(SMALL) BLOCKING SETS OF NON-DESARGUESIAN PROJECTIVE AND AFFINE PLANES

Finite Geometries Fifth Irsee Conference

September 14, 2017

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This is joint work with

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INTRODUCTION

Does the 1 *mod* p *results holds for small blocking sets in non-Desarguesian projective planes?*

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Let $\mathcal{S}$ be a projective plane or affine plane of order q.

Definition

A *blocking set* is a set $\mathcal{B}$ of Π_q such that every line of Π_q meets $\mathcal{B}$ in at least one point.

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Definition

A blocking set is called *minimal* if no smaller blocking set is contained in it.

SMALLEST MINIMAL BLOCKING SETS

Theorem (Bruen 1970,1971)

Let $\mathcal{B}$ be a minimal blocking set of Π_q. Then $|\mathcal{B}| \geq q + \sqrt{q} + 1$. In case of equality $\mathcal{B}$ is a Bear subplane.

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Theorem (Jamison 1977, Brouwer-Schrijver 1978)

Let $\mathcal{B}$ be a minimal blocking set of $\mathrm{AG}(2, q)$. Then $|\mathcal{B}| \geq 2q - 1$

SMALL MINIMAL BLOCKING SETS

Theorem (Blokhuis 1994)

Let q be prime. Then a minimal blocking set of $\mathrm{PG}(2, q)$ has size at least $\frac{3}{2}(p + 1)$.

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A blocking set $\mathcal{B}$ of $\mathrm{PG}(2,q)$ is *small* if $|\mathcal{B}| < \frac{3}{2}(q+1)$

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Theorem (Blokhuis 1994, Szőnyi 1997, Sziklai 2008)

Let $\mathcal{B}$ be a small blocking set of $\mathrm{PG}(2,q)$, $q = p^h$. Then every line of $\mathrm{PG}(2,q)$ meets $\mathcal{B}$ in $1 \pmod{p}$ points. If a line meets $\mathcal{B}$ in $p+1$ points, then it meets $\mathcal{B}$ in a subline.

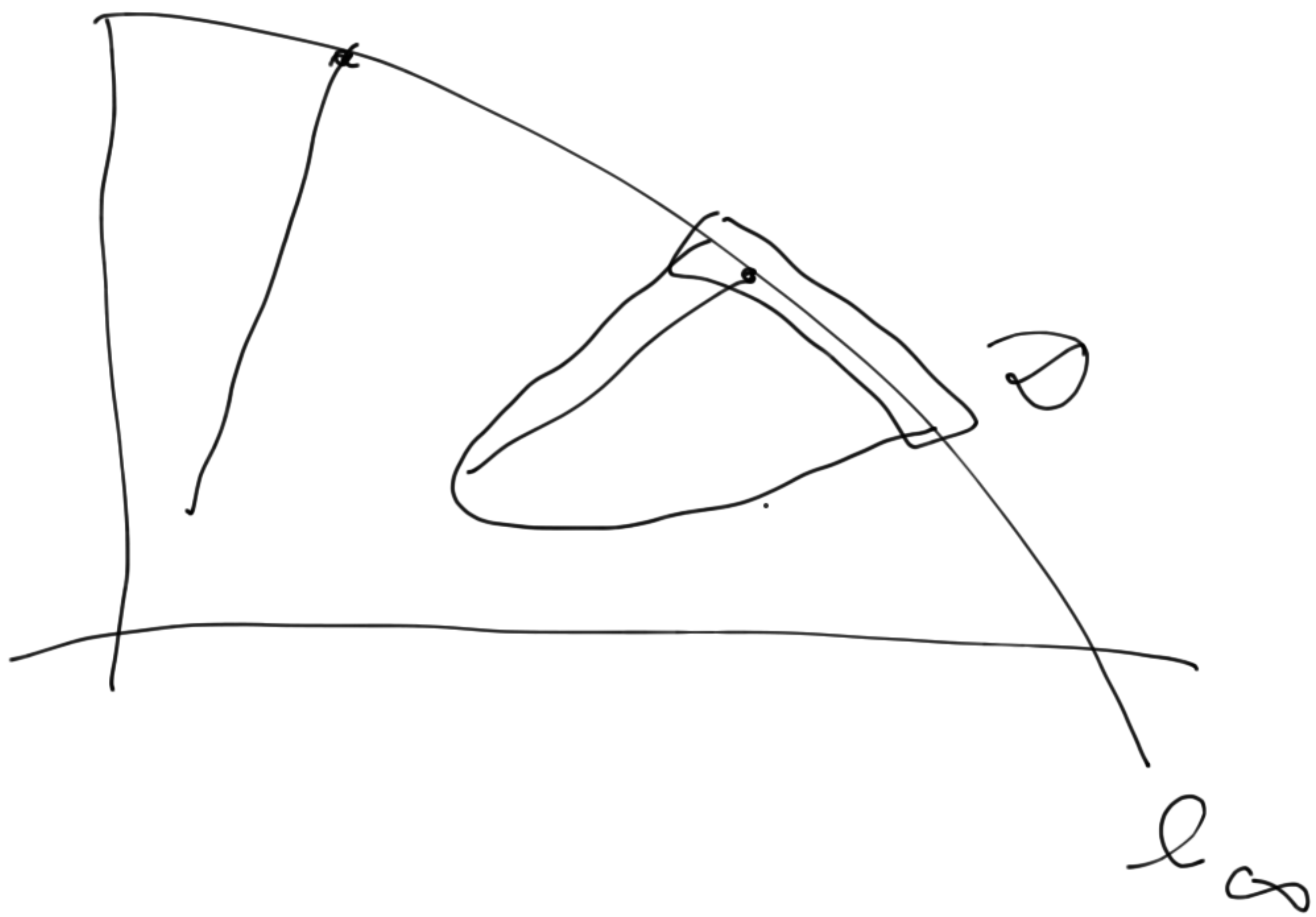
ANDRÉ-BRUCK-BOSE CONSTRUCTION

- Consider a regular spread $\mathcal{R}$ in $\mathrm{PG}(3, q)$. Replace one regulus of $\mathcal{R}$ by its opposite regulus, call the new spread $\mathcal{R}'$.
- Embed $\mathrm{PG}(3, q)$ as hyperplane π_∞ into $\mathrm{PG}(4, q)$. Consider the lines of $\mathcal{R}'$ in π_∞. Define:
  1. **points** as the points of $\mathrm{PG}(4, q) \setminus \pi_\infty$
  2. **lines** as the planes of $\mathrm{PG}(4, q) \setminus \pi_\infty$ meeting π_∞ in a line of $\mathcal{R}'$.
  3. inherited incidence of $\mathrm{PG}(4, q)$.

  This point line geometry is the *affine Hall plane of order q^2*. Its points at infinity can be represented by the lines of $\mathcal{R}'$.

BY DERIVATION

- Consider the Desarguesian projective plane $\mathrm{PG}(2, q^2)$ and a line at infinity l_∞. Choose a subline $\mathrm{PG}(1, q)$ of l_∞, with point set $\mathcal{D}$.



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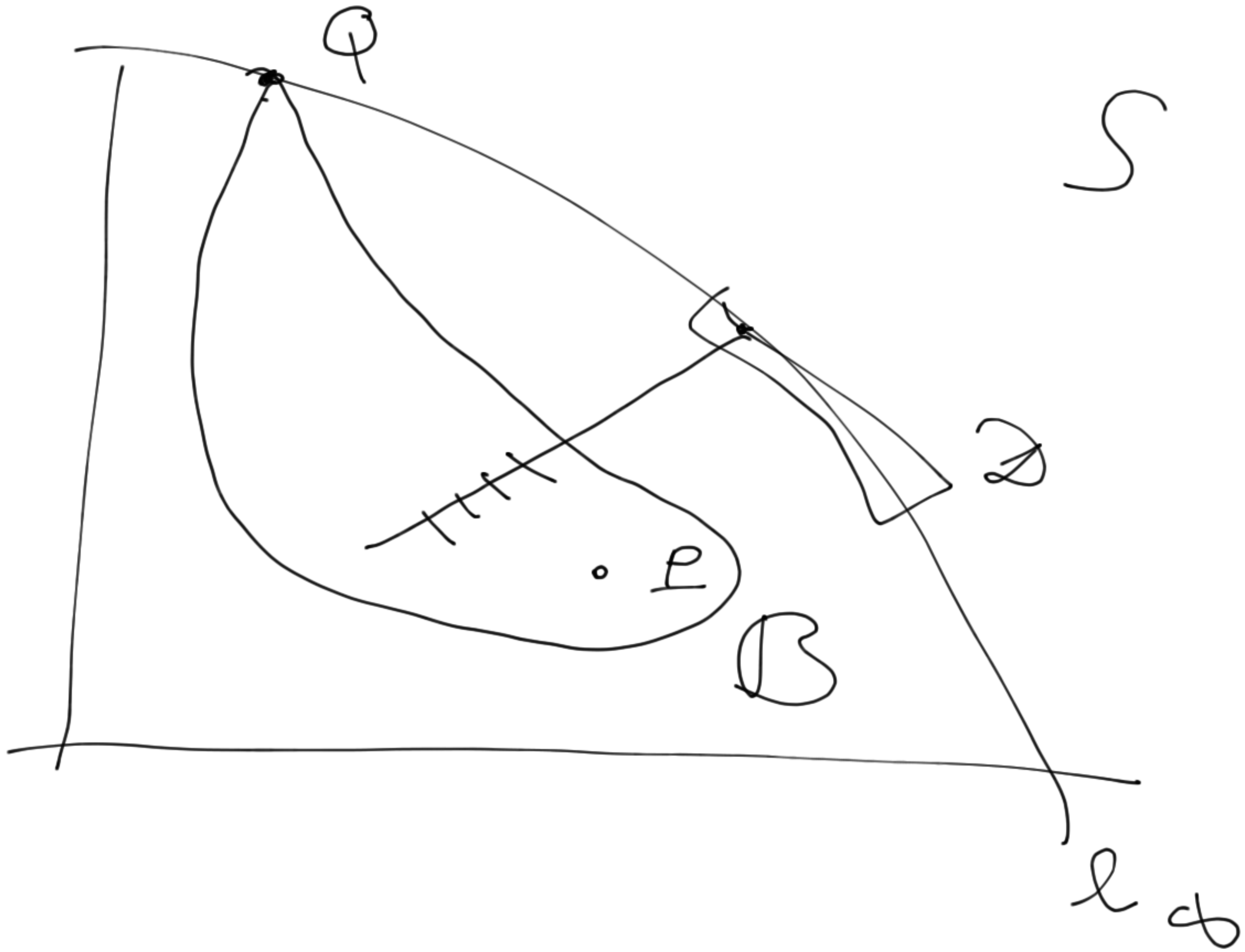
- Consider the Desarguesian projective plane $\mathrm{PG}(2, q^2)$ and a line at infinity l_∞. Choose a subline $\mathrm{PG}(1, q)$ of l_∞, with point set $\mathcal{D}$.
- Define:
  1. **points** as the points of $\mathrm{PG}(2, q^2) \setminus l_\infty$
  2. **lines** of type one as lines of $\mathrm{PG}(2, q^2)$ not meeting l_∞ in a point of $\mathcal{D}$.
  3. **lines** of type two as Baer subplanes of $\mathrm{PG}(2, q^2)$ meeting l_∞ in the points of $\mathcal{D}$.

This point line geometry is the affine Hall plane of order q^2. The points at infinity of the lines of type one can be represented by the points of $_\infty \setminus \mathcal{D}$. Denote by $\mathcal{D}'$ the points at infinity of the lines of type two.

MAIN IDEA TO CONSTRUCT A BLOCKING SET



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Lemma

The set $\{l \cap \mathcal{B} | l \in \mathcal{S}\}$ *is a dual oval of* $\mathcal{B}$.

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Call $\mathcal{O}$ the set of points of $\mathcal{B}$ that are contained in exactly one line of $\mathcal{S}$. Call $\mathcal{O}^+$ the set of points of $\mathcal{B}$ covered by exactly two lines of $\mathcal{S}$ and call $\mathcal{O}^-$ the set of points of $\mathcal{B} \setminus \{Q\}$ not covered by any line of $\mathcal{S}$.

MAIN IDEA TO CONSTRUCT A BLOCKING SET

Lemma

1. $|\mathcal{O}| = q + 1$, $|\mathcal{O}^+| = \frac{q(q+1)}{2}$, $|\mathcal{O}^-| = \frac{(q+1)(q-2)}{2}$. *If q is odd, then $\mathcal{O}$ is an oval of $\mathcal{B}$; if q is even, then $\mathcal{O}$ is a line of $\mathcal{B}$.*

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2. *If $P \in \mathcal{O}^-$, then each Baer subplane of $[P]$ meets $\mathcal{B}$ only in P.*

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2. *If $P \in \mathcal{O}^-$, then each Baer subplane of $[P]$ meets $\mathcal{B}$ only in P.*
3. *If $P \in \mathcal{O}$, then q Baer subplanes of $[P]$ meet $\mathcal{B}$ in two points, one of which is P and the other is contained in $\mathcal{O}^+$, and one Baer subplane of $[P]$ meets $\mathcal{B}$ only in P.*
4. *If $P \in \mathcal{O}^+$, then $q - 1$ Baer subplanes of $[P]$ meet $\mathcal{B}$ in three points of $\mathcal{O}^+$ (including P), and two Baer subplanes of $[P]$ meet $\mathcal{B}$ in two points, one of which is P and the other is contained in $\mathcal{O}$.*

MAIN IDEA TO CONSTRUCT A BLOCKING SET

Lemma

- *If* $q \not\equiv 2 \pmod 3$, *then* $\forall P \in \mathcal{D}'$, $t_0(P) = \frac{q^2-q}{3}$.
- *If* $q \equiv 2 \pmod 3$, *then for* $\frac{q+1}{3}$ *points* $P \in \mathcal{D}'$, $t_0(P) = \frac{q^2-q-2}{3}$, *and for* $\frac{2(q+1)}{3}$ *points* $P \in \mathcal{D}'$, $t_0(P) = \frac{q^2-q+1}{3}$.

BLOCKING SETS OF THE PROJECTIVE HALL PLANE

Theorem (DB, Héger, Szőnyi, Van de Voorde)

In the projective Hall plane of order q^2, $q > 2$, *there exists a minimal blocking set of size* $q^2 + 2q + 2$, *which admits* $1-$, $2-$, $3-$, $4-$, $(q+1)-$ *and* $(q+2)$*-secants.*

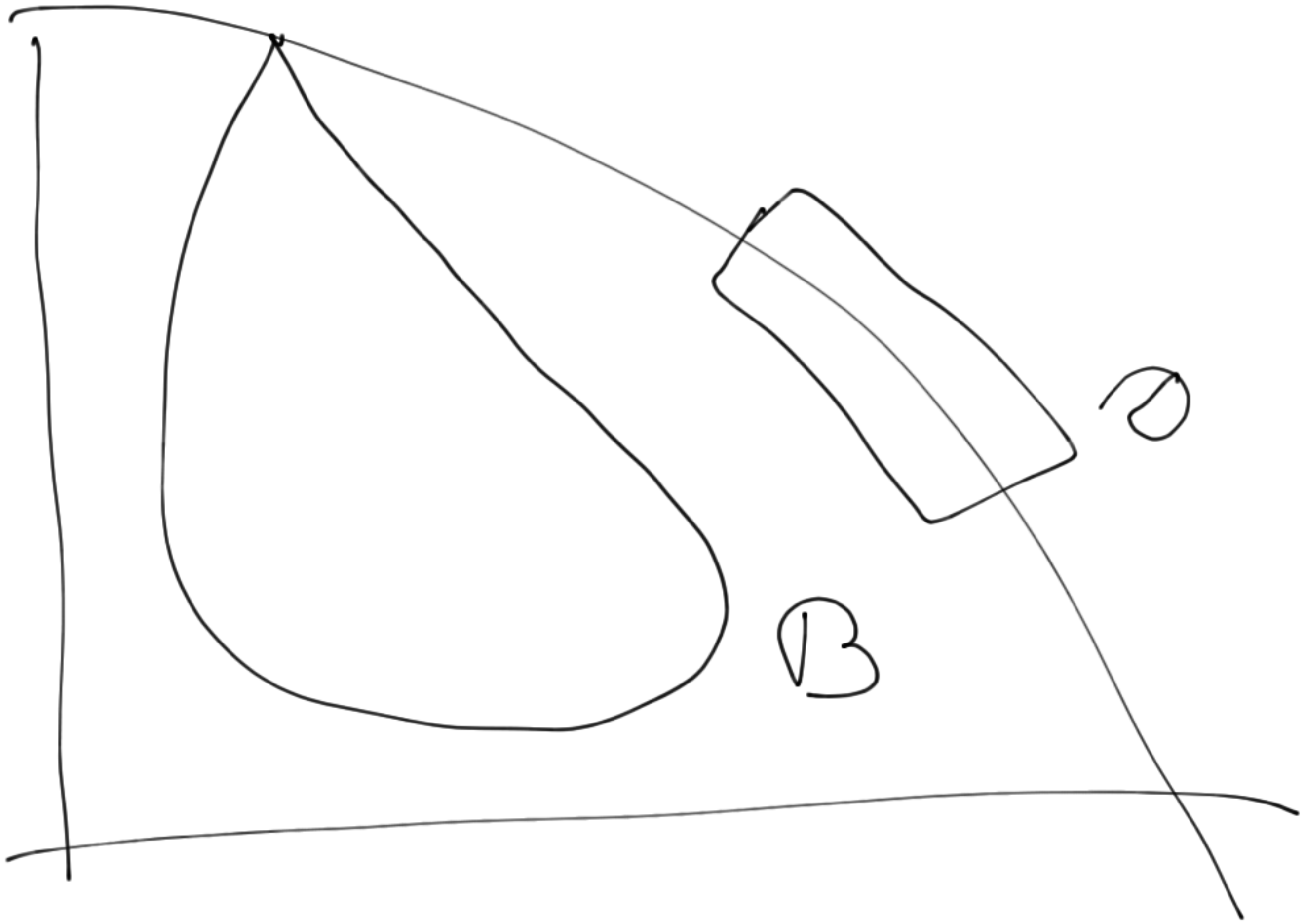
SMALL BLOCKING SETS ...

Theorem

Let q be a prime power. There exists a non-Desarguesian affine plane of order q^2 in which there is a blocking set of size at most $\frac{4q^2}{3} + \frac{5q}{3}$

A blocking set in the Hall plane of order q^2

ANDRÉ PLANES



$$PG(2, p^h)$$

$$\frac{p^h - 1}{p - 1}$$

ANDRÉ PLANES

Theorem (DB, Héger, Szőnyi, Van de Voorde)

If $\mathcal{B}$ is a blocking set in $\mathrm{PG}(2, p^h)$, $p > 5$ prime, of size at most $\frac{3}{2}(p^h - p^{h-1})$, then there exists a double blocking set in $\mathrm{PG}(2, p^h)$ of size $|\mathcal{B}| + p^h + p^{h-1} + 1$. In particular, if $p > 5$, then there exist double blocking sets in $\mathrm{PG}(2, p^h)$ of size $2p^h + 2p^{h-1} + 2$.

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Theorem (DB, Héger, Szőnyi, Van de Voorde)

Consider the André plane $\Pi_{\mathcal{D}}$ of order p^h derived from $\mathrm{PG}(2, p^h)$, p prime, $h \geq 2$. Then there exists a non-trivial minimal blocking set in $\Pi_{\mathcal{D}}$ of size at least $p^h + p^{h-1} + 2$ and at most $p^h + p^{h-1} + \frac{p^h - 1}{p-1} + 1$.

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Computer experiments confirm that blocking sets of non-Desarguesian affine planes of order q^h of size smaller than

$$\left\lfloor \frac{4}{3}q^h + \frac{5}{3}\sqrt{q^h} \right\rfloor$$

exist.