Lost in Hilbert Space

Matthew G. Parker

Selmer Centre, Inst. for Informatikk, Høyteknologisenteret i Bergen,
University of Bergen, Bergen 5020, Norway,
matthew@ii.uib.no

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LOST in Hilbert space

Important Sequence Families

- Periodic: m-sequences, univariate
- bent Boolean functions, multivariate
- Aperiodic: Golay sequences, univariate
- complementary arrays, multivariate
  occur in pairs.

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  occur in pairs.

Golay Pairs?

Golay sequence pairs.

Aperiodic autocorrelations sum to δ.

A: 1, 1, 1, −1

1 1 1 −1 : 4

B: 1, 1, −1, 1

1 1 −1 1 : 4

8

Golay Pairs?

Golay sequence pairs.

Aperiodic autocorrelations sum to δ.

A: 1, 1, 1, −1

1, 1, 1 : 4, 1

B: 1, 1, −1, 1

1, 1, −1 : 4, −1

8, 0

Golay Pairs?

Golay sequence pairs.

Aperiodic autocorrelations sum to δ.

A: 1, 1, 1, −1
      1, 1 : 4, 1, 0

B: 1, 1, −1, 1
      1, 1 : 4, −1, 0

8, 0, 0

Golay Pairs?

Golay sequence pairs.

Aperiodic autocorrelations sum to δ.

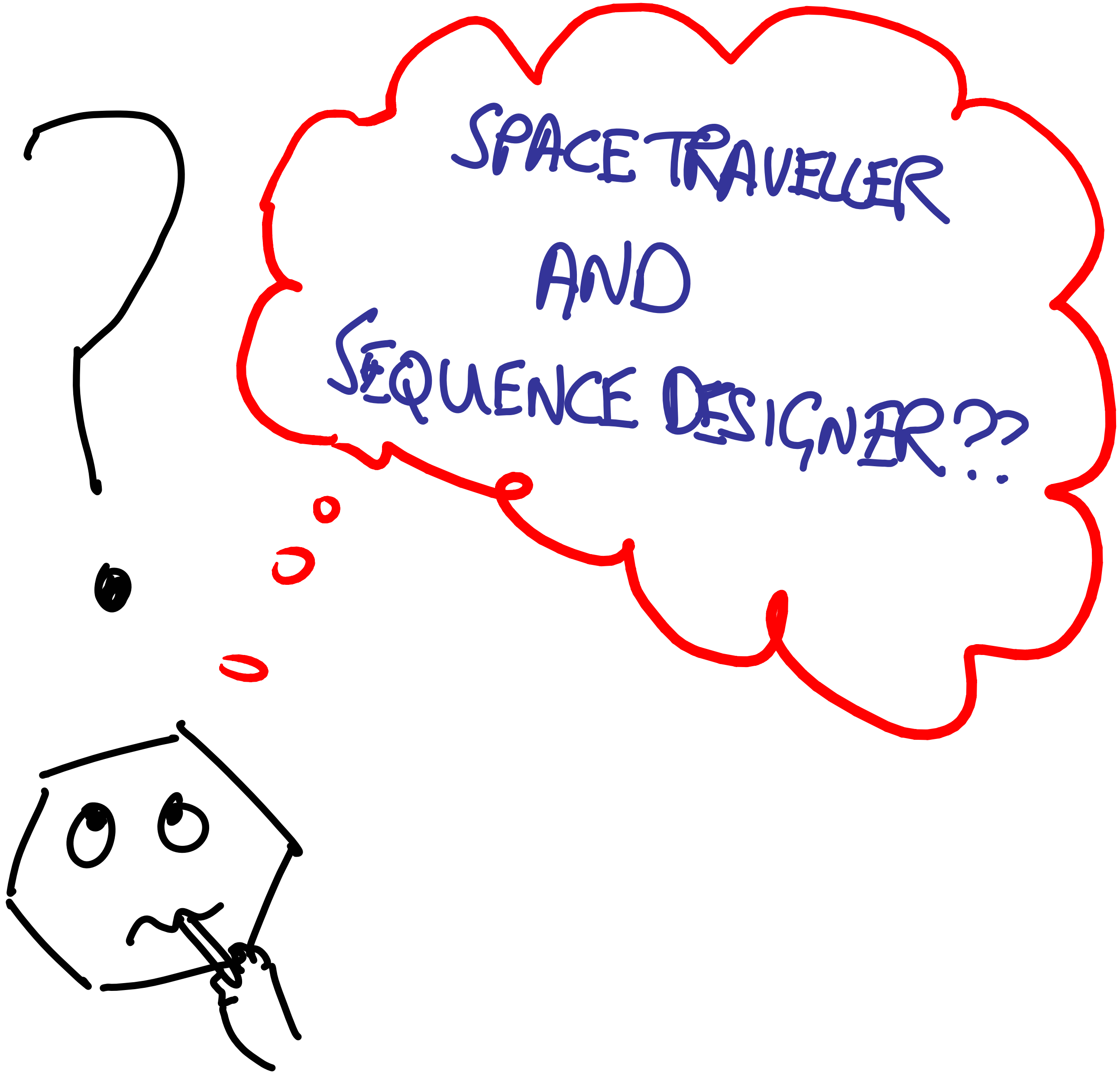
$A: \; 1, 1, 1, -1$

$: \; 4, 1, 0, -1$

$B: \; 1, 1, -1, 1$

$: \; 4, -1, 0, 1$

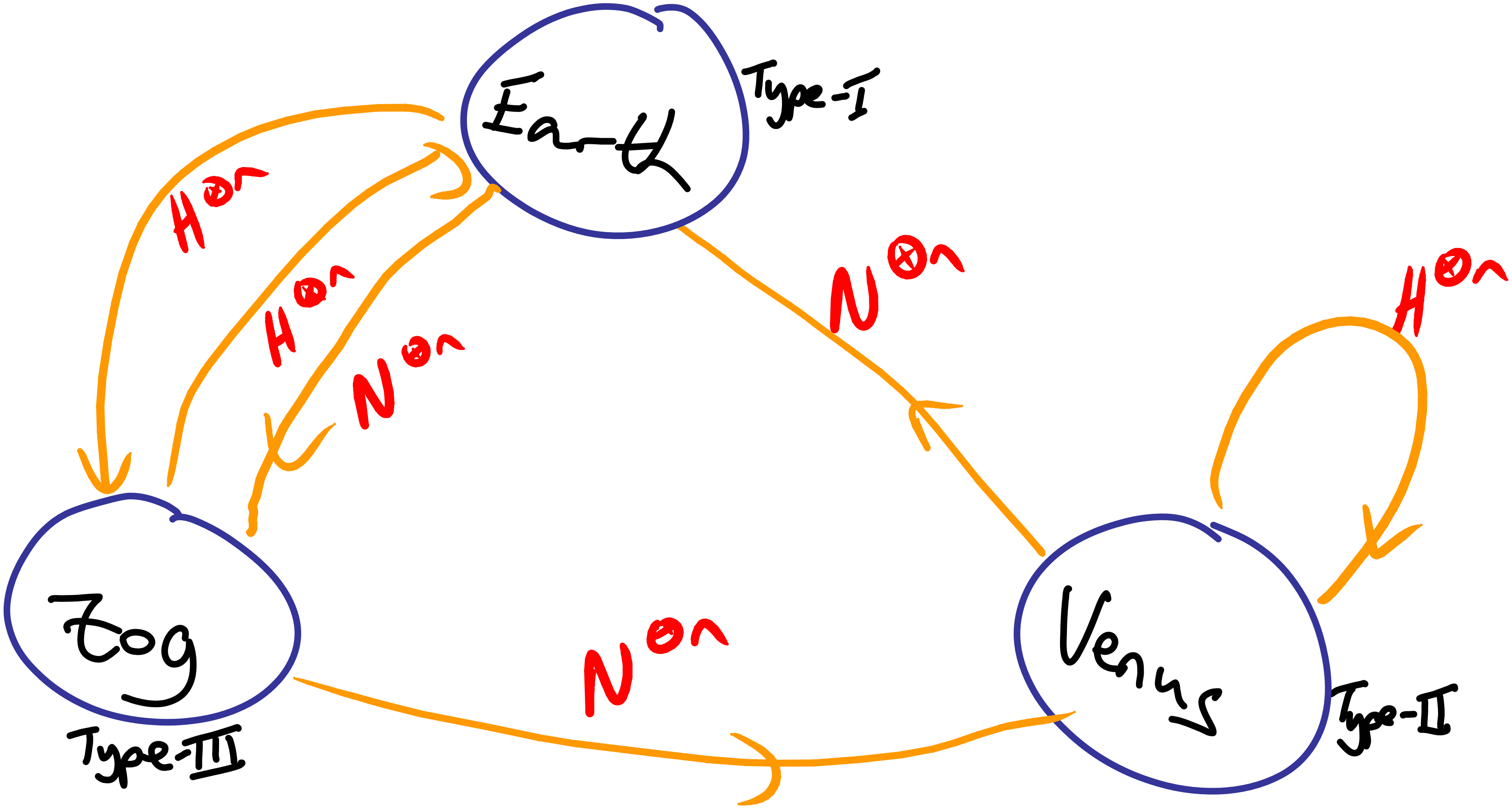
$8, 0, 0, 0 \leftarrow \delta$


SPACE TRAVELLER
AND
SEQUENCE DESIGNER??


Courtesy MPTV
Irwin Allen News Network Vaults

Space Map

$$H=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix},\quad N=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & i\\ 1 & -i\end{bmatrix}.$$



Earth $\to$ Venus

$$N = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}, \quad N^{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}.$$

$$(A_\Pi, B_\Pi) \leftarrow N^{\otimes n}(A, B).$$

$$A_\Pi = \frac{1}{2}\overbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ -i & i & -i & i \\ -i & -i & i & i \\ -1 & 1 & 1 & -1 \end{bmatrix}}^{(N^{-1})^{\otimes 2}} \overbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}}^{A} = \begin{bmatrix} 1 \\ -i \\ -i \\ 1 \end{bmatrix}$$

Similarly,

$$B_\Pi = \begin{bmatrix} 1 \\ i \\ -i \\ -1 \end{bmatrix}.$$

Venus

Complementary sequence pairs:
Aperiodic autoconvolutions sum to

$2, 0, 2, 0 \dots 2, 0, 2.$

A_{I} : $1, -i, -i, 1$

$1, -i, -i, 1$

B_{I} :

Venus

Complementary sequence pairs:
Aperiodic autoconvolutions sum to

2, 0, 2, 0 .... 2, 0, 2.

A_{I} : 1, −i, −i, 1

1, i, i, 1

B_{I} :

Venus

Complementary sequence pairs:
Aperiodic autoconvolutions sum to
2, 0, 2, 0 .... 2, 0, 2.

A_{II} :
$1, -i, -i, 1$
$1, i, i, 1$: 1

B_{II} :
$1, i, -i, -1$
$-1, i, -i, 1$: 1

2

Venus

Complementary sequence pairs:
Aperiodic autoconvolutions sum to

$2, 0, 2, 0 \ldots.\ 2, 0, 2.$

A_{I}: $\quad 1, -i, -i, 1$

$\quad 1, i, i, 1 \quad$: $1, 0$

B_{II}: $\quad 1, i, -i, -1$

$\quad -1, i, -i, 1 \quad$: $1, 0$

$2, 0$

Venus

Complementary sequence pairs:
Aperiodic autoconvolutions sum to
$2,0,2,0 \ldots 2,0,2.$

A_{I}: $1,-i,-i,1$
$1,i,i,1$: $1,0,1$

B_{II}: $1,i,-i,-1$
$-1,i,-i,1$: $1,0,1$
$2,0,2$

Venus

Complementary sequence pairs:
Aperiodic autoconvolutions sum to

$2, 0, 2, 0 \ldots 2, 0, 2.$

A_{I} :

$1, -i, -i, 1$
$1, i, i, 1$: $1, 0, 1, 4$

B_{I} :

$1, i, -i, -1$
$-1, i, -i, 1$: $1, 0, 1, -4$

$2, 0, 2, 0$

Venus

Complementary sequence pairs:
Aperiodic autoconvolutions sum to

$2, 0, 2, 0 \ldots\ 2, 0, 2.$

A_{I} : $\quad 1, -i, -i, 1$

$\qquad 1, i, i \quad : 1, 0, 1, 4, 1$

B_{I} : $\quad 1, i, -i, -1$

$\qquad -1, i, -i \quad : 1, 0, 1, -4, 1$

$2, 0, 2, 0, 2$

Venus

Complementary sequence pairs:
Aperiodic autoconvolutions sum to

$2, 0, 2, 0 \dots\ 2, 0, 2.$

A_{I} : $1, -i, -i, 1$: $1, 0, 1, 4, 1, 0$

$1, i$

B_{I} : $1, i, -i, -1$: $1, 0, 1, -4, 1, 0$

$-1, i$

$2, 0, 2, 0, 2$

Venus

Complementary sequence pairs:
Aperiodic autoconvolutions sum to

$2,0,2,0 \dots 2,0,2.$

A_{II} : $1, -i, -i, 1$: $1,0,1,4,1,0,1$

1

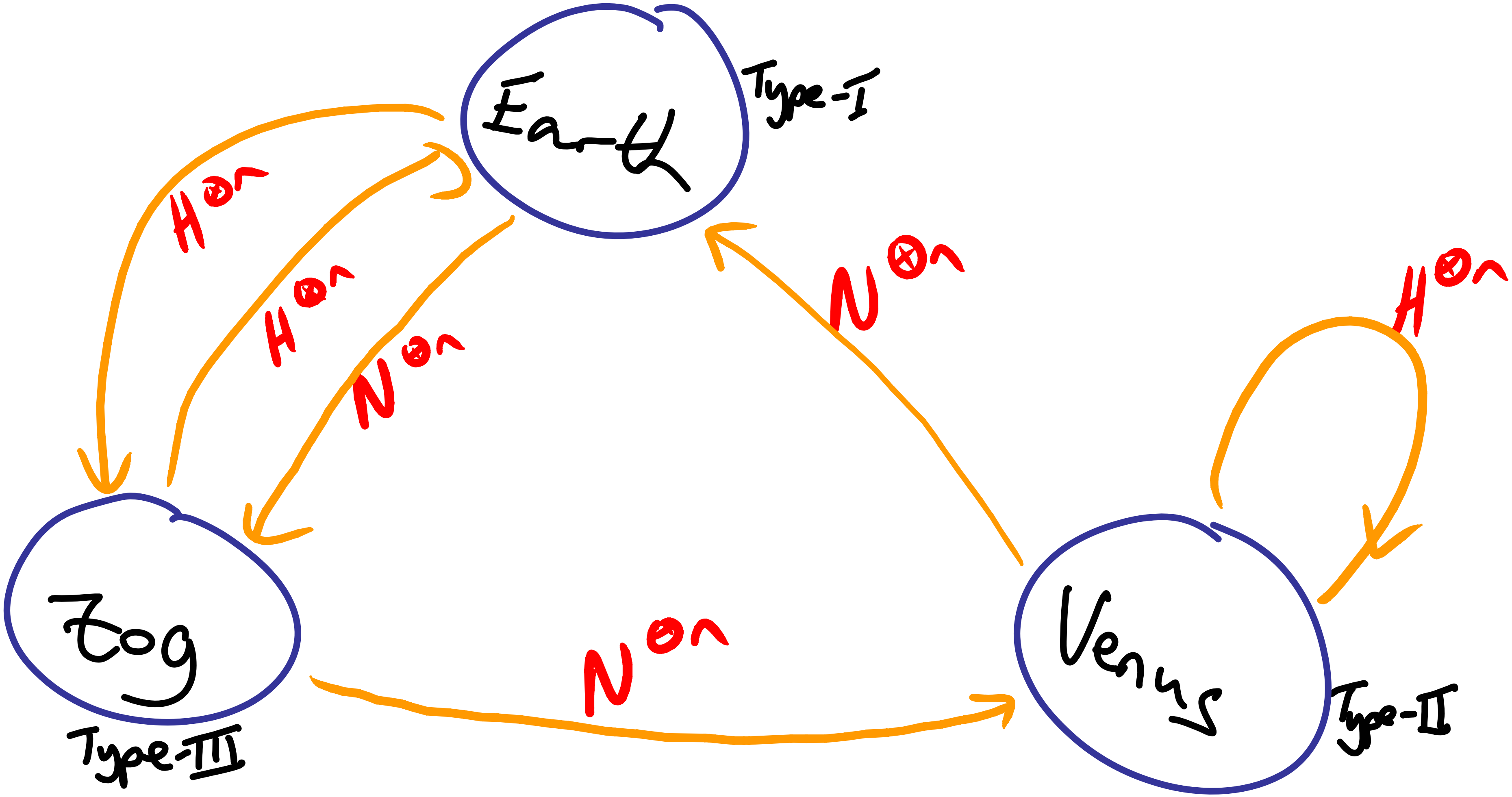
B_{II} : $1, i, -i, -1$: $1,0,1,-4,1,0,1$

-1

$2,0,2,0,2,0,2$

Space Map

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad N = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}.$$



Earth $\rightarrow$ Zog

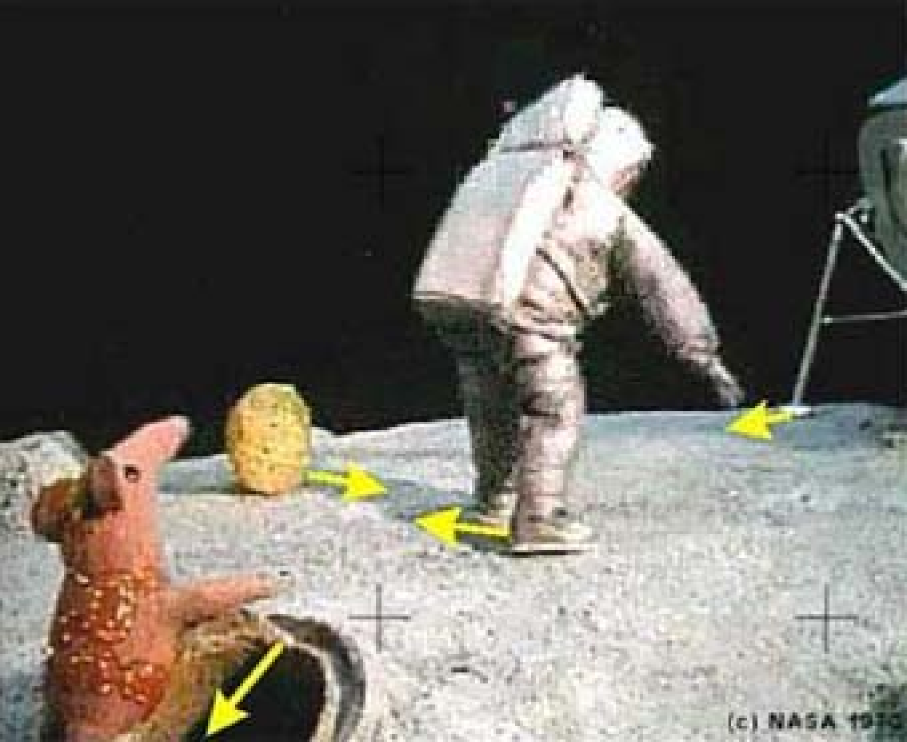
$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}.$$

$$(A_{III}, B_{III}) \leftarrow H^{\otimes n}(A, B).$$

$$A_{II} = \frac{1}{2}\overset{H^{\otimes 2}}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}}\overset{A}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

Similarly,

$$B_{II} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}.$$


(c) NASA 1970

Zog

Complementary sequence pairs:

Aperiodic autozogs sum to

2, 0, -2, 0, -2, 0, 2, -2, 0, ...

$A_{\underline{\pi}}$:

1, 1, 1, -1

1, 1, 1, -1

$B_{\underline{\pi}}$:

Zog

Complementary sequence pairs:
Aperiodic autozogs sum to

2, 0, -2, 0, -2, 0, 2, -2, 0, ...

A_{III} :   1, 1, 1, -1
       -1, 1, 1, 1

B_{III} :

Zog

Complementary sequence pairs:

Aperiodic autozogs sum to

2, 0, -2, 0, -2, 0, 2, -2, 0, ...

$A_{\underline{\text{III}}}$:

1, 1, 1, -1

-1, -1, -1, 1

$B_{\underline{\text{III}}}$:

Zog

Complementary sequence pairs:

Aperiodic autozogs sum to

2, 0, -2, 0, -2, 0, 2, -2, 0, ...

$A_{\underline{\pi}}$:

1, 1, 1, -1

-1, -1, -1, 1

$B_{\underline{\pi}}$:

Zog

Complementary sequence pairs:

Aperiodic autozogs sum to

2, 0, -2, 0, -2, 0, 2, -2, 0, ...

A_{III}: 1, 1, 1, -1

-1, -1, -1, 1

1

B_{III}: 1, -1, 1, 1

1, -1, 1, 1

1

2

Zog

Complementary sequence pairs:
Aperiodic autozogs sum to

2, 0, -2, 0, -2, 0, 2, -2, 0, ...

A_{III} :

1, 1, 1, -1
-1, -1, -1, 1

1, 0

B_{III} :

1, -1, 1, 1
1, -1 1, 1

1, 0
2, 0

Zog

Complementary sequence pairs:

Aperiodic autozogs sum to

2, 0, -2, 0, -2, 0, 2, -2, 0, ...

A_{III} :

1, 1, 1, -1

-1, -1, -1, 1

1, 0, -1

B_{III} :

1, -1, 1, 1

1, -1, 1, 1

1, 0, -1

2, 0, -2

Zog

Complementary sequence pairs:

Aperiodic autozogs sum to

2, 0, -2, 0, -2, 0, 2, -2, 0, ...

$A_{\underline{\text{III}}}$:

1, 1, 1, -1

-1, -1, -1, 1

1, 0, -1, -4

$B_{\underline{\text{III}}}$:

1, -1, 1, 1

1, -1, 1, 1

1, 0, -1, 4

2, 0, -2, 0,

Zog

Complementary sequence pairs:

Aperiodic autozogs sum to

2, 0, -2, 0, -2, 0, 2, -2, 0, ...

A_{III} :

1, 1, 1, -1

-1, -1, -1, 1

1, 0, -1, -4, -1

B_{III} :

1, -1, 1, 1

1, -1, 1, 1

1, 0, -1, 4, -1

2, 0, -2, 0, -2

Zog

Complementary sequence pairs:

Aperiodic autozogs sum to

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1, 0, -1, -4, -1, 0

B_{III} :

1, -1, 1, 1

1, -1, 1, 1

1, 0, -1, 4, -1, 0

2, 0, -2, 0, -2, 0

Zog

Complementary sequence pairs:

Aperiodic autozogs sum to

2, 0, -2, 0, -2, 0, 2, -2, 0, ...

$A_{\underline{\text{III}}}$:   1, 1, 1, -1

-1, -1, -1, 1   1, 0, -1, -4, -1, 0, 1

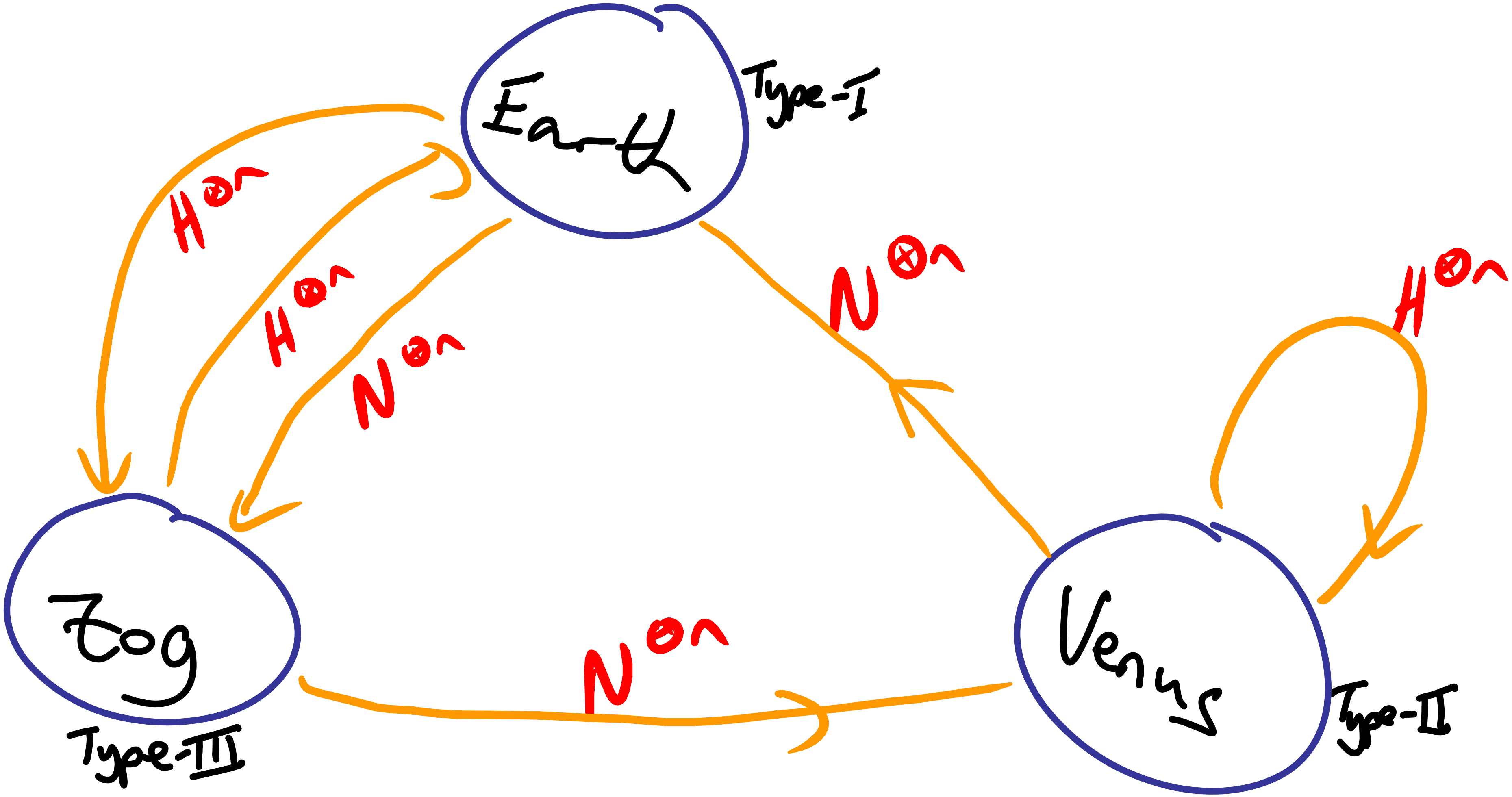
$B_{\underline{\text{III}}}$:   1, -1, 1, 1

1, -1, 1, 1   1, 0, -1, 4, -1, 0, 1

2, 0, -2, 0, -2, 0, 2

Space Map

$$H=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}, \quad N=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & i\\ 1 & -i\end{bmatrix}.$$



Planet Earth

Golay sequence pairs.

Aperiodic autocorrelations sum to δ.

$$A(y) = 1 + y + y^2 - y^3$$
$$B(y) = 1 + y - y^2 + y^3.$$

$|A(\text{circle})|^2 + |B(\text{circle})|^2 = 2.$

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$$A(y) = 1 + y + y^2 - y^3$$
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$$|A(\text{circle})|^2 + |B(\text{circle})|^2 = 2.$$

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Aperiodic autocorrelations sum to δ.

$$\begin{aligned} A(y) &= 1 + y + y^2 - y^3 \\ B(y) &= 1 + y - y^2 + y^3. \end{aligned}$$

$|A(\text{circle})|^2 + |B(\text{circle})|^2 = 2.$

Planet Earth

Golay sequence pairs.

Aperiodic autocorrelations sum to δ.

$$A(y) = 1 + y + y^2 - y^3$$
$$B(y) = 1 + y - y^2 + y^3.$$

$|A(e^{i2\pi/k})|^2 + |B(e^{i2\pi/k})|^2 = 2.$

Planet Venus

Type-II sequence pairs.

Normalised aperiodic auto-convolutions
sum to δ.

$A =$ 1, 1, 1, -1

-1, 1, 1, 1

: 1,

$B =$ 1, -1, -1, -1

-1, -1, -1, 1

: 1,

2,

Planet Venus

Type-II sequence pairs.

Normalised aperiodic auto-convolutions sum to δ.

$A =$ 1, 1, 1, −1

−1, 1 1, 1 : 1, 2,

$B =$ 1, −1, −1, −1

−1, −1, −1, 1 : 1, −2,

2, 0,

Planet Venus

Type-II sequence pairs.

Normalised aperiodic auto-convolutions sum to δ.

$A = \quad 1, 1, 1, -1$

$\qquad -1, 1, 1, 1 \qquad : 1, 2, 3$

$B = \quad 1, -1, -1, -1$

$\qquad -1, -1, -1, 1 \qquad : 1, -2, -1$

$\qquad\qquad\qquad\qquad 2, 0, 2$

Planet Venus

Type-II sequence pairs.

Normalised aperiodic auto-convolutions sum to δ.

$A =$ $1, 1, 1, -1$

$-1, 1, 1, 1$: $1, 2, 3, 0$

$B =$ $1, -1, -1, -1$

$-1, -1, -1, 1$: $1, -2, -1, 0$

$2, 0, 2, 0$

Planet Venus

Type-II sequence pairs.

Normalised aperiodic auto-convolutions sum to δ.

$A =$ 1, 1, 1, −1

−1, 1, 1, 1 : 1, 2, 3, 0, −1

$B =$ 1, −1, −1, −1

−1, −1, −1, 1 : 1, −2, −1, 0, 3

2, 0, 2, 0, 2

Planet Venus

Type-II sequence pairs.

Normalised aperiodic auto-convolutions sum to δ.

$A =$ 1, 1, 1, -1

-1, 1, 1, 1 : 1, 2, 3, 0, -1, -2

$B =$ 1, -1, -1, -1

-1, -1, -1, 1 : 1, -2, -1, 0, 3, 2

2, 0, 2, 0, 2, 0

Planet Venus

Type-II sequence pairs.

Normalised aperiodic auto-convolutions sum to δ.

$A =$ 1, 1, 1, -1

-1, 1, 1, 1 : 1, 2, 3, 0, -1, -2, 1

$B =$ 1, -1, -1, -1

-1, -1, -1, 1 : 1, -2, -1, 0, 3, 2, 1

2, 0, 2, 0, 2, 0, 2

Planet Venus

Type-II sequence pairs.

Normalised aperiodic auto-convolutions sum to δ.

$A = 1, 1, 1, -1$

$B = 1, -1, -1, -1$

normalise

$$\frac{2,0,2,0,2,0,2}{1,0,1,0,1,0,1} = \delta.$$

Planet Venus

Type II sequence pairs.

Normalised aperiodic autoconvolutions sum to δ.

$$A(y) = 1 + y + y^2 - y^3$$
$$B(y) = 1 - y - y^2 - y^3.$$

$|A(\text{real})|^2 + |B(\text{real})|^2/\text{norm.} = 2.$

Planet Venus

Type II sequence pairs.

Normalised aperiodic autoconvolutions sum to δ.

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$$A(y) = 1 + y + y^2 - y^3$$
$$B(y) = 1 - y - y^2 - y^3.$$

For $t \in \mathbb{R}$,

$|A(t)|^2 + |B(t)|^2/\text{norm.} = 2.$

Planet Venus

Type II sequence pairs.

Normalised aperiodic autoconvolutions sum to δ.

$$\begin{aligned} A(y) &= 1 + y + y^2 - y^3 \\ B(y) &= 1 - y - y^2 - y^3. \end{aligned}$$

For $t \in \mathbb{R}$,

$|A(t)|^2 + |B(t)|^2/\text{norm.} = 2.$

Planet Zog

Type-III sequence pairs

Normalised aperiodic auto-zogs sum to δ.

$A =$ $\quad\quad\quad$ $1, 1, 1, -1$ $\quad\quad$ $: 1$

$-1, -1, -1, 1$

$B =$ $\quad\quad\quad$ $1, 1, -1, 1$ $\quad\quad$ $: 1$

$1, 1, -1, 1$

2

Planet Zog

Type-III sequence pairs

Normalised aperiodic auto-zogs sum to δ.

$A =$ 1, 1, 1, −1 : 1,0

−1, −1 −1, 1

$B =$ 1, 1, −1, 1 : 1,0

1, 1, −1, 1

2,0

Planet Zog

Type-III sequence pairs

Normalised aperiodic auto-zogs sum to δ.

$A = \quad 1, 1, 1, -1 \qquad : 1, 0, -1$

$\qquad -1, -1, -1, 1$

$B = \quad 1, 1, -1, 1 \qquad : 1, 0, -1$

$\qquad 1, 1, -1, 1$

$\qquad\qquad\qquad 2, 0, -2$

Planet Zog

Type-II sequence pairs

Normalised aperiodic auto-zogs sum to δ.

$A =$ 1, 1, 1, -1 : 1, 0, -1, -4

-1, -1, -1, 1

$B =$ 1, 1, -1, 1 : 1, 0, -1, 4

1, 1, -1, 1

2, 0, -2, 0

Planet Zog

Type-III sequence pairs

Normalised aperiodic auto-zogs sum to δ.

$A =$ 1, 1, 1, -1 : 1, 0, -1, -4, -1
-1, -1, -1, 1

$B =$ 1, 1, -1, 1 : 1, 0, -1, 4, -1
1, 1, -1, 1

2, 0, -2, 0, -2

Planet Zog

Type-II sequence pairs

Normalised aperiodic auto-zogs sum to δ.

A = 1, 1, 1, -1 : 1, 0, -1, -4, -1, 0

-1, -1, -1, 1

B = 1, 1, -1, 1 : 1, 0, -1, 4, -1, 0

1, 1, -1, 1

2, 0, -2, 0, -2, 0

Planet Zog

Type-III sequence pairs

Normalised aperiodic auto-zogs sum to δ.

$A =$ 1, 1, 1, -1 : 1, 0, -1, -4, -1, 0, 1

-1, -1, -1, 1

$B =$ 1, 1, -1, 1 : 1, 0, -1, 4, -1, 0, 1

1, 1, -1, 1

2, 0, -2, 0, -2, 0, 2

Planet Zog

Type-II sequence pairs

Normalised aperiodic auto-zogs sum to δ.

$A = 1, 1, 1, -1$

$B = 1, 1, -1, 1$

normalise

$$\frac{2,0,-2,0,-2,0,2}{1,0,-1,0,-1,0,1} = \delta$$

Planet Zog

Type III sequence pairs.

Normalised aperiodic autozogs sum to δ.

$$A(y) = 1 + y + y^2 - y^3$$
$$B(y) = 1 - y - y^2 + y^3.$$

$|A(\text{imag})|^2 + |B(\text{imag})|^2/\text{norm.} = 2.$

Planet Zog

Type III sequence pairs.

Normalised aperiodic autozogs sum to δ.

$$A(y) = 1 + y + y^2 - y^3$$
$$B(y) = 1 - y - y^2 + y^3.$$

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Planet Zog

Type III sequence pairs.

Normalised aperiodic autozogs sum to δ.

$$A(y) = 1 + y + y^2 - y^3$$
$$B(y) = 1 + y - y^2 + y^3.$$

For $t \in \mathbb{I}$,

$|A(t)|^2 + |B(t)|^2/\text{norm.} = 2.$

Planet Zog

Type III sequence pairs.

Normalised aperiodic autozogs sum to δ.

$$
\begin{aligned}
A(y) &= 1 + y + y^2 - y^3 \\
B(y) &= 1 + y - y^2 + y^3.
\end{aligned}
$$

For $t \in \mathbb{I}$,

$|A(t)|^2 + |B(t)|^2/\text{norm.} = 2.$

Planet Zog

Type III sequence pairs.

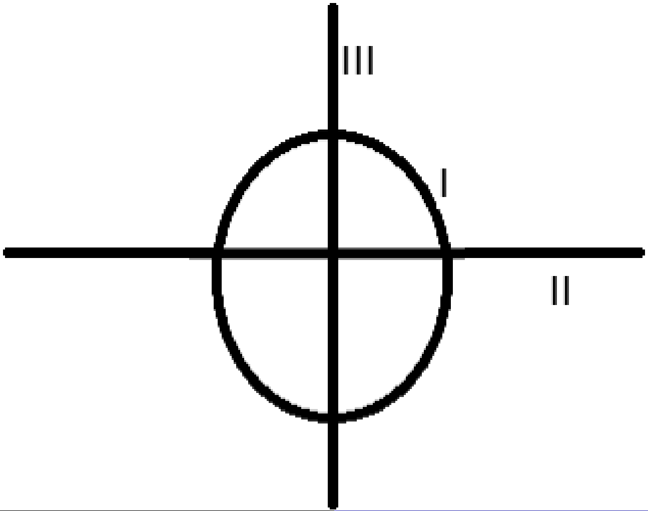
Normalised aperiodic autozogs sum to δ.

$$A(y) = 1 + y + y^2 - y^3$$
$$B(y) = 1 + y - y^2 + y^3.$$

For $t \in \mathbb{I}$,

$|A(t)|^2 + |B(t)|^2/\text{norm.} = 2.$

Type I, II, III Evaluation Sets



Planet Earth:

Only one Type-I (Golay) bipolar sequence pair known (of length 2^n).

Planet Earth:

Only one Type-I (Golay) bipolar sequence pair known (of length 2^n).

Planet Venus:

Only one Type-II bipolar sequence pair known.

Planet Earth:

Only one Type-I (Golay) sequence pair known (of length 2^n).

bipolar

Planet Venus:

Only one Type-II sequence pair known.

bipolar

Planet Zog:

Plenty of Type-III sequence pairs known.

bipolar

e.g. $n=5$: $\approx \frac{1}{4}$ of all "quadratics"

$n=6$: $\approx \frac{1}{16}$ " "

$n=7$: $\approx 1/100$ " "

Actually....

sequence pair property is, for 2^n elements, invariably an array pair property.

... for each of

Type-I,

Type-II,

and Type-III.

Planet Earth

Golay array pairs

Aperiodic autocorrelations sum to δ.

$$A: \begin{array}{cccc} 1 & 1 & & \\ 1 & -1\;1 & 1 & \\ & 1 & -1 & \end{array} \quad : -1$$

$$B: \begin{array}{cccc} 1 & 1 & & \\ -1 & 1\;1 & 1 & \\ & -1 & 1 & \end{array} \quad \begin{array}{l} : 1 \\ : 0 \end{array}$$

Planet Earth

Golay array pairs

Aperiodic autocorrelations sum to δ.

  1   1

A: 11 11   : $-1, 0$

  1 -1

  1   1

B: 11 11   : $1, 0$

  -1 1   : $0, 0$

Planet Earth

Golay array pairs

Aperiodic autocorrelations sum to δ.

A: $\begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix}$: $-1, 0, 1$

B: $\begin{matrix} 1 & 1 \\ -1 & 1 \end{matrix}$: $1, 0, -1$

: $0, 0, 0$

Planet Earth

Golay array pairs

Aperiodic autocorrelations sum to δ.

A: 1 11 1   : $-1, 0, 1, 0$
   1 -11 -1

B: 1 11 1   : $1, 0, -1, 0$
   -1 +11 1

: $0, 0, 0, 0$

Planet Earth

Golay array pairs

Aperiodic autocorrelations sum to δ.

A:

11 11
11 -11

: -1,0,1,0,4

B:

11 11
+11 11

: 1,0,-1,0,4

: 0,0,0,0,8

Planet Earth

Golay array pairs

Aperiodic autocorrelations sum to δ.

A: $\begin{matrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{matrix}$: $-1, 0, 1, 0, 4, 0$

B: $\begin{matrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{matrix}$: $1, 0, -1, 0, 4, 0$

: $0, 0, 0, 0, 8, 0$

Planet Earth

Golay array pairs

Aperiodic autocorrelations sum to δ.

A:
1 1
1 1 1 -1
1 -1

: -1, 0, 1, 0, 4, 0, 1

B:
1 1
1 -1 1 1
-1 1

: 1, 0, -1, 0, 4, 0, -1

: 0, 0, 0, 0, 8, 0, 0

Planet Earth

Golay array pairs

Aperiodic autocorrelations sum to δ.

A: $\begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix}$: −1, 0, 1, 0, 4, 0, 1, 0

B: $\begin{matrix} 1 & 1 \\ -1 & 1 \end{matrix}$: 1, 0, −1, 0, 4, 0, −1, 0

: 0, 0, 0, 0, 8, 0, 0, 0

Planet Earth

Golay array pairs

Aperiodic autocorrelations sum to δ.

$$A: \begin{array}{ccc} 1 & 1 & \\ 1 & +1\ 1 & 1 \\ & 1 & -1 \end{array} \quad : -1,0,1,0,4,0,1,0,-1$$

$$B: \begin{array}{ccc} 1 & 1 & \\ -1 & 1\ 1 & 1 \\ & -1 & 1 \end{array} \quad : 1,0,-1,0,4,0,-1,0,1$$

$$: 0,0,0,0,8,0,0,0,0$$

... similar array versions exist for

auto-convolution on planet Venus

auto-zog on planet Zog.

Multivariate Polynomials

For $A \in (\mathbb{C}^2)^{\otimes n} = (A_x \mid x \in \mathbb{F}_2^n)$

$$= (A_{0\dots00}, A_{0\dots01}, \dots, A_{1\dots11}),$$

$$A(z_0, z_1, \dots z_{n-1})$$

$$= \sum_{x \in \mathbb{F}_2^n} A_x z_0^{x_0} z_1^{x_1} \cdots z_{n-1}^{x_{n-1}}.$$

Similarly for $B(z_0, z_1, \dots, z_{n-1})$.

Earth Array Pairs

For Type-I multivariate polynomial pair

$$\left(A(z_0, z_1, \ldots z_{n-1}), B(z_0, z_1, \ldots z_{n-1}) \right),$$

Earth Array Pairs

For Type-I multivariate polynomial pair

$$\left(A(z_0, z_1, \ldots z_{n-1}), B(z_0, z_1, \ldots z_{n-1})\right),$$

$$|A(\text{circle}, \text{circle}, \ldots, \text{circle})|^2 + |B(\text{circle}, \text{circle}, \ldots, \text{circle})|^2 = 2.$$

Earth Array Pairs

i.e.

$$\left|A\left(e^{i\frac{2\pi}{k_0}}, e^{i\frac{2\pi}{k_1}}, \ldots\ldots, e^{i\frac{2\pi}{k_{n-1}}}\right)\right|^2$$

$$+\left|B\left(e^{i\frac{2\pi}{k_0}}, e^{i\frac{2\pi}{k_1}}, \ldots\ldots, e^{i\frac{2\pi}{k_{n-1}}}\right)\right|^2$$

$$= 2.$$

Venus Array Pairs

For Type-II multivariate polynomials,

$A(z_0, z_1, \dots z_{n-1})$, $B(z_0, z_1, \dots, z_{n-1})$,

Venus Array Pairs

For Type-II multivariate polynomials,

$A(z_0, z_1, \ldots z_{n-1})$, $B(z_0, z_1, \ldots, z_{n-1})$,

$$|A(\text{real}, \text{real}, \ldots, \text{real})|^2 + |B(\text{real}, \text{real}, \ldots, \text{real})|^2 \Big/ \text{norm}.$$

$$= 2.$$

Venus Array Pairs

i.e.

for,

$$t = (t_0, t_1, \ldots, t_{n-1}) \in \mathbb{R}^n,$$

$$\frac{|A(t_0, t_1, \ldots t_{n-1})|^2 + |B(t_0, t_1, \ldots, t_{n-1})|^2}{\text{norm.}}$$

$$= 2.$$

Zog Array Pairs

For Type-III multivariate polynomials,

$A(z_0, z_1, \ldots z_{n-1})$, $B(z_0, z_1, \ldots z_{n-1})$,

Zog Array Pairs

For Type-III multivariate polynomials,

$$A(z_0, z_1, \ldots z_{n-1}),\quad B(z_0, z_1, \ldots z_{n-1}),$$

$$\left|A(\text{imag}, \text{imag}, \ldots, \text{imag})\right|^2 + \left|B(\text{imag}, \text{imag}, \ldots, \text{imag})\right|^2 \Big/ \text{norm}.$$

$$= 2.$$

Zog Array Pairs

i.e.

for,

$$t = (t_0, t_1, \ldots, t_{n-1}) \in \mathcal{I}^n,$$

$$\frac{|A(t_0, t_1, \ldots, t_{n-1})|^2 + |B(t_0, t_1, \ldots, t_{n-1})|^2}{\text{norm.}}$$

$$= 2.$$

Polynomial Products

Type-I: Aperiodic autocorrelation:

$$:= A(z_0, z_1, \dots z_{n-1})\,\overline{A(z_0^{-1}, z_1^{-1}, \dots z_{n-1}^{-1})}$$

Polynomial Products

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$$:= A(z_0, z_1, \dots z_{n-1})\,\overline{A(z_0^{-1}, z_1^{-1}, \dots z_{n-1}^{-1})}$$

Type-II: Normalised aperiodic autoconvolution:

$$:= |A(z_0, z_1, \dots z_{n-1})|^2 \Big/ \prod_j (1 + z_j^2).$$

Polynomial Products

Type-I: Aperiodic autocorrelation:

$$:= A(z_0, z_1, \dots z_{n-1})\,\overline{A(z_0^{-1}, z_1^{-1}, \dots z_{n-1}^{-1})}$$

Type-II: Normalised aperiodic autoconvolution:

$$:= |A(z_0, z_1, \dots z_{n-1})|^2 / \prod_j (1 + z_j^2).$$

Type-III: Normalised aperiodic autozog:

$$:= A(z_0, z_1, \dots, z_{n-1})\,\overline{A(-z_0, -z_1, \dots, -z_{n-1})} / \prod_j (1 - z_j^2).$$

Alternative Boolean Function Description

$$A(z_0, z_1, \ldots z_{n-1})$$

$$= A_{0\ldots00} + A_{0\ldots01}z_0 + A_{0\ldots10}z_1 + A_{0\ldots11}z_0z_1$$

$$\cdots\cdots + A_{1\ldots11}z_0z_1\ldots z_{n-1}$$

Alternative Boolean Function Description

$$A(z_0, z_1, \dots z_{n-1})$$

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$$\dots\dots + A_{1\dots11} z_0 z_1 \dots z_{n-1}$$

$$= \sum_{x \in F_2^n} A_x z_0^{x_0} z_1^{x_1} \dots z_{n-1}^{x_{n-1}}.$$

Alternative Boolean Function Description

$$A(z_0, z_1, \ldots z_{n-1})$$

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$$\cdots\cdots + A_{1\ldots11} z_0 z_1 \ldots z_{n-1}$$

$$= \sum_{x \in F_2^n} A_x z_0^{x_0} z_1^{x_1} \ldots z_{n-1}^{x_{n-1}}.$$

So consider $A_x = (-1)^{a(x)}$, $a : F_2^n \to F_2$.

where $A_x : F_2^n \to C$.

Earth Array Pair

Only one bipolar Type-I pair known:

$$A = (-1)^{a(x)}, \quad B = (-1)^{b(x)},$$

$$a(x) = x_0 x_1 + x_1 x_2 + \cdots + x_{n-2} x_{n-1}$$

$$b(x) = a(x) + x_{n-1}.$$

Earth Array Pair

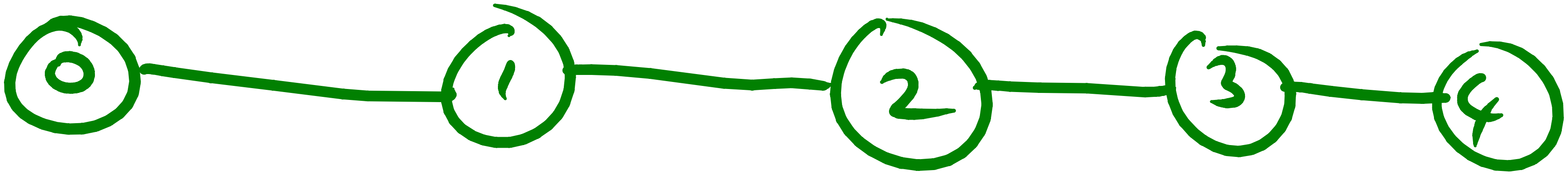
Only **one** bipolar Type-I pair known:

$$A = (-1)^{a(x)}, \quad B = (-1)^{b(x)},$$

$$a(x) = x_0 x_1 + x_1 x_2 + \dots + x_{n-2} x_{n-1}$$

$$b(x) = a(x) + x_{n-1}.$$

Path Graph (e.g. $n=5$)



Venus Array Pair

Only **one** bipolar Type-II pair known:

$A = (-1)^{a(x)}$, $B = (-1)^{b(x)}$

$$a(x) = \sum_{i<j} x_i x_j$$

$$b(x) = a(x) + x_0 + x_1 + \dots + x_{n-1}.$$

Venus Array Pair

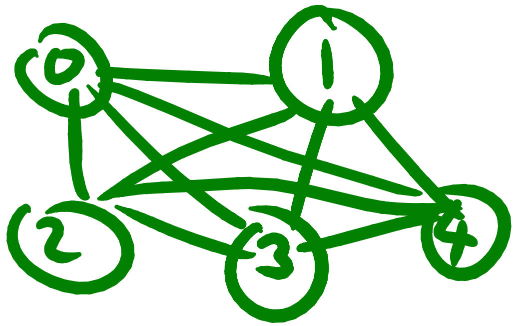
Only **one** bipolar Type-II pair known:

$A = (-1)^{a(x)}$, $B = (-1)^{b(x)}$

$$a(x) = \sum_{i<j} x_i x_j$$

$$b(x) = a(x) + x_0 + x_1 + \dots + x_{n-1}.$$

Clique Graph (e.g. $n=5$)



Zog Array Pairs

but all quadratic or affine

Many bipolar Type-III pairs known.

$A(x) = (-1)^{a(x)}$, $B(x) = (-1)^{b(x)}$,

e.g.

$$a(x) = x_0(x_1 + x_2 + \dots + x_{n-1})$$

$$b(x) = a(x) + x_1 + x_2 + \dots + x_{n-1}.$$

Zag Array Pairs

but all quadratic or affine

Many bipolar Type-III pairs known.

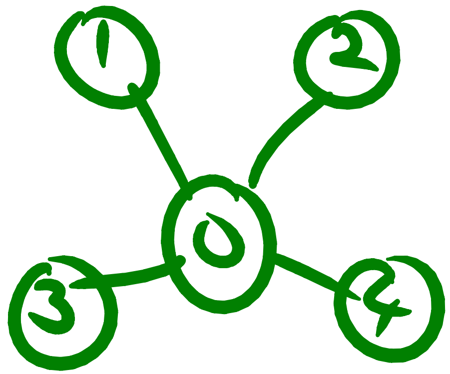
$A(x) = (-1)^{a(x)}$, $B(x) = (-1)^{b(x)}$,

e.g.

$$a(x) = x_0\left(x_1 + x_2 + \dots + x_{n-1}\right)$$

$$b(x) = a(x) + x_1 + x_2 + \dots + x_{n-1}.$$

Star Graph (e.g. $n=5$):



Zeg Array Pairs

Many more type-III quadratic (+affine) pairs.

$$\frac{\#\text{quad pairs}}{\text{all quads}} \quad \in \{1,-1\}^{\otimes 5} \approx \frac{1}{4},$$

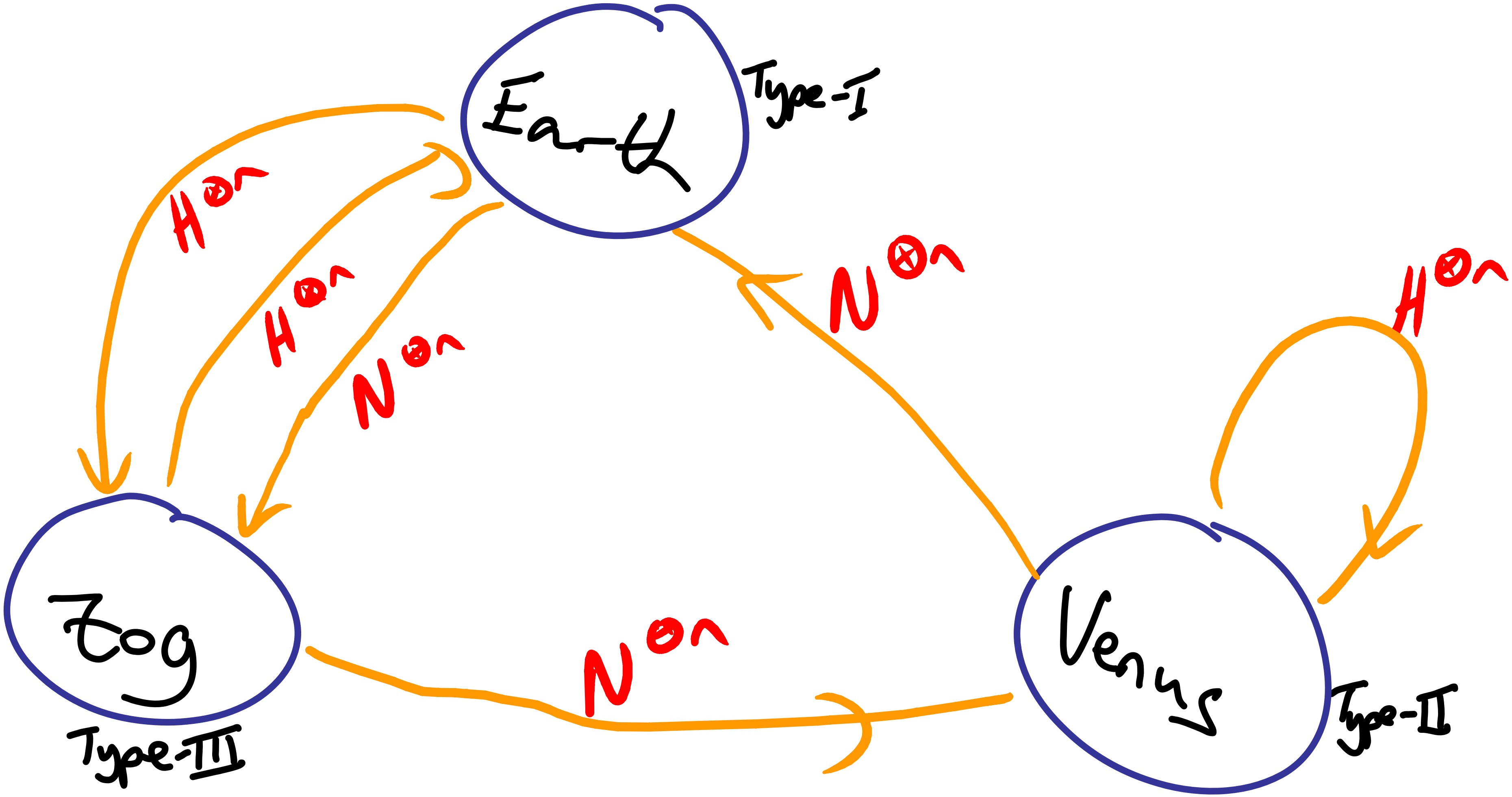
$$\in \{1,-1\}^{\otimes 6} \approx \frac{1}{16},$$

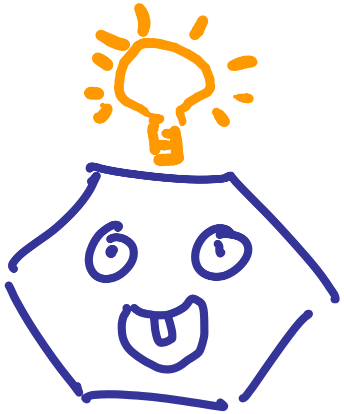
$$\in \{1,-1\}^{\otimes 7} \approx \frac{1}{100}.$$

No known type-III arrays of degree > 2.

Space Map

$$H=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix},\quad N=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & i\\ 1 & -i\end{bmatrix}.$$

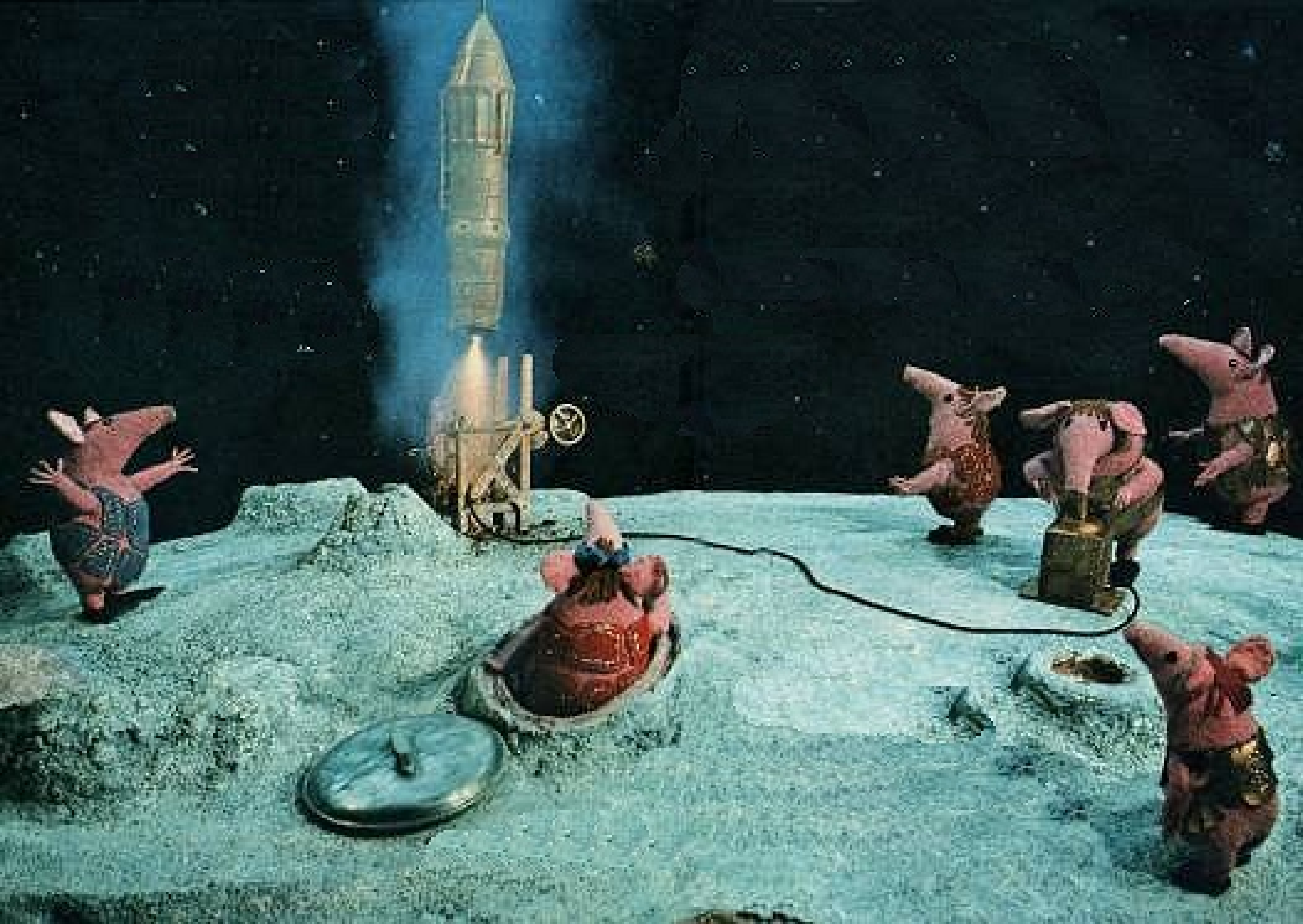




bipolar

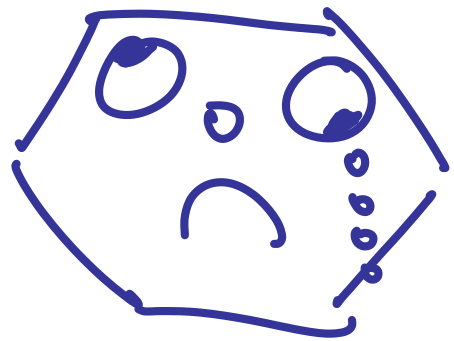
Why not bring Zog pairs back to Earth?...

... become rich and famous!!!



..... but..

a Type-III bipolar array is, on Earth,
with one known exception,
no longer bipolar or even having
unimodular entries.



Why?

.... Travelling through Hilbert space modifies my array alphabet.

Example

Array Pair

Type-III (A, B), $A = (-1)^{a}$, $B = (-1)^{b}$,

$$a(x) = x_0(x_1 + x_2 + x_3) = 0001010001000001$$

$$b(x) = a(x) + x_1 + x_2 + x_3 = 0010100010000010$$

Example

Array Pair

Type-III (A, B), $A = (-1)^a$, $B = (-1)^b$,

$$a(x) = x_0(x_1 + x_2 + x_3) = 0001010001000001$$

$$b(x) = a(x) + x_1 + x_2 + x_3 = 0010100010000010$$

$\Rightarrow$

$$A(z) = 1 + z_0 + z_1 - z_0 z_1 + z_2 - z_0 z_2 + z_1 z_2 + z_0 z_1 z_2 + z_3 - z_0 z_3 + z_1 z_3 + z_0 z_1 z_3 + z_2 z_3 + z_0 z_2 z_3 + z_1 z_2 z_3 - z_0 z_1 z_2 z_3.$$

$$B(z) = 1 + z_0 - z_1 + z_0 z_1 - z_2 + z_0 z_2 + z_1 z_2 + z_0 z_1 z_2 - z_3 + z_0 z_3 + z_1 z_3 + z_0 z_1 z_3 + z_2 z_3 + z_0 z_2 z_3 - z_1 z_2 z_3 + z_0 z_1 z_2 z_3.$$

Example

Array Pair → Sequence Pair

multivariate → univariate

$$\left(A(z_0, z_1, z_2, z_3), B(z_0, z_1, z_2, z_3)\right)$$

Using substitutions, $z_j = y^{2^j}$.

Example

Array Pair → Sequence Pair

multivariate → univariate

$$\left(A(z_0, z_1, z_2, z_3),\; B(z_0, z_1, z_2, z_3)\right)$$

Using substitutions, $z_j = y^{2^j}$

$\Rightarrow$

$$A(y) = 1 + y + y^2 - y^3 + y^4 - y^5 + y^6 + y^7 + y^8 - y^9 + y^{10} + y^{11} + y^{12} + y^{13} + y^{14} - y^{15}.$$

$$B(y) = 1 + y - y^2 + y^3 - y^4 + y^5 + y^6 + y^7 - y^8 + y^9 + y^{10} + y^{11} + y^{12} + y^{13} - y^{14} + y^{15}.$$

Example - Auto-Zog

Type-III pairing given by sum of auto-zogs.

$$A(z_0, z_1, z_2, z_3)\overline{A(-z_0, -z_1, -z_2, -z_3)}$$

$$+$$

$$B(z_0, z_1, z_2, z_3)\overline{B(-z_0, -z_1, -z_2, -z_3)}$$

$$= 2\prod_j (1 - z_j^2).$$

Example - Auto-Zog : Array → Sequence

multivariate

$\downarrow$

univariate

Assign, $z_j = y^{2^j}$, $\forall j$.

Then,

$$A(z_0, z_1, z_2, z_3)\overline{A(\bar{z}_0, \bar{z}_1, \bar{z}_2, \bar{z}_3)} + B(z_0, z_1, z_2, z_3)\overline{B(\bar{z}_0, \bar{z}_1, \bar{z}_2, \bar{z}_3)}$$

Example - Auto-Zog : Array → Sequence

multivariate
↓
univariate

Assign, $z_j = y^{2^j}$, $\forall j$.

Then,

$$A(z_0,z_1,z_2,z_3)\overline{A(z_0^{-2},z_1^{-2},z_2^{-2},z_3^{-2})} + B(z_0,z_1,z_2,z_3)\overline{B(z_0^{-2},z_1^{-2},z_2^{-2},z_3^{-2})}$$

↓

$$A(y)\overline{\tilde{A}(y)} + B(y)\overline{\tilde{B}(y)}$$

$$= 2\prod_j\left(1 - y^{2^{j+1}}\right).$$

Example - explicit sequence Auto-Zog.

$$A(y) = 1+y+y^2-y^3+y^4-y^5+y^6+y^7+y^8-y^9+y^{10}+y^{11}+y^{12}+y^{13}+y^{14}-y^{15}$$

$$\tilde{A}(y) = 1-y-y-y^3-y^4-y^5+y^6-y^7-y^8-y^9+y^{10}-y^{11}+y^{12}-y^{13}-y^{14}-y^{15}.$$

$$B(y) = 1+y-y^2+y^3-y^4+y^5+y^6+y^7-y^8+y^9+y^{10}+y^{11}+y^{12}+y^{13}-y^{14}+y^{15}.$$

$$\tilde{B}(y) = 1-y+y^2+y^3+y^4+y^5+y^6-y^7+y^8+y^9+y^{10}-y^{11}+y^{12}-y^{13}+y^{14}+y^{15}.$$

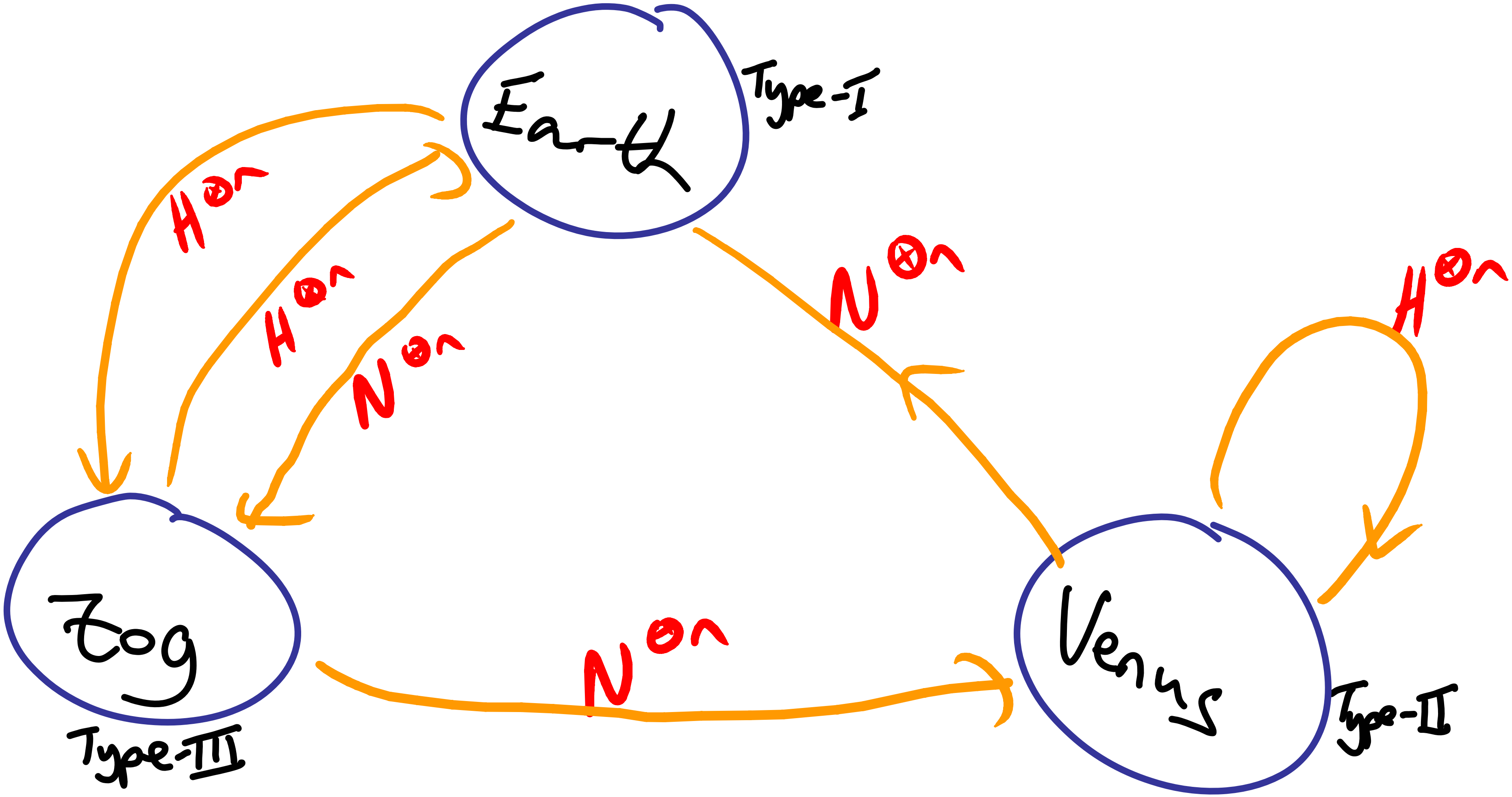
.... computations give ...

$$A(y)\overline{\tilde{A}(y)} + B(y)\overline{\tilde{B}(y)}$$

$$= 2\prod_j \left(1-y^{2^{j+1}}\right).$$

Consult Map

$$H=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix},\quad N=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & i\\ 1 & -i\end{bmatrix}.$$



Example

Take Type-III Zog pair $(A_{\mathrm{III}}, B_{\mathrm{III}})$ from

Zog $\to$ Earth

Example

Take Type-III Zog pair (A_{III}, B_{III})

from

Zog → Earth

to become

a Type-I Earth pair (A_I, B_I)

Example

Take Type-III Zog pair (A_{III}, B_{III})

from

Zog $\rightarrow$ Earth

to become

a Type-I Earth pair (A_I, B_I)

where,

$$A_I = H^{\otimes 4} A_{III}, \qquad B_I = H^{\otimes 4} B_{III}$$

Example - Type-III → Type-I

Zog → Earth

$$A_{I} = H^{\otimes 4} A_{III}$$

$$A_{I} = \overbrace{\left[\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right]}^{H^{\otimes 4}} \overbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ 1 \\ 1 \\ -1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}}^{A_{III}} = 2 \overbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}}^{A_{I}}$$

Example - Type-III → Type-I

Zog → Earth

$$A_I = H^{\otimes 4} A_{III}$$

$$A_I = \underbrace{\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right)}_{H^{\otimes 4}} \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ 1 \\ 1 \\ -1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}}_{A_{III}} = 2 \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}}_{A_I}$$

NOT BIPOLAR!!

Example Type-I array to sequence

multivariate
$\downarrow$
univariate

$$A_{I}(z_0, z_1, z_2, z_3) = 1 + z_0 + z_1 z_2 z_3 - z_0 z_1 z_2 z_3.$$

$$B_{I}(z_0, z_1, z_2, z_3) = 1 - z_0 + z_1 z_2 z_3 + z_0 z_1 z_2 z_3.$$

Example Type-I array to sequence

multivariate $\downarrow$ univariate

$$A_I(z_0, z_1, z_2, z_3) = 1 + z_0 + z_1 z_2 z_3 - z_0 z_1 z_2 z_3.$$

$$B_I(z_0, z_1, z_2, z_3) = 1 - z_0 + z_1 z_2 z_3 + z_0 z_1 z_2 z_3.$$

$\downarrow$ $z_j = y^{2^j}$

$$\Rightarrow A_I(y) = 1 + y + y^{14} - y^{15}, \quad B_I(y) = 1 - y + y^{14} + y^{15}.$$

Example Type-I array to sequence multivariate $\downarrow$ univariate

$$\grave{A}(z_0, z_1, z_2, z_3) = 1 + z_0 + z_1 z_2 z_3 - z_0 z_1 z_2 z_3.$$

$$\grave{B}(z_0, z_1, z_2, z_3) = 1 - z_0 + z_1 z_2 z_3 + z_0 z_1 z_2 z_3.$$

$\downarrow \quad z_j = y^{2^j}$

$$\Rightarrow A_I(y) = 1 + y + y^{14} - y^{15}, \quad B_I(y) = 1 - y + y^{14} + y^{15}.$$

$$A_I(y)\widetilde{A_I(y^{-1})} + B_I(y)\widetilde{B_I(y^{-1})} = 8 = \delta. \qquad (\text{Type-I}).$$

Consolation Prize

No new bipolar Golay pairs (apparently),

Consolation Prize

No new bipolar Golay pairs (apparently),

The first ternary $\{-1, 0, 1\}$ array pairs

(I suppose).

Consolation Prize

No new bipolar Golay pairs (apparently),

The first ternary $\{-1,0,1\}$ array pairs

(I suppose).

Characterize these new pairs.

Restricted Path Graphs (type I array pairs)

$$A(x) = m(x)(-1)^{a(x)}, \quad B(x) = m(x)(-1)^{b(x)},$$
$$x \in \mathbb{F}_2^n, a, b, m : \mathbb{F}_2^n \rightarrow \mathbb{F}_2.$$

Where,

$$a(x) = x_0x_1 + x_1x_2 + \ldots \; x_{q-2}x_{q-1},$$
$$b(x) = a(x) + x_{q-1},$$
$$m(x) = \prod_{k=q}^{n-1}(x_k + x_{r_k} + 1),$$

where,

$q \in \{0, 1, \ldots, n-1\}, r = (r_q, r_{q+1}, \ldots, r_{n-1}), r_k \in Q, \forall k$
$Q = \{0, 1, \ldots, q-1\}.$

Restricted Path Graphs (type I array pairs)

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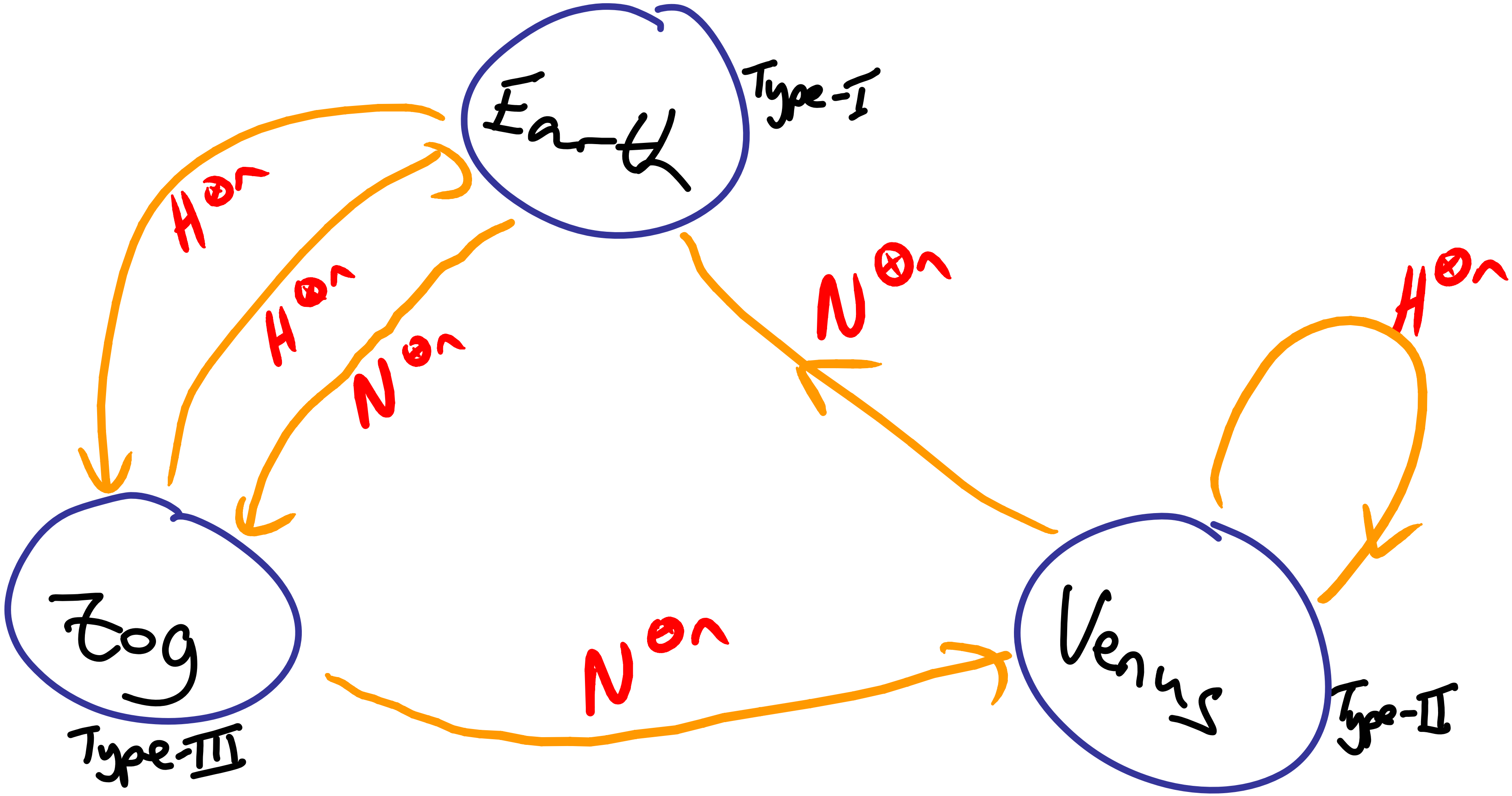
$H=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}, \quad N=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & i\\ 1 & -i\end{bmatrix}.$



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Restricted Path Graphs - type III characterisation

$$A(x) = m(x)(-1)^{a(x)}, \qquad B(x) = m(x)(-1)^{b(x)},$$

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$$a(x) = \sum_{j=0}^{q/2-1}(f_{2j+1} + x_{2j+1})\sum_{k=0}^{j}(f_{2k} + x_{2k}),$$
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where $f_j(x) = \sum_{q \le k < n, r_k = j} x_k$,

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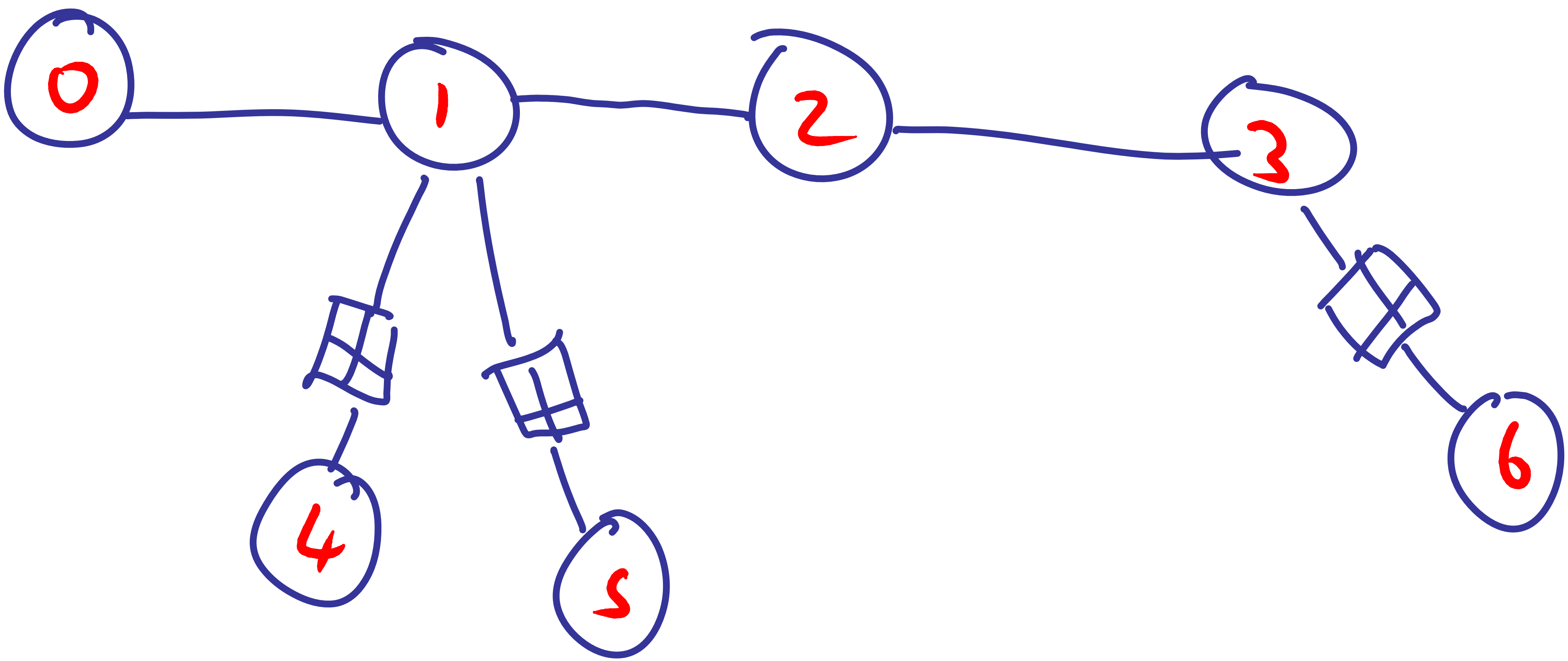
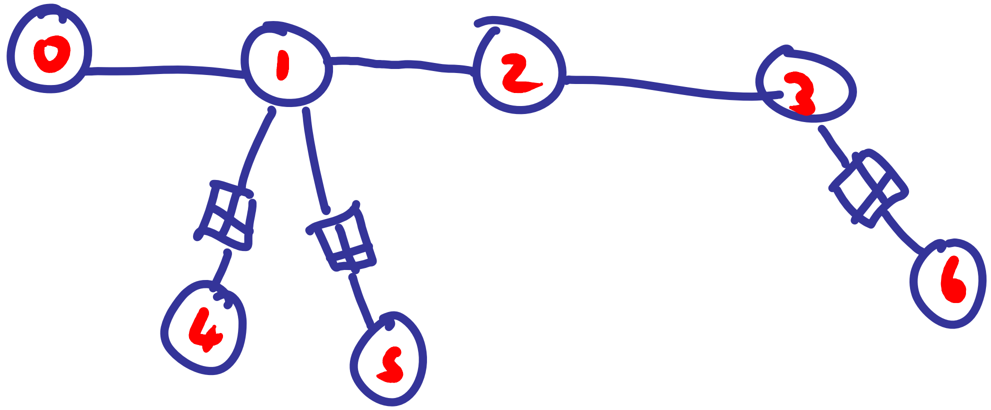
On Earth Restricted Path Graphs Look Like This



Enumerations



$$2^{n-2} + 2^{\lfloor n/2 \rfloor - 1}$$

structurally-distinct restricted path graphs

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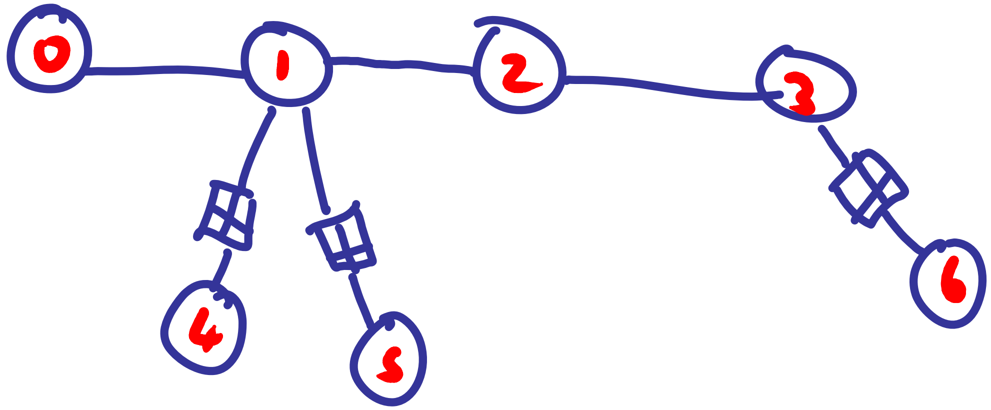

$2^{n-2} + 2^{\lfloor n/2 \rfloor - 1}$ structurally-distinct restricted path graphs

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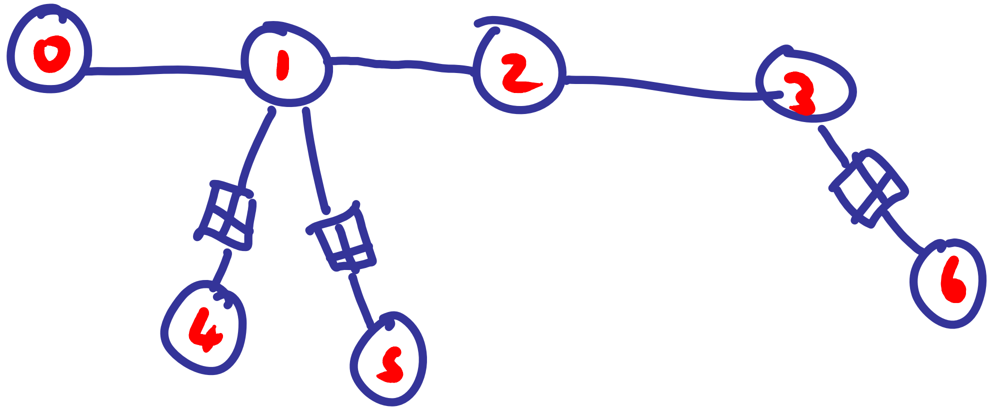

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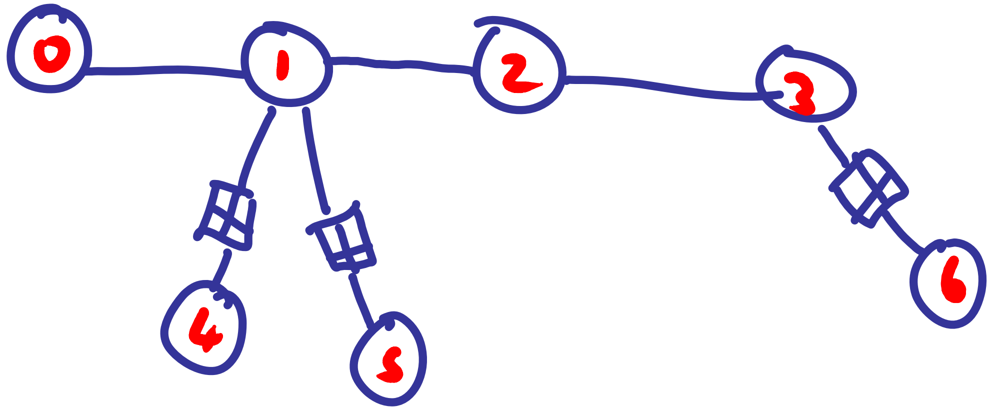

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Correspond to rows of Lozanic's triangle

- isomers of alkanes - hydrocarbons : C_nH_{2n+2}.

Our hero scrapes out a living commuting
between Earth and Zog,
and recently noticed the
following ...

A Recipe

- Take a self-dual, n-variable, Boolean function, f $\quad (f = H^{\otimes n} f.)$
- Let $g_k = f(x+k) + f(x+\bar{k}), \quad k \in \mathbb{F}_2^n$.
- For $k_o, k_e \in \mathbb{F}_2^n$, $\mathrm{wt}(k_o)$ odd, $\mathrm{wt}(k_e)$ even,

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Self-Dual Equivalence

- Duality preserved by extended orthogonal group.
$$f(x) \rightarrow f(Lx + d) + d \cdot x,$$
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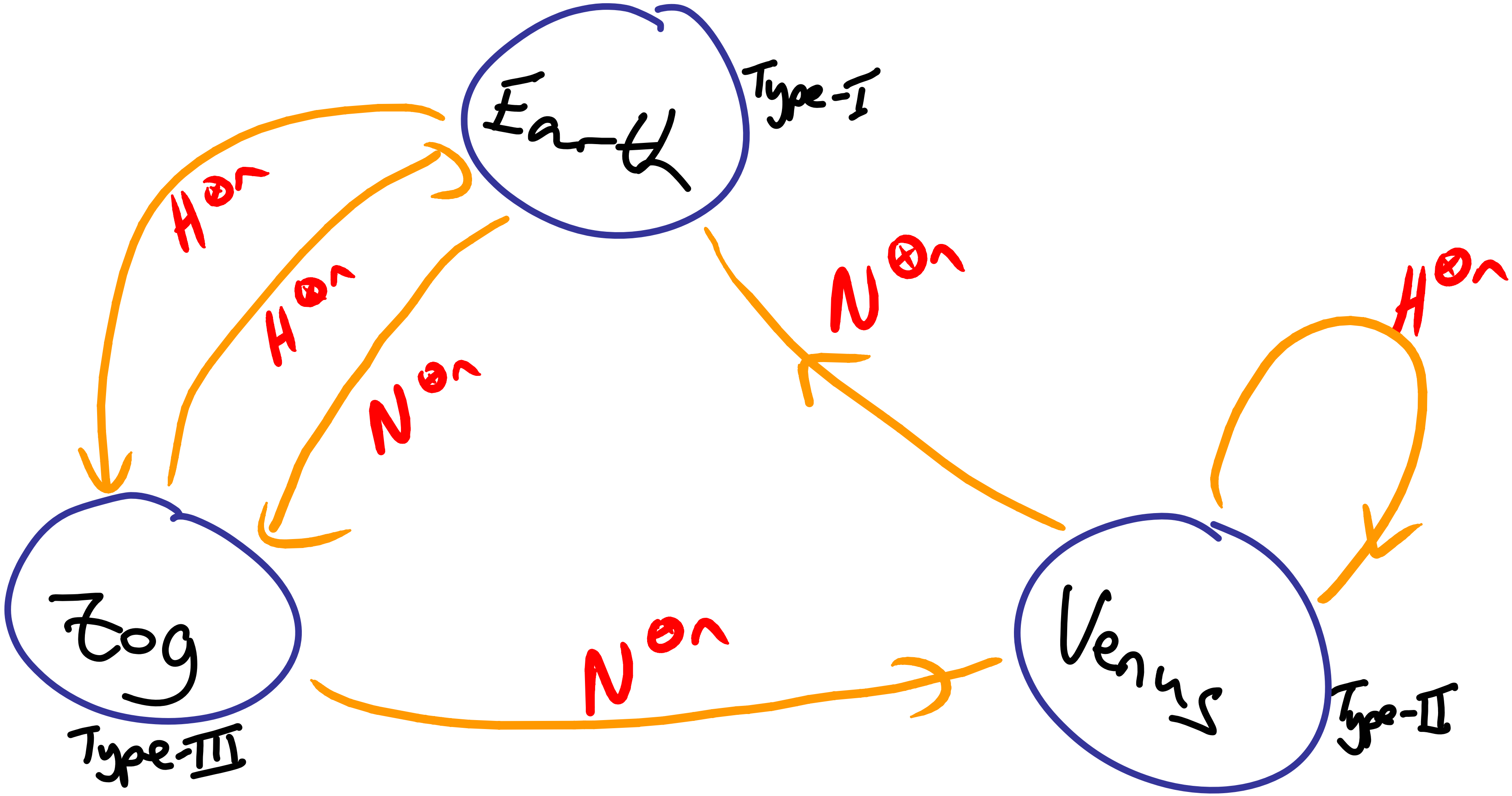

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Result

Aperiodic properties preserved by subgroup of affine group. (for some functions)

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Conjecture

$f(x+k)+f(x+\bar{k})$ is never bent if f is self-dual.

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and $F^{\perp}{}_i^{(x_0+x_1+\ldots+x_{n-1})}$ is type-I.

where $F^{\perp} = (-1)^{f^{\perp}}$.

Summary

- Golay complementary pairs - both sequence and array.
- Two new types of pair ; Type II and Type III.
- Essentially, one array pair for types I and II, but many array pairs for type III.
- Restricted path graphs characterised as type I array pairs over alphabet $\{-1, 0, 1\}$.

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