BLT-sets admitting the symmetric group S_4

Michel Lavrauw *
Dipartimento di Matematica e Applicazioni
Universitá degli studi di Napoli "Federico II"
Complesso Monte S. Angelo
80126 Napoli
Italia
lavrauw@unina.it

and

Maska Law
Dipartimento di Matematica
Universitá degli studi de Roma "La Sapienza"
Piazzale Aldo Moro 2
I 00185 Roma
Italy
maska@maths.uwa.edu.au

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Abstract

We describe a set of subgroups isomorphic to S_4 in $P\Gamma O(5,q)$, $q \equiv 5(6)$, and prove that they belong to exactly two different conjugacy classes. As an application we use a representative group in each of the conjugacy classes to construct a number of BLT-sets of $Q(4,q)$, some of which were previously found by computer in [5] (see also [4]), others of which are new.

1 Introduction

Since their introduction in [1] in 1990, BLT-sets have received a great deal of attention. Arising in the study of flocks of the quadratic cone of $\mathrm{PG}(3,q)$, q odd, they are therefore connected intimately with elation generalised quadrangles of order (q^2,q) and translation planes of order q^2 arising from linespreads of $\mathrm{PG}(3,q)$. These connections are well studied, and for a survey from the point of view of BLT-sets we refer to [4].

One main focus of research on BLT-sets has been their construction, and from [5] we have a long list of examples constructed with the use of a computer. The present article is an attempt to understand further the structure of some of these examples. We note that the translation planes associated with a flock can be of different types, arising from the flock via the ovoid of $Q^+(5,q)$ construction of Thas and Walker, or from the flock via the hyperbolic fibration of $\mathrm{PG}(3,q)$ construction of Baker, Ebert, Penttila. Hence, due to the construction of Hiramine, Matsumoto, Oyama of a linespread of $\mathrm{PG}(3,q^2)$ from a linespread of $\mathrm{PG}(3,q)$, a BLT-set can give rise to *many* infinite families of translation planes. This provides further motivation for attempting to unify a number of examples of BLT-sets within a common framework.

2 A model for $Q(4,q)$

In this section we fix our notation and the quadratic form that we will use throughout the article. Let q be a prime power, $q \equiv 5(6)$. Then $V = \{(x,y,z,\alpha) : x,y,z \in GF(q), \alpha \in GF(q^2)\}$ is a five-dimensional vector space over the finite field $GF(q)$. Define the quadratic form $Q : V \to GF(q)$ by

$$Q(x,y,z,\alpha) := x^2 + y^2 + z^2 + \alpha^{q+1}$$

which has associated with it the symmetric bilinear polar form

$$f(X_1, X_2) = 2(x_1x_2 + y_1y_2 + z_1z_2) + \alpha_1^q\alpha_2 + \alpha_1\alpha_2^q.$$

Then (V, Q) determines a nonsingular parabolic quadric $Q(4, q)$ of the projective space $PG(4, q)$ arising from V. This particular model for $Q(4, q)$ is chosen so that it allows us to give an elegant representation of the groups A_4 and S_4 as subgroups of $P\Gamma L(5, q)$, when $q \equiv 5(6)$. Throughout the article we will use the same notation for an element of $\Gamma L(5, q)$ and the element of $P\Gamma L(5, q)$ induced by it.

3 Subgroups of $P\Gamma O(5, q)$ isomorphic to A_4

Let $q \equiv 5$ (6) and consider the above model of $Q(4, q)$. Fix η to be an element of order 3 in $GF(q^2)$ (which is necessarily not in $GF(q)$ since $q \equiv 5(6)$), and define the following two maps from V to V :

$$\varphi : (x, y, z, \alpha) \mapsto (x, -y, -z, \alpha),$$

$$\psi : (x, y, z, \alpha) \mapsto (z, x, y, \eta\alpha).$$

Lemma 1 *The collineations φ and ψ together generate a subgroup of $P\Gamma O(5, q)$ isomorphic to A_4.*

Proof: Clearly both φ and ψ preserve $Q(4, q)$, and so are elements of the full collineation group $P\Gamma O(5, q)$ of $Q(4, q)$. Since the alternating group A_4 of degree 4 can be presented as the following set of generators and relations:

$$A_4 \cong \langle h_1, h_2 : h_1^2, h_2^3, (h_1h_2)^3\rangle,$$

taking $h_1 = \varphi$ and $h_2 = \psi$ as generators, one obtains a subgroup of $P\Gamma O(5, q)$ isomorphic to A_4. $\square$

We will denote by H_η this subgroup isomorphic to A_4, and list here explicitly its elements for ease of reference throughout the paper (where the notation for each element should be suggestive of its action, and $\varphi_{23} = \varphi$ and $\psi_\eta = \psi$):

$$
\begin{aligned}
\iota : (x,y,z,\alpha) &\mapsto \quad (x,y,z,\alpha) \\
\varphi_{12} : (x,y,z,\alpha) &\mapsto \quad (-x,-y,z,\alpha) \\
\varphi_{13} : (x,y,z,\alpha) &\mapsto \quad (-x,y,-z,\alpha) \\
\varphi_{23} : (x,y,z,\alpha) &\mapsto \quad (x,-y,-z,\alpha) \\
\psi_{\eta} : (x,y,z,\alpha) &\mapsto \quad (z,x,y,\eta\alpha) \\
\psi_{\eta 12} : (x,y,z,\alpha) &\mapsto \quad (-z,-x,y,\eta\alpha) \\
\psi_{\eta 13} : (x,y,z,\alpha) &\mapsto \quad (-z,x,-y,\eta\alpha) \\
\psi_{\eta 23} : (x,y,z,\alpha) &\mapsto \quad (z,-x,-y,\eta\alpha) \\
\psi_{\eta^2} : (x,y,z,\alpha) &\mapsto \quad (y,z,x,\eta^2\alpha) \\
\psi_{\eta^2 12} : (x,y,z,\alpha) &\mapsto \quad (-y,-z,x,\eta^2\alpha) \\
\psi_{\eta^2 13} : (x,y,z,\alpha) &\mapsto \quad (-y,z,-x,\eta^2\alpha) \\
\psi_{\eta^2 23} : (x,y,z,\alpha) &\mapsto \quad (y,-z,-x,\eta^2\alpha)
\end{aligned}
$$

Of course we could have taken η^2 as the element of order 3 of $GF(q^2)$, and obtained another subgroup H_{η^2} of $P\Gamma O(5,q)$ isomorphic to A_4. The following lemma says that these are conjugate in $P\Gamma O(5,q)$.

Lemma 2 *The groups H_η and H_{η^2} are conjugate subgroups of $P\Gamma O(5,q)$*

Proof: Let g be the map $(x,y,z,\alpha) \mapsto (x,y,z,\alpha^q)$. Then it is straightforward to check that $g^{-1}H_\eta g = H_{\eta^2}$ and g fixes the quadratic form (hence is an element of $P\Gamma O(5,q)$). $\square$

4 Subgroups of $P\Gamma O(5,q)$ isomorphic to S_4

Let $\epsilon \in GF(q^2)$ be an element of order $q+1$, $\sigma \in S_3$ a transposition, and $\eta \in GF(q^2) \setminus GF(q)$ an element of order 3 as before. Define the following map

$$\theta_{\sigma,\epsilon^i} : (x,y,z,\alpha) \mapsto (\sigma(x,y,z), \epsilon^i \alpha^q),$$

where we assume the natural action of S_3 on the first three coordinates, and let $G_{\eta,\sigma,\epsilon^i}$ denote the group generated by H_η and $\theta_{\sigma,\epsilon^i}$.

Theorem 1 *The groups $G_{\eta,\sigma,\epsilon^i}$ and $G_{\eta,\sigma,\epsilon^j}$ are conjugate subgroups of $P\Gamma O(5,q)$ if and only if both i,j are even or both i,j are odd. Moreover every one of these groups is isomorphic to the symmetric group S_4.*

Proof: Without loss of generality assume that $\sigma = (2,3) \in S_3$. To prove that each one of these groups is isomorphic to the symmetric group S_4, recall that S_4 can be presented as the following set of generators and relations:

$$S_4 \cong \langle g_1, g_2 : g_1^3, g_2^2, (g_1g_2)^4 \rangle,$$

and take $g_1 := \psi$ and $g_2 := \theta_{\sigma,\epsilon^i}\varphi_{23}$. It is straightforward to check that $G_{\eta,\sigma,\epsilon^i} \leq P\Gamma O(5,q)$. For every $\beta \in GF(q^2)$ of order dividing $q+1$ define the map

$$g_\beta : (x, y, z, \alpha) \mapsto (x, y, z, \beta\alpha).$$

Then ${g_\beta}^{-1} = g_{\beta^q}$, ${g_\beta}^{-1}H_\eta g_\beta = H_\eta$, and ${g_\beta}^{-1}\theta_{\sigma,\epsilon^i}g_\beta = \theta_{\sigma,\beta^2\epsilon^i}$. It follows that with $\beta = \epsilon$ we get ${g_\epsilon}^{-1}G_{\eta,\sigma,\epsilon^i}g_\epsilon = G_{\eta,\sigma,\epsilon^{i+2}}$. By repeating this conjugation from $G_{\eta,\sigma,\epsilon^i}$ one can obtain every $G_{\eta,\sigma,\epsilon^j}$ if j has the same parity as i. Clearly g_β belongs to $O(5,q)$.

To prove the converse we study the set of fixed points of the maps $\theta_\epsilon : (x, y, z, \alpha) \mapsto (x, z, y, \epsilon\alpha^q)$ and $\theta_1 : (x, y, z, \alpha) \mapsto (x, z, y, \alpha^q)$. Both of these maps have no fixed points in the planes $x = y = 0$ and $x = z = 0$ and clearly also in the plane $\alpha = 0$ the structure of their set of fixed points does not differ. From now on we only consider points with coordinates (x, y, z, α) with $\alpha \neq 0$, not both x and y equal to 0, and not both x and z equal to 0.

First we consider points contained in the hyperplane $y - z = 0$. Let P denote the point with coordinates (x, y, y, α). If $P^{\theta_1} = P$ then P lies in the hyperplane $\alpha = \alpha^q$ and hence $\alpha \in GF(q)^*$. It follows that the fixed points of θ_1 in the hyperplane $y - z = 0$ lie on the quadric defined by the equation $x^2 + 2y^2 + \alpha^2 = 0$ in the plane $\pi_1 :\ y - z = \alpha - \alpha^q = 0$, which is a non-degenerate conic on $Q(4,q)$. If $P^{\theta_\epsilon} = P$ then it follows that $y = z$, $1 + 2y^2 + \alpha^{q+1} = 0$, and $\alpha = \epsilon\alpha^q$. This defines a non-degenerate conic in the plane π_ϵ, the intersection of the hyperplanes $y - z = 0$ and $\alpha - \epsilon\alpha^q = 0$. It follows that the set of fixed points of θ_1, respectively θ_ϵ contained in the hyperplane $y - z = 0$, is exactly the set of points of a non-degenerate conic in the plane π_1, respectively π_ϵ.

Now we consider a point P not contained in the hyperplane $y - z = 0$. If P is fixed by θ_1 or θ_ϵ then P is contained in the hyperplane $x = 0$, and P can be normalised such that P has coordinates $(0, 1, z, \alpha)$ where $z^2 = 1$, $\alpha^{q+1} = -2$, and $\epsilon\alpha^q = z\alpha$, respectively $\alpha^q = z\alpha$, for a fixed point of θ_ϵ, respectively θ_1. Since P is not contained in the hyperplane $y - z = 0$ we obtain that $z = -1$, $\alpha^{q+1} = -2$, and $\epsilon\alpha^q = -\alpha$, respectively $\alpha^q = -\alpha$, for a fixed point of θ_ϵ, respectively θ_1. In the latter case this implies that $\alpha^{q-1} = -1$ and $\alpha^2 = 2$. If

2 is a square in $GF(q)$ (iff $q = \pm 1(8)$), then this is a contradiction. If 2 is a non-square in $GF(q)$ then there are two solutions $\alpha = \pm\sqrt{2}$ and hence two fixed points $(0, 1, -1, \sqrt{2})$ and $(0, 1, -1, -\sqrt{2})$ of θ_1. For θ_ϵ we get two fixed points $(0, 1, -1, \sqrt{2\epsilon})$ and $(0, 1, -1, -\sqrt{2\epsilon})$ if 2 is a square in $GF(q)$. Note that

$$(\pm\sqrt{2\epsilon})^{q+1} = -2,$$

and

$$(\pm\sqrt{2\epsilon})^{q-1} = -\epsilon^{-1},$$

since

$$(\sqrt{\epsilon})^{q-1} = (w^{(q-1)/2})^{q-1} = w^{(q^2-1)/2}w^{1-q} = (-1)(w^{q-1})^q = -\epsilon^{-1},$$

where w denotes a primitive element of $GF(q^2)$. If 2 is not a square in $GF(q)$ then the solutions of $\alpha^{q+1} = -2$ are $\alpha = \beta\sqrt{2}$, with $\beta^{q+1} = 1$. But then

$$2\epsilon = \alpha^{q+1}/\alpha^{q-1} = \alpha^2 = (\beta\sqrt{2})^2 = 2\beta^2$$

implying

$$\epsilon^{(q+1)/2} = \beta^{q+1} = 1,$$

contradicting that ϵ is of order $q + 1$.

We have shown that if $q = \pm 1(8)$ then θ_1 has $q + 1$ fixed points and θ_ϵ has $q + 3$ fixed points, and if $q = \pm 3(8)$ then θ_1 has $q + 3$ fixed points and θ_ϵ has $q + 1$ fixed points. The involutions in $G_{\eta,\sigma,\epsilon^i} \setminus H_\eta$ all have the same number of fixed points since they are conjugate in $G_{\eta,\sigma,\epsilon^i}$. It follows that there are exactly two conjugacy classes of subgroups $G_{\eta,\sigma,\epsilon^i}$ in $P\Gamma O(5, q)$. $\square$

5 BLT-sets of $Q(4, q)$

A *partial BLT-set* of $Q(4, q)$, q odd, is a set of at least three points of $Q(4, q)$ such that any three points of the set span a plane whose polar line with respect to $Q(4, q)$ is exterior to $Q(4, q)$. A partial BLT-set can have size at most $q + 1$, in which case it is called a *BLT-set*.

In the above model of $PG(4, q)$, consider the plane π defined by $\alpha = 0$. Then π meets $Q(4, q)$ in the conic defined by $x^2 + y^2 + z^2 = \alpha = 0$, which is a set $\mathcal{P}$ of $q + 1$ points. The polar line l to π is defined by the intersection of the three hyperplanes with equations $x = 0$, $y = 0$ and $z = 0$. Since $\alpha^{q+1} = 0$

has only the trivial solution, this line l is exterior to $Q(4,q)$. Hence $\mathcal{P}$ is a BLT-set, known as the *classical BLT-set.*

In general, Bader, O'Keefe and Penttila [2] have given the following algebraic condition for a set of points of $Q(4,q)$, q odd, to be a partial BLT-set. Here we denote by $\square$ the subgroup of index 2 in the multiplicative group $GF(q)^*$ of $GF(q)$ consisting of elements which are squares, and by $\boxtimes$ the other coset of $\square$ in the factor group $GF(q)^*/\square$. The *discriminant* $\mathrm{disc}(Q)$ of the non-degenerate quadratic form Q, defined to be $\det(B)\square \in GF(q)^*/\square$ where B is the matrix of the polar form f, is a complete invariant of such orthogonal spaces for q odd.

Lemma 3 *If $\langle X\rangle, \langle Y\rangle, \langle Z\rangle$ are three points of $Q(4,q)$, q odd, spanning a plane of $PG(4,q)$, then the polar line to $\langle X, Y, Z\rangle$ is exterior to $Q(4,q)$ if and only if*

$$\frac{-2f(X,Y)f(X,Z)f(Y,Z)\square}{\mathrm{disc(Q)}} = \boxtimes \in GF(q)^*/\square.$$

Furthermore, Johnson [3] has proved that we need only check all triples on a randomly chosen point of our set.

Lemma 4 *Let $\mathcal{P}$ be a set of at least three points of $Q(4,q)$, q odd. If there exists a point $\langle X\rangle$ of $\mathcal{P}$ such that $\{\langle X\rangle, \langle Y\rangle, \langle Z\rangle\}$ is a partial BLT-set for all distinct $\langle Y\rangle, \langle Z\rangle \in \mathcal{P}\backslash\{\langle X\rangle\}$, then $\mathcal{P}$ is a partial BLT-set.*

In the above model of $Q(4,q)$, the value of $\mathrm{disc}(Q)$ is the product of the values of the discriminant restricted to the first three coordinates and restricted to the last coordinate, since this restriction represents an orthogonal direct sum decomposition of V. The value of the discriminant restricted to the first three coordinates is clearly $2^3\square = 2\square$, whereas the value on the last coordinate, as this represents an external line, is $\boxtimes$ if $q \equiv 1$ (4), and $\square$ if $q \equiv 3$ (4) (see, e.g., [4]). Now -1 is a square exactly when $q \equiv 1$ (4) and is a nonsquare exactly when $q \equiv 3$ (4), hence combining the above two lemmas we have the following, where we use the shorter notation $F(X,Y,Z) = f(X,Y)f(X,Z)f(Y,Z)$.

Lemma 5 *In the above model of $Q(4,q)$, with q odd, let $\mathcal{P}$ be a set of at least three points of $Q(4,q)$. If there exists a point $\langle X\rangle$ of $\mathcal{P}$ such that*

$$F(X,Y,Z)\square = \square$$

for all distinct $\langle Y\rangle, \langle Z\rangle \in \mathcal{P}\backslash\{\langle X\rangle\}$, then $\mathcal{P}$ is a partial BLT-set.

5.1 Partial BLT-sets as orbits under A_4

Let $H = H_\eta$ for some fixed η. Then the following lemma is clear from the action of H on our model.

Lemma 6 *If ϕ is an element of H then $f(X, Y) = f(\phi(X), \phi(Y))$.*

The condition for an orbit under H_η to be a partial BLT-set turns out to be quite neat.

Lemma 7 *The orbit P^H under the subgroup H of a point $P = (x, y, z, \alpha)$ of $Q(4, q)$ forms a partial BLT-set if and only if each of $-(x^2+y^2)$, $-(x^2+z^2)$, $-(y^2+z^2)$ is a (possibly zero-valued) square in $GF(q)$.*

Proof: Let $P = (x, y, z, \alpha)$ be a point of $Q(4, q)$, and consider the orbit P^H of P under H. Any triple of distinct points of P^H containing P is of the form $\{P, \phi_1(P), \phi_2(P)\}$ for distinct $\phi_1, \phi_2 \in H \backslash \{\iota\}$. Hence, by Lemma 6,

$$F(P, \phi_1(P), \phi_2(P)) = f(P, \phi_1(P)) f(P, \phi_2(P)) f(P, \phi_1^{-1}\phi_2(P)).$$

Thus we need only calculate $f(P, \phi(P))$ for each $\phi \in H \backslash \{\iota\}$ in order to determine if P^H forms a partial BLT-set. Here we summarise these calculations. Note that $\eta^2 + \eta = -1$.

$$f(P, \varphi_{12}(P)) = 4(z^2 + \alpha^{q+1}) = -4(x^2 + y^2)$$

$$\begin{aligned}
f(P, \varphi_{13}(P)) &= 4(y^2 + \alpha^{q+1}) = -4(x^2 + z^2) \\
f(P, \varphi_{23}(P)) &= 4(x^2 + \alpha^{q+1}) = -4(y^2 + z^2) \\
f(P, \psi_\eta(P)) &= f(P, \psi_{\eta^2}(P)) = (x + y + z)^2 \\
f(P, \psi_{\eta 12}(P)) &= f(P, \psi_{\eta^2 13}(P)) = (-x + y + z)^2 \\
f(P, \psi_{\eta 13}(P)) &= f(P, \psi_{\eta^2 23}(P)) = (x + y - z)^2 \\
f(P, \psi_{\eta 23}(P)) &= f(P, \psi_{\eta^2 12}(P)) = (x - y + z)^2
\end{aligned}$$

Considering the various cases concludes the proof. □

Note that we do not require that each of the above be nonzero, since the possibility of starting with, for example, $P = (0, 0, z, \alpha)$, merely gives rise to an orbit of length 6 (and not 12).

Examples The orbit of the point with coordinates $(5, 10, 9, 1)$ in $PG(4, 23)$ is a partial BLT-set of size 12. Similarly $(14, 40, 41, 1)$, and $(12, 41, 17, 1)$ in $PG(4, 47)$.

5.2 Partial BLT-sets as orbits under S_4

Let $P = (x, y, z, \alpha)$ be a point of $Q(4, q)$ such that the orbit P^H under $H \cong A_4$ is a partial BLT-set, and consider the point $Q = (x, z, y, \varepsilon\alpha^q)$, where $\varepsilon^{q+1} = 1$. It then follows from Lemma 7 that Q^H is also a partial BLT-set. From the previous section we see that in fact $Q = \theta_\epsilon(P)$, and so considering the orbit of P under the subgroup $G \cong S_4$ generated by H and θ_ϵ, we need merely consider $P^H \cup Q^H$; that is, the orbit P^G of P under G is equal to the union of the orbits P^H and Q^H of P and Q under H.

Any triple of distinct points of $P^H \cup Q^H$ containing P is of one of the following forms : $\{P, \phi_1(P), \phi_2(P)\}$ for distinct $\phi_1, \phi_2 \in H\backslash\{\iota\}$; $\{P, \phi_1(P), \phi_2(Q)\}$ for $\phi_1 \in H\backslash\{\iota\}, \phi_2 \in H$; $\{P, \phi_1(Q), \phi_2(Q)\}$ for distinct $\phi_1, \phi_2 \in H$. Thus as before, by Lemma 6, the product

$$F(P, P_1, P_2) = f(P, P_1)f(P, P_2)f(P_1, P_2)$$

with $P_1, P_2 \in P^H \cup Q^H$, contains an even number of $f(P, \phi(Q))$, $\phi \in H$, and an odd number of $f(R, \phi(R))$, $R \in \{P, Q\}$, $\phi \in H$. It follows that in order to determine if $P^H \cup Q^H$ is a partial BLT-set, we need only further calculate $f(P, \phi(Q))$ for each $\phi \in H$, and verify that these are all nonsquares or all squares. Here we summarise these calculations, where $T : GF(q^2) \to GF(q), x \mapsto x + x^q$ is the trace function and $N : GF(q^2) \to GF(q), x \mapsto x^{q+1}$ is the norm function. Note that $\eta^q = \eta^2$.

$$f(P, Q) = 2(x^2 + 2yz) + T(\varepsilon^q\alpha^2) = -2(y - z)^2 - N(\alpha - \varepsilon\alpha^q)$$

$$f(P, \varphi_{23}(Q)) = 2(x^2 - 2yz) + T(\varepsilon^q\alpha^2) = -2(y + z)^2 - N(\alpha - \varepsilon\alpha^q)$$

$$f(P, \varphi_{12}(Q)) = f(P, \varphi_{13}(Q)) = -2x^2 + T(\varepsilon^q\alpha^2) = 2(y^2 + z^2) + N(\alpha + \varepsilon\alpha^q)$$

$$f(P, \psi_\eta(Q)) = 2(z^2 + 2xy) + T(\varepsilon^q\eta^2\alpha^2) = -2(x - y)^2 - N(\eta\alpha - \varepsilon\eta^q\alpha^q)$$

$$f(P,\psi_{\eta 12}(Q)) = 2(z^2-2xy)+T(\varepsilon^q\eta^2\alpha^2) = -2(x+y)^2 - N(\eta\alpha - \varepsilon\eta^q\alpha^q)$$
$$f(P,\psi_{\eta 13}(Q)) = f(P,\psi_{\eta 23}(Q)) = -2z^2+T(\varepsilon^q\eta^2\alpha^2) = 2(x^2+y^2)+N(\eta\alpha+\varepsilon\eta^q\alpha^q)$$
$$f(P,\psi_{\eta^2}(Q)) = 2(y^2+2xz)+T(\varepsilon^q\eta\alpha^2) = -2(x-z)^2 - N(\eta^2\alpha - \varepsilon\eta^{2q}\alpha^q)$$
$$f(P,\psi_{\eta^2 13}(Q)) = 2(y^2-2xz)+T(\varepsilon^q\eta\alpha^2) = -2(x+z)^2 - N(\eta^2\alpha - \varepsilon\eta^{2q}\alpha^q)$$
$$f(P,\psi_{\eta^2 12}(Q)) = f(P,\psi_{\eta^2 23}(Q)) = -2y^2+T(\varepsilon^q\eta\alpha^2) = 2(x^2+z^2)+N(\eta^2\alpha+\varepsilon\eta^{2q}\alpha^q)$$

Hence for $P^H \cup Q^H$ to be a partial BLT-set, we require that all of the above are squares, or they are all nonsquares.

Examples Here we list some examples of BLT-sets of $Q(4,q)$, $q \equiv -1(24)$. In all the examples $\eta = w^{(q^2-1)/3}$ and $\epsilon = w^{(q^2-1)/(q+1)}$, where w is a primitive element in $GF(q^2)$. We give the minimal polynomial $f(w)$ of w, and the power ϵ^i which determines the group $G_{\eta,\sigma,\epsilon^i}$ in each case; for σ we always take $(2,3)$. We do not list the classical BLT-set. For each q we give two examples, the first one listed is fixed by the group $G_{\eta,\sigma,\epsilon^i}$, for some odd i; the second one for some even i. The number of fixed points of the involutions of $G_{\eta,\sigma,\epsilon^i} \setminus H_\eta$ follows from Theorem 1 ($q+3$ for odd powers of ϵ and $q+1$ for even powers of ϵ). The second example listed is the FTWK-example (see [4]), and the first is the example found by computer, listed in [4], or is new (for $q = 167$).

- $q = 23$: $(f(w) = w^2 - 2w + 5)$
  1. $\{(5,10,9,1)\}$ and $\epsilon^9 = w^{198}$.
  2. $(5,20,17,w^{11})$ and $\epsilon^4 = w^{88}$.
- $q = 47$: $(f(w) = w^2 - 2w + 5)$
  1. $\{(12,41,17,1),(8,12,13,w^{943})\}$ and $\epsilon^{43} = w^{1978}$.
  2. $\{(14,40,41,1),(8,12,13,w^{1679})\}$ and $\epsilon^{22} = w^{1012}$.
- $q = 71$: $(f(w) = w^2 - 2w + 7)$
  1. $\{(46,41,6,1),(27,28,11,w^{1785}),(45,62,5,w^{2205})\}$ and $\epsilon^{59} = w^{4130}$.
  2. $\{(45,21,35,1),(49,2,62,w^{315}),(47,54,46,w^{2100})\}$ and $\epsilon^{56} = w^{3920}$.
- $q = 167$: $(f(w) = w^2 + 104w + 109)$

1. $\{(28, 127, 17, w^{83}),\ (141, 116, 79, w^{17098}),\ (69, 95, 110, w^{6557}),$ $(129, 144, 52, w^{1079}),\ (154, 160, 99, w^{21580}),\ (105, 110, 151, w^{15189}),$ $(160, 79, 97, w^{8549})\}$ and $\epsilon^{129} = w^{21414}$.
2. $\{(144, 71, 21, 1),\ (52, 131, 95, w^{10956}),\ (123, 39, 160, w^{17430}),$ $(58, 99, 19, w^{25564}),\ (28, 160, 105, w^{19007}),\ (140, 127, 141, w^{16434}),$ $(124, 123, 97, w^{22327})\}$ and $\epsilon^{166} = w^{27556}$.

Final remarks

Many more examples of BLT-sets can be constructed using this method. Referring to the list of examples in [5], the following further BLT-sets are found to arise naturally within the above model : for $q = 17$, the examples DCH and PR with groups of orders 144 and 24 respectively; for $q = 29$, the examples with groups of orders 720 and 48; for $q = 41$, the examples with groups of orders 60 and 24; for $q = 53$, the examples with groups of orders 24 and 12; and for $q = 59$, the examples with groups of orders 120 (not Penttila) and 24. For $q = 11$, the example arising from the family of Penttila, and for $q = 83$ and $q = 89$ the examples listed in [4], both with groups of order 24, can also be constructed. Furthermore, for all $q \equiv 5(6)$, $q \leq 71$ (and presumably for greater q), the example arising from the family of Fisher-Thas/Walker/Kantor can be described naturally within the above model. The example in $PG(4, 167)$ listed first is new and one could definitely go on and construct more new examples. However it is our hope that some of these examples will be shown to be members of an infinite family of BLT-sets of $Q(4, q)$, with S_4 as automorphism group.

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