Small weight codewords in projective geometric codes

Sam Adriaensen

Vrije Universiteit Brussel

(Joint work with Lins Denaux, Leo Storme, Zsuzsa Weiner)

Let $q = p^h$ be a prime power, with p prime, and let $\mathbb{F}_q$ be the finite field of order q. Denote the n-dimensional projective space over $\mathbb{F}_q$, arising from the vector space $\mathbb{F}_q^{n+1}$, as $\mathrm{PG}(n, q)$. Let $\mathcal{P}$ denote the set of points of $\mathrm{PG}(n, q)$. For each subspace π of $\mathrm{PG}(n, q)$, define its *characteristic function* as

$$\mathcal{P} \to \{0, 1\} : P \mapsto \begin{cases} 1 & \text{if } P \in \pi, \\ 0 & \text{if } P \notin \pi. \end{cases}$$

Now choose an integer $k \in [1, n-1]$ and let $\mathcal{C}_k(n, q)$ denote the subspace of the vector space of functions $\mathcal{P} \to \mathbb{F}_p$ generated by the characteristic functions of the k-dimensional subspaces of $\mathrm{PG}(n, q)$.

We are interested in the codewords of $\mathcal{C}_k(n, q)$ of small weight. Codewords which are linear combinations of relatively few characteristic functions will have relatively small weight. Are there other small weight codewords? As it turns out this heavily depends on whether q is prime. A summary of the state of the art, due to many different authors, is as follows.

Theorem.

- *Codewords of $\mathcal{C}_k(n, q)$ of weight at most $2q^k$ are linear combinations of at most two characteristic functions.*
- *If $q > 3$ is prime, then $\mathcal{C}_k(n, q)$ has codewords of weight $3(q-1)q^{k-1}$ which are not linear combinations of few characteristic functions.*
- *If $q = p^h$, with $h > 1$, is large enough, then all codewords of $\mathcal{C}_k(n, q)$ of weight at most $\left(1 - \frac{\delta_{h,2}}{2} - o(1)\right)\sqrt{q}q^k$ are linear combinations of few characteristic functions. If q is square and $h > 2$, this bound is essentially tight.*

Also small weight codewords of the dual code $\mathcal{C}_k(n, q)^\perp$ are of interest. This is a much harder problem. In general, determining the minimum weight of $\mathcal{C}_k(n, q)^\perp$ is an open problem, although good lower and upper bounds are known. It has been solved for q prime or even, and minimum weight codewords are characterised if q is prime or $q \in \{4, 8\}$.

Vrije Universiteit Brussel, Department of Mathematics and Data Science
Pleinlaan 2, 1050 Elsene, Belgium
sam.adriaensen@vub.be