$\mathbb{F}_{p^2}$-maximal curves with many automorphisms are Galois-covered by the Hermitian curve

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Outline

1. Maximal curves

2. $\mathbb{F}_{p^2}$-maximal curves with many automorphisms: main result

3. The Fricke-MacBeath curve over finite fields and the $\mathbb{F}_{11^2}$-maximal Wiman's sextic

4. Related questions

Notation and terminology

- $\mathcal{X} \subseteq \mathbb{P}^r(\bar{\mathbb{F}}_q)$ projective, geometrically irreducible, non-singular algebraic curve defined over $\mathbb{F}_q$

- g genus of $\mathcal{X}$

  If $r = 2$ then $g = \frac{(d-1)(d-2)}{2}$ where $d = deg(\mathcal{X})$

- $\mathcal{X}(\mathbb{F}_q) = \mathcal{X} \cap \mathbb{P}^r(\mathbb{F}_q)$

Maximal Curves

$\mathcal{X}$ defined over $\mathbb{F}_q$

Hasse-Weil Bound

$|\mathcal{X}(\mathbb{F}_q)| \leq q + 1 + 2g\sqrt{q}$

Definition

$\mathcal{X}$ is $\mathbb{F}_q$-maximal if $|\mathcal{X}(\mathbb{F}_q)| = q + 1 + 2g\sqrt{q}$

Example

Hermitian curve:

$$\mathcal{H}_q : X^q + X = Y^{q+1}, \quad q = p^h$$

$$g = q(q-1)/2, \quad Aut(\mathcal{H}_q) \cong PGU(3, q), \quad |\mathcal{H}_q(\mathbb{F}_{q^2})| = q^3 + 1$$

Rational maps and pull-back

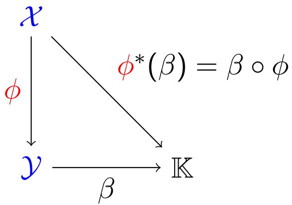
$\mathcal{X} \subseteq \mathbb{P}^r(\mathbb{K})$ and $\mathcal{Y} \subseteq \mathbb{P}^s(\mathbb{K})$

- Function field of $\mathcal{X}$

$$\mathbb{K}(\mathcal{X}) = \left\{ \frac{F + \mathbb{I}}{G + \mathbb{I}} \;\middle|\; G \notin \mathbb{I} = \mathbb{I}(\mathcal{X}) \right\}$$

- Rational map $\phi : \mathcal{X} \to \mathcal{Y}$: is a map given by rational functions

$$\phi = (\alpha_0 : \ldots : \alpha_s), \; \textit{for almost all } P \in \mathcal{X}$$



Coverings and Galois-coverings

$\mathcal{Y}$ is covered by $\mathcal{X}$ if there exists a non-constant rational map

$$\phi : \mathcal{X} \to \mathcal{Y}$$

$\mathbb{K}(\mathcal{X}) : \phi^*(\mathbb{K}(\mathcal{Y}))$ is a finite field extension

$\mathcal{Y}$ is a Galois-covered by $\mathcal{X}$ $\iff$ $\mathbb{K}(\mathcal{X}) : \phi^*(\mathbb{K}(\mathcal{Y}))$ Galois extension

Basic properties of coverings

$\mathcal{X}$ $\mathbb{F}_q$-maximal
$\phi : \mathcal{X} \to \mathcal{Y}$ non-constant rational map defined over $\mathbb{F}_q$ $\Longrightarrow$ $\mathcal{Y}$ is $\mathbb{F}_q$-maximal

(Serre, Kleiman)

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- $\mathcal{C}$ non-singular algebraic curve
- G finite automorphism group acting on $\mathcal{C}$
- $\mathcal{X}$ quotient curve of $\mathcal{C}$ by G

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Riemann-Hurwitz Formula

$$2g(\mathcal{C}) - 2 = |G|(2g(\mathcal{X}) - 2) + D$$

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Situation up to $p = 5$

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- $p = 2, 3$: trivial (Every $\mathbb{F}_{p^2}$-maximal curve is Galois-covered by $\mathcal{H}_p$)
- $p = 5$: Every $\mathbb{F}_{25}$-maximal curve is Galois-covered by $\mathcal{H}_5$

$$M(5) := \{g(\mathcal{X}) \mid \mathcal{X} \text{ is } \mathbb{F}_{25}\text{-maximal}\} = \{0, 1, 2, 3, 4, 10\}$$

1. $g(\mathcal{X}) = 10 \iff \mathcal{X} \cong \mathcal{H}_5 : y^6 = x^5 + x$
2. $g(\mathcal{X}) = 4 \iff \mathcal{X} \cong \mathcal{Y}_4 : y^3 = x^5 + x$
3. $g(\mathcal{X}) = 3 \iff \mathcal{X} \cong \mathcal{Y}_3 : y^6 = x^5 + 2x^4 + 3x^3 + 4x^2 + 3xy^3$
4. $g(\mathcal{X}) = 2 \iff \mathcal{X} \cong \mathcal{Y}_2 : y^2 = x^5 + x$
5. $g(\mathcal{X}) = 1 \iff \mathcal{X} \cong \mathcal{Y}_1 : x^3 + y^3 + 1 = 0$

Main Result: a partial answer

Almost all the known examples of maximal curves not Galois-covered by $\mathcal{H}_q$ have large automorphism group

$$\Downarrow$$

We investigated curves with $|Aut(\mathcal{X})| > 84(g(\mathcal{X}) - 1)$

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Theorem (B.-Montanucci-Torres, 2017)

$\mathcal{X}$ $\mathbb{F}_{p^2}$-maximal curve, $p \geq 7$, $g(\mathcal{X}) \geq 2$,

$$|\mathrm{Aut}(\mathcal{X})| > 84(g(\mathcal{X}) - 1) \Longrightarrow \mathcal{X} \text{ Galois-covered by } \mathcal{H}_p$$

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- (B. Gunby, A. Smith, A. Yuan, 2015): $\mathcal{X}$ defined over $\mathbb{F}_{p^2}$, $p \geq 7$, $g(\mathcal{X}) \geq 2$, $|Aut(\mathcal{X})| \geq \max\{84(g-1), g^2\}$

$$\Longrightarrow \mathcal{X} \cong \mathcal{X}_m \ : \ y^m = x^p - x$$

($\mathcal{X}_m$ is not $\mathbb{F}_{p^2}$-maximal for each $m \ldots$)

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- An $\mathbb{F}_{p^2}$-maximal "GK-curve" cannot exist

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- An $\mathbb{F}_{p^2}$-maximal "GK-curve" cannot exist
- Find $\mathbb{F}_{p^2}$-maximal curves not Galois-covered by the Hermitian curve
- Can we extend it to $|\mathrm{Aut}(\mathcal{X})| \leq 84(g - 1)$?

Sketch of the proof

Lemma

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$\mathcal{X} \not\cong_{\mathbb{F}_{p^2}} \mathcal{H}_p$
$G \leq Aut(\mathcal{X})$, $p \mid |G|$

$\Longrightarrow$ $p^2 \nmid |G|$

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Theorem (Garcia-Tafazolian, 2008)

- $q = p^h$, $\mathcal{X}$ $\mathbb{F}_{q^2}$-maximal curve
- $\exists$ $H \leq \mathrm{Aut}(\mathcal{X})$ abelian, $|H| = q$, $\mathcal{X}/H$ rational

$$\exists m \mid q+1 \quad : \quad \mathcal{X} \cong_{\mathbb{F}_{q^2}} \mathcal{H}_m : x^q + x = y^m$$

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$\mathcal{H}_m$ is covered by $\mathcal{H}_q : x^q + x = y^{q+1}$:

$$\mathcal{H}_m \cong \mathcal{H}_p/G, \qquad G = \{\varphi_\lambda : (x,y) \mapsto (x, \lambda y) \mid \lambda^{(q+1)/m} = 1\}$$

How many automorphisms?

- $g(\mathcal{C}) \geq 2 \Longrightarrow |\mathrm{Aut}(\mathcal{C})| < \infty$

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  Example: Klein quartic: $\mathcal{K} : X^3 + Y + XY^3 = 0$,
  $g = 3$, $\mathrm{Aut}(\mathcal{K}) = PSL(2,7)$, $|\mathrm{Aut}(\mathcal{K})| = 168 = 84(3-1)$

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- $gcd(p, |\mathrm{Aut}(\mathcal{C})|) = 1 \Longrightarrow |\mathrm{Aut}(\mathcal{C})| \leq 84(g-1)$

What if $|Aut(\mathcal{X})| \leq 84(g-1)$?

- $p > 0$: No classifications of Hurwitz groups $|G| = 84(g-1)$
- Partial classification if $\mathcal{X}$ is classical (Schoeneberg's Lemma)

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Lemma (B.-Montanucci-Torres, 2017)

$p \geq 7$, $\mathcal{X}$ $\mathbb{F}_{p^2}$-maximal, $g(\mathcal{X}) \geq 2$
$40(g(\mathcal{X})-1) < |\mathrm{Aut}(\mathcal{X})| \leq 84(g(\mathcal{X})-1)$

Then $\mathcal{X}$ is Galois-covered by $\mathcal{H}_p$ unless $g(\mathcal{X}/\mathrm{Aut}(\mathcal{X})) = 0$ and

- $p \geq 11, \quad (|O_1|, |O_2|, |O_3|) = \left(\frac{|\mathrm{Aut}(\mathcal{X})|}{2}, \frac{|\mathrm{Aut}(\mathcal{X})|}{3}, \frac{|\mathrm{Aut}(\mathcal{X})|}{7}\right)$
- $(|O_1|, |O_2|, |O_3|) = \left(\frac{|\mathrm{Aut}(\mathcal{X})|}{2}, \frac{|\mathrm{Aut}(\mathcal{X})|}{3}, \frac{|\mathrm{Aut}(\mathcal{X})|}{8}\right)$

Search for $\mathbb{F}_{p^2}$-maximal curves not Galois-Covered: Hurwitz curves

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$$\mathcal{F} : 1 + 7xy + 21x^2y^2 + 35x^3y^3 + 28x^4y^4 + 2x^7 + 2y^7 = 0$$

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- (J. Top, C. Verschoor, 2016): Criterion to count the points of the Fricke-MacBeath curve over finite fields

The Fricke-MacBeath curve

Proposition (Top-Verschoor, 2016)

$p \equiv \pm 1 \pmod{14}$

$\mathcal{F}$ is $\mathbb{F}_{p^2}$-maximal $\iff$ $y^2 = (x^3 + x^2 - 114x - 127)$ is $\mathbb{F}_{p^2}$-maximal

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MAGMA: $\mathcal{F}$ is $\mathbb{F}_{p^2}$-maximal for $p \in \{71, 251, 503, 2591\}$ and $\mathrm{Aut}(\mathcal{X}) \cong PSL(2, 8)$

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Theorem (B.-Montanucci-Torres, 2017)

For $p = 71$ $\mathcal{F}$ is not a Galois subcover of $\mathcal{H}_{71}$

A natural question

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- Positive answer $\rightarrow$ First example of an $\mathbb{F}_{q^2}$-maximal curve which is covered but not Galois-covered by $\mathcal{H}_q$

Thank you for your attention