On the Mathon bound for regular near hexagons

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Near $2d$-gons

A near $2d$-gon is a point-line geometry satisfying:

- Every two distinct points are incident with at most one line.
- Diameter collinearity graph = d
- For every point x and every line L, there is a unique point on L nearest to x.

A near polygon has order (s, t) if

- Every line has $s + 1$ points.
- Every point is contained in $t + 1$ lines.

Generalized $2d$-gons

A generalized $2d$-gon is a near $2d$-gon satisfying:

- every point is incident with at least two lines;
- for every two distinct nonopposite points x and y, we have $|\Gamma_1(y) \cap \Gamma_{i-1}(x)| = 1$ where $i = \mathrm{d}(x, y)$.

Consider a finite generalized $2d$-gon of order (s, t) with $s > 1$.

Higman inequality $t \leq s^2$ for generalized quadrangles/octagons

Haemers-Roos inequality $t \leq s^3$ for generalized hexagons.

In case of equality: Extremal generalized polygon

Generalization to regular near hexagons

A finite near hexagon is called regular with parameters (s, t, t_2) if it has order (s, t) and if every two points at distance 2 have precisely $t_2 + 1$ common neighbours. The collinearity graph is then a distance-regular graph.

The regular near hexagons with parameters $(s, t, t_2) = (s, t, 0)$ are precisely the finite generalized hexagons of order (s, t).

Mathon bound: if $s \neq 1$, then $t \leq s^3 + t_2(s^2 - s + 1)$.

In case of equality: Extremal regular near hexagon

Examples: $GH(s, s^3)$, $DH(2n-1, q^2)$, M_{12} and M_{24} near hexagons

Generalization to near hexagons with order (s, t)

Theorem (BDB)

Let $\mathcal{S}$ be a finite near hexagon with order (s,t), $s \neq 1$, and suppose x and y are two opposite points of $\mathcal{S}$. Then

$$t \leq s^3 + (\frac{G}{t+1} - 1)(s^2 - s + 1),$$

where G is the number of geodesics connecting x and y.

Theorem (BDB)

Let $\mathcal{S}$ be a finite near hexagon with order (s,t), $s \neq 1$. Then

$$t \leq s^4 + s^2,$$

wiith equality if and only if $\mathcal{S}$ is a Hermitian dual polar space.

Interesting problem:

Determine necessary and sufficient combinatorial conditions that would imply that a generalized quadrangle/hexagon/octagon of order (s, t) with $s \neq 1$ is extremal.

Solved for $d = 2$ (Bose-Shrikhande 1972) and $d = 4$ (Neumaier 1990)

Haemers (1979) already finds combinatorial conditions satisfied by extremal hexagons, but the problem whether these (or any other) conditions are sufficient remained open.

BDB (2016): Solution for generalized hexagons, as a special case of a more general result on regular near hexagons.

The case of regular near hexagons: I

Suppose $\mathcal{S}$ is a regular near hexagon with parameters (s, t, t_2). Let x and y be two opposite points of $\mathcal{S}$. Put

$$Z := \Big(\Gamma_2(x) \cap \Gamma_3(y)\Big) \cup \Big(\Gamma_3(x) \cap \Gamma_2(y)\Big) \cup \Big(\Gamma_3(x) \cap \Gamma_3(y)\Big).$$

- For every $z \in \Gamma_2(x) \cap \Gamma_3(y)$, put $N_z = N(x, y, z)$ equal to

  $$s \cdot |\Gamma_1(x) \cap \Gamma_2(y) \cap \Gamma_1(z)| + |\Gamma_2(x) \cap \Gamma_1(y) \cap \Gamma_2(z)| - (s+1)(s+t_2) - 1.$$

- For every $z \in \Gamma_3(x) \cap \Gamma_2(y)$, put $N_z = N(x, y, z) := N(y, x, z)$.
- For every $z \in \Gamma_3(x) \cap \Gamma_3(y)$, put N_z equal to

  $$|\Gamma_1(x) \cap \Gamma_2(y) \cap \Gamma_2(z)| - |\Gamma_2(x) \cap \Gamma_1(y) \cap \Gamma_2(z)|.$$

The case of regular near hexagons: II

Theorem (BDB)

We have

$$\sum_{z \in Z} N_z^2 = 2 \cdot \Big(s^3 + t_2(s^2 - s + 1) - t \Big) \cdot \Omega,$$

where

$$\Omega := \Big(s^3 + t_2(s^2 - s + 1) - t \Big) \cdot \Big(\frac{t(t - t_2)}{t_2 + 1} - s \Big)$$

$$+ \Big(s^2 + st_2 - t_2 - 1 \Big) \cdot \Big(s^2 + st + \frac{t(t - t_2)}{t_2 + 1} \Big).$$

The case of regular near hexagons: III

Theorem (BDB)

The following are equivalent for a regular near hexagon with parameters (s, t, t_2):

- $t = s^3 + t_2(s^2 - s + 1)$;
- *all N_z's are equal to 0.*

The proof: I

Ideas already in two papers:

- [1] B. De Bruyn and F. Vanhove. Inequalities for regular near polygons, with applications to m-ovoids. *European J. Combin.* 34 (2013), 522–538.
- [2] B. De Bruyn and F. Vanhove. On Q-polynomial regular near $2d$-gons. *Combinatorica* 35 (2015), 181–208.

Extremal hexagon $\Rightarrow$ N_z's are 0

[1]: Case $z \in \Gamma_3(x) \cap \Gamma_3(y)$
[2]: Cases $z \in \Gamma_2(x) \cap \Gamma_3(y)$ and $z \in \Gamma_3(x) \cap \Gamma_2(y)$

The proof: II

$$A_x := \Gamma_1(x) \cap \Gamma_2(y), \qquad A_y := \Gamma_1(y) \cap \Gamma_2(x).$$

Let $p_1, p_2, \ldots, p_v$ be an ordering of the points. Put $M = (M_{ij})$, where

$$M_{ij} := (-\frac{1}{s})^{d(p_i,p_j)}, \qquad \forall i,j \in \{1,2,\ldots,v\}.$$

Then $M^2 = \alpha \cdot M$, with

$$\alpha = \frac{s+1}{s^3}(s^2 + st + \frac{t(t-t_2)}{t_2+1}).$$

χ_X denotes characteristic vector of set X of points.

$$\eta := s(s+t_2+1) \cdot (\chi_x - \chi_y) + \chi_{A_x} - \chi_{A_y}.$$

The proof: III

For every point z, put

$$U_z := \chi_z \cdot M \cdot \eta^T \in \mathbb{Q}.$$

Using $M^2 = \alpha M$, we find

$$\sum_{z \in \mathcal{P}} U_z^2 = \sum_{z \in \mathcal{P}} (\chi_z \cdot M \cdot \eta^T)^2 = \eta \cdot M \cdot M \cdot \eta^T = \alpha \cdot \eta \cdot M \cdot \eta^T$$

$$= \alpha \cdot \Big(s(s + t_2 + 1) \cdot (U_x - U_y) + \sum_{z \in A_x} U_z - \sum_{z \in A_y} U_z \Big).$$

The proof: IV

Seven possible cases for points z:

(1) $z = x$ or $z = y$;

(2) $z \in A_x$ or $z \in A_y$;

(3) $z \in \Gamma_1(x) \setminus A_x$ or $z \in \Gamma_1(y) \setminus A_y$;

(4) $z \in \Gamma_2(x) \cap \Gamma_2(y)$ is contained on a line joining a point of A_x with a point of A_y;

(5) $z \in \Gamma_2(x) \cap \Gamma_2(y)$ is not contained on a line joining a point of A_x with a point of A_y;

(6) $z \in \Gamma_2(x) \cap \Gamma_3(y)$ or $z \in \Gamma_3(x) \cap \Gamma_2(y)$;

(7) $z \in \Gamma_3(x) \cap \Gamma_3(y)$.

The proof: V

The corresponding values for U_z:

(1) $U_z = \pm\frac{s+1}{s^2} \cdot (s^3 + t_2(s^2 - s + 1) - t)$

(2) $U_z = \pm\frac{s+1}{s^3} \cdot (s^3 + t_2(s^2 - s + 1) - t)$

(3) $U_z = \pm\frac{s+1}{s^3} \cdot (s^3 + t_2(s^2 - s + 1) - t)$

(4) $U_z = 0$

(5) $U_z = 0$

(6) $U_z = \pm\frac{s+1}{s^3} \cdot N(z)$

(7) $U_z = \frac{s+1}{s^3} \cdot N(z)$

A second proof of the Mathon inequality (Vanhove – BDB, 2013)

As $M^2 = \alpha M$, the matrix M is positive-semidefinite. Hence,

$$(X_1\chi_x+X_2\chi_y+X_3\chi_{A_x}+X_4\chi_{A_y})\cdot M\cdot(X_1\chi_x+X_2\chi_y+X_3\chi_{A_x}+X_4\chi_{A_y})^T \geq 0.$$

We thus obtain a positive semidefinite quadratic form in the variables X_1, X_2, X_3 and X_4.

Sylvester's Criterion implies that $t \leq s^3 + t_2(s^2 - s + 1)$.

A third proof of the Mathon inequality (Haemers-Mathon)

Let $\mathcal{S}$ be a regular near hexagon with parameters (s, t, t_2), $s \geq 2$, having v points.

Γ: collinearity graph.

Γ_2: graph defined on the point set by the distance 2 relation.

A and A_2 are adjacency matrices of Γ and Γ_2.

$$C := A_2 - (s-1)A + (s^2 - s + 1)I_v.$$

A third proof of the Mathon inequality

Let L be line of $\mathcal{S}$. Let $\widetilde{C}$ be the square principle submatrix of C whose rows and columns correspond to the points of $\Gamma_1(L)$, the set of points at distance 1 from L.

Theorem (Haemers-Mathon, 1979)

- $rank(C) = 1 + s^3 \frac{(t_2+1)+st(t_2+1)+s^2t(t-t_2)}{(t_2+1)s^2+(t_2+1)st+t(t-t_2)}$.
- $rank(\widetilde{C}) = s + 1 + \frac{(s^2-1)st}{s+t_2}$

From $rank(\widetilde{C}) \leq rank(C)$, we deduce:

Theorem

We have $t \leq s^3 + t_2(s^2 - s + 1)$ with equality if and only if $rank(\widetilde{C}) = rank(C)$.

Another generalisation to near hexagons of order (s, t)

Theorem (BDB)

Suppose $\mathcal{S}$ is a finite near hexagon with order (s, t), $s \geq 2$, having v points. Let L be a line of $\mathcal{S}$ and let $Q_1, Q_2, \ldots, Q_k$ with $k \in \mathbb{N}$ denote all quads through L. Suppose Q_i with $i \in \{1, 2, \ldots, k\}$ has order $(s, t_2^{(i)})$. Then

$$\sum_{i=1}^{k} \frac{(t_2^{(i)})^2}{s + t_2^{(i)}} \geq t - \frac{s(s^2+1)v - s(s+1)(s^2+1) - s^2 t(s+1)}{(s+1)(s^4-1) + st(s-1)(s+1)^2 + v}.$$