is an n-set with at most N points in any hyperplane of $PG(N, q)$ and is called an *n-arc*. For $N = 2$, the two definitions of n-arc coincide. Here, $m(N, N-1; N, q)$, $m'(N, N-1; N, q)$ and $t(N, N-1; N, q)$ are denoted by $m(N, q)$, $m'(N, q)$ and $t(N, q)$.

(3) An $(n; r, 1; N, q)$-set K is a set of n points with at most r on a line. When $N = 2$, the set K is called a *(plane) (n, r)-arc*; when $N > 2$, the set K is called an (n, r)-cap. For these sets, the parameters $m(r, 1; N, q), m'(r, 1; N, q)$, $t(r, 1; N, q)$ are respectively denoted by $m_r(N, q), m'_r(N, q)$, $t_r(N, q)$.

(4) An $(n; r, r-1; N, q)$-set K is a set of n points at most r of which lie in a Π_{r-1} but some $r+2$ lie in a Π_r; that is, $r+1$ points are always linearly independent, but some $r+2$ points are linearly dependent. Such a set K is called an *(n, r)-set* or *n-set of kind r*. The value $m(r, r-1; N, q)$ is denoted by $M_r(N, q)$. Hence $M_N(N, q) = m(N, q)$ and $M_2(N, q) = m_2(N, q)$.

For n-arcs, Problem (I) has been studied in detail. The main conjecture for this problem is given in Table 1.1.

$m(N, q)$	Conditions
$N+2$	$N \geq q-1$
$q+2$	q even, $N \in \{2, q-2\}$
$q+1$	in all other cases

Table 1.1: Main conjecture for $m(N, q)$

The results for this problem are gathered in Tables 2.1, 3.1, 3.3, 3.4, 3.5, and 3.7.

The sets above are discussed in the following sections: §2: n-arcs in $PG(2, q)$; §3: n-arcs in $PG(N, q)$, $N \geq 3$; §4: n-caps in $PG(N, q)$, $N \geq 3$; §5: (n, r)-arcs in $PG(2, q)$; §6: Multiple blocking sets in $PG(2, q)$; §7: Blocking sets in $PG(N, q)$; §8: n-tracks and almost MDS codes; §9: Minihypers.

Throughout the article, $\lfloor x \rfloor$ denotes the greatest integer smaller than or equal to x, while $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

2 n-arcs in $PG(2,q)$

q	$m(2,q)$	
q odd	$q+1$	[37]
q even	$q+2$	[37]

Table 2.1: $m(2,q)$

An $m(2,q)$-arc in $PG(2,q)$, q odd, is called an *oval* and an $m(2,q)$-arc in $PG(2,q)$, q even, is called a *hyperoval*. Formerly, the term *oval* was used in both cases.

Theorem 2.1 (Segre [191, 192]) *For q odd, an oval is the set of rational points of a conic.*

Bose [37] showed that, for q even, a conic plus its nucleus (the intersection point of its tangents) is a hyperoval. A hyperoval of this type is called *regular*. As shown by Segre [194], for $q = 2,4,8$, every hyperoval is regular.

For $q = 2^h, h \geq 4$, there exist *irregular* hyperovals, that is, hyperovals which are not the union of a conic and its nucleus. Several infinite classes of irregular hyperovals are known. In general, the following result is valid.

Theorem 2.2 (Segre [194], [129, §8.4]) *Any hyperoval of $PG(2,q)$, $q = 2^h$ and $h > 1$, is projectively equivalent to a hyperoval*

$$\mathcal{D}(F) = \{(1,t,F(t)) \parallel t \in \mathbf{F}_q\} \cup \{\, e_1, e_2\},$$

where F is a permutation polynomial over $\mathbf{F}_q$ of degree at most $q-2$, satisfying $F(0) = 0$, $F(1) = 1$, and such that $F_s(X) = (F(X+s)+F(s))/X$ is a permutation polynomial for each s in $\mathbf{F}_q$, satisfying $F_s(0) = 0$.

A polynomial $F(X)$ of this type is called an *o-polynomial* [56] and, conversely, every o-polynomial gives rise to a hyperoval. Table 2.2 presents the known infinite classes of hyperovals. For each infinite class, the simplest form of the corresponding o-polynomial is given [59, 182]. The references for these infinite classes of hyperovals are: (1) regular (Bose [37]), (2) translation (Segre [194]), (3) Segre [197], (4) Glynn I and Glynn II [98], (4) Payne [173], (5) Cherowitzo [57], (6) Subiaco (Cherowitzo, Penttila, Pinneri and Royle [59], Payne [174], Payne, Penttila and Pinneri [175]), (7) Adelaide (Cherowitzo, O'Keefe and Penttila [58]).

In Table 2.2,

$$\begin{aligned}
P(X) &= X^{1/6} + X^{3/6} + X^{5/6};\\
C(X) &= X^{\sigma} + X^{\sigma+2} + X^{3\sigma+4};\\
S_1(X) &= \frac{\omega^2(X^4+X)}{X^4+\omega^2X^2+1} + X^{1/2};\\
S_2(X) &= \frac{\delta^2X^4+\delta^5X^3+\delta^2X^2+\delta^3X}{X^4+\delta^2X^2+1} + \left(\frac{X}{\delta}\right)^{1/2};\\
S_3(X) &= \frac{(\delta^4+\delta^2)X^3+\delta^3X^3+\delta^2X}{X^4+\delta^2X^2+1} + \left(\frac{X}{\delta}\right)^{1/2};\\
S(X) &= \frac{T(\beta^m)(X+1)}{T(\beta)} + \frac{T((\beta X+\beta^q)^m)}{T(\beta)(X+T(\beta)X^{1/2}+1)^{m-1}} + X^{1/2}.
\end{aligned}$$

In fact, the derivation of the Subiaco and Adelaide hyperovals is more complicated [59, 174, 175, 58]. They come in a set of $q+1$, called a *herd*, which is defined by two o-polynomials f, g satisfying the property that

$$T_2\left(\frac{c(f(s)+f(t))(g(s)+g(t))}{s+t}\right) = 1$$

for all $s, t \in \mathbf{F}_q$ with $s \neq t$, and a fixed c with $T_2(c) = 1$, where T_2 is the trace function from $\mathbf{F}_q$ to $\mathbf{F}_2$.

Name	$F(X)$	$q = 2^h$	Conditions
Regular	X^2		
Translation	X^{2^i}		$(h,i) = 1$
Segre	X^6	h odd	
Glynn I	$X^{3\sigma+4}$	h odd	$\sigma = 2^{(h+1)/2}$
Glynn II	$X^{\sigma+\lambda}$	h odd	$\sigma = 2^{(h+1)/2}$; $\lambda = 2^m$ if $h = 4m-1$; $\lambda = 2^{3m+1}$ if $h = 4m+1$
Payne	$P(X)$	h odd	
Cherowitzo	$C(X)$	h odd	$\sigma = 2^{(h+1)/2}$
Subiaco	$S_1(X)$	$h = 4r+2$	$\omega^2+\omega+1 = 0$
Subiaco	$S_2(X)$	$h = 4r+2$	$\delta = \zeta^{q-1}+\zeta^{1-q}$, ζ primitive in $\mathbf{F}_{q^2}$
Subiaco	$S_3(X)$	$h \neq 4r+2$	$T_2(1/\delta) = 1$
Adelaide	$S(X)$	h even, $h \geq 4$	$\beta \in \mathbf{F}_{q^2} \setminus \{1\}$, $\beta^{q+1} = 1$, $T(x) = x + x^q$, $m \equiv \pm(q-1)/3 \pmod{q+1}$

Table 2.2: Hyperovals in $PG(2,q), q$ even

Then, with

$$f_s(X) = \frac{f(X)+csg(X)+s^{1/2}X^{1/2}}{1+cs+s^{1/2}},$$

the herd is given by the o-polynomials g and f_s for $s \in \mathbf{F}_q$. For $q = 2^h,\ h \not\equiv 2 \pmod 4$, all Subiaco hyperovals are projectively equivalent to $\mathcal{D}(S_3)$, whereas for $q = 2^h,\ h \equiv 2 \pmod 4$, there are two classes of hyperovals $\mathcal{D}(S_1),\ \mathcal{D}(S_2)$.

Apart from the infinite classes of Table 2.2, there is one other sporadic hyperoval known, namely the O'Keefe–Penttila hyperoval in $PG(2, 32)$ [164].

It is interesting to remark that the translation hyperoval with $F(X) = X^4$, the Subiaco hyperovals and Adelaide hyperovals also arise from a unified construction of Cherowitzo, O'Keefe and Penttila [58]. In a personal communication, T. Penttila [179] confirmed that the Adelaide hyperovals are distinct from the Subiaco hyperovals.

Except for the O'Keefe–Penttila hyperoval in $PG(2, 32)$, every known hyperoval arises from an α-flock [57, 58].

In $PG(2, q)$, $q \leq 32$, the complete classification of hyperovals has been determined and is given in the next theorem.

Theorem 2.3 (i) (Segre [194]) *In* $PG(2, q)$, $q = 2, 4, 8$, *the only hyperovals are the regular hyperovals.*

(ii) (Hall [109], O'Keefe and Penttila [163]) *In* $PG(2, 16)$, *there are exactly two distinct hyperovals. They are the regular hyperoval and the Subiaco hyperoval. This latter hyperoval is also called the* Lunelli-Sce *hyperoval* [151].

(iii) (Penttila and Royle [181]) *In* $PG(2, 32)$, *there are exactly six distinct hyperovals. They are the regular hyperoval, the translation hyperoval* $\mathcal{D}(X^4)$, *the Segre hyperoval, the Payne hyperoval, the Cherowitzo hyperoval and the O'Keefe–Penttila hyperoval.*

Finding the values of n for which an n-arc is always contained in an oval, for q odd, or hyperoval, for q even, is relevant for solving problems in higher-dimensional spaces. Table 2.3 surveys the known results on this problem. In $PG(2, q)$, for q subject to the conditions in the second column, the third column gives an upper bound for $m'(2, q)$; the fourth column indicates when this upper bound is sharp in the case that equality does not hold in general. The fifth column gives the value of $m(2, q)$. So any n-arc with $n > m'(2, q)$ is contained in an $m(2, q)$-arc. This latter arc is a conic when q is odd and a hyperoval when q is even.

	q	$m'(2, q)$	Sharp	$m(2, q)$
(1)	$q = p^{2e}$, $p > 2,\ e \geq 1$	$\leq q - \sqrt{q}/4 + 25/16$		$q + 1$
(2)	$q = p^{2e+1}$, $p > 2,\ e \geq 1$	$\leq q - \sqrt{pq}/4$ $+29p/16 + 1$		$q + 1$
(3)	q prime, $q > 2$	$\leq 44q/45 + 8/9$		$q + 1$
(4)	$q = p^h,\ p \geq 5$	$\leq q - \sqrt{q}/2 + 5$		$q + 1$
(5)	$q = p^h,\ p \geq 3$, $q \geq 23^2,\ q \neq 5^5,\ 3^6$, h even for $p = 3$	$\leq q - \sqrt{q}/2 + 3$		$q + 1$
(6)	$q = 2^{2e},\ e > 1$	$= q - \sqrt{q} + 1$		$q + 2$
(7)	$q = 2^{2e+1},\ e \geq 1$	$\leq q - \sqrt{2q} + 2$	$q = 8$	$q + 2$

Table 2.3: Upper bounds for $m'(2, q)$

Result (1) is due to Thas [220]; (2) and (7) to Voloch [225]; (3) to Voloch [223]; (4) and (5) to Hirschfeld and Korchmáros [130, 131]; (6) as an upper bound is due to Segre [198].

The sharpness of (6) was shown independently by Boros and Szőnyi [36], Fisher, Hirschfeld and Thas [87], and Kestenband [142]. That these arcs are unique follows from Hirschfeld [124, §14.8], [129, §8.7], Lisonek [146, p.193], Chao and Kaneta [54, 55].

Theorem 2.4 *In $PG(2,q)$, $q = p^{2e}$, $q > 4$, a complete $(q-\sqrt{q}+1)$-arc exists. For $q = 9, 16, 25$, a complete $(q-\sqrt{q}+1)$-arc is projectively unique.*

Conjecture $m'(2,q) = q - \sqrt{q} + 1$ for $q = p^{2e}$, $q > 9$.

For arcs in $PG(2,q)$, $q = 2^{2e}$, $q > 4$, of size smaller than $q - \sqrt{q} + 1$, there is the following result of Hirschfeld and Korchmáros [132].

Theorem 2.5 *A complete n-arc of $PG(2,q)$, $q = 2^{2e}$, $e > 2$, has size either $q+2$, or $q-\sqrt{q}+1$, or at most $q - 2\sqrt{q} + 6$.*

For $q = 16$, the list of sizes n of complete n-arcs in $PG(2,16)$ is $9, 10, 11, 12, 13, 18$.

For small q, the values $m'(2,q)$ for the size of the second largest complete arc are given in Table 2.4. For $q \leq 5$, there is no second largest complete arc, so we do not consider these values of q. For $q \leq 9$, see [124, 129, Ch.14]; for $q = 11$, see [189], [129, §14.8]; for $q = 13$, see [5], [100], [190], [6], [7], [129, §14.9]; for $q = 17, 19$, see [54, 77, 159]; for $q = 23, 25, 27$, see [55]; for $q = 29$, see [53]. See also [82], [146, p.193], [183], [212].

q	7	8	9	11	13	16	17	19	23	25	27	29
$m'(2,q)$	6	6	8	10	12	13	14	14	17	21	22	24

Table 2.4: $m'(2,q)$ in small planes

Similarly, for small q, the size $t(2,q)$ of the smallest complete arc in $PG(2,q)$ has been calculated and is given in Table 2.5; see [77], [82], [146, p.193], [183], [212].

q	2	3	4	5	7	8	9	11	13	16	17	19	23	25	27	29
$t(2,q)$	4	4	6	6	6	6	6	7	8	9	10	10	10	12	12	13

Table 2.5: $t(2,q)$ in small planes

General lower and upper bounds are given in Table 2.6. In the upper bound of Kim and Vu [143], d and c are absolute constants. Giulietti [96] has shown that, for q an odd square with $q \leq 1681$ or $q = 2401$, there exists a complete $4(\sqrt{q}-1)$-arc in $PG(2,q)$.

q	$t(2,q)$	
q	$> \sqrt{2q}+1$	[195]
$q = p^h$, p prime, $h = 1,2,3$	$> \sqrt{3q}+\frac{1}{2}$	[27, 11, 185]
q	$\leq d\sqrt{q}\log^c q$	[143]

Table 2.6: Lower and upper bounds for $t(2,q)$

To conclude this section on arcs in $PG(2,q)$, the following theorem presents results on the spectrum of the values n for which a complete n-arc in $PG(2,q)$ exists.

Theorem 2.6 (i) (Bose [37]) *In $PG(2,q)$, a complete $(q+1)$-arc exists for q odd and a complete $(q+2)$-arc exists for q even.*

(ii) (Boros and Szőnyi [36], Fisher, Hirschfeld and Thas [87], Kestenband [142]) *In $PG(2,q)$, q square, $q > 4$, there exists a complete $(q - \sqrt{q} + 1)$-arc.*

(iii) (Szőnyi [209]) *Let $A = \{n/q \,\|\,$ there exists a complete n-arc in $PG(2,q)$, for some prime power $q\}$. Then $A \cap [0, 1/2]$ is dense in $[0, 1/2]$.*

(iv) (Voloch [222]) *If $E \subseteq PG(2,q)$ is an elliptic cubic curve with $2n$ rational points, then there exists an n-arc contained in E, which is complete if the modular (or j-)invariant $j(E) \neq 0$* ([117, Ch.IV, §4]) *and $q \geq 175$ for q odd or $q \geq 256$ for q even.*

(v) (Giulietti [97]) *When $j(E) = 0$ and $q > 9887$, then the n-arc can be completed to an arc of size at most $n + 3$.*

(vi) (Hadnagy [106], Voloch [224] and Szőnyi [210]) *Let $p \geq 5^{10}$ be a prime number. Then there exists a complete n-arc in $PG(2,p)$ for every integer n satisfying $\lfloor 2.46 \cdot \log p \cdot p^{3/4} \rfloor \leq n < (\sqrt{p} + 1)^2/2$.*

Many constructions of complete arcs are based on an idea of Segre [197] and Lombardo-Radice [147]; the points of the arc are chosen, with some exceptions, among the points of a conic or cubic curve.

This leads to complete arcs with approximately $q/2$ [124, §9.4], [129, §9.3], [198], $q/3$ [2, 144, 208, 224], $q/4$ [144], $2q^{9/10}$ [208] points.

3 n-arcs in $PG(N,q)$

In Tables 3.1–3.3, NRC stands for *normal rational curve*, that is, a $(q+1)$-arc in $PG(N,q)$, $2 \leq N \leq q-2$, projectively equivalent to the $(q+1)$-arc $\{(1, t, \ldots, t^N) \,\|\, t \in \mathbf{F}_q^+\}$, where $\mathbf{F}_q^+ = \mathbf{F}_q \cup \{\infty\}$ and where $t = \infty$ defines the point e_N. In $PG(3,q)$, $q = 2^h$, $h > 2$, $L_e = \{(1, t, t^e, t^{e+1}) \,\|\, t \in \mathbf{F}_q^+\}$, with $e = 2^v$, $(v,h) = 1$, and with $t = \infty$ defining the point e_3.

Table 3.1 shows the value of $m(N,q)$ for small dimensions N. The characterization of the $m(N,q)$-arcs L in $PG(N,q)$ is given in the last column.

	q	N	$m(N,q)$	L
(1)	$q = p^h$	$N \geq q-1$	$N+2$	$\{e_0, \ldots, e_N, e\}$
(2)	q odd, $q > 3$	3	$q+1$	NRC
(3)	$q = 2^h$, $q > 4$	3	$q+1$	$L_e, e = 2^v, (v,h) = 1$
(4)	q odd, $q > 5$	4	$q+1$	NRC for $q > 83$
(5)	$q = 2^h$, $q > 4$	4	$q+1$	NRC
(6)	$q = 2^h$, $q \geq 8$	5	$q+1$	NRC for $q \geq 16$
(7)	$q = 2^h$, $q \geq 16$	6	$q+1$	NRC for $q > 64$

Table 3.1: $m(N,q)$ and $m(N,q)$-arcs

Result (1) is due to Bush [46] and (2) to Segre [193]. The value of $m(N, q)$ in (3) is due to Casse [48], as well as to Gulati and Kounias [105], while the characterization of the $(q+1)$-arcs is by Casse and Glynn [49]. In (4), the value of $m(N, q)$ is due to Segre [193]; the characterization of L follows from Table 3.2. The bound in (5) is by Casse [48], and by Gulati and Kounias [105], while the characterization of the $(q+1)$-arcs is due to Casse and Glynn [50]. Result (6) is by Maruta and Kaneta [155], and (7) by the same authors [141, 155] for the bound and by Storme and Thas [203] for the characterization of the $(q+1)$-arcs.

A problem that has been intensively studied is the extendability of n-arcs in $PG(N, q)$ to larger arcs. Table 3.2 summarizes the main results on this topic. An n-arc in $PG(N, q)$, satisfying the condition on n in the third column, can be extended uniquely to a $(q+1)$-arc L, whose description is given in column 4. Alternatively, the third column gives an upper bound for $m'(N, q)$. The results are respectively due to Thas [217, 220], Voloch [225], Voloch [223], Hirschfeld and Korchmáros for the next two formulas [130, 131], and Storme and Thas [203] for the latter two formulas.

Theorem 3.1 (Kaneta and Maruta [141]) *If every $(q+1)$-arc of the space $PG(N, q)$ is a normal rational curve, then $q+1$ is the maximum value of n for which n-arcs exist in $PG(N+1, q)$.*

N	q	$n >$	L
≥ 2	$q = p^{2e},\ p$ odd, $e \geq 1$	$q - \sqrt{q}/4 + N - 7/16$	NRC
≥ 2	$q = p^{2e+1},\ p$ odd, $e \geq 1$	$q - \sqrt{pq}/4 + 29p/16$ $+N - 1$	NRC
≥ 2	q prime, $q > 2$	$44q/45 + N - 10/9$	NRC
≥ 2	$q = p^h,\ p \geq 5$	$q - \sqrt{q}/2 + N + 3$	NRC
≥ 2	$q = p^h,\ p \geq 3,\ q \geq 23^2$, h even for $p = 3$, $q \neq 5^5,\ 3^6$	$q - \sqrt{q}/2 + N + 1$	NRC
3	$q = 2^h, h > 1$	$q - \sqrt{q}/2 + 9/4$	L_e
≥ 4	$q = 2^h,\ h > 2$	$q - \sqrt{q}/2 + N - 3/4$	NRC

Table 3.2: Upper bounds for $m'(N, q)$

Applying the results in Table 3.2 for $n = q+1$ gives an upper bound on the dimension N for the spaces $PG(N, q),\ q = p^h,\ p$ prime, in which every $(q+1)$-arc is a normal rational curve. The upper bound is given in the second column of Table 3.3. Column 3 of Table 3.3 gives the upper bound, obtained from Theorem 3.1, on the dimension N of the spaces $PG(N, q)$ in which $m(N, q) = q+1$.

As indicated in [133, §1.3], an n-arc in $PG(n-k-1, q)$ defines an $[n, k, n-k+1]$ MDS code C. But the dual code $C^{\perp}$ of an $[n, k, n-k+1]$ MDS code is an $[n, n-k, k+1]$ MDS code [152, Ch.11, §2]. Hence, to $C^{\perp}$ also corresponds an n-arc $\hat{K}$ in $PG(k-1, q)$.

This theorem, proved independently by Thas [218] and by Halder and Heise [108], makes it possible to link n-arcs in $PG(n-k-1, q)$ to n-arcs in $PG(k-1, q)$. An n-arc K in $PG(n-k-1, q)$ and an n-arc $\hat{K}$ in $PG(k-1, q)$ are called C-*dual arcs* if and only if they define dual MDS codes.

q	$(q+1)$-arc = NRC	$m(N,q)=q+1$
$q=p^{2e},\ p$ odd, $e\geq 1$	$2\leq N<$ $\sqrt{q}/4+23/16$	$2\leq N<$ $\sqrt{q}/4+39/16$
$q=p^{2e+1},\ p$ odd, $e\geq 1$	$2\leq N<$ $\sqrt{pq}/4-29p/16+2$	$2\leq N<$ $\sqrt{pq}/4-29p/16+3$
q prime, $q>2$	$2\leq N<q/45+19/9$	$2\leq N<q/45+28/9$
$q=p^h,\ p\geq 5$	$2\leq N<\sqrt{q}/2-2$	$2\leq N<\sqrt{q}/2-1$
$q=p^h,\ p\geq 3$, h even for $p=3$, $q\geq 23^2,\ q\neq 5^5,\ 3^6$	$2\leq N<\sqrt{q}/2$	$2\leq N<\sqrt{q}/2+1$
$q=2^h,\ h>2$	$4\leq N<\sqrt{q}/2+7/4$	$4\leq N<\sqrt{q}/2+11/4$

Table 3.3: $(q+1)$-arcs in $PG(N,q)$

Combining this C-duality principle on arcs with the result stating that an n-arc of $PG(n-k-1,q)$ is contained in a normal rational curve if and only if its C-dual n-arc in $PG(k-1,q)$ is contained in a normal rational curve [152, Ch.10, §8], the results in the preceding tables on arcs in $PG(N,q)$, with N small, immediately imply other results on arcs in $PG(N,q)$, where N is close to q.

In Table 3.4, in the spaces $PG(N,q)$, with N satisfying the bounds in the table, any n-arc, where n satisfies the bound in the second column, is contained in a normal rational curve. So, in these spaces, $m(N,q)=q+1$ and every $(q+1)$-arc is a normal rational curve.

Only n-arcs in $PG(N,q)$, with $n\geq N+4$, are considered since an $(N+3)$-arc of $PG(N,q)$, $2\leq N\leq q-2$, is always contained in a unique normal rational curve [127, §21.1].

q	$n\geq$	$m(N,q)=q+1$
$q=p^{2e},\ p>2,\ e\geq 1$	$N+4$	$q-3\geq N>q-\sqrt{q}/4-39/16$
$q=p^{2e+1},\ p>2,\ e\geq 1$	$N+4$	$q-3\geq N>$ $q-\sqrt{pq}/4+29p/16-3$
q prime, $q>2$	$N+4$	$q-3\geq N>44q/45-28/9$
$q=p^h,\ p\geq 5$	$N+4$	$q-3\geq N>q-\sqrt{q}/2+1$
$q=p^h,\ p\geq 3$, h even for $p=3$, $q\geq 23^2,\ q\neq 5^5,\ 3^6$	$N+4$	$q-3\geq N>q-\sqrt{q}/2-1$
$q=2^h,\ h>2$	$N+6$	$q-5\geq N>q-\sqrt{q}/2-11/4$

Table 3.4: Arcs in $PG(N,q), N$ close to q

Theorem 3.2 (Glynn [99]) *In $PG(4,9)$, a 10-arc is one of two types; it is either a normal rational curve or is equivalent to* $L=\{(1,t,t^2+\eta t^6,t^3,t^4)\parallel t\in\mathbf{F}_9\}\cup\{(0,0,0,0,1)\}$, *where* $\eta^4=-1$.

The preceding result is of particular interest since this 10-arc L is the only known $(q+1)$-arc in $PG(N,q)$, q odd, $2\leq N\leq q-2$, which is not a normal rational curve.

There are some fundamental differences between arcs in spaces of even characteristic and odd characteristic; the existence and characterization of $(q+2)$-arcs are given by the next two results.

Theorem 3.3 (Thas [218]) *In $PG(q-2,q)$, q even, $m(q-2,q) = q+2$.*

Theorem 3.4 (Storme and Thas [204]) *A point $P = (a_0, \ldots, a_{q-2})$ extends the normal rational curve $K = \{(1, t, \ldots, t^{q-2}) \,\|\, t \in \mathbf{F}_q^+\}$ to a $(q+2)$-arc if and only if $F(X) = \sum_{i=0}^{q-2} a_{q-2-i} X^{i+1}$ defines a $(q+2)$-arc $K' = \{(1, t, F(t)) \,\|\, t \in \mathbf{F}_q\} \cup \{e_1, e_2\}$ in $PG(2,q)$; in this case, K' is a C-dual $(q+2)$-arc of $K \cup \{P\}$.*

This correspondence between the point P and the $(q+2)$-arc K' is one-to-one if the condition $F(1) = 1$ is added.

Table 3.5 presents the results by Storme and Thas [202] for the values of n for which there exist complete n-arcs in the respective spaces $PG(N,q)$. For more information on complete $(N+4)$-arcs and $(N+5)$-arcs, see [202].

q	N	$n \in$
$q = p^{2e}$, $e \geq 1$, $p > 2$	$q-2 \geq N > q - \sqrt{q}/4 - 39/16$	$\{q+1\}$
$q = p^{2e+1}$, $e \geq 1$, $p > 2$	$q-2 \geq N > q - \sqrt{pq}/4 + 29p/16 - 3$	$\{q+1\}$
q prime, $q > 2$	$q-2 \geq N > 44q/45 - 28/9$	$\{q+1\}$
$q = p^h$, $p \geq 5$	$q-2 \geq N > q - \sqrt{q}/2 + 1$	$\{q+1\}$
$q = p^h$, $p \geq 3$, h even for $p = 3$, $q \geq 23^2$, $q \neq 5^5$, 3^6	$q-2 \geq N > q - \sqrt{q}/2 - 1$	$\{q+1\}$
$q = 2^h$, $q \geq 32$, $q \neq 64$	$q-5 \geq N > q - \sqrt{q}/2 - 11/4$	$\{N+4, N+5, q+1\}$
$q = 64$	$N = 58$ or $N = 59$	$\{N+4, q+1\}$
$q = 2^h$, $q \geq 8$	$q-4$	$\{q, q+1\}$
$q = 2^h$, $q \geq 8$	$q-3$	$\{q+1\}$
$q = 2^h$, $q \geq 4$	$q-2$	$\{q+2\}$

Table 3.5: Spectrum of complete arcs

Table 3.6 presents the known values for the size $m'(3,q)$ of the second largest complete n-arc in $PG(3,q)$. These results are by Kaneta [139], Chao and Kaneta [54], in combination with O'Keefe and Storme [167] for $q = 13$, and Kaneta [140].

q	8	9	11	13	16	17	19	25
$m'(3,q)$	7	9	9	10	12	13	14	18

Table 3.6: $m'(3,q)$

To conclude this section, Table 3.7 gives known results on arcs in small projective spaces. For the corresponding spaces $PG(N,q)$, $2 \leq N \leq q-2$, it is indicated if every $(q+1)$-arc is a normal rational curve, the values of N for which $m(N,q)=q+1$, and if there exist $(q+2)$-arcs. For $q \geq 11$, these results are by Chao and Kaneta [54, 55].

q	$(q+1)$-arc=NRC	$m(N,q)=q+1$	$(q+2)$-arcs
$5 \leq q \leq 27$, $q \neq 9$, q odd	$N \in \{2,\ldots,q-2\}$	$N \in \{2,\ldots,q-2\}$	
$8,16$	$N \in \{3,\ldots,q-4\}$	$N \in \{3,\ldots,q-3\}$	$N \in \{2,q-2\}$
9	$N \in \{2,3\} \cup \{5,6,7\}$	$N \in \{2,\ldots,7\}$	

Table 3.7: n-arcs in small spaces $PG(N,q)$

As a consequence of Tables 3.1, 3.2, 3.3, 3.4, 3.7,
(1) $m(5,q)=q+1$ for $q \geq 7$, except possibly for q odd and

$$q \in \{29,\ldots,83\};$$

(2) $m(6,q)=q+1$ for $q \geq 9$, except possibly for q odd and

$$q \in \{29,\ldots,127\} \cup \{169\}.$$

For an alternative summary on n-arcs, see [128].

4 n-caps in $PG(N,q)$

q	N	$m_2(N,q)$	
q odd	3	q^2+1	[37]
q even, $q>2$	3	q^2+1	[188]
$q=2$	N	2^N	[37]
$q=3$	4	20	[177]
$q=3$	5	56	[119]
$q=4$	4	41	[70]

Table 4.1: $m_2(N,q)$

The maximum size $m_2(N,q)$ of a cap is known only for the values given in Table 4.1. An $m_2(3,q)$-cap, $q>2$, is called an *ovoid* or *ovaloid*.

Theorem 4.1 (i) (Barlotti [21], Panella [172]) *For q odd or $q=4$, an ovoid in $PG(3,q)$ is an elliptic quadric.*

(ii) (Tits [221], [127, §16.4]) *There exists an ovoid in $PG(3,2^{2e+1})$, $e \geq 1$, which is not an elliptic quadric. This ovoid, the* Tits ovoid, *is projectively equivalent to the set $K = \{(0,1,0,0)\} \cup \{(1,z,y,x) \| z = xy + x^{\sigma+2} + y^{\sigma}$, with $(x,y) \in \mathbf{F}_q^2$ and $\sigma = 2^{e+1}\}$.*

(iii) (Fellegara [84], Penttila and Praeger [180]) *In $PG(3,8)$, an ovoid is an elliptic quadric or a Tits ovoid.*

(iv) (O'Keefe and Penttila [162, 165]) *In $PG(3,16)$, an ovoid always is an elliptic quadric.*

(v) (O'Keefe, Penttila and Royle [166]) *In $PG(3,32)$, an ovoid is an elliptic quadric or a Tits ovoid.*

(vi) (Segre [195]) *In $PG(N,2)$, a 2^N-cap is the complement of a hyperplane.*
(vii) (Hill [121]) *There are* 9 *projectively distinct* 20*-caps in $PG(4,3)$.*
(viii) (Hill [120]) *The* 56*-cap in $PG(5,3)$ is projectively unique.*
(ix) (Edel and Bierbrauer [70]) *There exist at least two types of* 41*-cap in $PG(4,4)$.*

Table 4.2 displays the values of n for which an n-cap in $PG(N,q)$ is contained in an $m_2(N,q)$-cap. The table is in two parts. The first part shows the exact values of $m'_2(N,q)$ for small q and upper bounds for q even. The second part of the table displays the upper bounds on $m'_2(3,q)$ for q odd, using the following relation between $m'_2(2,q)$ and $m'_2(3,q)$. For the corresponding value of $m'_2(2,q)$, see Table 2.3.

Theorem 4.2 (Nagy and Szőnyi [161]) *If, for q odd, $m'_2(2,q) \geq (5q+19)/6$, then*

$$m'_2(3,q) < qm'_2(2,q) + \tfrac{3}{4}(q + \tfrac{10}{3} - m'_2(2,q))^2 - q - 1.$$

N	q	$m'_2(N,q)$	$m_2(N,q)$-cap	
3	q even, $q \geq 8$	$\leq q^2 - q + 5$	ovoid	[51]
3	3	$= 8$	elliptic quadric	[78]
3	4	$= 14$	elliptic quadric	[135]
3	5	$= 20$	elliptic quadric	[4]
3	7	$= 32$	elliptic quadric	[75]
4	3	$= 19$	20-cap	[216]
5	3	$= 48$	56-cap	[19]
N	2	$= 2^{N-1} + 2^{N-3}$	complement of a hyperplane	[65]

Table 4.2(i): Upper bounds for and values of $m'_2(N,q)$

q odd, $q \geq 17$	$m'_2(3,q)$
$q = p^{2e},\ p > 2,\ e \geq 1$	$< q^2 - q^{3/2}/4 + 39q/64 + O(q^{1/2})$
$q = p^{2e+1},\ p > 2,\ e \geq 1$	$< q^2 - p^{3e+2}/4 + 119p^{2e+2}/64 - O(p^{e+2})$
q prime	$< 2641q^2/2700 - 4q/135 + 94/27$
$q = p^h,\ p \geq 5$	$< q^2 - q^{3/2}/2 + 67q/16 - O(q^{1/2})$
$q = p^h,\ p \geq 3$, h even for $p = 3$, $q \geq 23^2,\ q \neq 5^5,\ 3^6$	$< q^2 - q^{3/2}/2 + 35q/16 + O(q^{1/2})$

Table 4.2(ii): Upper bounds for $m'_2(3,q)$

The spectrum of the known values n for which there exists a complete n-cap in $PG(N,q)$ is given in the next result.

Theorem 4.3 (i) (Bose [37], Qvist [188]) *In $PG(3,q)$, $q > 2$, a complete (q^2+1)-cap exists.*
(ii) (Davydov and Tombak [65]) *In $PG(N,2)$, $N \geq 3$, a complete n-cap K, with $n \geq 2^{N-1} + 1$, has order $n = 2^{N-1} + 2^{N-1-g}$ for some $g = 0,2,3,\ldots,N-1$. For each $g = 0,2,3,\ldots,N-1$, there exists a complete $(2^{N-1} + 2^{N-1-g})$-cap in $PG(N,2)$.*

(iii) (Faina [76]) *Let* $A = \{n/q^2 \parallel$ *there exists a complete* n*-cap in* $PG(3,q)$*, for some prime power* $q\}$*. Then* $A \cap [1/3, 1/2]$ *is dense in* $[1/3, 1/2]$.

(iv) (Ferret and Storme [85]) *There does not exist a complete* n*-cap in* $PG(3,q)$*,* q *even,* $q \geq 1024$*, for which*

$$n \in \left[q^2 - (c-1)q + (2c^3 + c^2 - 5c + 6)/2,\ q^2 - (c-2)q - 2c^2 + 3c\right]$$

where $2 \leq c \leq \sqrt[4]{q}$.

Generalizing the idea of Segre [197] and Lombardo-Radice [147] for constructing plane arcs, complete caps containing a large number of points of an elliptic quadric have been constructed. This leads to complete caps with approximately $q^2/2$ [1, 76, 127, 178], $q^2/3$ [3, 76, 80] and $q\sqrt{q/2}$ [178] points.

In Tables 4.3(i) and 4.3(ii), the known upper bounds on $m_2(4,q)$ are presented. The bounds for $q = 9, 11, 13$ are obtained by applying the ideas of Gronchi [104] to the results of Heim [118]. The entries in Table 4.3(ii) are a consequence of the results in Theorem 4.4. Since the result by Storme, Thas and Vereecke is better than the bound by Nagy and Szőnyi for large values of q, we only state the bounds following from the second bound on $m_2(4,q)$, q odd.

Theorem 4.4 (i) (Nagy and Szőnyi [161]) *For* q *odd, let* $m = \max\{(q^2 + 5q + 2)/2, m_2'(3,q)\}$*. Then*

$$m_2(4,q) < qm + 2q^2.$$

(ii) (Storme, Thas and Vereecke [205]) *Let* K *be an* n*-cap in* $PG(4,q)$*,* q *odd,* $n > (41q^3 + 202q^2 - 47q)/48$*, and assume that every plane section of* K*, of size bigger than* θ_K*, with* $\theta_K \geq (5q + 25)/6$*, is contained in a conic, then*

$$n < (q+1)\left(q \cdot \theta_K + \frac{3}{4}\left(q + \frac{10}{3} - \theta_K\right)^2 - q - 1 - \theta_K\right) + \theta_K.$$

In particular, when $m_2'(2,q) \geq (5q + 25)/6$*, this result can be applied.*

(iii) (Hirschfeld and Thas [135]) *If* $\gamma \geq q^2 - q + 4$ *for* q *even and* $q \geq 8$*, and if every* n*-cap of* $PG(3,q)$ *with* $n > \gamma$ *is contained in an ovoid of* $PG(3,q)$*, then*

$$m_2(4,q) \leq (q-1)\gamma + q^2 + 2.$$

In particular, when $m_2'(3,q) \geq q^2 - q + 4$*, this result can be applied.*

q	$m_2(4,q)$	
$q = 5$	≤ 96	[75]
$q = 7$	≤ 285	[75]
$q = 9$	≤ 703	
$q = 11$	≤ 1266	
$q = 13$	≤ 2107	
q odd, $q \geq 7$	$\leq q^3 - q^2 + 8q - 14$	[23]
q odd, $q \geq 67$	$< q^3 - q^{5/2}/4 + 31q^2/16 + O(q^{3/2})$	[125]
q even, $q \geq 8$	$\leq q^3 - q^2 + 6q - 3$	[51, 135]

Table 4.3(i): Upper bounds for $m_2(4,q)$

q odd	$m_2(4,q)$
$q = p^{2e},\ p > 2,\ e \geq 1,$ $q \geq 49$	$< q^3 - q^{5/2}/4 + 39q^2/64 + O(q^{3/2})$
$q = p^{2e+1},\ p > 2,\ e \geq 1$	$< q^3 - p^{e+1}q^2/4 + 119pq^2/64 - O(p^{e+2}q)$
q prime, $q \geq 37$	$< 2641q^3/2700 - 79q^2/2700 + O(q)$
$q = p^h,\ p \geq 5,\ q \geq 19$	$< q^3 - q^{5/2}/2 + 67q^2/16 - O(q^{3/2})$
$q = p^h,\ p \geq 3,$ h even for $p = 3,$ $q \neq 5^5,\ 3^6,\ q \geq 23^2$	$< q^3 - q^{5/2}/2 + 35q^2/16 + O(q^{3/2})$

Table 4.3(ii): General upper bounds for $m_2(4,q),\ q$ odd

We now concentrate on $m_2(N,q)$. Tables 4.4(i) and (ii) give the upper bounds for $m_2(N,q)$, $N \geq 5$. The results in Table 4.4(ii) follow from those in Table 4.3(ii).

The latter formulas in Table 4.4(i), 4.4(ii) are of Bierbrauer and Edel [26]. This is an improvement to the result, for q odd, of Meshulam [158]. These are the best upper bounds, when the dimension N is large. More precisely, the larger the dimension N, the bigger the difference between the formula of Bierbrauer and Edel, and the other formulas of Table 4.4(i) and 4.4(ii). We remark that the bounds of Bierbrauer and Edel, and Meshulam, were originally bounds on caps in affine spaces $AG(N,q)$; bounds which have been rewritten into bounds on caps in projective spaces. In Table 4.4(i), the reference for $q = 3$ is Barát *et al.* [19], for $q = 5, 7$ is Edel, Storme and Sziklai [75], for q odd, $q > 7$, is Hill [120], and for q even, $q \geq 16$, is Storme, Thas and Vereecke [205].

Table 4.5 now gives general inequalities on $m_2(N,q)$. We first present formulas which are crucial in the determination of lower bounds on $m_2(N,q)$. This is then followed by Hill's recurrence relation [120], which makes it possible to calculate upper bounds on $m_2(N,q)$. To this formula of Hill, small improvements have been made by Storme, Thas and Vereecke. We refer to [205] for the exact improvements.

The references for the formulas in Table 4.5 are: Segre [195] for the first two formulas, Edel and Bierbrauer [71] for the third formula, and Hill [120] for the latter two formulas.

$q,\ N$	$m_2(N,q)$
$q=3,\ N\geq 6$	$\leq 2\cdot 3^{N-2}-3^{N-3}+2$
$q=4,\ N\geq 4$	$\leq 4^{N-1}-4^{N-2}-2\cdot 4^{N-3}-2\cdot 4^{N-4}/3+5/3$
$q=5,\ N\geq 4$	$\leq 5^{N-1}-5^{N-2}-5^{N-3}-5^{N-4}/2+3/2$
$q=7,\ N\geq 4$	$\leq 7^{N-1}-7^{N-2}-4\cdot 7^{N-3}/3+4/3$
$q=9,\ N\geq 4$	$\leq 9^{N-1}-109\cdot 9^{N-4}/4+5/4$
$q=11,\ N\geq 4$	$\leq 11^{N-1}-331\cdot 11^{N-4}/5+6/5$
$q=13,\ N\geq 4$	$\leq 13^{N-1}-547\cdot 13^{N-4}/6+7/6$
q odd, $q>7$	$\leq q^{N-1}-q^{N-2}+8q^{N-3}-15q^{N-4}-2\theta_{N-5}+1$
q even, $q\geq 16$	$m_2(5,q)<q^4-q^3+5q^2+3q-1$
q even, $q\geq 16$, $N\geq 6$	$<q^{N-1}-q^{N-2}+5q^{N-3}+2q^{N-4}+O(q^{N-5})$
$q=Q^h, Q>2$	$(Nh+1)q^N/(Nh)^2+m_2(N-1,q)$

Table 4.4(i): Upper bounds for $m_2(N,q)$

$q=p^h$ odd, p prime	$m_2(N,q)$
$q=p^{2e},\ e\geq 1,\ q\geq 49$	$<q^{N-1}-q^{N-3/2}/4+39q^{N-2}/64+O(q^{N-5/2})$
$q=p^{2e+1},\ e\geq 1$	$<q^{N-4}(q^3-p^{e+1}q^2/4+119pq^2/64-O(p^{e+2}q))$
q prime, $q\geq 37$	$<2641q^{N-1}/2700-79q^{N-2}/2700+O(q^{N-3})$
$q=p^h,\ p\geq 5,\ q\geq 19$	$<q^{N-1}-q^{N-3/2}/2+67q^{N-2}/16-O(q^{N-5/2})$
$q=p^h,\ p\geq 3$, h even for $p=3$, $q\geq 23^2,\ q\neq 5^5,\ 3^6$	$<q^{N-1}-q^{N-3/2}/2+35q^{N-2}/16+O(q^{N-5/2})$
$q=Q^h, Q>2$	$(Nh+1)q^N/(Nh)^2+m_2(N-1,q)$

Table 4.4(ii): General upper bounds for $m_2(N,q)$, q odd

$q,\ N$	$m_2(N,q)$
q	$m_2(N+3,q)\geq q^2m_2(N,q)+1$
q	$m_2(N+3r,q)\geq q^{2r}m_2(N,q)+(q^{2r}-1)/(q^2-1)$
q	$m_2(N+6,q)\geq (q^4+q^2-1)m_2(N,q)+q^2+1$
$q>2, N\geq 4$	$m_2(N,q)\leq qm_2(N-1,q)-(q+1)$
$q>2, N\geq 5$	$m_2(N,q)\leq q^{N-4}m_2(4,q)-q^{N-4}-2\theta_{N-5}+1$

Table 4.5: General inequalities

We now present lower bounds on $m_2(N,q)$. The lower bounds for arbitrary values of the dimension N follow from an inductive application of the formulas of Table 4.5. To obtain such a lower bound, in general, we start with the cardinality of a known n-cap in a particular projective space of small dimension. These cardinalities are listed in Table 4.6(i).

q	N	$m_2(N,q) \geq$	
q even, $q > 4$	4	$2q^2 + q + 9$	[72]
$q \equiv 1 \pmod 8$	4	$(5q^2 - 2q - 7)/2$	[73]
$3 < q \equiv 3 \pmod 8$	4	$(5q^2 - 8q - 13)/2$	[73]
$q \equiv 5 \pmod 8$	4	$(5q^2 - 6q - 11)/2$	[73]
$q \equiv 7 \pmod 8$	4	$(5q^2 - 4q - 9)/2$	[73]
q odd	5	$q^3 + q^2 + 3q + 3$	[71]
q even	5	$q^3 + 2q^2 + q + 2$	[160]
q	6	$q^4 + 2q^2$	[71]

Table 4.6(i): Lower bounds on $m_2(N,q)$, $N = 4, 5, 6$

In Tables 4.6(ii) and 4.6(iii), we use the general inequalities to obtain lower bounds for $m_2(N,q)$ for arbitrary dimensions N. The bound for $N = 9$ follows from Edel and Bierbrauer [71].

q	N	$m_2(N,q) \geq$	Condition
$q > 4$	7	$aq^4 + bq^3 + cq^2 + 1$	$m_2(4,q) \geq aq^2 + bq + c$
q odd	8	$q^5 + q^4 + 3q^3 + 3q^2 + 1$	
q even	8	$q^5 + 2q^4 + 2q^3 + 4q^2 + q + 2$	
q	9	$q^6 + 2q^4 + q^2$	

Table 4.6(ii): Lower bounds on $m_2(N,q)$, $N = 7, 8, 9$

The bounds in Table 4.6(iii) follow from an inductive application of the bounds of Edel and Bierbrauer. Namely, for $q > 3$, the formulas for $N = 1 + 6l$ and $N = 4 + 6l$ start from $m_2(4,q) \geq aq^2 + bq + c$, the one for $N = 2 + 6l$ from $m_2(8,q) \geq aq^5 + bq^4 + cq^3 + dq^2 + eq + f$, and the one for $N = 5 + 6l$ from $m_2(5,q) \geq aq^3 + bq^2 + cq + d$. For exact values of the parameters a, b, c, d, e, f, we refer to Tables 4.6(i) and 4.6(ii).

In the first formula for $q = 3$, the corresponding lower bounds on $m_2(v,3)$ can be found in Table 4.7. The second formula for $q = 3$ follows from Calderbank and Fishburn [47].

Table 4.7 presents the sizes of the largest known caps in $PG(N,q)$, for N and q small. This table is from Edel and Bierbrauer [74].

For the size $t_2(N,q)$ of the smallest complete n-cap in $PG(N,q)$, the trivial lower bound is $t_2(N,q) > \sqrt{2q^{N-1}}$. Upper bounds on this parameter are given in Table 4.8. Other small caps are given by Östergard [168].

q	N	$m_2(N,q) \geq$
3	$6l+v$, $v=7,\ldots,12$	$112^l m_2(v,3)$
3	$6l$	$32^l((3l/8)(7/2)^{l-1}+1)$
>3	$6l$	$q^{4l}+(1+l)q^{4l-2}+(l^2-2l+1)q^{4l-4}$
>3	$6l+1$	$aq^{4l}+bq^{4l-1}+(c-a+al)q^{4l-2}+(bl-l)q^{4l-3}$
>3	$6l+2$	$aq^{4l+1}+bq^{4l}+(c-a+al)q^{4l-1}+(d-b+bl)q^{4l-2}$
>3	$6l+3$	$q^{4l+2}+(l+1)q^{4l}+(l^2-2l+3)q^{4l-2}/2$
>3	$6l+4$	$aq^{4l+2}+bq^{4l+1}+(al+c)q^{4l}+blq^{4l-1}$
>3	$6l+5$	$aq^{4l+3}+bq^{4l+2}+(c+al)q^{4l+1}+(d+bl)q^{4l}$

Table 4.6(iii): Lower bounds on $m_2(N,q)$, $N \geq 10$

N \ q	3	4	5	7	8	9	11	13
2	4	6	6	8	10	10	12	14
3	10	17	26	50	65	82	122	170
4	20	41	66	132	208	212	316	388
5	56	126	186	434	695	840		
6	112	288	675	2499	4224	6723		
7	248	756	1715	6472	13520	17220		
8	532	2110	4700	21555	45174	68070		
9	1120	4938	17124	122500	270400	544644		
10	2744	15423	43876	323318	878800	1411830		
11	6272	34566	120740	1067080	2812160	5580100		
12	13312							

Table 4.7: Large caps in small projective spaces

N	q	$t_2(N,q)$	
3	2	$= 5$	[127, §18.2]
3	3	$= 8$	[127, §18.4]
3	4	$= 10$	[81], [183]
3	5	$= 12$	[80], [183]
3	7	≤ 17	[168]
3	8	≤ 20	[168]
3	9	≤ 24	[168]
3	11	≤ 30	[168]
3	13	≤ 37	[168]
3	16	≤ 41	[170]
3	17	≤ 52	[168]
4	2	$= 9$	[90]
4	3	$= 11$	[78], [183]
4	4	≤ 20	[170]
4	5	≤ 31	[168]
4	7	≤ 57	[168]
4	8	≤ 72	[168]
4	9	≤ 88	[168]
5	2	$= 13$	[90]
5	3	≤ 22	[170]
5	4	≤ 50	[168]
5	5	≤ 83	[168]
$2m-1 \geq 9$	2	$\leq 15 \cdot 2^{m-3} - 3$	[90]
$2m-2 \geq 10$	2	$\leq 23 \cdot 2^{m-4} - 3$	[90]
3	q even	$\leq 2q + t_2(2,q)$	[171]
$2r$	q even, $q > 2$	$\leq q^r + 3(q^{r-1} + \cdots + q) + 2$	[170]
$2r+1$	q even, $q > 2$	$\leq 3(q^r + \cdots + q) + 2$	[170]
$4k+2$	q odd, $q \geq 5$	$\leq q^{2k+1} + t_2(2k,q)$	[64]

Table 4.8: Upper bounds for $t_2(N,q)$

5 (n,r)-arcs in $PG(2,q)$

To begin this section on (plane) (n,r)-arcs K, Table 5.1 presents the known exact values of $m_r(2,q)$, while Tables 5.2 and 5.3 give upper and lower bounds on $m_r(2,q)$.

Theorem 5.1 (i) (Ball and Blokhuis [15]) *In $PG(2,q)$, q even, a (k,r)-arc, with $r|q$ and with $k \geq rq - q + r/2$ when $r < q/2$ or with $k > rq - r + 523r/1000$ when $r = q/2$, can be extended in a unique way to an $(rq-q+r,r)$-arc.*

(ii) (Hadnagy and Szőnyi [107]) *An $(qn-q+n-\varepsilon,n)-$arc can be embedded into a maximal arc, provided that $\varepsilon \leq c \cdot n$, c is a constant satisfying $0 < c < 2/3$, n divides q, and $K = q/n$ is large enough. More precisely, $K = 2$, when $0 \leq c \leq 1/3$, $K = 3$, when $0 < c \leq 0.449$, $K = 4$, when $0.449 < c \leq 1/2$, $K \geq (1+2c)(1-c)/(1-c-c^2)$, when $1/2 < c \leq 3/5$ and $K \geq (1+2c)(4-5c)/(4-6c)$, when $3/5 < c < 2/3$.*

(iii) (Szőnyi [214]) *If $\varepsilon \leq q^{1/4}/2$, then any (k,p)-arc of $PG(2,q)$, $q = p^h$, p prime, with $k = qp - q + p - \varepsilon$, can be embedded in a maximal arc.*

r	$q = p^h$, p prime	$m_r(2,q)$	
2	q odd	$q+1$	[37]
2^e	2^h	$(2^e-1)q+2^e$	[68]
$\sqrt{q}+1$	q square, $q \geq 25$	$q\sqrt{q}+1$	[52]
$(q+1)/2$	q odd prime	$(q^2-q)/2+1$	[10]
$(q+3)/2$	q odd prime	$(q^2+q)/2+1$	[10]
q	q	q^2	
$q-1$	q square, $q > 4$	$q^2-q-2\sqrt{q}-1$	[13]
$q-2$	q odd square, $q > 121$	$q^2-2q-3\sqrt{q}-2$	[9]
$q+1-t$, $t > 1$	$q = p^{2e}$, $p > 3$, $t < \min(q^{1/6}, q^{1/4}/2)$	$q^2+q+1-t(q+\sqrt{q}+1)$	[33]
$q+1-t$, $t > 1$	$q = p^{2e}$, $p = 2,3$, $t < \min(2^{-1/3}q^{1/6}, q^{1/4}/2)$	$q^2+q+1-t(q+\sqrt{q}+1)$	[33]

Table 5.1: Exact values of $m_r(2,q)$

The second row of Table 5.2 is implied by the result of Ball, Blokhuis and Mazzocca [16], which states that a maximal arc, that is, an (n,r)-arc with $n = (r-1)q+r$, does not exist for q odd. This had previously been shown for $q = 3^h$ with $r = 3$ by Thas [219] and for $q = 9$ by Cossu [63]. The proof of [16] has been simplified by Ball and Blokhuis [14].

In Table 5.3, the integer $q = p^h$ is *exceptional* if h is odd, $h \geq 3$ and p divides $\lfloor 2\sqrt{q} \rfloor$.

For small r and q, the known values and bounds on $m_r(2,q)$ are enumerated in Table 5.4. The values $m_3(2,11)$ and $m_3(2,13)$ were determined by Marcugini, Milani and Pambianco ([153] and [154], respectively). See Ball [10] for the exact references for the other values.

For recent results on *multi-arcs* in $PG(2,q)$, that is, (n,r)-arcs in $PG(2,q)$ which are allowed to have multiple points, we refer to Ball [12] and Ball *et al.* [17]. In the latter article, a detailed study of $(q^2+q+2, q+2)$-arcs in $PG(2,q)$ is made.

r	Conditions	$m_r(2,q) \leq$	
r		$(r-1)q+r$	[22]
r	q odd	$(r-1)q+r-2$	[16]
r	q odd, $r \mid q$	$(r-1)q+r/2$	[15]
r	$4 \leq r < q,\ r \nmid q$	$(r-1)q+r-3$	[151]
r	$9 \leq r < q,\ r \nmid q$	$(r-1)q+r-4$	[151]
r	q prime, $r \geq (q+3)/2$	$(r-1)q+r-(q+1)/2$	[10]
r	$r \leq 2q/3$	$(r-1)q+q-r$	[122]
r	$r \geq q/2,\ (r,q)>1$, K has a skew line	$(r-1)q+q-r$	[10]
r	$r > 2q/3$	$(r-1)q+(q+r-r/q)/2- \sqrt{(q+r-r/q)^2/4-(r^2-r)}$	[122]
r	K does not have a skew line	$rq-\sqrt{(q+1-r)q}$	[10]
r	K does not have a skew line	$m_r(2,q) < (r-1)q+\lfloor r^2/q \rfloor$	[12]
r	$(r,q)=1,\ r<\sqrt{q}+1$	$(r-1)q+1$	[8]
r	$(r,q)=1$, $6 \leq r \leq \sqrt{2q}+1 \leq q-1$	$(r-1)q+1$	[52]
r	$(r,q)=p^e$, K has a skew line	$(r-1)q+p^e$	[12]
r	q prime, $r \leq (q+1)/2$	$(r-1)q+1$	[10]
$q-1$	$q=17,19$	$q^2-3q/2-5/2$	[10, 13]
$q-1$	$q=p^3$, p prime, $q \geq q_0$	q^2-q-2p^2+5p-1	[149]
$q-1$	$q=p^{2e+1},\ q>19$	$q^2-q-p^e\lceil \frac{p^{e+1}+1}{p^e+1} \rceil -1$	[10, 13]
$q-2$	$q=p^{2e+1},\ q>17$	$q^2-2q-p^e\lceil \frac{p^{e+1}+1}{p^e+1} \rceil -2$	[9, 10]
$q-2$	$q=2^{2e},\ q>4$, or $q \in \{25,49,81,121\}$	$q^2-2q-2\sqrt{q}-2$	[9, 10]
$q-2$	$q=17$	$q^2-5q/2-7/2$	[9, 10]

Table 5.2: Upper bounds for $m_r(2,q)$

r	q	$m_r(2,q) \geq$	
$r=3$	q non-exceptional	$q+1+\lfloor 2\sqrt{q} \rfloor$	[226]
$r=3$	q exceptional	$q+\lfloor 2\sqrt{q} \rfloor$	[226]
$r=\sqrt{q}+1$	q square	$(r-1)q+1$	[38]
$r=(q+1)/2$	q odd	$(r-1)q+1$	[23]
$r=(q+3)/2$	q odd	$(r-1)q+1$	[23]
$r=q-1$	q	$(r-1)q+1$	[123]
$r=q-2$	q even	$(r-1)q+2$	[123]
r	q square	$(r-1)q+r-\sqrt{q}+$ $\sqrt{q}(r-q)$	[123]
$r=q-\sqrt{q}$	q square	$(r-1)q+\sqrt{q}$	[156]
$(q-r)\|q$	q	$(r-1)q+q-r$	[157]

Table 5.3: Lower bounds for $m_r(2,q)$

q / r	2	3	4	5	7	8	9	11	13
2	4	4	6	6	8	10	10	12	14
3	7	9	9	11	15	15	17	21	23
4		13	16	16	22	28	28	$31-34$	$35-40$
5			21	25	29	33	37	$43-45$	$47-53$
6				31	36	42	48	56	$63-66$
7					49	49	55	67	79
8					57	64	65	$77-78$	92
9						73	81	$89-90$	105
10							91	$100-102$	$117-119$
11								121	$131-133$
12								132	$144-147$
13									169
14									183

Table 5.4: Small values of $m_r(2,q)$

$q=p^h,\ p$ prime	$k \geq$	Sharpness
$q=p,\ p$ odd	$3(p+1)/2$	sharp
$q=p^{2e},\ e>0$	$q+\sqrt{q}+1$	sharp
$q=p^{2e+1},\ e>0$	$q+p^{e+1}+1$	sharp for $e=1$
$q=p^{2e+1},\ e>0,\ p>3$	$q+q^{2/3}+1$	sharp for q cube
$q=p^{2e+1},\ e>0,\ p=2,3$	$q+2^{-1/3}q^{2/3}+1$	

Table 6.1: Lower bounds for the size of a blocking set

6 Multiple blocking sets in $PG(2,q)$

A *t-blocking k-set* S in $PG(2,q)$ is a set of k points such that every line of $PG(2,q)$ intersects S in at least t points, and there is a line intersecting S in exactly t points.

For $t = 1$, a 1-blocking set is simply called a *blocking set*. A *trivial* blocking set S is a blocking set containing a line of $PG(2,q)$. A t-blocking set is called *minimal* or *irreducible* when no proper subset of it is a t-blocking set. A 2-blocking set is also called a *double* blocking set and a 3-blocking set a *triple* blocking set.

A subset S of $PG(2,q)$ is a t-blocking set if and only if its complement $S' = PG(2,q) \setminus S$ is a $(k', q+1-t)$-arc. So the study of t-blocking sets is equivalent to the study of $(n, q+1-t)$-arcs.

First of all, lower bounds on the sizes of minimal non-trivial blocking sets S are given in Table 6.1. The third column indicates when this bound is sharp. These results are respectively of Blokhuis [27], Bruen [42, 43], Blokhuis [29], and Blokhuis, Storme and Szőnyi [33].

Theorem 6.1 (Bruen [42]) *In $PG(2,q)$, q square, a non-trivial blocking set of cardinality $q + \sqrt{q} + 1$ is a Baer subplane.*

The following two theorems present results on the spectrum of the values k for which there exists a minimal blocking k-set in $PG(2,q)$.

Theorem 6.2 (i) (Bruen and Thas [45]) *Let S be a minimal blocking k-set in $PG(2,q)$. Then $k \leq q\sqrt{q} + 1$, with equality holding if and only if S is a unital in $PG(2,q)$, q square.*

(ii) (Blokhuis and Metsch [31], Innamorati [137]) *In $PG(2,q)$, with q square and $q \geq 25$ or $q = 9$, there is no minimal blocking set of size $q\sqrt{q}$.*

(iii) (Ball and Blokhuis [13]) *For q square, $q \geq 16$, there is no minimal blocking k-set S with $q + \sqrt{q} + 1 < k < q + 2\sqrt{q} + 1$.*

(iv) (Innamorati and Maturo [138]) *In $PG(2,q)$, $q \geq 4$, there exists a blocking k-set for every k with $2q - 1 \leq k \leq 3q - 3$.*

Let S be a minimal non-trivial blocking k-set in $PG(2,q)$. Then $|S \setminus L| \geq q$ for every line L of $PG(2,q)$. If $|S \setminus L| = q$ for some line L, then the blocking set S is *of Rédei type* with respect to L.

A blocking k-set S in $PG(2,q)$ is called *small* when $k < 3(q+1)/2$. We note that the number e described in Theorem 6.3 (i)(2) is called the *exponent* of the minimal blocking k-set.

Theorem 6.3 (i) (Polverino [185], Szőnyi [213]) *Let S be a small minimal blocking k-set in $PG(2,q)$, $q = p^h$.*

(1) *Then*

$$q + 1 + p^e \left\lceil \frac{q/p^e + 1}{p^e + 1} \right\rceil \leq k \leq \frac{1 + (p^e+1)(q+1) - \sqrt{\Delta}}{2},$$

where $\Delta = (1 + (p^e+1)(q+1))^2 - 4(p^e+1)(q^2+q+1)$, for some integer e, $1 \leq e \leq n/2$.

(2) *If k lies in the interval belonging to e and $p^e \neq 4, 8$, then each line intersects S in 1 $(\mathrm{mod}\ p^e)$ points.*

(ii) (Szőnyi [213]) *Let* $q = p^2$, *p prime, and let S be a minimal blocking k-set which is not a Baer subplane. Then* $k \geq 3(q+1)/2$.

(iii) (Polverino [185], Szőnyi [213]) *For p prime,* $p \geq 7$, *let S be a small minimal blocking k-set in* $PG(2, p^3)$. *Then* $k = p^3 + p^2 + 1$ *or* $k = p^3 + p^2 + p + 1$.

In $PG(2, q)$, q odd, there exists a minimal blocking k-set of Rédei type, the *projective triangle*, of cardinality $3(q+1)/2$. This is a blocking k-set projectively equivalent to the set containing the points $(1, 0, -c)$, $(0, -c, 1)$, $(-c, 1, 0)$, where c is a square of $\mathbf{F}_q$. In $PG(2, 7)$, there also exists a sporadic example of a blocking k-set of cardinality $3(7+1)/2 = 12$; this is the dual of the Hesse configuration $(9_4, 12_3)$ of the nine inflexions of a cubic curve.

Theorem 6.4 (i) (Blokhuis, Ball, Brouwer, Storme, and Szőnyi [30]) *Let S be a minimal blocking k-set of Rédei type in* $PG(2, q)$, *with* $q = p^h$, *p prime. Suppose that e, with* $0 \leq e < h$, *is the largest integer such that each line intersects S in* $1 \pmod{p^e}$ *points. Then* $k \equiv 1 \pmod{p^e}$ *and one of the following holds*:

(1) $e = 0$ *and* $3(q+1)/2 \leq k \leq 2q$;

(2) $e = 1$, $p = 2$ *and* $(4q+5)/3 \leq k \leq 2q - 1$;

(3) $p^e > 2$, $e|h$, *and* $q + q/p^e + 1 \leq k \leq q + (q-1)/(p^e - 1)$.

Also, if $p^e > 3$ *or* $(p^e, k) = (3, q + q/3 + 1)$, *and L is a line such that* $|S \setminus L| = q$, *then the subset* $U = S \setminus L$ *of the affine plane* $AG(2, q) \cong PG(2, q) \setminus L \cong \mathbf{F}_{q^2}$ *is an* $\mathbf{F}_{p^e}$*-linear subspace of* $\mathbf{F}_{q^2}$.

(ii) (Polito and Polverino [184]) *There exist small minimal blocking k-sets, not of Rédei type, in every projective plane* $PG(2, q)$, $q = p^t$, *p prime,* $t \geq 4$.

(iii) (Polverino [186]) *In* $PG(2, p^3)$, *p prime,* $p \geq 7$, *every small minimal blocking k-set is of Rédei type.*

(iv) (Polverino and Storme [187]) *In* $PG(2, q^3)$, $q = p^h$, $h \geq 1$, *p prime,* $p \geq 7$, *the non-trivial minimal blocking k-sets with exponent* $e \geq h$ *are as follows:*

(1) *A Baer subplane* $PG(2, q^{3/2})$ *of cardinality* $q^3 + q^{3/2} + 1$ *when q is a square.*

(2) *A minimal blocking k-set of size* $q^3 + q^2 + 1$, *projectively equivalent to the set* $K = \{(x, \mathrm{T}(x), 1) || x \in \mathbf{F}_{q^3}\} \cup \{ (x, \mathrm{T}(x), 0) || \ x \in \mathbf{F}_{q^3} \setminus \{0\}\}$, *with* $\mathrm{T} : \mathbf{F}_{q^3} \rightarrow \mathbf{F}_q : x \mapsto x + x^q + x^{q^2}$.

(3) *A minimal blocking k-set of size* $q^3 + q^2 + q + 1$, *projectively equivalent to the set* $K = \{(x, x^q, 1) || x \in \mathbf{F}_{q^3}\} \cup \{ (x, x^q, 0) || x \in \mathbf{F}_{q^3} \setminus \{0\}\}$.

(v) (Lovász and Schrijver [148]) *Let S be a blocking k-set of cardinality* $3(p+1)/2$ *in* $PG(2, p)$, *p odd prime, of Rédei type. Then S is projectively equivalent to the projective triangle.*

(vi) (Sziklai and Szőnyi [207]) *Let S be a minimal non-trivial blocking k-set of cardinality* $3(p+1)/2$ *in* $PG(2, p)$, $p > 2$ *prime. Then every point of S lies on exactly* $(p-1)/2$ *tangents and there are at least* $(p+1)/2$ *lines intersecting S in exactly 2 points.*

(vii) (Gács [92]) *For a Rédei type blocking k-set S in* $PG(2, p)$, *p odd prime, which is not a projective triangle,* $|S| \geq p + 2(p-1)/3 + 1$.

(viii) (Gács [93]) *In* $PG(2, 11)$, *every minimal blocking* 18*-set is a projective triangle.*

(ix) (Gács [93]) *In* $PG(2, 7)$, *a minimal blocking* 12*-set is a projective triangle or is the sporadic example.*

A blocking k-set S of $PG(2, q)$ is of *almost Rédei type* when there exists a line L of $PG(2, q)$ such that $|S \setminus L|$ is close to q; more precisely, $|S \setminus L| = q + m$ with $m \leq \sqrt{q}/2$.

Theorem 6.5 (i) (Blokhuis, Pellikaan and Szőnyi [32]) *Let S be a minimal non-trivial blocking k-set in $PG(2,q)$, $q = p^h$, p prime, such that $|S \cap L| = k - q - m$, with $0 < m \leq \sqrt{q}/2$. Then one of the following possibilities occurs*:

(1) *if* $k < 3(q+1)/2$, *then* $m \equiv 0 \pmod{p}$;

(2) *if* $m = 1$, *then* $k \geq 3(q+1)/2$;

(3) *if* $m = 2$ *and* $p \neq 2$, *then* $k \geq 3(q+1)/2$;

(4) *if* $m > 2$ *and* $m \not\equiv 0 \pmod{p}$, *then* $k \geq q + (q+1)/2 + m$.

(ii) *Let $q = p$ be a prime, let S be a blocking k-set, not of Rédei type, with $k = 3(p+1)/2$ in $PG(2,p)$, and let L be a line of $PG(2,p)$. Then, there are three cases*:

(1) (Blokhuis, Pellikaan and Szőnyi [32]) *when* $|S \cap L| \leq (p+3)/2 - 3$, *then* $|S \cap L| \leq (9p - 15)/20$;

(2) (Gács, Sziklai and Szőnyi [94]) *if* $|S \setminus L| = p + 1$, *then* $p = 3$ *or* $p = 5$ *and S is of Rédei type with respect to an other line, or* $p = 7$ *and S is the sporadic blocking 12-set in* $PG(2,7)$;

(3) (Gács [91]) *if* $|S \setminus L| = p + 2$, *then* $p \leq 7$ *and S is of Rédei type with respect to an other line.*

Table 6.2 shows the values of n for which there exists a minimal blocking k-set in $PG(2,q)$ (Blokhuis [29]). The non-existence of a minimal blocking 14-set in $PG(2,8)$ was proved by Barát, Del Fra, Innamorati and Storme [18], and the non-existence of minimal blocking k-sets in $PG(2,8)$ for $k = 22, 23$ was proved by Del Fra and Innamorati [67].

q	$n \in$
4	$\{7,8,9\}$
5	$\{9,\ldots,12\}$
7	$\{12,\ldots,19\}$
8	$\{13,15,16,\ldots,21\}$
9	$\{13,15,16,\ldots,26,28\}$

Table 6.2: The sizes of a blocking k-set in small planes

To give an idea of how good Theorem 6.2(i) is in general, there are asymptotic results on large minimal blocking k-sets.

Theorem 6.6 (i) (Szőnyi [211]) *In* $PG(2,q)$, $q > q_0$, *there exist minimal blocking k-sets of size* $cq \log q$.

(ii) (Hirschfeld and Szőnyi [134]) *In* $PG(2,q)$, *q square*, $q > q_0(\lambda)$, *there exist minimal blocking k-sets of size* $cq^{1+\lambda}$, $1/4 < \lambda \leq 1/2$.

$q = p^h,\ p$ prime	t	$\|S\| \geq$	Sharp	
$q = p^{2e},\ p > 3,$ $e \geq 2$	$t <$ $\min(q^{1/6}, q^{1/4}/2)$	$t(q+\sqrt{q}+1)$	yes	[33]
$q = p^{2e},\ p = 2, 3,$ $e \geq 2$	$t <$ $\min(2^{-1/3}q^{1/6}, q^{1/4}/2)$	$t(q+\sqrt{q}+1)$	yes	[33]
$q = p^2$	$t < q^{1/4}/2$	$t(q+\sqrt{q}+1)$	yes	[33]

$q = p^h,\ p$ prime	t	$\|S\| \geq$	Sharp for	
$q = p$ prime, $p > 3$	$t <$ $p/2$	$(2t+1)(p+1)/2$	$t = 1,$ $(p-1)/2$	[10]
$q = p$ prime, $p > 3$	$t >$ $p/2$	$(t+1)p$	$t =$ $(p+1)/2$	[10]
$q < 9$	2	$3q$	$t = 2$	[25, 60] [151]
$q = 11, 13, 17, 19$	2	$(5q+7)/2$		[10, 13]
q square, $q > 4$	2	$2q + 2\sqrt{q} + 2$	$t = 2$	[10, 13]
$q = p^3,\ q \geq q_0$	2	$2q + 2p^2 - 5p + 2$		[149]
$q = p^{2e+1} > 19$	2	$2q + p^e\lceil \frac{p^{e+1}+1}{p^e+1} \rceil + 2$		[10, 13]
$q = 5, 7, 9$	3	$4q$	$t = 3$	[9, 60] [151]
$q = 8$	3	31	$t = 3$	[123]
$q = 11, 13, 17$	3	$(7q+9)/2$		[9, 10]
q odd square, $q > 121$	3	$3q + 3\sqrt{q} + 3$	$t = 3$	[9, 10]
$q = p^{2e+1},\ q > 17$	3	$3q + p^e\lceil \frac{p^{e+1}+1}{p^e+1} \rceil + 3$		[9, 10]
q even square, $q > 4$, or $q \in \{25, 49, 81, 121\}$	3	$3q + 2\sqrt{q} + 3$		[9, 10]

q	t	Condition	$\|S\| \geq$	
q	t	S does not contain a line	$tq + \sqrt{tq} + 1$	[10]
q	t	S contains a line and $(t-1, q) = 1$	$q(t+1)$	[28]
q	t	S contains a line and $(t-1, q) > 1,\ t \leq q/2 + 1$	$tq + q - t + 2$	[44]
q	t	S contains a line and $(t-1, q) > 1,\ t \geq q/2 + 1$	$t(q+1)$	[10]

Table 6.3: Lower bounds for t-blocking k-sets

Now, Table 6.3 gives lower bounds on the number of points in a t-blocking k-set S of $PG(2, q)$. The table is subdivided into three parts. In the first and second part, lower bounds on t-blocking k-sets, with no condition on the t-blocking k-set, are given. The third part gives lower bounds on t-blocking k-sets with conditions on the t-blocking k-sets.

There are various characterizations of the smallest t-blocking k-sets.

Theorem 6.7 (i) (Gács and Szőnyi [95]) *In* $PG(2,q)$, *with* q *an odd square and* $q \geq 169$, *a double blocking* k*-set containing* $2q+2\sqrt{q}+2$ *points is the union of two disjoint Baer subplanes.*

(ii) (Blokhuis, Storme and Szőnyi [33]) *Let* S *be a* t*-blocking* k*-set in* $PG(2,q)$, $q = p^h$, p *prime, of size* $t(q+1)+c$. *Let* $c_2 = c_3 = 2^{-1/3}$ *and* $c_p = 1$ *for* $p > 3$.

(1) *If* $q = p^{2d+1}$ *and* $t < \frac{1}{2}(q - c_p q^{2/3})$, *then* $c \geq c_p q^{2/3}$.

(2) *If* q *is a square with* $q > 4$, *and* $t < \min(c_p q^{1/6}, q^{1/4}/2)$, $c < c_p q^{2/3}$, *then* $c \geq t\sqrt{q}$ *and* S *contains the union of* t *disjoint Baer subplanes.*

(3) *If* $q = p^2$, $t < q^{1/4}/2$ *and* $c < p\left\lceil \frac{1}{4} + \sqrt{\frac{p+1}{2}} \right\rceil$, *then* $c \geq t\sqrt{q}$ *and* S *contains the union of* t *disjoint Baer subplanes.*

A t-blocking k-set is called *small* when $k < tq + (q+3)/2$. Similarly to Theorem 6.3, there are the following results on small minimal t-blocking k-sets. These results are based upon the abstract [149].

Theorem 6.8 (Lovász and Szőnyi [149]) (i) *Let* S *be a small minimal* t*-blocking* k*-set in* $PG(2,q)$, $q = p^h$, p *prime,* $t < p$, *with* $q > q_0(t)$.

(1) *Then*

$$tq + t + p^e \left\lceil \frac{q/p^e + 1}{p^e + 1} \right\rceil \leq k \leq \frac{1 + (q+1)(2t - 1 + p^e) - \sqrt{\Delta}}{2},$$

where $\Delta = (1 + (q+1)(2t - 1 + p^e))^2 - 4(q^2 + q + 1)(t^2 + tp^e)$, *for some integer* $1 \leq e \leq h/2$.

(2) *If* k *lies in the interval belonging to* e, *then each line intersects* S *in* $t \pmod{p^e}$ *points.*

(ii) *Let* S *be a small minimal* t*-blocking* k*-set in* $PG(2,p^2)$, p *prime,* $t < \sqrt{p}/2$, $p > p_0(t)$. *Then* S *is the disjoint union of* t *Baer subplanes.*

7 Blocking sets in $PG(N,q)$

The concepts of Section 6 can be extended to $PG(N,q)$. A *blocking set in* $PG(N,q)$ *with respect to* k*-dimensional subspaces of* $PG(N,q)$ is a set of points K such that any k-dimensional subspace intersects K in at least one point.

Theorem 7.1 (Bose and Burton [40]) *The smallest blocking sets of* $PG(N,q)$ *with respect to* k*-dimensional subspaces are the* $(N-k)$*-dimensional subspaces of* $PG(N,q)$.

A blocking set with respect to k-dimensional subspaces in $PG(N,q)$ is called *non-trivial* when no $(N-k)$-dimensional subspace is contained in K. *Minimality* is defined in the usual way.

We now present a characterization result on the smallest such non-trivial minimal blocking sets.

Theorem 7.2 (Beutelspacher [24], Heim [118]) *In* $PG(N,q)$, *the smallest non-trivial minimal blocking sets with respect to* k*-dimensional subspaces are cones with an* $(N-k-2)$*-dimensional vertex and a non-trivial* 1*-blocking set of minimum cardinality in a plane of* $PG(N,q)$ *as base curve.*

Improvements to this characterization for q square have been obtained by Bokler and Metsch.

Theorem 7.3 (Bokler and Metsch [35], Bokler [34]) *Let B be a minimal non-trivial blocking set with respect to k-dimensional subspaces of $PG(N,q)$, $N \geq 2$, q square, $q \geq 16$, $N \geq k$. If $|B| \leq \theta_{N-k} + \sqrt{q}\theta_{N-k-1}$, then B contains a cone with t-dimensional vertex and with base a subgeometry $PG(2(N-k-t-1), \sqrt{q})$, for some t with* $\max\{-1, N-2k-1\} \leq t \leq N-k-1$.

A significant result on small minimal non-trivial 1-blocking sets of $PG(2,q)$ was shown by Szőnyi (Theorem 6.3(i)(2)); this states that every line intersects S in $1 \pmod{p^e}$ points. This result gives precise disjoint intervals in which the sizes of small minimal non-trivial 1-blocking sets of $PG(2,q)$ must lie.

These results were extended to blocking sets with respect to k-dimensional subspaces of $PG(N,q)$ by Szőnyi and Weiner [215]. In the following theorem, let $S(q)$ denote the set of sizes k of the small minimal blocking k-sets in $PG(2,q)$; see also Theorem 6.3(i)(1).

Theorem 7.4 (Szőnyi and Weiner [215]) *Let B be a minimal blocking set B of $PG(N,q)$ with respect to k-dimensional subspaces, $q = p^h$, $p > 2$ prime, and assume that $|B| < 3(q^{N-k}+1)/2$. Then any subspace that intersects B intersects it in $1 \pmod{p}$ points.*

Regarding the sizes of minimal blocking sets of $PG(N,q)$ with respect to k-dimensional subspaces, the following results are known.

Theorem 7.5 (Szőnyi and Weiner [215]) *Let B be a minimal blocking set of $PG(N,q)$ with respect to k-dimensional subspaces, $q = p^h$, p prime. If $p = 2$ let $|B| < \sqrt{2}q^{N-k}$, otherwise let $|B| < 3(q^{N-k}+1)/2$. Then*

(i) $|B| \in S(q^{N-k})$;

(ii) *if $p > 2$, then* $((|B|-1)(q^{N-k})^{N-2}+1) \in S((q^{N-k})^{N-1})$.

In the following theorems, let $l(q,e)$ and $u(q,e)$ denote respectively the smallest and largest value so that for any minimal non-trivial blocking k-set B in $PG(2,q)$, $q = p^h$, p prime, and of size $l(q,e) \leq |B| \leq u(q,e)$, e is the largest integer such that each line intersects B in $1 \pmod{p^e}$ points; see also Theorem 6.3(i).

Theorem 7.6 (Szőnyi and Weiner [215]) *Let B be a minimal blocking set in $PG(N,q)$, $q = p^h$, $p > 2$ prime, with respect to k-dimensional subspaces. Assume that $|B| < 3(q^{N-k}+1)/2$ and let e be the integer for which*

$$l((q^{N-k})^{N-1}, e) \leq (|B|-1)(q^{N-k})^{N-2}+1 \leq u((q^{N-k})^{N-1}, e).$$

Then each subspace that intersects B in at least one point intersects it in $1 \pmod{p^e}$ points.

For characterizations of small minimal blocking sets in $PG(N,q)$, the following results have been obtained. The first result is on *Rédei-type* blocking sets with respect to k-dimensional subspaces of $PG(N,q)$. A blocking set with respect to k-dimensional subspaces of $PG(N,q)$, of size q^k+d, is called a *Rédei-type* blocking set with respect to k-dimensional subspaces, when it intersects at least one hyperplane Π_{N-1} in exactly d points. We describe the theorem via the corresponding affine space $AG(N,q) = PG(N,q) \setminus \Pi_{N-1}$.

Theorem 7.7 (Storme and Sziklai [200]) *Let $U \subset AG(N,q)$, $N \geq 3$, $|U| = q^k$. Suppose U determines $|D| \leq \frac{q+3}{2}q^{k-1} + q^{k-2} + q^{k-3} + ... + q^2 + q$ directions and suppose that U is a $\mathbf{F}_p$-linear set of points, where $q = p^h$, $p > 3$. If $N-1 \geq (N-k)h$, then U is a cone with an $(N-1-h(N-k))$-dimensional vertex in Π_{N-1} and with base a $\mathbf{F}_p$-linear point set $U_{(N-k)h}$ of size $q^{(N-k)(h-1)}$, contained in some affine $(N-k)h$-dimensional subspace of $AG(N,q)$.*

Theorem 7.8 (Szőnyi and Weiner [215]) (i) *Let B be a minimal blocking set of $PG(N,q)$ with respect to k-dimensional subspaces, $q = p^h$ and $p > 2$ prime. Suppose that $|B| < 3(q^{N-k}+1)/2$ and $h(N-k) \leq N$. Assume that B is not contained in an $(h(N-k)-1)$-dimensional subspace of $PG(N,q)$, then B is projectively equivalent to a subgeometry $PG(h(N-k),p)$.*

(ii) *Let B be a minimal blocking set of $PG(N,q)$ with respect to k-dimensional subspaces, $q = p^h$ and $p > 2$ prime. Assume that e is an integer, $1 \leq e \leq h/2$, and $h(N-k)/e \leq N$. Suppose also that $(|B|-1)(q^{N-k})^{N-2} + 1 \leq u((q^{N-k})^{N-1}, e)$. Then B is contained in an $(\lceil h(N-k)/e \rceil - 1)$-dimensional subspace of $PG(N,q)$ or B is projectively equivalent to $PG(h(N-k)/e, p^e)$.*

The following theorem presents more specific characterizations. Parts (i), (iii) and (iv) are from Storme and Weiner [206], and Part (ii) from Szőnyi and Weiner [215].

Theorem 7.9 (i) *A non-trivial minimal blocking set of $PG(N,p^2)$, $p > 3$ prime, with respect to hyperplanes and of size less than or equal to $3(p^2+1)/2$, is a Baer subplane or a planar blocking set of size $3(p^2+1)/2$.*

(ii) *A non-trivial minimal blocking set of $PG(N,p^3)$, $p > 2$ prime, with respect to hyperplanes and of size less than $3(p^3+1)/2$, is a planar blocking set or a subgeometry $PG(3,p)$.*

(iii) *Let $s(q^2)$ be the size of the smallest non-trivial blocking set in $PG(2,q^2)$ not equal to a Baer subplane. Let B be a minimal blocking set in $PG(N,q^2)$, $q = p^h$, $h \geq 1$, $p > 3$ prime, $N \geq 3$, with respect to hyperplanes and with $|B| \leq s(q^2)$. Then B is a line or a minimal planar blocking set of $PG(N,q^2)$.*

(iv) *In $PG(N,q^3)$, $q = p^h$, $h \geq 1$, p prime, $p \geq 7$, $N \geq 3$, a minimal blocking set B, with respect to hyperplanes, of cardinality at most q^3+q^2+q+1 is one of the following:*

(1) *a line;*
(2) *a Baer subplane when q is square;*
(3) *a minimal blocking set of cardinality q^3+q^2+1 in a plane of $PG(N,q^3)$;*
(4) *a minimal blocking set of cardinality q^3+q^2+q+1 in a plane of $PG(N,q^3)$;*
(5) *a subgeometry $PG(3,q)$ in a solid of $PG(N,q^3)$.*

For an article containing general information on a particular class of blocking sets in $PG(N,q)$ with respect to k-dimensional subspaces, called *linear* blocking sets, we refer to Lunardon [150].

8 n-tracks and almost MDS codes

An *n-track* in $PG(N,q)$ is an n-set of kind $N-1$. In other words, it is a set of n points such that every N of them are linearly independent, but some $N+1$ of them are linearly dependent.

An $[n,k,d]$ code C over $\mathbf{F}_q$ is an *almost MDS code* if its minimum distance d is equal to $n-k$.

N	q	$M_{N-1}(N,q)$
2		q^2+q+1
3	$q>2$	q^2+1
$2q-2$	$q>3$	$2q+1$
$2q-1$	$q>3$	$2q+2$
$\geq 2q$		0

Table 8.1: Exact values of $M_{N-1}(N,q)$

N	q	$M_{N-1}(N,q) \geq$	Conditions
5		$m_3(2,q)$	
	$q=p^h$	$q+\lfloor 2\sqrt{q} \rfloor$	q exceptional
	$q=p^h$	$q+\lfloor 2\sqrt{q} \rfloor + 1$	q non-exceptional

Table 8.2: Lower bounds for $M_{N-1}(N,q)$

$N,\ q$	$M_{N-1}(N,q) \leq$	Conditions
	$M_{N-2}(N-1,q)+1$	
$q=p^h$	$q(q-N+3)+1$	$m(N,q)=q+1$ and $N=q-p^l+3$ for some $l \leq h$
$q=p^h$	$q(q-N+3)$	$m(N,q)=q+1$ but $N \neq q-p^l+3$ for all $l \leq h$
$N \geq q$	$2q+N-1$	$N \geq 4$ when $q=2$ and $N \geq 6$ when $q=3$

Table 8.3: Upper bounds for $M_{N-1}(N,q)$

N \ q	2	3	4	5	7	8	9	11
2	7	13	21	31	57	73	91	133
3	8	10	17	26	50	65	82	122
4		11	11	$12-20$	$16-30$	$14-36$	$16-43$	$22-57$
5		12	12	$12-14$	$15-31$	$15-37$	$17-44$	$23-58$
6			9	$10-15$	$13-28$	$14-34$	$17-39$	$18-49$
7			10	$11-16$	$13-20$	$14-35$	$18-40$	$18-50$
8				11	$13-21$	$14-23$	$19-36$	$19-50$
9				12	$13-22$	$14-24$	$20-26$	$20-51$
10					$14-23$	$14-25$	$16-27$	$18-44$
11					$15-24$	$15-26$	$16-28$	$18-32$
12					15	$15-27$	$16-29$	$18-33$
13					16	$16-28$	$17-30$	$18-34$

Table 8.4: Small values of $M_{N-1}(N,q)$

Almost MDS codes and n-tracks are equivalent objects since an n-track in the space $PG(n-k-1,q)$ defines a parity check matrix of an $[n,k,n-k]$ code over $\mathbf{F}_q$.

These two equivalent structures have been studied in detail by De Boer [66] and by Dodunekov and Landjev [69], resulting in Tables 8.1, 8.2, 8.3, which give the known exact values, as well as upper and lower bounds on the maximum number $M_{N-1}(N,q)$ of points of an n-track in $PG(N,q)$. The integer $q=p^h$ is *exceptional* if h is odd, $h \geq 3$ and p divides $\lfloor 2\sqrt{q} \rfloor$.

Table 8.4 gives values for $M_{N-1}(N,q)$ for small N and q; see De Boer [66].

9 Minihypers

A subconfiguration closely related to (multiple) blocking sets with respect to k-dimensional subspaces of $PG(N,q)$, is the subconfiguration *minihyper*.

An $\{n,m;N,q\}$-*minihyper* K is the complement of a $(\theta_N - n; \theta_{N-1} - m, N-1; N,q)$-set; that is, K is a set of n points in $PG(N,q)$, $N \geq 2, n \geq 1$, such that (a) $|K \cap \Pi_{N-1}| \geq m$ for every hyperplane Π_{N-1}, and (b) $|K \cap \Pi_{N-1}| = m$ for some hyperplane Π_{N-1}.

When constructing a code, from an economical point of view, it is desirable to obtain an $[n,k,d]$ code C over $\mathbf{F}_q$ whose length n is minimal for given values of k, d, and q. The Griesmer bound [103, 199] shows that if there is an $[n,k,d]$ code C over $\mathbf{F}_q$ for given k,d, and q, then $n \geq \sum_{i=0}^{k-1} \lceil d/q^i \rceil$.

Theorem 9.1 (Hamada [110, 111]) *For $k \geq 3$ and $1 \leq d < q^{k-1}$, there is a one-to-one correspondence between the set of all non-equivalent $[n,k,d]$ codes C over $\mathbf{F}_q$ meeting the Griesmer bound and the set of all $\{\theta_{k-1}-n, \theta_{k-2}-n+d; k-1, q\}$-minihypers.*

There are a number of articles studying this correspondence. For surveys on recent work, see Hamada [112, 113, 114]. The general results are presented in the following theorems. We also wish to remark that this notion of minihypers is linked to that of *anticodes*, introduced by Farrell [83].

Theorem 9.2 (Hamada and Helleseth [115], Hamada and Maekawa [116]) *Let t,q,h and λ_i, $i=1,\ldots,h$, be any integers such that $t \geq 2, h \geq 1$, $q > (h-1)^2$ and $0 \leq \lambda_1 \leq \cdots \leq \lambda_h < t$.*

(i) *If $t < \lambda_{h-1} + \lambda_h + 1$, there is no $\{\sum_{i=1}^h \theta_{\lambda_i}, \sum_{i=1}^h \theta_{\lambda_i - 1}; t,q\}$-minihyper.*

(ii) *If $t \geq \lambda_{h-1} + \lambda_h + 1$, then F is a $\{\sum_{i=1}^h \theta_{\lambda_i}, \sum_{i=1}^h \theta_{\lambda_i - 1}; t,q\}$-minihyper if and only if F is the pairwise disjoint union of a subspace $PG(\lambda_1,q)$, a subspace $PG(\lambda_2,q), \ldots, PG(\lambda_h,q)$.*

Theorem 9.3 (Barát and Storme [20]) *If F is a $\{\varepsilon_0 + \varepsilon_1(q+1), \epsilon_1; t,q\}$-minihyper of $PG(t,q)$, $t \geq 3$, $q=p^h$, p prime, $q \geq 2^{12}$, $c_2 = c_3 = 2^{-1/3}$ and $c_p = 1$ when $p > 3$, with $\varepsilon_1 < c_p q^{1/6}$ and with $\varepsilon_0 + \varepsilon_1 < c_p q^{2/3} - (\varepsilon_1 - 1)(\varepsilon_1 - 2)/2$, then F contains the union of a collection of s disjoint lines and/or Baer subplanes.*

Theorem 9.4 (Ferret and Storme [86]) *Assume that the set F is a $\{\sum_{i=0}^s \epsilon_i \theta_i, \sum_{i=0}^s \epsilon_i \theta_{i-1}; t,q\}$-minihyper, where $\sum_{i=0}^s \epsilon_i < 2\sqrt{q}$, q square, $q=p^f$, p prime, with $q > 2^{18}$ when $p > 3$ and $q > 2^{20}$ when $p=2,3$. Then F consists of the disjoint union of one of the following:*

(i) *ϵ_s spaces $PG(s,q), \epsilon_{s-1}$ spaces $PG(s-1,q), \ldots, \epsilon_0$ points;*

(ii) *one subgeometry* $PG(2l{+}1, \sqrt{q})$*, for some integer* l *with* $1 \le l \le s$*,* ϵ_s *spaces* $PG(s,q), \ldots,$ ϵ_{l+1} *spaces* $PG(l+1,q), \epsilon_l - \sqrt{q} - 1$ *spaces* $PG(l,q), \epsilon_{l-1}$ *spaces* $PG(l-1,q), \ldots, \epsilon_0$ *points;*

(iii) *one subgeometry* $PG(2l, \sqrt{q})$*, for some integer* l *with* $1 \le l \le s$*,* ϵ_s *spaces* $PG(s,q), \ldots,$ ϵ_{l+1} *spaces* $PG(l+1,q), \epsilon_l - 1$ *spaces* $PG(l,q), \epsilon_{l-1} - \sqrt{q}$ *spaces* $PG(l-1,q), \epsilon_{l-2}$ *spaces* $PG(l-2,q), \ldots, \epsilon_0$ *points.*

Theorem 9.5 (Ferret and Storme [86]) *Assume that the set* F *is a* $\{\sum_{i=0}^{s} \epsilon_i\theta_i, \sum_{i=0}^{s} \epsilon_i\theta_{i-1}; t, q\}$*-minihyper,* $t \ge 3$*, where*

(a) $\sum_{i=0}^{s} \epsilon_i \le q^{6/9}/(1+q^{1/9})$*,* $q = p^f$*,* f *odd,* p *prime,* $p > 3$*,* $q \ge 2^{12}$*;*

(b) $\sum_{i=0}^{s} \epsilon_i \le c_p q^{5/9}$*,* $q = p^f$*,* f *odd,* $p = 2, 3$*,* $q > 2^{12}$*,* $c_p = 2^{-1/3}$*.*

Then F *is the disjoint union of* ϵ_s *spaces* $PG(s,q)$*,* ϵ_{s-1} *spaces* $PG(s-1,q)$*, …,* ϵ_0 *points.*

Theorem 9.6 (Govaerts and Storme [101]) *Let* $q > 2$ *and* $\delta < \epsilon$*, where* $q + \epsilon$ *is the size of the smallest non-trivial blocking sets in* $PG(2,q)$*. If* F *is a* $\{\delta\theta_\mu, \delta\theta_{\mu-1}; N, q\}$*-minihyper satisfying* $\mu \le N - 1$*, then* F *is the disjoint union of* δ *subspaces* $PG(\mu, q)$*.*

Theorem 9.7 (Govaerts and Storme [102]) *A* $\{\delta\theta_\mu, \delta\theta_{\mu-1}; N, q\}$*-minihyper* F*,* $q > 16$ *square,* $\delta < q^{5/8}/\sqrt{2}{+}1$*,* $2\mu{+}1 \le N$*, is a unique disjoint union of subspaces* $PG(\mu,q)$ *and subgeometries* $PG(2\mu+1, \sqrt{q})$*.*

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