Planar Arcs

Simeon Ball

joint work with Michel Lavrauw

12 September 2017

An *arc* S in $\mathrm{PG}(k-1,q)$ is a set of points with the property that every subset of S of size k spans the whole space.

A *planar arc* S is an arc of $\mathrm{PG}(2,q)$.

An *arc* S in $\mathrm{PG}(k-1,q)$ is a set of points with the property that every subset of S of size k spans the whole space.

A *planar arc* S is an arc of $\mathrm{PG}(2,q)$.

A conic is an example of a planar arc of size $q+1$.

An *arc* S in $\mathrm{PG}(k-1,q)$ is a set of points with the property that every subset of S of size k spans the whole space.

A *planar arc* S is an arc of $\mathrm{PG}(2,q)$.

A conic is an example of a planar arc of size $q+1$.

[Segre 1955] If q is odd then a planar arc of size $q+1$ is a conic.

An *arc* S in $\mathrm{PG}(k-1,q)$ is a set of points with the property that every subset of S of size k spans the whole space.

A *planar arc* S is an arc of $\mathrm{PG}(2,q)$.

A conic is an example of a planar arc of size $q+1$.

[Segre 1955] If q is odd then a planar arc of size $q+1$ is a conic.

[Qvist 1952] If q is even then a planar arc of size $q+1$ is extendable to an arc of size $q+2$.

Let S be a planar arc of size $q+1$.

There is a symmetric bilinear form $F(X,Y)$ with the property that $F(X,y)$ is the tangent to S at y, for all $y \in S$.

Let S be a planar arc of size $q+1$.

There is a symmetric bilinear form $F(X,Y)$ with the property that $F(X,y)$ is the tangent to S at y, for all $y \in S$.

We can deduce both [Segre 1955] and [Qvist 1952] from this.

Let S be a planar arc of size $q+1$.

There is a symmetric bilinear form $F(X,Y)$ with the property that $F(X,y)$ is the tangent to S at y, for all $y \in S$.

We can deduce both [Segre 1955] and [Qvist 1952] from this.

If q is odd then $F(X,X) \neq 0$ and so S is a conic.

Let S be a planar arc of size $q+1$.

There is a symmetric bilinear form $F(X,Y)$ with the property that $F(X,y)$ is the tangent to S at y, for all $y \in S$.

We can deduce both [Segre 1955] and [Qvist 1952] from this.

If q is odd then $F(X,X) \neq 0$ and so S is a conic.

If q is even then $F(X,Y)$ is alternating, so it is $\det(X,Y,u)$ for some u, which is the nucleus of S.

Let S be a planar arc of size $q+2-t$.

Let S be a planar arc of size $q+2-t$.

[Theorem 1]

Modulo the subspace of polynomials of degree t that are zero on S,

there is a skew-symmetric polynomial $F(X,Y)$, homogeneous of degree t in both X and Y with the property that

$F(X,y)$ is the product of the tangents to S at y, for all $y \in S$.

The polynomial $F(X,Y)$ is unique.

[Theorem 2] Let S be a planar arc of size $q+2-t$ not contained in a conic.

If q is odd then S is contained in the intersection of two curves, which do not have a common component, and have degree at most $t+p^{\lfloor \log_p t \rfloor}$.

[Theorem 2] Let S be a planar arc of size $q+2-t$ not contained in a conic.

If q is odd then S is contained in the intersection of two curves, which do not have a common component, and have degree at most $t + p^{\lfloor \log_p t \rfloor}$.

This has the following corollary.

[Theorem 3] If q is odd then this implies that an arc of size at least $q - \sqrt{q} + 3 + \max\{\sqrt{q}/p, \frac{1}{2}\}$ is contained in a conic.

[Theorem 2] Let S be a planar arc of size $q+2-t$ not contained in a conic.

If q is odd then S is contained in the intersection of two curves, which do not have a common component, and have degree at most $t + p^{\lfloor \log_p t \rfloor}$.

This has the following corollary.

[Theorem 3] If q is odd then this implies that an arc of size at least $q - \sqrt{q} + 3 + \max\{\sqrt{q}/p, \frac{1}{2}\}$ is contained in a conic.

[Kestenband 1981] If q is square then there are examples of planar arcs, not contained in a conic, of size $q - \sqrt{q} + 1$.

Arcs contained in curves

Let S be a planar arc of size $q+2-t$ contained in a curve of degree d and not contained in a conic.

If q is odd then $(t+p^{\lfloor \log_p t \rfloor})d+t \geqslant q+2$.

Let S be a planar arc of size $q+2-t$ contained in a curve of degree d and not contained in a conic.

If q is odd then $(t+p^{\lfloor \log_p t \rfloor})d+t \geqslant q+2$.

If q is prime then $|S| \leqslant d(q+3)/(d+1)$.

Let S be a planar arc of size $q+2-t$ contained in a curve of degree d and not contained in a conic.

If q is odd then $(t+p^{\lfloor \log_p t \rfloor})d+t \geqslant q+2$.

If q is prime then $|S| \leqslant d(q+3)/(d+1)$.

There is an arc of size 24 in $\mathrm{PG}_2(\mathbb{F}_{29})$ which is a curve of degree 4.

Classification of very large planar arcs

$|S| = q + 2$.

[Bose 1947] they do not exist for q odd. [Caullery and Schmidt 2015] classify those with o-polynomial of degree less than $\frac{1}{2}q^{\frac{1}{4}}$.

$|S| = q + 2$.

[Bose 1947] they do not exist for q odd. [Caullery and Schmidt 2015] classify those with o-polynomial of degree less than $\frac{1}{2}q^{\frac{1}{4}}$.

$|S| = q + 1$.

[Segre 1955] is a conic for q odd, incomplete for q even.

$|S| = q + 2$.

[Bose 1947] they do not exist for q odd. [Caullery and Schmidt 2015] classify those with o-polynomial of degree less than $\frac{1}{2}q^{\frac{1}{4}}$.

$|S| = q + 1$.

[Segre 1955] is a conic for q odd, incomplete for q even.

$|S| = q$.

incomplete for q odd [Segre 1955], incomplete for q even [Tallini 1957].

$|S| = q + 2$.

[Bose 1947] they do not exist for q odd. [Caullery and Schmidt 2015] classify those with o-polynomial of degree less than $\frac{1}{2}q^{\frac{1}{4}}$.

$|S| = q + 1$.

[Segre 1955] is a conic for q odd, incomplete for q even.

$|S| = q$.

incomplete for q odd [Segre 1955], incomplete for q even [Tallini 1957].

$|S| = q - 1$.

Complete planar arcs occur for only $q = 7, 9, 11$ and 13.

$|S| = q - 2$.

Complete planar arcs occur for only $q = 8, 9$ and 11.

$|S| = q - 2$.

Complete planar arcs occur for only $q = 8, 9$ and 11.

$|S| = q - 3$.

The only examples of complete planar arcs of size $q - 3$ are for $q = 9, 13, 16$ and 17 and possibly $q = 37$.

If there is an arc of size 34 in $\mathrm{PG}_2(\mathbb{F}_{37})$ then it is contained in the intersection of two sextic curves, which do not share a common component.

Classifications use [Theorem 3] and [Coolsaet 2013], [C-S 2009] and [Coolsaet-Sticker 2011] and other computational results.

Segre's lemma of tangents

For each point $a \in S$ let

$$f_a(X) = \prod_{i=1}^{t} \alpha_i(X),$$

where $\alpha_i(X)$ are the tangents to S at a.

For each point $a \in S$ let

$$f_a(X) = \prod_{i=1}^{t} \alpha_i(X),$$

where $\alpha_i(X)$ are the tangents to S at a.

Fix a point $e \in S$ and scale $f_a(X)$ so that

$$f_a(e) = (-1)^{t+1} f_e(a).$$

For each point $a \in S$ let

$$f_a(X) = \prod_{i=1}^{t} \alpha_i(X),$$

where $\alpha_i(X)$ are the tangents to S at a.

Fix a point $e \in S$ and scale $f_a(X)$ so that

$$f_a(e) = (-1)^{t+1} f_e(a).$$

Segre's lemma of tangents implies that for all $x, y \in S$

$$f_x(y) = (-1)^{t+1} f_y(x).$$

For all $x, a \in S$

$$f_a(x) = (-1)^{t+1} f_x(a).$$

For all $x, a \in S$

$$f_a(x) = (-1)^{t+1} f_x(a).$$

Let λ be a vector of $\mathbb{F}_q^{|S|}$ which is orthogonal to the vectors which are the evaluations of polynomials of degree t at the points of S.

Then, by Segre's lemma of tangents, for all $x \in S$

$$\sum_{a \in S} \lambda_a f_a(x) = 0.$$

For all $x, a \in S$

$$f_a(x) = (-1)^{t+1} f_x(a).$$

Let λ be a vector of $\mathbb{F}_q^{|S|}$ which is orthogonal to the vectors which are the evaluations of polynomials of degree t at the points of S.

Then, by Segre's lemma of tangents, for all $x \in S$

$$\sum_{a \in S} \lambda_a f_a(x) = 0.$$

The polynomial

$$\sum_{a \in S} \lambda_a f_a(X)$$

is either zero or it is a polynomial of degree t which is zero on S.

For all $a \in S$, $f_a(X) = f_a(X + a)$.

For all $a \in S$, $f_a(X) = f_a(X + a)$.

We prove that there is a polynomial $F(X, Y)$ such that

$$F(X, a) = f_a(X).$$

modulo the subspace of polynomials of degree t that are zero on S.

For all $a \in S$, $f_a(X) = f_a(X + a)$.

We prove that there is a polynomial $F(X, Y)$ such that

$$F(X, a) = f_a(X).$$

modulo the subspace of polynomials of degree t that are zero on S.

The non-zero coefficients of

$$F(X + Y, Y) - F(X, Y)$$

are polynomials in Y of degree between $t + 1$ and $2t$ which are zero on S.

Final Comments

It is possible that the following statement is true.

It is possible that the following statement is true.

Let S be a planar arc of size $q+2-t$ not contained in an arc of maximum size.

Then S is contained in the intersection of two curves, which do not have a common component, and have degree at most $t+p^{\lfloor \log_p t \rfloor}$.

The smallest possible counterexample would be in $\mathrm{PG}(2,128)$.

It is even possible that the following stronger statement is true.

It is even possible that the following stronger statement is true.

Let S be a planar arc of size $q+2-t$ not contained in an arc of maximum size.

Then S is contained in the intersection of two curves, which do not have a common component, and have degree at most $t+1$.

The smallest possible counterexample would be in $\mathrm{PG}(2,49)$.

It would be good to have more constructions of complete arcs of size larger than $(5q+15)/7$.

Below this bound it is trivial that an arc is covered by conics and lines in two disjoint ways.

As far as I am aware the only known examples are

the hyperovals, the conic, the Kestenband examples and two small examples, the 12-arc in $\mathrm{PG}(2,13)$ and the 24-arc in $\mathrm{PG}(2,29)$.