Classifying Cubic Surfaces over Small Finite Fields

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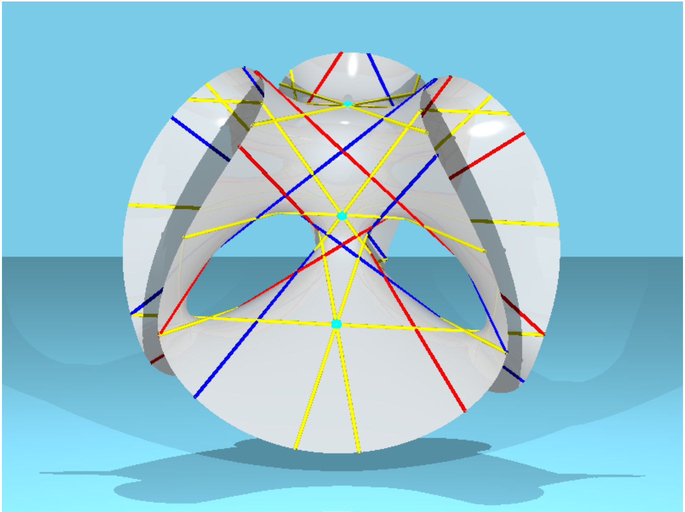
Cubic Surfaces with 27 Lines

Let $\mathbb{F}$ be a field.

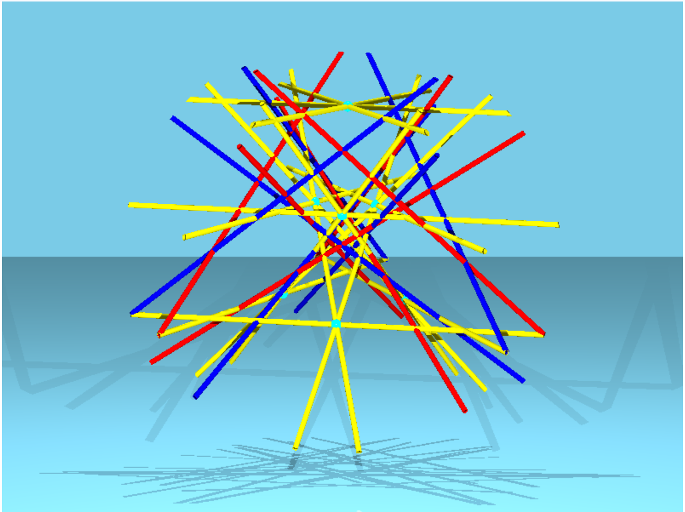
A cubic surface in $\mathrm{PG}(3, \mathbb{F})$ is defined by an irreducible homogeneous cubic polynomial in 4 variables.

Cayley (1849): A cubic surface has at most 27 lines.

Cubic Surfaces with 27 Lines



The Clebsch surface. Animation! (picture based on work of Alain Esculier).



The 27 lines (surface removed). Animation! We'll talk about the colors later. Why 27?

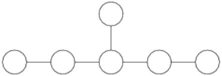
The Schläfli Graph

The intersection graph of the 27 lines on a cubic surface is the complement of the Schläfli graph.

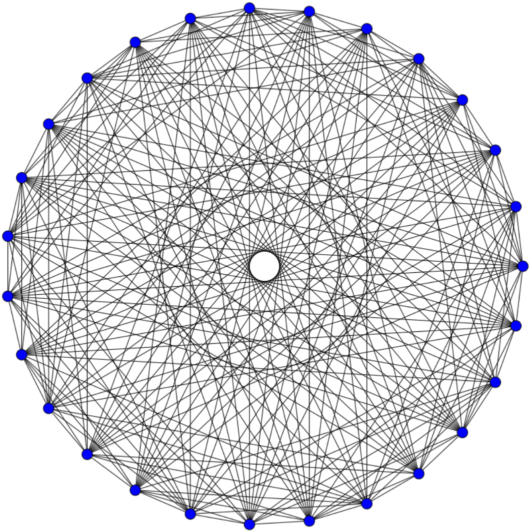
That is, two vertices are adjacent in the Schläfli graph if and only if the corresponding pair of lines are skew.

The Schläfli graph is a strongly regular graph with parameters srg$(27, 16, 10, 8)$.

The automorphism group is the Weyl group of type E_6 of order 51840.

Dynkin diagram: 

The Schläfli graph



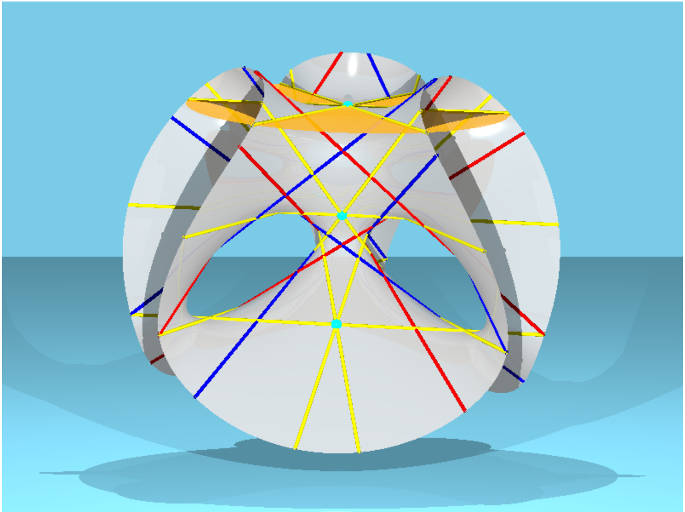
picture credit: Claudio Rocchini.

Tritangent Planes

A tritangent plane is a plane which intersects the surface in three lines.

Example:

Tritangent Planes



Animation!

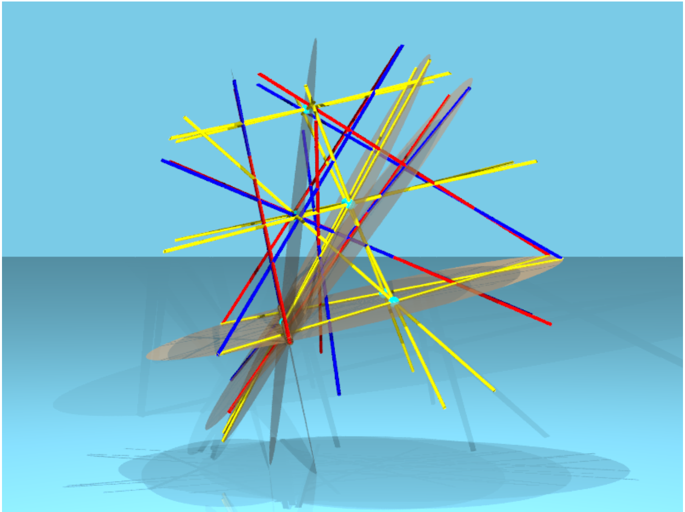
Why 27 lines?

There are 5 tritangent planes through each line.

So, by considering one tritangent plane, we count the number of lines

$$3 + (5 - 1) \times 3 \times 2 = 27.$$

Five tritangent planes through a line



Animation!

Five tritangent planes through a line

Information:

The incidence structure:

Points = lines of the surface

Lines = tritangent planes

is a Generalized Quadrangle of type GQ(4,2).

I would like to thank Jef Thas for pointing this out to me.

Classification

We are interested in classifying cubic surfaces with 27 lines up to projective equivalence over finite fields.

So, $\mathbb{F} = \mathbb{F}_q$ for the remainder of this talk (except for the pictures, which show the Clebsch surface over $\mathbb{R}$).

There are two approaches:

- Use double sixes (this talk)
- Use the blow-up construction of six points in the plane (Karaoglu)

Cubic Surfaces with 27 Lines

Question:

What is a double six?

A **Schlaefli double six** (or double six, for short) is a bijection between two sets of six pairwise disjoint lines

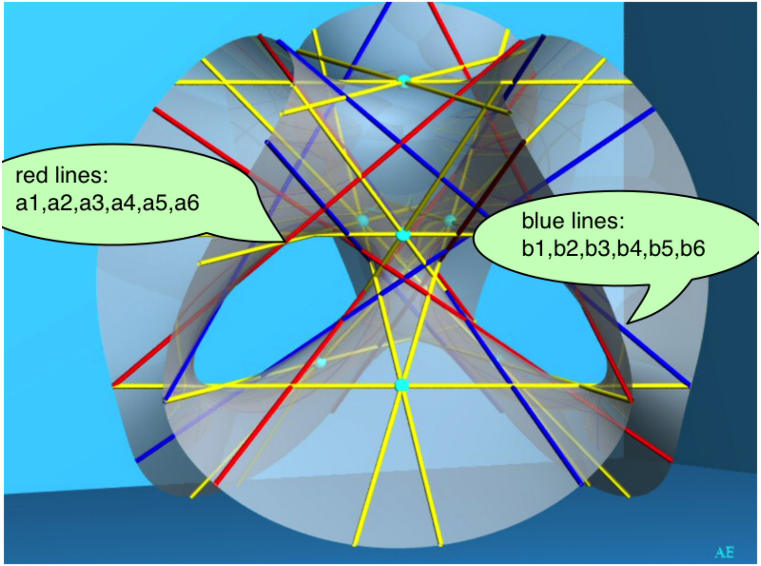
$$a_i \quad \leftrightarrow \quad b_i, \qquad i = 1, \ldots, 6$$

such that a_i intersects b_j iff $i \neq j$.

Often, the notation

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{bmatrix}$$

is used.

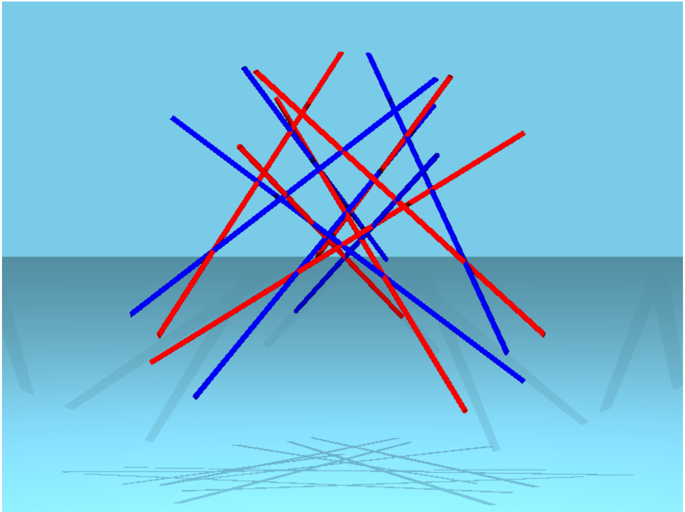

red lines:
a1,a2,a3,a4,a5,a6
blue lines:
b1,b2,b3,b4,b5,b6
AE

There is a 36 to 1 relation between double sixes and cubic surfaces with 27 lines:

A double six defines exactly one cubic surface with 27 lines.

A cubic surface with 27 lines has exactly 36 double sixes.

A Double Six



A double six Animation!

The c_{ij} lines

Fifteen further lines are defined using the formula

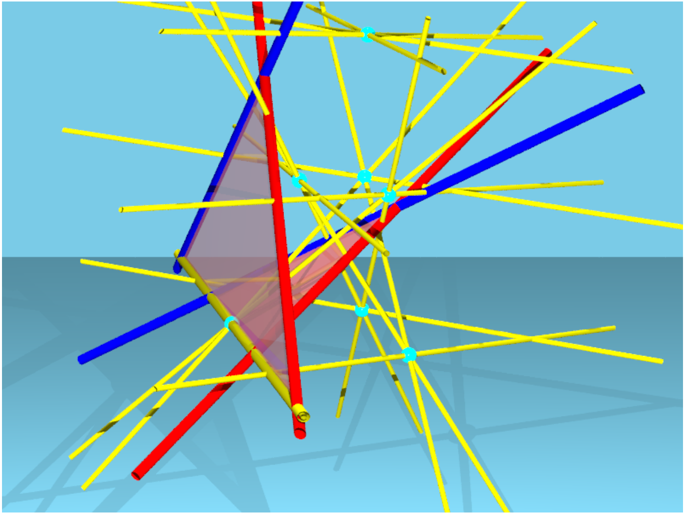
$$c_{ij} = a_i b_j \cap a_j b_i.$$

Here, $a_i b_j$ is the tritangent plane spanned by a_i and b_j.

Likewise, $a_j b_i$ is the tritangent plane spanned by a_j and b_i.

These fifteen lines c_{ij} are drawn in yellow.

The c_{ij} lines



The line $c_{14} = a_1 b_4 \cap a_4 b_1$ Animation!

Schläfli's Theorem

Schläfli 1858

Given five skew lines a_1, a_2, a_3, a_4, a_5 with a single transversal b_6 such that each set of four a_i omitting a_j ($j = 1, \ldots, 5$) has a unique further transversal b_j, then the five lines b_1, b_2, b_3, b_4, b_5 also have a transversal a_6.

A Schläfli double six determines a unique cubic surface with 27 lines.

Classification

We can classify cubic surfaces with 27 lines in $\mathrm{PG}(3, q)$ by classifying all double sixes in $\mathrm{PG}(3, q)$ up to equivalence.

Each double six leads to a unique surface with 27 lines.

However, each surface contains exactly 36 double sixes. Some of these may come from different double sixes.

Isomorphism classification:

The isomorphism types of double sixes involved with a surface correspond to the orbits of the stabilizer of a surface on the 36 double sixes of the surface.

Classification

The "$5 + 1$" configurations:

To classify the double sixes, we classify configurations of 5 pairwise skew lines with a common transversal in $\mathrm{PG}(3, q)$.

From these, we use Schläfli's theorem to create a unique double six (if the generality condition holds).

To classify configurations of five pairwise skew lines with a common transversal, we use the Klein correspondence:

Classification

Under the Klein correspondence, 5 pairwise skew lines with a common transversal become a partial ovoid of size 5 in the perp of a point.

It helps to observe that the group is transitive on lines of $\mathrm{PG}(3, q)$ and hence on points of the Klein quadric.

Classification

It remains to classify the partial ovoids of size 5 in the perp of a point.

To perform this classification, we use the software package Orbiter.

We test the generality condition after we have classified the "$5 + 1$" configurations.

Classification

Theorem

The number of isomorphism types of cubic surfaces with 27 lines in $\mathrm{PG}(3, q)$ are known for $q \leq 97$:

q	#
2	0
3	0
4	1
5	0
7	1
8	1
9	2
11	2
13	4

q	#
16	5
17	7
19	10
23	16
25	18
27	11
29	34
31	43
32	11

q	#
37	77
41	107
43	126
47	169
49	121
53	258
59	376
61	427
64	101

q	#
67	595
71	731
73	813
79	1081
81	331
83	1292
89	1673
97	2304

The numbers for $q \geq 16$ are new.

Classification

For $q = 97$, there are

$$1,338,525$$

orbits of $5 + 1$ configurations.

Of these,

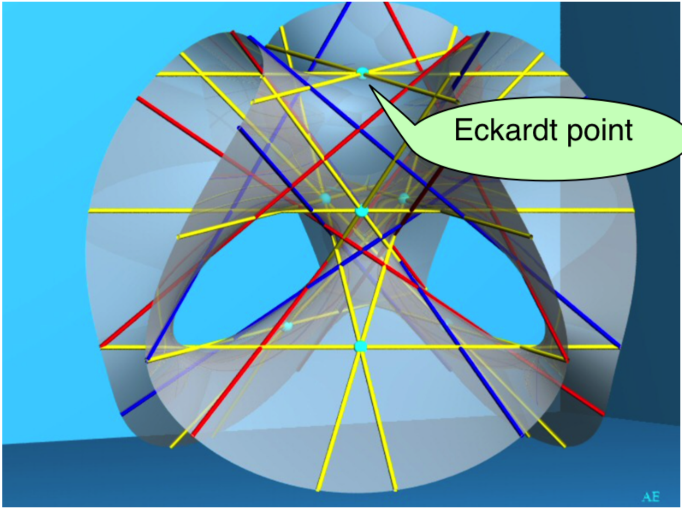
$$665,560$$

satisfy the generality conditions and hence correspond to double sixes in $\mathrm{PG}(3,97)$.

From these, the algorithm finds 2304 isomorphism classes of cubic surfaces with 27 lines.

The classification for $q = 97$ took 17 days on a single machine.

An **Eckardt point** of a surface $\mathcal{F}$ is a point P which lies on exactly three lines of the surface.



Let $\#E$ denote the number of Eckardt points of $\mathcal{F}$.

Known:

$$0 \leq \#E \leq 45.$$

Eckardt Points

Here is the classification of cubic surfaces with 27 lines in $\mathrm{PG}(3, q)$ by Eckardt points:

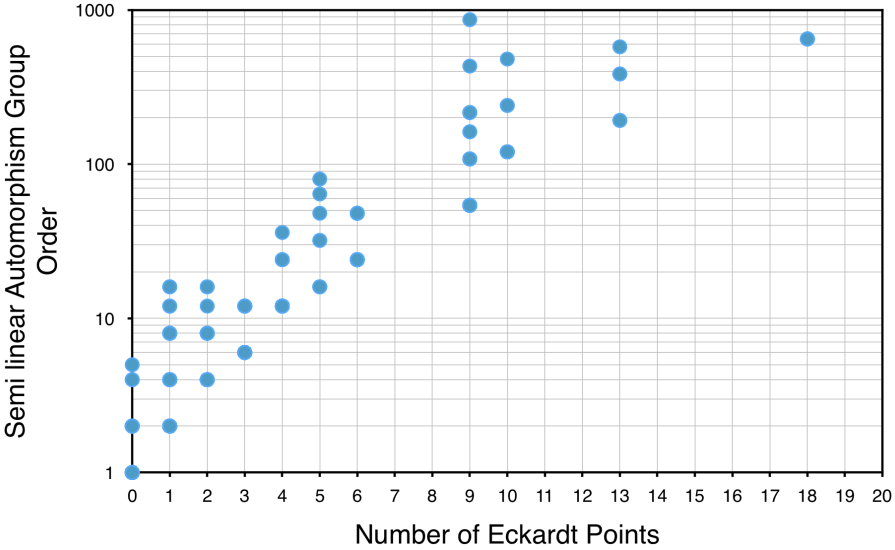
q	0	1	2	3	4	5	6	9	10	13	18	45
4	0	0	0	0	0	0	0	0	0	0	0	**1**
7	0	0	0	0	0	0	0	0	0	0	1	0
8	0	0	0	0	0	0	0	0	0	1	0	0
9	0	0	0	0	0	0	0	1	1	0	0	0
11	0	0	0	0	0	0	1	0	1	0	0	0
13	0	0	0	0	**1**	0	**1**	**1**	0	0	**1**	0
16	0	0	0	1	0	1	0	1	0	1	0	**1**
17	0	**1**	0	1	**2**	0	**3**	0	0	0	0	0
19	0	0	2	2	1	0	2	1	1	0	1	0
23	0	2	2	4	3	0	5	0	0	0	0	0
25	0	4	3	3	2	0	3	2	0	0	1	0

q	0	1	2	3	4	5	6	9	10	13	18	45
27	0	2	2	2	2	0	2	1	0	0	0	0
29	1	6	7	11	3	0	5	0	1	0	0	0
31	1	10	9	11	3	0	5	2	1	0	1	0
32	1	3	0	4	0	2	0	0	0	1	0	0
37	4	25	14	18	5	0	7	3	0	0	1	0
41	9	37	19	28	5	0	8	0	1	0	0	0
43	11	48	21	27	6	0	9	3	0	0	1	0
47	20	67	26	38	7	0	11	0	0	0	0	0
49	16	46	19	25	4	0	6	3	1	0	1	0
53	40	110	36	52	8	0	12	0	0	0	0	0
59	72	166	48	68	8	0	13	0	1	0	0	0

q	0	1	2	3	4	5	6	9	10	13	18	45
61	85	193	53	69	8	0	12	5	1	0	1	0
64	20	51	0	17	0	7	0	2	0	3	0	1
67	139	275	65	85	10	0	15	5	0	0	1	0
71	189	335	75	105	10	0	16	0	1	0	0	0
73	216	378	80	105	11	0	16	6	0	0	1	0
79	321	500	97	127	11	0	17	6	1	0	1	0
81	100	149	30	38	4	0	6	3	1	0	0	0
83	411	592	107	149	13	0	20	0	0	0	0	0
89	577	759	127	176	13	0	20	0	1	0	0	0
97	868	1033	154	203	15	0	22	8	0	0	1	0

Results:

The distribution of automorphism groups of these surfaces with respect to the number of Eckardt points is:


Semi linear Automorphism Group Order
Number of Eckardt Points
1000
100
10
1
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20

A Family of Cubic Surfaces

Next, we will describe a family of cubic surfaces with $\#E = 6$ invariant under a group isomorphic to Sym_4.

A Family of Cubic Surfaces

Consider the 4×4 matrices

$$S_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, S_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, S_3 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Let s_i be the projective transformation induced by S_i.

A Family of Cubic Surfaces

Lemma

The subgroup G of $\mathrm{PGL}(4, q)$ generated by s_1, s_2, s_3 is isomorphic to Sym_4 if q is odd and isomorphic to Sym_3 if q is even.

Proof:
Verify the relations of Sym_4 projectively:

$$s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^3 = (s_2 s_3)^3 = (s_1 s_3)^2 = 1.$$

Note 1: For q even, $S_3 = S_1$ and hence $G \simeq \mathrm{Sym}_3$.

Note 2: One of the relations only holds modulo scalars, so this is a projective representation of Sym_4.

A Family of Cubic Surfaces

Let q be odd.

For $a, b \in \mathbb{F}_q \setminus \{0\}$, $a^4 \neq 1$, consider the line

$$\ell_{a,b} = \begin{bmatrix} 1 & a & 0 & 0 \\ 0 & 0 & 1 & b \end{bmatrix}.$$

Let G be the group from above.

Let $\mathcal{O}_{a,b}$ be the orbit under G of the line $\ell_{a,b}$.

A Family of Cubic Surfaces

Lemma

$\mathcal{O}_{a,b}$ is a double six.

Let

$$\mathcal{S}_{a,b}$$

be the surface defined by $\mathcal{O}_{a,b}$.

A Family of Cubic Surfaces

Lemma

The equation of $\mathcal{S}_{a,b}$ is

$$x_3^3 - b^2(x_0^2 + x_1^2 + x_2^2)x_3 + \frac{b^3(a^2+1)}{a}x_0x_1x_2 = 0$$

A Family of Cubic Surfaces

Lemma

The surface $\mathcal{S}_{a,b}$ in $\mathrm{PG}(3, q)$ is invariant under $G \ltimes \mathrm{Aut}(\mathbb{F}_q)$.

If $q = p^h$ for some prime p, this group has order $24 \cdot h$.

The group G has three orbits on lines, with orbit structure $27 = 3 + 12 + 12$.

A Family of Cubic Surfaces

Let α be a primitive element of $\mathbb{F}_q$. Let

$$n_\alpha = \mathrm{diag}(1, 1, 1, \alpha).$$

The group $N := \langle G, n_\alpha \rangle$ normalizes G and hence N/G acts regularly on the set of G-orbits

$$\{\mathcal{S}_{a,b} \mid b \neq 0\}.$$

Hence

$$\mathcal{S}_{a,b} \simeq \mathcal{S}_{a,1} \quad \text{for all } b \neq 0.$$

Hence we may restrict ourselves to the surfaces

$$\mathcal{S}_a := \mathcal{S}_{a,1}.$$

A Family of Cubic Surfaces

Lemma

The surface $\mathcal{S}_a$ has at least 6 Eckardt points. Six lie in pairs on the three lines of the line-orbit of size 3.

If $\sqrt{5} \in \mathbb{F}_q$ and $a = -2 \pm \sqrt{5}$, then $\mathcal{S}_a$ has at least 10 Eckardt points.

If $a = \pm\sqrt{-3} \in \mathbb{F}_q$ or $a = \pm\sqrt{-\frac{1}{3}} \in \mathbb{F}_q$, then $\mathcal{S}_a$ has at least 18 Eckardt points.

In summary...

A Family of Cubic Surfaces

Theorem

For q odd, and for $a \in \mathbb{F}_q \setminus \{0\}$, and $a^4 \neq 1$, the surface $\mathcal{S}_a$ in $\mathrm{PG}(3, q)$, given by

$$x_3^3 - (x_0^2 + x_1^2 + x_2^2)x_3 + \frac{a^2+1}{a}x_0x_1x_2 = 0$$

has at least 6 Eckardt points and is invariant under $\mathrm{Sym}_4 \ltimes \mathrm{Aut}(\mathbb{F}_q)$.

If $\sqrt{5} \in \mathbb{F}_q$ and $a = -2 \pm \sqrt{5}$ then $\mathcal{S}_a$ has at least 10 Eckardt points.

If $a = \pm\sqrt{-3} \in \mathbb{F}_q$ or $a = \pm\sqrt{-\frac{1}{3}} \in \mathbb{F}_q$, then $\mathcal{S}_a$ has at least 18 Eckardt points.

To Do List

(a) Verify the classification using the blow-up approach (Karaoglu).

(b) Find more interesting families of cubic surfaces.

(c) Solve the isomorphism problem for the members of the family: For which $a, b \in \mathbb{F}_q$ is

$$\mathcal{S}_a \simeq \mathcal{S}_b.$$

(d) Classify related things like quartic curves with 28 bitangents.

Thank you for your attention!