The Final Four

Jim Davis

Irsee conference

September 2014

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The Final Result

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Statement of Problem

Which groups of order 256 contain a (256, 120, 56) difference set?

History lesson #1

- Groups of order 16
- 1960s Kibler does computer search.
- 12 of 14 have difference sets

History lesson #2

- Groups of order 64
- 1990s Dillon does computer search.
- 259 of 267 have difference sets

Modular group

- Original proof: didn't exist
- Ken Smith: construction!

Statement of Problem

Which groups of order 256 contain a (256, 120, 56) difference set?

Issue: there are 56,092 nonisomorphic groups!

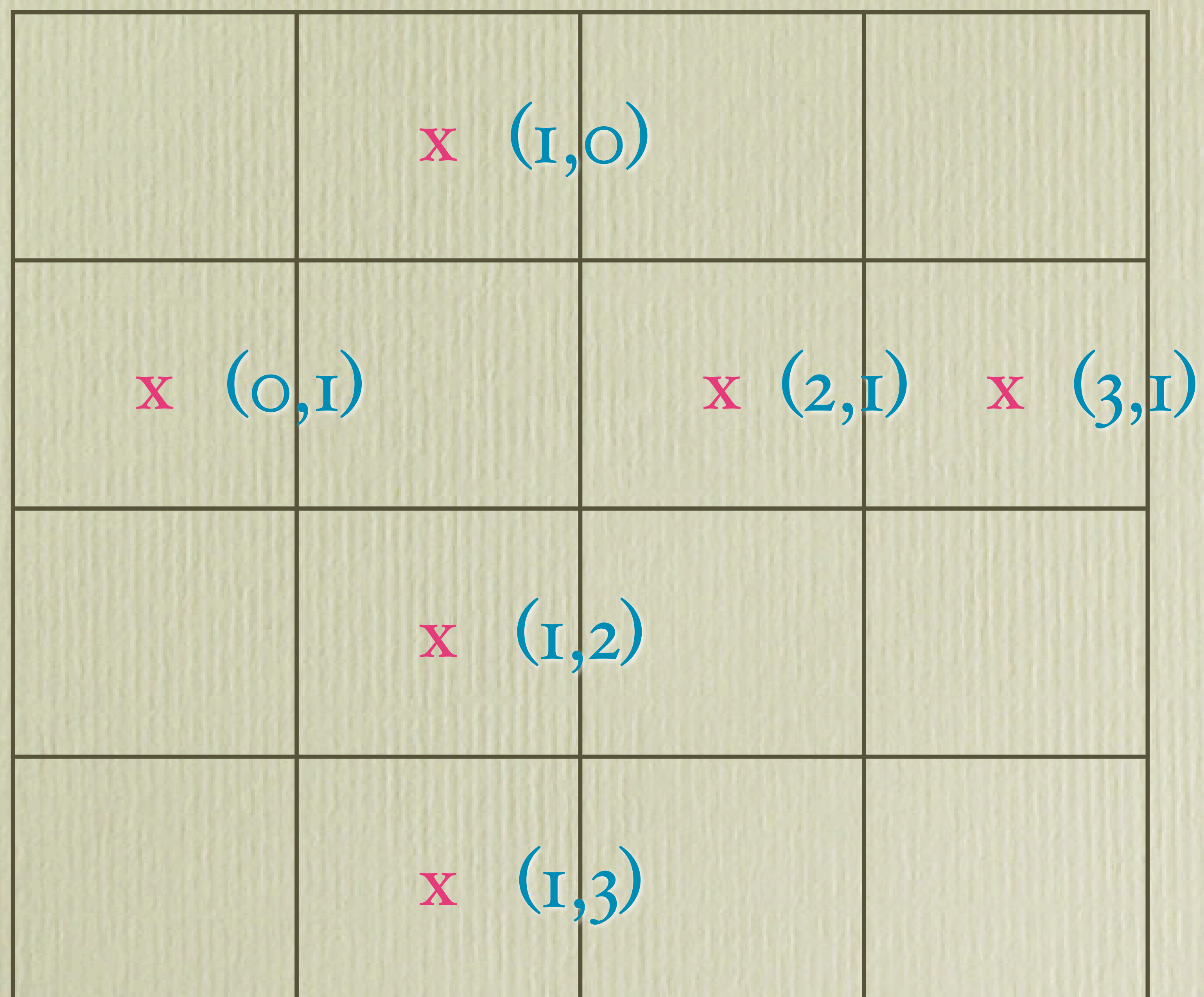
Outline of talk

- Motivation
- Easy examples
- Nonexistence
- Constructions
- Computer searches
- Final thoughts

Key example

	x		
x		x	x
	x		
	x		

Key example



Key example

	X		
X		X	X
	X		
	X		

Similar group

X	X	X		X			
		X			X		

Move it around!

	X	X	X		X		
			X			X	

Independently move pieces

X	X	X		X			
		X			X		

Independently move pieces

X		X		X			X
		X	X				

Independently move pieces

X				X		X	X
			X			X	

Independently move pieces

						X	X
X			X	X		X	

Works here too

	X		
X		X	X
	X		
	X		

Works here too

	X	X	
X		X	
X	X		

Works here too

	X	X	X
X			
X			
X			

Nonexistence

G/H large and cyclic $\Rightarrow$ no (256,120,56) DS

Nonexistence

G/H large and cyclic ⇒ no (256,120,56) DS

G/H large and dihedral ⇒ no (256,120,56) DS

Nonexistence

That is it!! Rules out 43 groups.

G/H large and cyclic $\Rightarrow$ no (256,120,56) DS

G/H large and dihedral $\Rightarrow$ no (256,120,56) DS

Nonexistence

That is it!! Rules out 43 groups.

56,092-43 = 56,049

G/H large and cyclic $\Rightarrow$ no (256,120,56) DS

G/H large and dihedral $\Rightarrow$ no (256,120,56) DS

Dillon-McFarland (Drisko)

If $H < G$ is normal, $H = (Z_2)^4$, then G has a DS

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~42,300 groups have such a normal subgroup

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If $H < G$ is normal, $H=(Z_2)^4$, then G has a DS

~42,300 groups have such a normal subgroup

~56000-42300 = ~13700

Product constructions

G, H have DSs $\Rightarrow$ GxH has DS

Product constructions

G, H have DSs → GxH has DS

Handles ~9500 of remaining groups

Product constructions

G, H have DSs ⇒ GxH has DS

Handles ~9500 of remaining groups

~13700-9500 = ~4200

$[G:H] = 4$

~3500 groups

795 groups remaining!

(Down to 714 a little later)

$Z_4 \times Z_4 \times Z_2$

- 649 of the remaining groups had a normal subgroup
- (16,8,8,-) covering EBSs

Why does this work?

- $B_iB_j^{-1} = cG$ for $i \neq j$
- $g_iB_iB_i^{(-1)}g_i^{-1}$ nice?

Modification of other methods

- K-Matrices
- Representation Theory

Final Four

- SmallGroup(256,408)
- SmallGroup(256,501)
- SmallGroup(256,536)
- SmallGroup(256,6700)

Final Four

- SmallGroup(256,408) $\longrightarrow$ $a^2 = b^{32} = c^2 = d^2 = 1,$ $cbc^{-1}=ba,\ dbd^{-1}=b^{23}c,$ $dcd^{-1}=b^{16}ac$
- SmallGroup(256,501)
- SmallGroup(256,536)
- SmallGroup(256,6700)

Final Four

- SmallGroup(256,408)
- SmallGroup(256,501) → $b^{64} = a^2 = c^2 = 1$, $aba^{-1}=b^{33}$, $cbc^{-1} = ba$
- SmallGroup(256,536)
- SmallGroup(256,6700)

Final Four

- SmallGroup(256,408)
- SmallGroup(256,501)
- SmallGroup(256,536) → $b^{64}=a^4=1$, $aba^{-1}=b^{-17}$
- SmallGroup(256,6700)

Final Four

- SmallGroup(256,408)
- SmallGroup(256,501)
- SmallGroup(256,536)
- SmallGroup(256,6700) ⟶ $b^{32}=a^{4}=c^{2}=1$,
  $aba^{-1}=b^{-17}$, $cbc^{-1}=b^{17}a^{2}$

Where now?

Conjecture???: the large cyclic and large dihedral quotient nonexistence criteria are necessary and sufficient for a difference set in a 2-group to exist.

Related work

- Bent functions!
- Relative difference sets in nonabelian groups.
- # of distinct difference sets in a given group
- Inequivalent designs