Semifield flocks, eggs, and ovoids of $Q(4, q)$

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Abstract

In this article we consider the connection between semifield flocks of a quadratic cone in $\mathrm{PG}(3, q^n)$, eggs in $\mathrm{PG}(4n-1, q)$ and ovoids of $Q(4, q^n)$, when q is odd. Starting from a semifield flock of a quadratic cone in $\mathrm{PG}(3, q^n)$, q odd, $\mathcal{F}$ one can obtain an ovoid $\mathcal{O}(\mathcal{F})$ of $Q(4, q^n)$ using the construction of Thas [9]. With a semifield flock there also corresponds a good egg $\mathcal{E}$ of $\mathrm{PG}(4n-1, q)$ (see, e.g., [2]) and the TGQ $T(\mathcal{E})$ contains at least $q^{3n} + q^{2n}$ subquadrangles all isomorphic to $Q(4, q^n)$ (Thas [7]). Hence by subtending one can obtain ovoids of $Q(4, q^n)$ (consider the set of points in the subquadrangle collinear with a point not in the subquadrangle). Here we prove that all the ovoids subtended from points of type (ii) are isomorphic to $\mathcal{O}(\mathcal{F})$, and that in at least $2q^n$ subGQ's the ovoids subtended from points of type (i) are isomorphic to the ovoids subtended from points of type (ii).

1. Definitions and motivation

Throughout the article we assume that q is an odd prime power. With $\mathrm{PG}(m, q)$ we denote the m-dimensional projective space arising from the $(m+1)$-dimensional vectorspace over the finite field $\mathrm{GF}(q)$ of order q. A *flock of a quadratic cone* $\mathcal{K}$ of $\mathrm{PG}(3, q)$ with vertex v is a partition of $\mathcal{K} \setminus \{v\}$ into irreducible conics. The planes containing the conics of the flock are called the *planes of the flock*.

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Put $F = \mathrm{GF}(q^n)$ and consider the quadratic cone $\mathcal{K}$ in $\mathrm{PG}(3, q^n)$ with vertex $v = \langle 0, 0, 0, 1\rangle$ and base the conic C with equation $X_0X_1 = X_2^2$. The planes of a flock of $\mathcal{K}$ can be written as $\pi_t : tX_0 + f(t)X_1 + g(t)X_2 + X_3 = 0$, $t \in F$, for some $f, g : F \to F$. We denote this flock with $\mathcal{F}(f, g)$. If f and g are linear over a subfield of $\mathrm{GF}(q^n)$ then the flock is called a *semifield* flock. The maximal subfield with this property is called the *kernel* of the flock.

An egg $\mathcal{E}$ of $\mathrm{PG}(2n + m - 1, q)$ is a partial $(n - 1)$-spread of size $q^m + 1$ such that every 3 different egg elements span a $(3n - 1)$-dimensional space and such that for every egg element E there is an $(n+m-1)$-dimensional space T_E, called the *tangent space of* $\mathcal{E}$ *at the element* E, containing E and skew from all the other egg elements.

If $n = m$ then $\mathcal{E}$ is called a *pseudo-oval* or a *generalized oval*. The only known examples of pseudo-ovals are ovals of $\mathrm{PG}(2, q^n)$, seen over $\mathrm{GF}(q)$. If $2n = m$ then $\mathcal{E}$ is called a *pseudo-ovoid* or a *generalized ovoid*. An ovoid of $\mathrm{PG}(3, q^n)$ seen over $\mathrm{GF}(q)$ is an example of a pseudo-ovoid. In this case more examples are known, see [2]. We call the examples of eggs which are ovals of $\mathrm{PG}(2, q^n)$ or ovoids of $\mathrm{PG}(3, q^n)$ seen over $\mathrm{GF}(q)$ *elementary*. In case the oval is a conic or the ovoid is an elliptic quadric, the egg is called *classical*.

A pseudo-ovoid is said to be *good at an element* E if every $(3n - 1)$-space on that element containing at least two other egg elements contains exactly $q^n + 1$ egg elements. A pseudo-ovoid $\mathcal{E}$ is called *good* if there exists at least one egg element E such that $\mathcal{E}$ is good at E.

A *generalized quadrangle of order* (s,t) $(GQ(s,t))$, $s > 1$, $t > 1$, is an incidence structure of points and lines with the properties that any two points are incident with at most one common line, any two lines are incident with at most one common point, every line is incident with $s+1$ points, every point is incident with $t+1$ lines, and given a line l and a point P not incident with l, there is a unique line m and a unique point Q, such that m is incident with P and Q and Q is incident with l. If $s = t$ then we speak of a generalized quadrangle of *order* s $(GQ(s))$. From a $GQ(s,t)$ we get a $GQ(t,s)$ by interchanging the labels point and line, called the *point-line dual* of the generalized quadrangle of order (s,t). For more on generalized quadrangles we refer to [5].

A *translation generalized quadrangle* (TGQ) *with base point* P is a generalized quadrangle for which there is an abelian group T acting regularly on the points not collinear with P, while fixing every line through P.

Let $\mathcal{E}$ be an egg of $\mathrm{PG}(2n+m-1, q)$. Now embed $\mathrm{PG}(2n+m-1, q)$ in a $\mathrm{PG}(2n+m, q)$ and construct an incidence structure $T(\mathcal{E})$ as follows. Points are of three types: (i) the points of $\mathrm{PG}(2n + m, q) - \mathrm{PG}(2n + m - 1, q)$; (ii) the $(n + m)$-dimensional subspaces of $\mathrm{PG}(2n + m, q)$ which intersect $\mathrm{PG}(2n + m - 1, q)$ in a tangent space of $\mathcal{E}$; (iii) the symbol (∞). Lines are of two types: (a) the n-dimensional subspaces of $\mathrm{PG}(2n + m, q)$ which intersect $\mathrm{PG}(2n + m - 1, q)$ in an egg element; (b) the egg elements. Incidence is defined as follows: lines of type (b) are incident with points of type (ii) which contain them and with the point (∞); lines of type (a) are incident with points of type (i) contained in it and with the point of type (ii) that contains it.

Theorem 1.1 ([5, 8.7.1])
The incidence structure $T(\mathcal{E})$ is a translation generalized quadrangle of order (q^n, q^m) with base point (∞). Conversely, every TGQ is isomorphic to a $T(\mathcal{E})$ for some egg $\mathcal{E}$ of $\mathrm{PG}(2n+m-1, q)$. It follows that the theory of TGQ is equivalent to the theory of eggs.

An *ovoid* of a generalized quadrangle Γ is a set of points such that every line of the Γ contains exactly one of these points. An ovoid $\mathcal{O}$ is called a *translation ovoid* or *semifield ovoid* if there is a group G of collineations of Γ fixing $\mathcal{O}$, and a point P in $\mathcal{O}$ such that G fixes P and every line incident with P, and G acts regular on the points not collinear with P. In 1997 Thas [9] gave a method of constructing a translation ovoid $\mathcal{O}(\mathcal{F})$ of $Q(4, q^n)$ from a semifield flock $\mathcal{F}$, and conversely. Lunardon [3] proved that two semifield flocks are isomorphic if and only if the ovoids are isomorphic. In [1] the authors construct the semifield flock corresponding with the translation ovoid of $Q(4, 3^5)$ found with the help of a computer in 1999 by Penttila and Williams [6].

If a GQ of order (s, t) contains a subGQ of order (s', t') then the set of points in the subGQ collinear with a point not on a line of the subGQ has the property that no two of these points are collinear. If $s = s'$ then every line of the subGQ contains one of these points, i.e., these points form an ovoid of the subGQ. Such an ovoid is called *subtended*.

By 8.7.2 of [5] the $q^{2n}+1$ tangent spaces of an egg $\mathcal{E}$ in $\mathrm{PG}(4n-1, q)$ form an egg $\mathcal{E}^D$ in the dual space to $\mathrm{PG}(4n-1, q)$, called the *dual egg of* $\mathcal{E}$. With a semifield flock there corresponds a pseudo-ovoid $\mathcal{E}$, such that the dual egg $\mathcal{E}^D$ is good at an element, [8], see also [2]. So there are at least $q^{2n} + q^n$ pseudo-ovals on that good element, contained in the pseudo-ovoid $\mathcal{E}$. This implies that in the corresponding translation generalized quadrangle $T^*(\mathcal{E}) = T(\mathcal{E}^D)$, i.e., the *translation dual* of the TGQ $T(\mathcal{E})$, we have at least $q^{3n} + q^{2n}$ subquadrangles of order q^n, see [7]. Thas [7] proved that every such subquadrangle is isomorphic to the classical GQ $Q(4, q^n)$. Hence by the method of subtending there arise many ovoids of $Q(4, q^n)$. In 1994 Thas and Payne [10] used this method to construct a new ovoid of $Q(4, q^n)$, using the so-called Roman GQ, [4], arising from the Cohen-Ganley semifield flock, by using one subGQ. The question remained if by using different subGQ's, new ovoids of $Q(4, q^n)$ could be obtained. For the translation ovoid of $Q(4, 3^5)$ found in 1999 by Penttila and Williams [6] it was an open question if new ovoids of $Q(4, 3^5)$ could be obtained by subtending in the corresponding TGQ. In this paper we solve both of these questions for all ovoids subtended by points of type (ii) and in at least q^{2n} subGQ's for all ovoids subtended by points of type (i). They are isomorphic to the translation ovoid $\mathcal{O}(\mathcal{F})$ which arises from the semifield flock $\mathcal{F}$ using the construction of Thas [9].

2. The classical generalized quadrangle $Q(4, q)$

Theorem 2.1 (Payne and Thas [5, 3.2.2])
If $\mathcal{O}$ is an oval in $\mathrm{PG}(2, q)$, then the GQ $T(\mathcal{O})$ is isomorphic to the classical GQ $Q(4, q)$ if and only if $\mathcal{O}$ is an irreducible conic.

Let us take a closer look at the isomorphism between these two generalized quadrangles $T(\mathcal{O})$ and $Q(4,q)$. To find the isomorphism we have to let the point (∞) of $T(\mathcal{O})$ correspond with a point of $Q(4,q)$. Since the collineation group of $Q(4,q)$ acts transitively on the points of $Q(4,q)$, we may choose any point P. The lines incident with P should correspond with the lines incident with (∞), i.e., the points of a conic. Intersecting the polar space of P with $Q(4,q)$ we get a quadratic cone $\mathcal{K}$ with vertex P. The base of the cone $\mathcal{K}$ is a conic $\mathcal{C}$ and hence there arises a natural way of making the necessary correspondence between the lines incident with P and the points of a conic, by projecting the cone $\mathcal{K}$ onto its base $\mathcal{C}$. Let π be the plane containing the conic $\mathcal{C}$. Take a hyperplane H of $\mathrm{PG}(4,q)$, containing π but not incident with P. Now we have the setting to construct the TGQ $T(\mathcal{C})$ in the hyperplane H. Again a natural correspondence arises between the q^3 points of H not in π, i.e., the points of type (i) of $T(\mathcal{C})$, and the q^3 points of $Q(4,q)$ not collinear with P, by projecting $Q(4,q)$ from P onto H. The lines incident with a point Q not collinear with P meet the cone $\mathcal{K}$ in a point, and hence they are projected from P onto a line of H meeting the plane π in a point of $\mathcal{C}$, this is a line of type (a) of $T(\mathcal{C})$. The points collinear with P now have to correspond with planes of H intersecting π in a tangent line to the conic $\mathcal{C}$. We can deduce this by considering the lines not on P and incident with a point Q collinear with P. All these lines are projected onto lines of H intersecting π in the same point of $\mathcal{C}$, and contained in a plane, namely the intersection of the polar space of Q with H. This plane is a point of type (ii) of $T(\mathcal{C})$, and hence by the above we obtained a bijection between the points collinear with P and the points of type (ii) of $T(\mathcal{C})$. It is straightforward to prove that the deduced correspondence defines an isomorphism ϕ between $Q(4,q)$ and $T(\mathcal{C})$.

Now we introduce coordinates, in order to give this isomorphism explicitly. For $Q(4,q)$ we take the non-degenerate quadric with equation $X_2^2 = X_0X_1 + X_3X_4$, for P we take the point $\langle 0,0,0,0,1\rangle$, and for the hyperplane H we choose the hyperplane defined by the equation $X_4 = 0$. Then the plane π has equation $X_3 = X_4 = 0$ and the conic $\mathcal{C}$ has equation $X_2^2 = X_0X_1$. We denote the tangent line at the point $\langle x_0,x_1,x_2,x_3,x_4\rangle$ of the conic $\mathcal{C}$ by $T_{\mathcal{C}}(x_0,x_1,x_2,x_3,x_4)$. Then the isomorphism $\phi\ :\ Q(4,q)\ \rightarrow T(\mathcal{C})$ can be defined by its action on the points of $Q(4,q)$:

$$\begin{array}{lcl}
\langle 0,0,0,0,1\rangle & \mapsto & (\infty), \\
\langle a,b,c,1,c^2-ab\rangle & \mapsto & \langle a,b,c,1,0\rangle, \\
\langle a^2,1,a,0,b\rangle & \mapsto & \langle T_{\mathcal{C}}(a^2,1,a,0,0),(-b,0,0,1,0)\rangle, \\
\langle 1,0,0,0,a\rangle & \mapsto & \langle T_{\mathcal{C}}(1,0,0,0,0),(0,-a,0,1,0)\rangle.
\end{array}$$

Theorem 2.1 can be extended to TGQs corresponding with classical pseudo-ovals, in the case where q is odd. The pseudo-oval then arises from a conic of $\mathrm{PG}(2,q^n)$, and the corresponding TGQ is isomorphic to $Q(4,q^n)$. It is clear that the isomorphism can easily be deduced from the isomorphism ϕ between $Q(4,q)$ and $T(\mathcal{C})$.

In the following section we need this isomorphism in detail. Suppose $\mathcal{F}(f,g)$ is a semifield flock. Then we can write f and g as

$$f(t) = \sum_{i=0}^{n-1} c_i t^{q^i},\ \text{and}\ g(t) = \sum_{i=0}^{n-1} b_i t^{q^i},$$

for some $b_i, c_i \in F$, $i = 0, 1, \ldots, n-1$. In [2] it was shown that the elements of the good egg $\mathcal{E}$ corresponding with the semifield flock $\mathcal{F}(f, g)$, can be written as

$$\begin{aligned} E(\gamma) &= \{\langle -g_t(\gamma), t, -\gamma t\rangle \| \ t \in F^*\}, \ \forall \gamma \in F^2, \\ E(\infty) &= \{\langle t, 0, (0,0)\rangle \| \ t \in F^*\}, \\ T_E(\gamma) &= \{\langle h(\gamma, \delta) + g_t(\gamma), t, \delta\rangle \| \ (t, \delta) \in F \times F^2 \setminus \{(0,0)\}\}, \ \forall \gamma \in F^2, \\ T_E(\infty) &= \{\langle t, 0, \delta\rangle \| \ (t, \delta) \in F \times F^2 \setminus \{(0,0)\}\}, \end{aligned}$$

with

$$g_t(a, b) = a^2 t + \sum_{i=0}^{n-1} (b_i ab + c_i b^2)^{1/q^i} t^{1/q^i},$$

and

$$h((a, b), (c, d)) = 2ac + \sum_{i=0}^{n-1} (b_i(ad + bc) + 2c_i bd)^{1/q^i}.$$

With these notations the pseudo-ovoid $\mathcal{E}$ is good at its element $E(\infty)$.

Remark 2.2 In [4] Payne calculates the 4-gonal family for the so called Roman GQ (see also [10, 5.1]) and in [10, 5.2,5.3] the authors use this 4-gonal family to study certain collineations of these GQ's. Because of the form of the model of good eggs corresponding to a semifield flock from [2] as stated above, it is immediate that the results obtained in [10, 5.2,5.3] can be generalised to every good egg presented in this model. We note that the collineations (ii) in [10, 5.2] imply that the stabilizer of the good element acts transitively on the other egg elements. When we use some of these collineations we will refer to this remark and [10]

Consider the pseudo-oval $\mathcal{O}$ determined by the triple $(E(\infty), E(0,0), E(1,0))$. So $\mathcal{O}$ consists of the elements $E(\gamma)$, with $\gamma \in \{(a, 0) \| \ a \in F\} \cup \{\infty\}$. From the coordinates we see that this pseudo-oval is classical. It is the conic with equation $X_0 X_1 + X_2^2 = 0$ seen over $\mathrm{GF}(q)$.

Consider the projective space $\mathrm{PG}(4n, q) = \{\langle r, s, t, u, x_{4n}\rangle \| \ (r, s, t, u, x_{4n}) \in (F^4 \times \mathrm{GF}(q)) \setminus \{(0,0,0,0,0)\}\}$, and suppose that the good egg is contained in the hyperplane with equation $X_{4n} = 0$. The pseudo-oval $\mathcal{O}$ is then contained in the $(3n-1)$-dimensional subspace $\rho = \{\langle r, s, t, 0, 0\rangle \| \ (r, s, t) \in F^3 \setminus \{(0,0,0)\}\}$. We construct $T(\mathcal{O})$ in the $3n$-dimensional subspace $\mathcal{G} = \{\langle r, s, t, 0, x_{4n}\rangle \| \ (r, s, t, x_{4n}) \in (F^3 \times \mathrm{GF}(q)) \setminus \{(0,0,0,0)\}\}$. Now we can define, in a similar way as we defined the isomorphism $\phi : Q(4, q^n) \rightarrow T(\mathcal{C})$, an isomorphism $\psi \ : \ Q(4, q^n) \rightarrow T(\mathcal{O})$:

$$\begin{array}{lcl} \langle 0,0,0,0,1\rangle & \mapsto & (\infty), \\ \langle a, b, c, 1, c^2 - ab\rangle & \mapsto & \langle -a, b, -c, 0, 1\rangle, \\ \langle a^2, 1, a, 0, b\rangle & \mapsto & \langle T_E(a, 0) \cap \rho, (b, 0, 0, 0, 1)\rangle, \\ \langle 1, 0, 0, 0, a\rangle & \mapsto & \langle T_E(\infty) \cap \rho, (0, -a, 0, 0, 1)\rangle. \end{array}$$

So points collinear with x are mapped onto points of type (ii) of $T(\mathcal{O})$, i.e., the span of a tangent space of $\mathcal{O}$ with a point of $\mathcal{G} \setminus \rho$, and points not collinear with x are mapped onto points of type (i) of $T(\mathcal{O})$, i.e., points of $\mathcal{G} \setminus \rho$.

3. Semifield flocks and translation ovoids

In this section we will give the connection between a semifield flock $\mathcal{F}$ of a quadratic cone in $\mathrm{PG}(3, q^n)$ and a translation ovoid $\mathcal{O}(\mathcal{F})$ of $Q(4, q^n)$, first explained by Thas in [9] in 1997, and later on by Lunardon [3] in more detail. We will need the following lemma.

Lemma 3.1 (see [2])
Let tr *be the trace map from* F *to* $\mathrm{GF}(q)$, *and* $\alpha_i \in F$, $i = 0, \ldots, n-1$. *Then*

$$\mathrm{tr}(\sum_{i=0}^{n-1} \alpha_i t^{q^i}) = 0,$$

for all $t \in F$ *if and only if*

$$\sum_{i=0}^{n-1} \alpha_i^{q^{n-1-i}} = 0.$$

Consider the semifield flock $\mathcal{F}(f, g)$ as before. Now we look at the dual space of $\mathrm{PG}(3, q^n)$ with respect to the standard inner product, i.e., a point $\langle a, b, c, d \rangle$ gets mapped to the plane with equation $aX_0 + bX_1 + cX_2 + dX_3 = 0$. The lines of the cone $\mathcal{K}$ become lines of (the dual space of) $\mathrm{PG}(3, q^n)$ all contained in the plane $\pi : X_3 = 0$ corresponding with the vertex of $\mathcal{K}$. In $\mathrm{PG}(3, q^n)$, they had the property that no three of them were contained in a plane, so now they form a dual oval of π. Since q is odd, this dual oval is a dual conic, i.e., the set of lines of the cone corresponds with the set of tangents of some conic $\mathcal{C}'$. The equation of the conic $\mathcal{C}'$ in π is $4X_0X_1 - X_2^2 = 0$. Two planes π_t and π_s of the flock $\mathcal{F}$ correspond with the points $\langle t, f(t), g(t), 1 \rangle$ and $\langle s, f(s), g(s), 1 \rangle$. Since π_t and π_s do not intersect on the cone $\mathcal{K}$, the line $\langle \langle t, f(t), g(t), 1 \rangle, \langle s, f(s), g(s), 1 \rangle \rangle$ intersects π in an internal point $\langle t - s, f(t) - f(s), g(t) - g(s), 0 \rangle$ of $\mathcal{C}'$. Since f and g are additive, we obtain a set $\{\langle t, f(t), g(t), 0 \rangle \parallel t \in F\}$ of internal points of $\mathcal{C}'$. Over $\mathrm{GF}(q)$ the plane π becomes a $(3n-1)$-dimensional space, the conic $\mathcal{C}'$ becomes a classical pseudo-oval $\mathcal{O}$ and the set of internal points, becomes an $(n-1)$- space skew to all the tangent spaces of $\mathcal{O}$.

In the dual space, this $(n-1)$-space becomes a $(2n-1)$-space U skew to the elements of the pseudo-oval which is the dual of $\mathcal{O}$, and isomorphic to it. To find U we use the inner product corresponding with the polarity defined by the conic $\mathcal{C}'$:

$$((x, y, z), (u, v, w)) \mapsto \mathrm{tr}(4xu + 4yv - 2zw),$$

where tr is the trace map from $\mathrm{GF}(q^n)$ to $\mathrm{GF}(q)$. So the point $\langle u, v, w \rangle \in U$ if $\mathrm{tr}(2uf(t) + 2vt + wg(t)) = 0$, for all $t \in F$. Using the expressions for f and g we obtain the condition

$$\operatorname{tr}\left[(2v + 2uc_0 + wb_0)t + \sum_{i=1}^{n-1}(2c_i u + b_i w)t^{q^i}\right] = 0, \text{ for all } t \in F.$$

Using Lemma 3.1, it follows that we can write U as $\{\langle u, -\tilde{F}(u,w), w\rangle \| \ (u,w) \in F^2 \setminus \{0\}\}$, with

$$\tilde{F}(u,w) = \sum_{i=0}^{n-1}(c_i u + \frac{1}{2}b_i w)^{1/q^i}.$$

Let ρ be the $(3n-1)$-space containing $\mathcal{O}$ and consider the construction of $T(\mathcal{O})$ in $\mathrm{PG}(3n,q)$. If we extend U with a point not contained in ρ and we apply the isomorphism ψ^{-1}, then we get a $2n$-dimensional space containing q^{2n} points of type (i) of $T(\mathcal{O})$. Because U is skew to the pseudo-oval $\mathcal{O}$, no two of these points are collinear in $T(\mathcal{O})$. Adding the point (∞) we get an ovoid of $T(\mathcal{O})$. Since $\mathcal{O}$ is a classical pseudo-oval this gives us an ovoid of $Q(4,q^n)$. In order to give the coordinates of the points of the ovoid of $Q(4,q^n)$, we have to apply a coordinate transformation such that the conic $\mathcal{C}'$ with equation $4X_0X_1 - X_2^2 = 0$ is mapped onto the conic with equation $X_0X_1 + X_2^2 = 0$, and then apply the isomorphism ψ^{-1}. After this transformation U becomes the subspace $\{\langle u, F(u,w), w\rangle \| \ (u,w) \in F^2 \setminus \{0\}\}$, with

$$F(u,w) = \sum_{i=0}^{n-1}(c_i u + b_i w)^{1/q^i}.$$

If we extend U with the point $\langle 0, \ldots, 0, 1\rangle$, we can write the ovoid as the set of points of $\mathrm{PG}(4,q^n)$

$$\{\langle -u, F(u,v), -v, 1, v^2 - uF(u,v)\rangle \| \ u,v \in F\} \cup \{\langle 0,0,0,0,1\rangle\}.$$

After a coordinate transformation fixing $Q(4,q^n)$, we get the ovoid $\mathcal{O}(\mathcal{F})$

$$\{\langle u, -F(u,v), v, 1, v^2 - uF(u,v)\rangle \| \ u,v \in F\} \cup \{\langle 0,0,0,0,1\rangle\}.$$

This construction also works starting with a translation ovoid of $Q(4,q^n)$ to obtain a semifield flock of a quadratic cone in $\mathrm{PG}(3,q^n)$.

4. Subtended ovoids

Theorem 4.1 *Let $\mathcal{E}$ be a good egg of $\mathrm{PG}(4n-1,q)$, q odd, represented as above. Then all the ovoids of any subquadrangle S determined by the elements $E(\infty)$, $E(0,0)$, and $E(1,0)$ of the good egg $\mathcal{E}$, obtained by subtending from points of $T(\mathcal{E})\setminus S$, are isomorphic translation ovoids of $Q(4,q^n)$. Moreover, these ovoids are isomorphic to the ovoid of $Q(4,q^n)$ arising from the semifield flock which corresponds with the good egg $\mathcal{E}$.*

Proof : Suppose the good egg $\mathcal{E}$ of $\mathrm{PG}(4n-1,q)$ is contained in the hyperplane with equation $X_{4n} = 0$ as before. We construct the TGQ $T(\mathcal{E})$ in $\mathrm{PG}(4n,q)$. Let $\mathcal{O}$

be the pseudo-oval $\{E(a,0) \| \ a \in F\} \cup \{E(\infty)\}$, let ρ denote the $(3n-1)$-space $\{\langle r,s,t,0,0\rangle \| \ (r,s,t) \in F^3 \setminus \{0\}\}$ and $\mathcal{G}$ the $3n$-space $\{\langle r,s,t,0,x_{4n}\rangle \| \ r,s,t \in F, \ x_{4n} \in \mathrm{GF}(q), (r,s,t,x_{4n}) \neq 0\}$. We construct $T(\mathcal{O})$ in $\mathcal{G}$. We see $T(\mathcal{O})$ as a subGQ of $T(\mathcal{E})$, i.e., we identify the points $\langle T_E(a,0) \cap \rho, x\rangle$ of type (ii) of $T(\mathcal{O})$ with the points $\langle T_E(a,0), x\rangle$ of type (ii) of $T(\mathcal{E})$. Since $T(\mathcal{E})$ has order (q^n, q^{2n}) and $T(\mathcal{O})$ has order (q^n, q^n), the above method yields subtended ovoids of $T(\mathcal{O})$.

First we consider the ovoids subtended from a point of type (ii) of $T(\mathcal{E})$. The obtained ovoids are translation ovoids determined by a $(2n-1)$-space which is skew to the elements of the pseudo-oval $\mathcal{O}$. Let $Q = \langle T_E(a,b), \langle x_0, \ldots, x_{4n-1}, 1\rangle\rangle$ be a point of type (ii) of $T(\mathcal{E})$ not contained in $T(\mathcal{O})$. It follows that $b \neq 0$. We may assume that $a = 0$, since there is a collineation of $T(\mathcal{E})$ mapping $E(a,b)$ to $E((a,b)+d(1,0))$ for all $d \in F$, and fixing $T(\mathcal{O})$, see [10, 5.3] and Remark 2.2. Then

$$T_E(0,b) \cap \rho = \{\langle \sum_{i=0}^{n-1} (b_i bc + c_i b^2 t)^{1/q^i}, t, c, 0\rangle \| \ (t,c) \in F^2 \setminus \{0\}\}$$

is a $(2n-1)$-space skew to the classical pseudo-oval arising from the conic of $\mathrm{PG}(2,q^n)$ with equation $X_0X_1 + X_2^2 = 0$. From the previous section it follows that the semifield flock corresponding with this ovoid is $\mathcal{F}(\tilde{f}, \tilde{g})$, with

$$\tilde{f}(t) = b^2 \sum_{i=0}^{n-1} c_i t^{q^i}, \text{ and } \tilde{g}(t) = b \sum_{i=0}^{n-1} b_i t^{q^i}.$$

In $\mathrm{PG}(3,q^n)$ we can apply a coordinate transformation fixing the cone $\mathcal{K}$ such that the planes of the flock $\mathcal{F}(\tilde{f}, \tilde{g})$ are mapped onto the planes of the flock $\mathcal{F}(f,g)$. The matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & d^2 & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

induces such a coordinate transformation. It follows that the subtended ovoids obtained by subtending from points of type (ii) are isomorphic to the translation ovoid $\mathcal{O}(\mathcal{F})$.

Next we consider the ovoids subtended from a point Q of type (i) of $T(\mathcal{E})$. We may assume that $Q = \langle 0,0,0,d,1\rangle$, with $d \in F^*$, since we can apply a translation fixing $T(\mathcal{O})$ if necessary, see [10, 5.3] and Remark 2.2. Points of type (i) of $T(\mathcal{O})$ collinear with Q are $\langle E(a,b), Q\rangle \cap \mathcal{G}$, with $b \neq 0$. We get the following $q^{2n} - q^n$ points

$$\langle -g_t(a,b), t, -at, 0, 1\rangle, \ a \in F, \ b \in F^*, \ t = \frac{d}{b}$$

of type (i). Points of type (ii) collinear with Q are $\langle T_E(a,0) \cap \rho, Q\rangle$, $a \in F$ and $\langle T_E(\infty) \cap \rho, Q\rangle$. We want to use the isomorphism ψ given earlier between $Q(4,q^n)$ and $T(\mathcal{O})$. First we remark that $\langle T_E(a,0) \cap \rho, Q\rangle = \langle T_E(a,0) \cap \rho, \langle h((a,0),(0,-d)), 0,0,0,1\rangle\rangle$, and $\langle T_E(\infty) \cap \rho, Q\rangle = \langle T_E(\infty) \cap \rho, \langle 0,0,0,0,1\rangle\rangle$. So applying ψ^{-1} we obtain the q^n points

$$\langle a^2, 1, a, 0, -\sum_{i=0}^{n-1}(b_i ad)^{1/q^i}\rangle,\ a \in F,$$

and the point $\langle 1,0,0,0,0\rangle$ of $Q(4,q^n)$. Applying ψ^{-1} to the $q^{2n}-q^n$ points of type (i) we obtain the points

$$\begin{aligned}&\langle g_{\frac{d}{b}}(a,b), \frac{a}{b}, \frac{ad}{b}, 1, (\frac{ad}{b})^2 - \frac{d}{b} g_{\frac{d}{b}}(a,b)\rangle \\ = \quad & \langle a^2 + \frac{b}{d}\sum_{i=0}^{n-1}(b_i ad + c_i bd)^{1/q^i}, 1, a, \frac{b}{d}, -\sum_{i=0}^{n-1}(b_i ad + c_i bd)^{1/q^i}\rangle\end{aligned}$$

for $a \in F$, and $b \in F^*$ of $Q(4,q^n)$. So the ovoid can be written as the set of points of $\mathrm{PG}(4,q^n)$

$$\left\{\langle a^2 + b\sum_{i=0}^{n-1}(b_i da + c_i d^2 b)^{1/q^i}, 1, a, b, -\sum_{i=0}^{n-1}(b_i da + c_i d^2 b)^{1/q^i}\rangle \parallel a, b \in F\right\}$$
$$\cup\{\langle 1,0,0,0,0\rangle\}.$$

It follows that the subtended ovoid of $Q(4,q^n)$ is the translation ovoid corresponding with the semifield flock determined by the functions

$$\tilde{f}(t) = d^2\sum_{i=0}^{n-1} c_i t^{q^i},\ \text{and}\ \tilde{g}(t) = d\sum_{i=0}^{n-1} b_i t^{q^i}.$$

From the previous section together with the above it follows that the ovoid is a translation ovoid and the corresponding semifield flock is isomorphic to the semifield flock $\mathcal{F}(f,g)$ we started with.

So for every $d \in F^*$ we obtain an ovoid of $Q(4,q^n)$, by subtending from a point $\langle 0,0,0,d,1\rangle$ of type (i). Also for every $b \in F^*$ we obtained an ovoid by subtending from a point $\langle T_E(0,b), \langle 0,0,0,0,1\rangle\rangle$ of type (ii), and in the above we have shown that all these ovoids are isomorphic translation ovoids of $Q(4,q^n)$.

We have now proved the theorem for one particular subGQ determined by the elements $E(\infty), E(0,0), E(1,0)$ (by choosing the $3n$-space in which we construct the subGQ). By considering the translation group of the TGQ it is clear the same holds for every such subGQ (constructed in a $3n$-space intersecting $\mathrm{PG}(4n-1,q)$ in $\langle E(\infty), E(0,0), E(1,0)\rangle$). □

Remark Note that it now follows easily that the above holds for at least $2q^n$ subGQ's, since we could have chosen the egg element $E(0,1)$ instead of $E(1,0)$, and we could have chosen another $3n$-dimensional subspace containing the pseudo-oval determined by the three egg elements.

Next we will show that the ovoids subtended from points of type (ii) in all the subGQs induced by the good element are equivalent. They all arise from the same semifield flock $\mathcal{F}(f,g)$.

Theorem 4.2 *Let $\mathcal{E}$ be a good egg of* $\mathrm{PG}(4n-1,q)$*, q odd, represented as above. Then all the ovoids of a subquadrangle S, determined by a pseudo-oval on $E(\infty)$ contained in $\mathcal{E}$, obtained by subtending from points of type (ii) of $T(\mathcal{E}) \setminus S$ are isomorphic translation ovoids of $Q(4,q^n)$. Moreover, these ovoids are isomorphic to the ovoid of $Q(4,q^n)$ arising from the semifield flock which corresponds with the good egg $\mathcal{E}$.*

Proof : Consider the egg $\mathcal{E}$ of $\mathrm{PG}(4n-1,q)$ from above and let $\rho_{a,b}$ be the $(3n-1)$-space spanned by the elements $E(\infty), E(0,0)$ and $E(a,b)$ and put $\rho = \rho_{1,0}$. We will construct the $(2n-1)$-space $T_E(c,d) \cap \rho_{a,b}$, where $E(c,d)$ is not contained in $\rho_{a,b}$, i.e., (c,d) is not a multiple of (a,b), or $ad - bc \neq 0$. This condition implies that the matrix

$$\begin{bmatrix} a^2 & c^2 & 2ac & 0 \\ b^2 & d^2 & 2bd & 0 \\ ab & cd & ad+bc & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

induces a collineation of $\mathrm{PG}(3,q^n)$ fixing the cone $\mathcal{K}$. Applying to the planes of the flock we get the planes with equations

$$(a^2t + b^2f(t) + abg(t))\ X_0 + \ (c^2t + d^2f(t) + cdg(t))\ X_1$$

$$+\ (2act + 2bdf(t) + (ad+bc)g(t))\ X_2 + \ X_3 = 0,\ t \in F.$$

In the dual flock model we get the $(n-1)$-space (over $\mathrm{GF}(q)$)

$$\{\langle a^2t + b^2f(t) + abg(t), c^2t + d^2f(t) + cdg(t),$$

$$2act + 2bdf(t) + (ad+bc)g(t), 0\rangle \parallel t \in F^*\}$$

which is skew to the tangent spaces of the pseudo-oval corresponding with the conic $\mathcal{C}$ with equation $4X_0X_1 - X_2^2 = 0$ in the plane with equation $X_3 = 0$. Let A be the bijection mapping $t \mapsto a^2t + b^2f(t) + abg(t)$ (this is a bijection since the functions f and g induce a flock), and let ρ be the $(3n-1)$-space corresponding with the plane with equation $X_3 = 0$. Applying the collineation induced by the matrix

$$\begin{bmatrix} A^{-1} & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_n \end{bmatrix},$$

to ρ the pseudo-oval corresponding with the conic $\mathcal{C}$ is mapped onto the pseudo-oval with elements $\{\{\langle A^{-1}t, r^2t, 2rt, 0\ \rangle \parallel t \in F^*\} \parallel r \in F\} \cup \{\langle 0,t,0,0\rangle \parallel t \in F^*\}$ or rewriting the coordinates we obtain the pseudo-oval with elements

$$\{\{\langle t, r^2At, 2rAt, 0\ \rangle \parallel t \in F^*\} \parallel r \in F\} \cup \{\langle 0,t,0,0\rangle \parallel t \in F^*\}.$$

The $(n-1)$-space skew to the tangent spaces of this pseudo-oval becomes the $(n-1)$-space

$$\{\langle t, c^2t + d^2f(t) + cdg(t), 2act + 2bdf(t) + (ad+cb)g(t), 0\rangle \parallel t \in F^*\}.$$

Now we dualise with respect to the inproduct

$$((x, y, z), (u, v, w)) = \mathrm{tr}(xu + yv + zw),$$

where tr is the trace map from $F \to \mathrm{GF}(q)$. The dual space of the pseudo-oval element $\{\langle t, r^2At, 2rAt, 0\ \rangle \parallel t \in F^*\}$ becomes

$$\{\langle -\sum_{i=0}^{n-1} \left[(a_i a^2 + b_i ab + c_i b^2)(r^2 v + 2rw)\right]^{1/q^i}, v, w, 0\rangle \parallel (v, w) \in F^2 \setminus \{0\}\},$$

where we introduced $(a_0, \ldots, a_{n-1}) = (1, 0, \ldots, 0)$ for convenience of notation. The dual of the $(n-1)$-space skew to the tangent spaces of the pseudo-oval becomes

$$\{\langle -\sum_{i=0}^{n-1} [(2a_i ac + b_i(ad + cb) + 2c_i bd)w$$

$$+(a_i c^2 + b_i cd + c_i d^2)v\big]^{1/q^i}, v, w, 0\rangle \parallel (v, w) \in F^2 \setminus \{0\}\}$$

skew to the elements of the new pseudo-oval in ρ. Now we apply the coordinate transformation mapping ρ to the $(3n-1)$-space $\rho_{a,b} = \{\langle r, s, at, bt\rangle \parallel (r, s, t) \in F^3 \setminus \{0\}\}$. This transformation maps the tangent space

$$\{\langle -\sum_{i=0}^{n-1} \left[(a_i a^2 + b_i ab + c_i b^2)(r^2 v + 2rw)\right]^{1/q^i}, v, w, 0\rangle \parallel (v, w) \in F^2 \setminus \{0\}\}.$$

of the pseudo-oval to the space

$$\{\langle \sum_{i=0}^{n-1} \left[(a_i a^2 + b_i ab + c_i b^2)(r^2 v + 2rw)\right]^{1/q^i}, v, wa, wb\rangle \parallel (v, w) \in F^2 \setminus \{0\}\},$$

which is the tangent space $T_E(ra, rb) \cap \rho_{a,b}$ of the pseudo-oval in $\rho_{a,b}$ at the element $E(ra, rb)$. (Note that we applied an extra coordinate transformation $X_0 \mapsto -X_0$ to get rid of the minus sign in the first coordinate.) The $(2n-1)$-space skew to the pseudo-oval in ρ is mapped to the $(2n-1)$-space $T_E(c, d) \cap \rho_{a,b}$, i.e., the $(2n-1)$-space which induces the translation ovoid subtended from a point of type (ii) on the tangent space $T_E(c, d)$ of the egg $\mathcal{E}$. Since the elements of a pseudo-oval are determined by the tangent spaces it follows that the obtained pseudo-oval is the one determined by the elements $E(\infty)$, $E(0, 0)$ and $E(a, b)$ of the good egg corresponding with the semifield flock. We have shown that all the ovoids of $Q(4, q^n)$ obtained by subtending from points of type (ii) in the subquadrangles induced by the egg elements $E(\infty), E(0, 0)$ and one other egg element are isomorphic to the ovoid arising from the semifield flock corresponding with the good egg. Since the stabilizer of $E(\infty)$ in the automorphism group of the egg $\mathcal{E}$ acts transitively on the elements of $\mathcal{E} \setminus \{E(\infty)\}$ (see Remark 2.2), we have proved the theorem. $\square$

Remark 4.3 After the writing of this paper was completed, in [11] it was shown that there exists a symmetry about a line on the point (∞) (of the form calculated in

[4, 5.4]), from which it then follows that every point of the line $E(\infty)$ is a translation point of $T(\mathcal{E})$, and hence that for a fixed subGQ S and a fixed point of type (i) z of $T(\mathcal{E}) \setminus S$, there exists an automorphism of $T(\mathcal{E})$ fixing S and mapping the point z to a point of type (ii). This implies that the ovoid of S subtended from z is isomophic to an ovoid subtended from a point of type (ii). In this way the solution of the isormophism problem was completed in [11].

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