Covering codes arising from small saturating sets in a projective space of arbitrary dimension

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Let $\mathcal{C}$ be a q-ary linear $[n, k, d]$-code. We call $\mathcal{C}$ a *covering code with covering radius* $R > 0$ if every vector of $\mathbb{F}_q^n$ lies within Hamming distance R of a codeword in $\mathcal{C}$. On the other hand, a *ϱ-saturating set* $\mathcal{S}$ of the projective space $\mathrm{PG}(m, q)$ is a point set such that every point of $\mathrm{PG}(m, q)$ is contained in a subspace spanned by at most $\varrho + 1$ points of $\mathcal{S}$.

These two objects are closely related to each other, as every ϱ-saturating set in $\mathrm{PG}(n - k - 1, q)$ gives rise to a parity check matrix of a q-ary linear $[n, k, d]$-covering code with covering radius $R = \varrho + 1$. It is naturally interesting to construct small ϱ-saturating sets of $\mathrm{PG}(n, q)$.

Let $\boldsymbol{\kappa}(n, q, \varrho)$ be the cardinality of a *smallest possible* ϱ-saturating set of $\mathrm{PG}(n, q)$. One can rather easily prove that there exists a lower bound for $\boldsymbol{\kappa}(n, q, \varrho)$ of size roughly $\sim \varrho \cdot q^{\frac{n-\varrho}{\varrho+1}}$. A lot of research has been done in finding upper bounds for $\boldsymbol{\kappa}(n, q, \varrho)$, mainly when $n = 2$ and $\varrho = 1$; a nice survey on this can be found in [2]. However, there is not a lot known if n and ϱ are arbitrary. Bartoli et al. [1] proved that $\boldsymbol{\kappa}(n, q, 1) \lesssim 2q^{\frac{n-1}{2}}\sqrt{\ln(q)}$ if q is large and n is even and either equal to 2 or relatively large.

Our main result is that, if q is a $(\varrho + 1)^{\text{th}}$ power, there exists an upper bound on $\boldsymbol{\kappa}(n, q, \varrho)$ of size roughly $\sim \varrho^2 \cdot q^{\frac{n-\varrho}{\varrho+1}}$. Moreover, for general $q \geqslant 4$, we find an upper bound of size roughly $(\varrho + 1)\theta_k - \varrho$ if $\frac{n-\varrho}{\varrho+1} = k \in \mathbb{N}$, slightly improving the known, naive bound of $(\varrho + 1)\theta_k$.

Keywords: Covering code, Saturating set, General construction, Baer subgeometry
MSC: 05B25, 51E20, 51E22

References

[1] D. Bartoli, A. A. Davydov, M. Giulietti, S. Marcugini and F. Pambianco, “New upper bounds on the smallest size of a saturating set in a projective plane”, *XV International Symposium Problems of Redundancy in Information and Control Systems (REDUNDANCY)*, St. Petersburg, pp. 18–22 (2016).

[2] A. A. Davydov and P. Östergård, "On saturating sets in small projective geometries", *European J. Combin.* **21**(5) (2000), pp. 563–570.

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