On two-intersection sets with respect to hyperplanes in projective spaces

Aart Blokhuis
Technische Universiteit Eindhoven,
Postbox 513, 5600 MB Eindhoven,
The Netherlands
and

Michel Lavrauw
Technische Universiteit Eindhoven,
Postbox 513, 5600 MB Eindhoven,
The Netherlands

January 14, 2004

Abstract

In [2] a construction of a class of two-intersection sets with respect to hyperplanes in $PG(r-1, q^t)$, rt even, is given, with the same parameters as the union of $(q^{t/2}-1)/(q-1)$ disjoint Baer subgeometries if t is even and the union of $(q^t-1)/(q-1)$ elements of an $(r/2-1)$-spread in $PG(r-1, q^t)$ if t is odd. In this paper we prove that although they have the same parameters, they are different. This was previously proved in [1] in the special case where $r=3$ and $t=4$.

1. Introduction and motivation

In [2] the notion of a *scattered space with respect to a spread* in a projective space is defined as a subspace intersecting every spread element in at most a point. The origin of this idea is a paper by P. Polito and O. Polverino [5] on blocking sets, where they give the first construction of small minimal non-Rédei blocking sets, called *linear* blocking sets. They use the correspondence between a normal spread in a Desarguesian projective space over a finite field and the points of a lower dimensional projective space over an extension field. In [2] the authors prove upper and lower bounds for the maximum dimension of a scattered space and it is shown that in the case of a normal spread, scattered spaces of maximal dimension give rise to two-intersection sets with respect to hyperplanes in projective spaces. The parameters of these two-intersection sets are not new. Sets with the same parameters can be obtained by taking the disjoint union of embedded subgeometries or subspaces. The first non-trivial case are two-intersection sets with intersection numbers $q+1$ and q^2+q+1 in $PG(2,q^4)$. They arise from a 5-dimensional scattered space with respect to a normal 3-spread in $PG(11,q)$. These sets have so called *standard parameters.* It is known that the union of $q+1$ disjoint Baer subplanes gives a two-intersection set with the same parameters. In [1] the existence of such a scattered spread is proved and it is shown that the corresponding two-intersection set can not contain a Baer subline, and so it gives a new example of such sets. In this article the authors are able to prove the general result. Namely that all two-intersection sets arising from scattered spaces with respect to a normal spread, give new examples. The proof is given in Section 3. In Section 2 we give some necessary definitions to state the precise result. We do not explain all the details of the connection between normal spreads and the points of a lower dimensional projective space over an extension field, for which we refer to [2],[4]. For more information about two-intersection sets we refer to [3]

2. Preliminaries

First we give some definitions, which are necessary to state the result. Let $t \geq 2$, $r \geq 3$, with rt even, and let $PG(rt-1,q)$ be the Desarguesian projective space of dimension $rt-1$ over the finite field of order q, $GF(q)$, where $q=p^h$, p prime, $h \geq 1$. Let S be a set of $(t-1)$-dimensional subspaces of $PG(rt-1,q)$. Then S is called a $(t-1)$-*spread* if every point of $PG(rt-1,q)$ is contained in exactly one element of S. A subspace of $PG(rt-1,q)$ is called *scattered with respect to a spread* S if it intersects every element of S in at most a point. A spread S is called *normal*, (*geometric*), if every subspace generated by two elements of S is also partitioned by elements of S. Let S be a normal $(t-1)$-spread in $PG(rt-1,q)$. We recall the main result of [2].

Theorem 2.1 *If* W *is scattered with respect to a normal* $t-1$ *spread* S *in* $PG(rt-1,q)$*, then* $\dim(W) \leq rt/2-1$.

So let W be a subspace of dimension $rt/2-1$ which is scattered with respect to S. Using the one to one correspondence between the elements of the normal spread S and the points of $PG(r-1,q^t)$, we define a set of points $B(W)$ in $PG(r-1,q^t)$ corresponding with the elements of S that intersect W. Moreover, the set $B(W)$ is a two-intersection set in $PG(r-1,q^t)$ with respect to hyperplanes, with intersection numbers

$$m = \frac{q^{\frac{rt}{2}-t}-1}{q-1} \text{ and } n = \frac{q^{\frac{rt}{2}-t+1}-1}{q-1}.$$

If t is even this set has the same parameters as the disjoint union of $(q^{t/2}-1)/(q-1)$ Baer subgeometries isomorphic to $PG(r-1,q^{t/2})$. We say that a two-intersection set isomorphic to such a union of subgeometries is of *type I.* If t is odd this set has the same parameters as the union of $(q^t-1)/(q-1)$ elements of an $(r/2-1)$-spread in $PG(r-1,q^t)$. We call these two-intersection sets of *type II.* We will prove that the sets arising from a scattered space are not of type I neither of type II.

Theorem 2.2 *The two-intersection sets arising from scattered spaces of dimension $rt/2$ with respect to a normal $(t-1)$-spread in $PG(rt-1,q)$ are not isomorphic with the two-intersection sets of type I or type II.*

3. Proof of the Theorem

First suppose that t is odd. An element $\mathcal{E}$ of an $(r/2-1)$-spread in $PG(r-1,q^t)$ induces an $(rt/2-t)$-dimensional space in $PG(rt-1,q)$, partitioned by a subset of the $(t-1)$-spread S. Theorem 2.1 implies that W intersects this subspace in a subspace of dimension at most $rt/4-1$, since the intersection is scattered with respect to the restriction of S to this subspace. Hence $B(W)$ can not contain this spread element $\mathcal{E}$. Note that using the same argument, it is easy to show that $B(W)$ can not contain a line of $PG(r-1,q^t)$.
Now suppose that t is even. We will prove that $B(W)$ can not contain a Baer hyperplane $\mathcal{B}$, i.e., a subgeometry of $PG(r-1,q^t)$ isomorphic with $PG(r-2,q^{t/2})$. Note that this is again, as in the case where t is odd, a stronger property than needed to prove the Theorem.
To avoid confusion in what follows $P(\vec{\alpha})$ will denote a point in $PG(r-1,q^t)$, while $\langle\vec{\lambda}\rangle$ will denote a point in $PG(rt-1,q)$.
Suppose $\mathcal{B}$ is contained in $B(W)$ and let $\mathcal{H}$ be the hyperplane of $PG(r-1,q^t)$, that contains $\mathcal{B}$. Without loss of generality we can assume that $\mathcal{B}$ and $\mathcal{H}$ are generated by the same points. So

$$\mathcal{B}=\{P(\alpha_1\vec{u}_1+\ldots+\alpha_{r-1}\vec{u}_{r-1}) \parallel \alpha_1,\ldots,\alpha_{r-1}\in GF(q^{t/2})\}$$

and

$$\mathcal{H}=\{P(a_1\vec{u}_1+\ldots+a_{r-1}\vec{u}_{r-1}) \parallel a_1,\ldots,a_{r-1}\in GF(q^t)\}.$$

Since $\mathcal{B}$ is contained in $B(W)$, the hyperplane $\mathcal{H}$ intersects $B(W)$ in n points, where $n=(q^{rt/2-t+1}-1)/(q-1)$ is the larger of the two intersection numbers. So the subspace in $PG(rt-1,q)$ induced by $\mathcal{H}$ intersects W in a subspace of dimension $k-1:=rt/2-t$. We denote the set of points in $PG(r-1,q^t)$ corresponding with spreadelements intersecting this subspace with $\mathcal{W}$. Put

$$\mathcal{W}=\{P(\lambda_1\vec{v}_1+\ldots+\lambda_k\vec{v}_k) \parallel \lambda_1,\ldots,\lambda_k\in GF(q)\}.$$

Moreover we can express the vectors $\vec{v}_i$, $(i=1,\ldots,k)$, as a linear combination of $\vec{u}_1,\ldots,\vec{u}_{r-1}$ over $GF(q^t)$. Let C be the matrix over $GF(q^t)$ such that

$$\begin{pmatrix}\vec{v}_1\\ \vec{v}_2\\ \ldots\\ \vec{v}_k\end{pmatrix}=C^t\begin{pmatrix}\vec{u}_1\\ \vec{u}_2\\ \ldots\\ \vec{u}_{r-1}\end{pmatrix}.$$

Then $\mathcal{B}$ will be contained in $B(W)$ if

$$\forall\alpha_1,\ldots,\alpha_{r-1}\in GF(q^{t/2}):\exists\lambda_1,\ldots,\lambda_k\in GF(q),\exists a\in GF(q^t)^*$$

such that

$$a \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \dots \\ \alpha_{r-1} \end{pmatrix} = C \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \dots \\ \lambda_k \end{pmatrix}.$$

Putting $\vec{\alpha} := (\alpha_1, \dots, \alpha_{r-1})^T$, and $\vec{\lambda} := (\lambda_1, \dots, \lambda_k)^T$ this equation becomes

$$a\vec{\alpha} = C\vec{\lambda}.$$

Let

$$T = \{(a, \vec{\alpha}, \vec{\lambda}) \in GF(q^t)^* \times GF(q^{t/2})^{r-1} \times GF(q)^k \ : \ a\vec{\alpha} = C\vec{\lambda}\}.$$

If $(a, \vec{\alpha}, \vec{\lambda})$, $(b, \vec{\alpha}, \mu) \in T$, then $C(b\vec{\lambda} - a\vec{\mu}) = \vec{0}$. This implies that

$$b\vec{\lambda}^T(\vec{v}_1, \dots, \vec{v}_k)^T = a\vec{\mu}^T(\vec{v}_1, \dots, \vec{v}_k)^T,$$

or $(\vec{v}_1, \dots, \vec{v}_k)\vec{\lambda} = a/b \ (\vec{v}_1, \dots, \vec{v}_k)\vec{\mu}$. Since W is scattered with respect to S and $\langle\vec{\lambda}\rangle, \langle\vec{\mu}\rangle \in W$, we must have that $a/b \in GF(q)$ and so $\langle\vec{\lambda}\rangle = \langle\vec{\mu}\rangle$. Let

$$T_a = \{\langle\vec{\lambda}\rangle, : \exists\vec{\alpha} \ : \ (a, \vec{\alpha}, \vec{\lambda}) \in T\}.$$

Note that if $a/b \in GF(q^{t/2})$ then $T_a = T_b$ and that T_a is a subspace of $PG(rt-1, q)$. Now if $T_a \neq \emptyset$ and $\langle\vec{\mu}\rangle \in T_b \setminus T_a$, $\langle\vec{\nu}\rangle \in T_c \setminus T_a$, $\langle\vec{\mu}\rangle \neq \langle\vec{\nu}\rangle$, and $\langle T_a, \langle\vec{\mu}\rangle\rangle = \langle T_a, \langle\vec{\nu}\rangle\rangle$, then the line joining $\langle\vec{\mu}\rangle$ and $\langle\vec{\nu}\rangle$ intersects T_a, so without loss of generality $\vec{\lambda}+\vec{\mu}+\vec{\nu} = \vec{0}$ and

$$(a, \vec{\alpha}, \vec{\lambda}), (b, \vec{\beta}, \vec{\mu}), (c, \vec{\gamma}, \vec{\nu}) \in T,$$

for certain $\vec{\beta}$ and $\vec{\gamma}$. It follows that

$$a\vec{\alpha} + b\vec{\beta} + c\vec{\gamma} = \vec{0}.$$

Let $\vec{\delta} \in GF(q^{t/2})^{r-1}$ be such that $\vec{\delta}^T\vec{\alpha} = 0 \neq \vec{\delta}^T\vec{\beta}$. This is possible since we saw that if $P(\vec{\alpha}) = P(\vec{\beta})$ then $\langle\vec{\lambda}\rangle = \langle\vec{\mu}\rangle$, but $\langle\vec{\mu}\rangle \notin T_a$. We get $b\vec{\delta}^t\vec{\beta} + c\vec{\delta}^t\vec{\gamma} = 0$, and $b/c \in GF(q^{t/2})$. This implies that $T_b = T_c$. Thus

$$(a, \vec{\alpha}, \vec{\lambda}), (b, \vec{\beta}, \vec{\mu}), (b, \vec{\gamma}, \vec{\nu}) \in T,$$

for certain $\vec{\beta}$ and $\vec{\gamma}$. Now we have that

$$b(\vec{\beta} + \vec{\gamma}) + a\vec{\alpha} = 0.$$

So $b/a \in GF(q^{t/2})$ or $T_a = T_b$, which is a contradiction. This shows that if T_a has dimension $d-1$, then there is at most one point in every subspace of dimension d, containing T_a. So the set

$$\{\langle\vec{\mu}\rangle \ : \ \exists b, \vec{\beta} : (b, \vec{\beta}, \vec{\mu}) \in T\}$$

contains at most

$$\frac{q^d - 1}{q - 1} + \frac{q^{k-d} - 1}{q - 1}$$

points. Every $P(\vec{\alpha})$ determines a different $\langle\vec{\mu}\rangle$, so we must have

$$\frac{q^{(r-1)t/2}-1}{q^{t/2}-1} \leq \frac{q^d-1}{q-1} + \frac{q^{k-d}-1}{q-1}.$$

Recall that $k = rt/2 - t + 1$. Since we assumed $d \geq 1$ this implies that $d = k$, but this is clearly impossible, since that would imply that $\mathcal{W}$ is completely contained in the smaller set $\mathcal{B}$.

References

[1] S. Ball, A. Blokhuis, M. Lavrauw, Linear $(q+1)$-fold blocking sets in $PG(2,q^4)$. *Finite Fields Appl.*, to appear.

[2] A. Blokhuis, M. Lavrauw, Scattered spaces with respect to a spread in $PG(n,q)$. *Geom. Dedicata*, to appear.

[3] R. Calderbank, W. M. Kantor, The geometry of two-weight codes. *Bull. London Math. Soc.* **18** (1986), 97–122.

[4] G. Lunardon, Normal spreads. *Geom. Dedicata* **75** (1999), 245–261.

[5] P. Polito, O. Polverino, On small blocking sets. *Conbinatorica* **18** (1998), 1–5.