Minimal multiple blocking sets

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Blocking Sets

A subset B of points in a projective plane π_n of order n s.t. for all lines ℓ we have $|\ell \cap B| \geq 1$. It is *minimal* iff $\forall X \in B$, $\exists \ell_X$ s.t. $\ell_X \cap B = \{X\}$.

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Trivially, a line is a blocking set of size $n+1$. A vertex-less triangle forms a blocking set of size $3(n-1)$.

Possible Sizes

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Theorem (Bruen 1970, Bruen and Thas 1977)

A non-trivial minimal blocking set B in π_n satisfies

$$n + \sqrt{n} + 1 \leq |B| \leq n\sqrt{n} + 1.$$

Baer subplanes and Hermitian curves prove sharpness for $n = p^{2k}$.

A. Blokhuis, **P. Sziklai** and **T. Szőnyi**. Blocking sets in projective spaces. In *Current Research Topics in Galois Geometry*, 2011.

Our results

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Main Result: *A generalization of the Bruen-Thas upper bound to minimal t-fold blocking sets.*

Spectral graph theory

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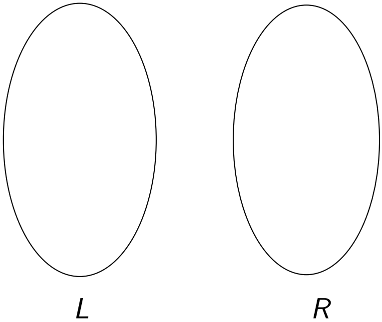
Let λ be the second largest eigenvalue in absolute terms.

Expander Mixing Lemma



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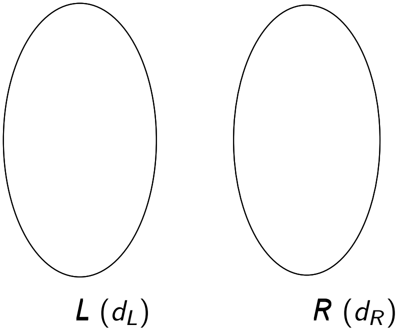

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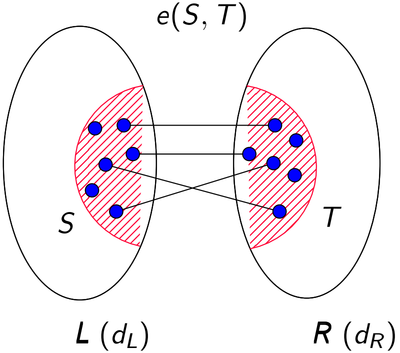

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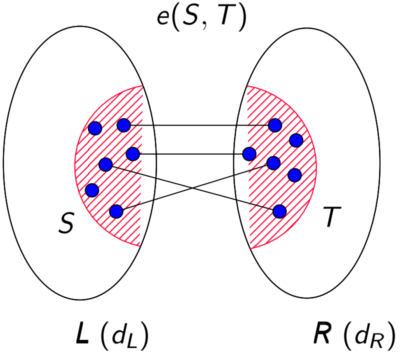

$$\left| e(S, T) - \frac{d_L |S| |T|}{|R|} \right| \leq$$

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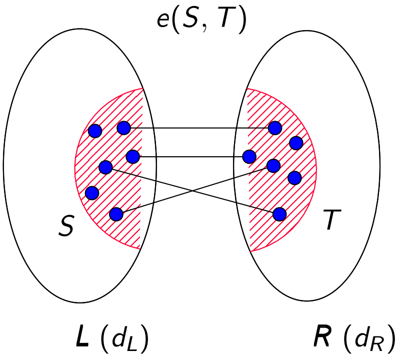

$$\left| e(S,T) - \frac{d_L |S||T|}{|R|} \right| \leq$$

$$\lambda \sqrt{|S||T| \left(1 - \frac{|S|}{|L|}\right) \left(1 - \frac{|T|}{|R|}\right)}$$

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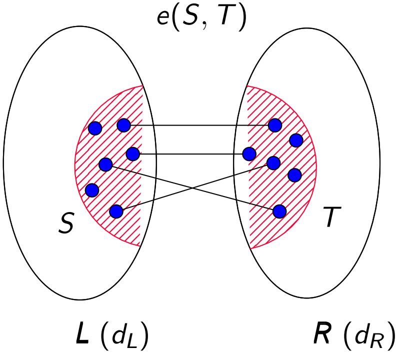

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$$\leq \lambda \sqrt{|S||T|}$$

The proof

For each point X of the blocking set S pick a line ℓ_X such that $|\ell_X \cap S| = 1$. This gives us a set T of lines such that $|T| = |S|$ and $e(S, T) = |S|$.

[1] https://www.win.tue.nl/~aeb/graphs/cages/cages.html

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The eigenvalues of π_n are $n + 1 \geq \sqrt{n} \geq -\sqrt{n} \geq -n - 1$. [1]

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Plug it in

$$\left| e(S, T) - \frac{d_L |S||T|}{|R|} \right| \leq \lambda \sqrt{|S||T| \left(1 - \frac{|S|}{|L|}\right) \left(1 - \frac{|T|}{|R|}\right)}.$$

and get

$$|S| \leq n\sqrt{n} + 1.$$

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Minimal multiple blocking sets

Theorem

Let B be a minimal t-fold blocking set in π_n. Then

$$|B| \leq \frac{1}{2}n\sqrt{4tn-(3t+1)(t-1)}+\frac{1}{2}(t-1)n+t \ = \Theta(\sqrt{t}n^{3/2}).$$

Case of Equality

This bound is sharp for:

1. $t = 1$ and $n =$ an even power of a prime. (Unitals)
2. $t = n$ and n arbitrary. (Full plane minus a point)
3. $t = n - \sqrt{n}$ and $n =$ an even power of prime. (Complement of a Baer subplane)

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Theorem

If equality occurs and n is a prime power, then B is one of the three types.

A construction

There exists such a set of size $q\sqrt{q}+1+(t-1)(q-\sqrt{q}+1)$ in $\mathrm{PG}(2,q)$ for every square q and $t \leq \sqrt{q}+1$.

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Take $t-1$ secant lines $\ell_1,\ldots,\ell_{t-1}$ through a point of a unital $\mathcal{U}$ and let

$$B = \mathcal{U} \cup \ell_1 \cup \cdots \cup \ell_{t-1} \cup \{\ell_1^{\perp} \cup \cdots \cup \ell_{t-1}^{\perp}\}.$$

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Remark: for $t = 2$ we can do better (Pavese)

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- Symmetric 2-(v, k, λ) designs
- Point-hyperplane designs in $\mathrm{PG}(k, q)$ (recovers a result of Bruen and Thas)
- Semiarcs (recovers a result of Csajbók and Kiss)
- Any set of points P which "determines" a set of lines L with $|L| = f(|P|)$ such that $e(P, L)$ can be computed in terms of $|P|$

Open Problems

1. Find better constructions.
2. Improve the upper bound when n is not a square.
3. Study multiple blocking sets with respect to hyperplanes in $\mathrm{PG}(k, q)$.
4. How large can a minimal blocking set with respect to lines in $\mathrm{PG}(3, q)$ be?

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